Putnam Problems I

This list of Putnam problems from the last 20 years was selected from among the "easy" problems based on the number of contestants solving the problems.

1. Let $f(n)$ be the sum of the first n terms of the sequence $0,1,1,2,2,3,3,4,\ldots$, where the nth term is given by

$$a_n = \begin{cases} n/2 & \text{if } n \text{ is even.} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Show that if x and y are positive integers and $x > y$ then $xy = f(x+y) - f(x-y)$.

2. Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

3. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

4. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \le |\sin x|$ for all real x, prove that

$$|a_1 + 2a_2 + \ldots + na_n| \le 1.$$

5. Prove

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx.$$

6. Given integers a,b,e,c,d, and f with $ad \neq bc$, and given a real number $\epsilon > 0$, show that there exist rational numbers r and s for which

$$0 < |ra + sb - e| < \epsilon,$$
$$0 < |rc + sd - f| < \epsilon.$$

7. Prove that a list can be made of all the subsets of a finite set in such a way that (i) the empty set is first in the list, (ii) each subset occurs exactly once, (iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

8. The temperatures in Chicago and Detroit are x^o and y^o, respectively. These temperatures are not assumed to be independent; namely, we are given:

(i) $P(x^o = 70^o)$, the probability that the the temperature in Chicago is 70^o,

(ii) $P(y^o = 70^o)$, and

(iii) $P(\max(x^o,y^o) = 70^o)$.

Determine $P(\min(x^o,y^o)= 70^o)$.

9. A is a subset of a finite group G (with group operation called multiplication), and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A.

10. Let p be a prime number. Let J be the set of all 2x2 matrices $\binom{ab}{cd}$ whose entries are chosen from $\{0,1,2,\ldots,p-1\}$ and satisfy the conditions $a + d \equiv 1 \pmod p$, $ad - bc \equiv 0 \pmod p$. Determine how many members J has.

11. Let D_n be the determinant of order n of which the element in the ith row and the jth column is the absolute value of the difference of i and j. Show that D_n is equal to

$$(-1)^{n-1}(n-1)2^{n-2}.$$

12. Let n be a positive integer such that n + 1 is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24.

13. The time–varying temperature of a certain body is given by a polynomial in the time of degree at most three. Show that the average temperature of the body between 9 a.m. and 3 p.m. can always be found by taking the average of the temperatures at two fixed times, which are independent of which polynomial occurs. Also, show that these two times are 10:16 a.m. and 1:44 p.m. to the nearest minute.

14. Let S be a set and let o be a binary operation on S satisfying the two laws

$$x \circ x = x \text{ for all } x \text{ in } S, \text{ and}$$

$$(x \circ y) \circ z = (y \circ z) \circ x \text{ for all } x,y,z \text{ in } S.$$

Show that o is associative and commutative.

15. Let F(x) be a real valued function defined for all real x except for x = 0 and x = 1 and satisfying the functional equation

$F(x) + F\{(x - 1)/x\} = 1 + x$. Find all functions F(x) satisfying these conditions.

16. Let S be a set and let * be a binary operation on S satisfying the laws

$$x * (x * y) = y \text{ for all } x,y \text{ in } S.$$

$$(y * x) * x = y \text{ for all } x,y, \text{ in } S.$$

Show that * is commutative but not necessarily associative.

17. A particle moving on a straight line starts from rest and attains a velocity v_0 after traversing a distance s_0. If the motion is such that the acceleration was never increasing, find the maximum time for the traverse.

18. Supposing that an integer n is the sum of two triangular numbers,

$$n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2},$$

write $4n + 1$ as the sum of two squares, $4n + 1 = x^2 + y^2$, and show how x and y can be expressed in terms of a and b.

Show that, conversely, if $4n + 1 = x^2 + y^2$, then n is the sum of two triangular numbers. [Of course, a, b, x, y are understood to be integers.]

19. For which ordered pairs of <u>real</u> numbers b, c do both roots of the quadratic equation

$$z^2 + bz + c = 0$$

lie inside the unit disk $\{|z| < 1\}$ in the complex plane?

Draw a reasonably accurate picture (i.e., 'graph') of the region in the real bc-plane for which the above condition holds. Identify precisely the boundary curves of this region.

20. Does there exist a subset B of the unit circle $x^2 + y^2 = 1$ such that (i) B is topologically closed, and (ii) B contains exactly one point from each pair of diametrically opposite points on the circle?

[A set B is topologically <u>closed</u> if it contains the limit of every convergent sequence of points in B.]

21. Consider all lines which meet the graph of

$$y = 2x^4 + 7x^3 + 3x - 5$$

is four distinct points, say (x_i, y_i), $i = 1,2,3,4$. Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line and find its value.

22. Let u,f, and g be functions, defined for all real numbers x, such that

$$\frac{u(x+1) + u(x-1)}{2} = f(x) \text{ and } \frac{u(x+4) + u(x-4)}{2} = g(x).$$

Determine $u(x)$ in terms of f and g.

23. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1,4,7,...,100. Prove that there must be two distinct integers in A whose sum is 104.

24. Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$ with r,s, and t positive integers.

25. Find positive integers n and $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \ldots + a_n = 1979$$

and the product $a_1 a_2 \ldots a_n$ is as large as possible.

26. Let r and s be positive integers. Derive a formula for the number of ordered quadruples (a,b,c,d) of positive integers such that

$$3^r \cdot 7^s = \text{lcm}[a,b,c] = \text{lcm}[a,b,d] = \text{lcm}[a,c,d] = \text{lcm}[b,c,d].$$

The answer should be a function of r and s.

(Note that lcm[x,y,z] denotes the least common multiple of x,y,z.)

27. Let V be the region in the cartesian plane consisting of all points (x,y) satisfying the simultaneous conditions

$$|x| \leq y \leq |x| + 3 \text{ and } y \leq 4.$$

Find the centroid $(\bar{x},\bar{y})$ of V.

28. Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a,b,c > 0$. Let B be the set of all points which are a distance at most one from some point of A (in particular, B contains A). Express the volume of B as a polynomial in a, b, and c.

29. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \dots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n + 2) = P(n)f(n + 1) + Q(n)f(n),$$

for all $n \geq 1$.

30. How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{40}, 20^{30}?

31. Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v than to any other vertex of the cube. Find the volume of R.

32. Given $n(\geq 3)$ distinct points in the plane, no three of which are on the same straight line, prove that there exists a simple closed polygon with these points as vertices.

33. Consider polynomial forms $ax^2 - bx + c$ with integer coefficients which have two distinct zeros in the open interval $0 < x < 1$. Exhibit with a proof the least positive integer value for a for which such a polynomial exists.

34. (a) A certain locker room contains n lockers numbered $1,2,3...,n$ and all are originally locked. An attendant performs a sequence of operations $T_1, T_2, ..., T_n$ whereby with the operation T_k, $1 \leq k \leq n$, the condition of being locked or unlocked is changed for all those lockers and only those lockers whose numbers are multiples of k. After all the n operations have been performed it is observed that all lockers whoses numbers are perfect squares (and only thoses lockers) are now open or unlocked. Prove this mathematically.

(b) Investigate in a meaningful mathematical way a procedure or set of operations similar to those above which will produce the set of cubes, or set of numbers of the form $2m^2$, or the set of numbers of the form $m^2 + 1$, or some nontrivial similar set of your own selection.

35. A set of real numbers is called compact if it is closed and bounded. Show that there does not exist a sequence $\{K_n\}_{n=o}^{\infty}$ of compact sets of rational numbers such that each compact set of rationals is contained in at least one K_n.

36. The terms of a sequence T_n satisfy

$$T_n T_{n+1} = n \quad (n = 1,2,3,...) \text{ and } \lim_{n\to\infty} \frac{T_n}{T_{n+1}} = 1.$$

Show that $\pi T_1^2 = 2$.

37. Show that the power series for the function

$$e^{ax} \cos bx \quad (a > 0,\ b > 0)$$

in powers of x has either no zero coefficients or inifintely many zero coefficients.

38. Given a sequence $\{x_n\}$, $n = 1,2,\ldots,$ such that $\text{limit}_{n\to\infty} \{x_n - x_{n-2}\} = 0$.

Prove that: $$\underset{n\to\infty}{\text{limit}} \frac{x_n - x_{n-1}}{n} = 0.$$

39. A closed subset S of R^2 lies in $a < x < b$. Show that its projection on the y-axis is closed.

40. Let u_n denote the "ramp" function

$$u_n(x) = \begin{cases} -n & \text{for } x \le -n \\ x & \text{for } -n < x \le n, \\ n & \text{for } x > n, \end{cases}$$

and let F denote a real function of a real variable. Show that F is continuous if and only if $u_n \circ F$ is continuous for all n. (Note: $(u_n \circ F)(x) = u_n[F(x)]$).

41. Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

42. Determine all polynomials P(x) such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

43. Show that there are no four consecutive binomial coefficients $\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3}$ (n,r integers > 0 and $r + 3 \le n$) which are in arithmetic progression.

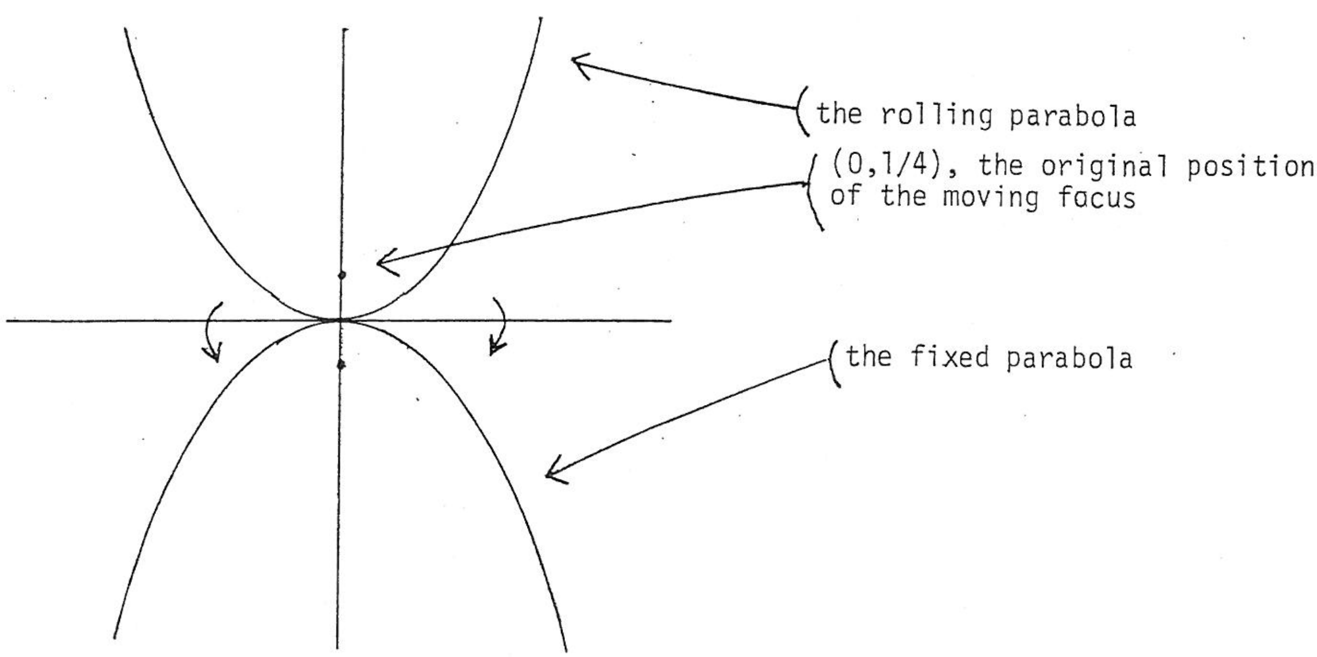
44. Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.

45. Let n be an integer greater than 1. Show that there exists a polynomial $P(x,y,z)$ with integral coefficients such that $x \equiv P(x^n, x^{n+1}, x + x^{n+2})$.

46. Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. (A set of integers is "pairwise relatively prime" if no pair of them has a common divisor greater than 1.) What is the largest number of elements in any "conspiratorial" subset of the integers 1 through 16?

47. A well-known theorem asserts that a prime $p > 2$ can be written as the sum of two perfect squares ($p = m^2 + n^2$, with m and n integers) if and only if $p \equiv 1 \pmod 4$. Assuming this result, find which primes $p > 2$ can be written in each of the following forms, using (not necessarily positive) integers x and y: (a) $x^2 + 16y^2$, (b) $4x^2 + 4xy + 5y^2$.

48. Consider the two mutually tangent parabolas $y = x^2$ and $y = -x^2$. [These have foci at (0, 1/4) and (0, −1/4), and directrices $y = -1/4$ and $y = 1/4$, respectively.] The upper parabola rolls without slipping around the fixed lower parabola. Find the locus of the focus of the moving parabola. (See figure.)



49. In three-dimensional Euclidean space, define a slab to be the open set of points lying between two parallel planes. The distance between the planes is called the thickness of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$, respectively, such that $\sum_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.

50. P is an interior point of the angle whose sides are the rays OA and OB. Locate X on OA and Y on OB so that the line segment $\overline{XY}$ contains P and so that the product of distances (PX)(PY) is a minimum.

51. In the (x,y)-plane, if R is the set of points inside and on a convex polygon, let D(x,y) be the distance from (x,y) to the nearest point of R. (a) Show that there exist constants a,b, and c, independent of R, such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-D(x,y)}\,dx\,dy = a + bL + cA,$$

where L is the perimeter of R and A is the area of R.

(b) Find the values of a,b, and c.

52. Suppose that G is a group generated by elements A and B, that is, every element of G can be written as a finite "word" $A^{n_1}B^{n_2}A^{n_3}\ldots B^{n_k}$, where $n_1, \ldots, n_k$ are any integers, and $A^0 = B^0 = 1$ as usual. Also, suppose that $A^4 = B^7 = ABA^{-1}B = 1$, $A^2 \neq 1$, and $B \neq 1$.

(a) How many elements of G are of the form C^2 with C in G?

(b) Write each such square as a word in A and B.

53. Determine all solutions in real numbers x, y, z, w of the system

$$x + y + z = w,$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$

54. Prove that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

for all integers $p, a,$ and b with p a prime, $p > 0$, and $a > b > 0$.

Notation: $\binom{m}{n}$ denotes the binomial coefficient $\frac{m!}{m!(m - n)!}$.

55. Let $0 < a < b$. Evaluate

$$\lim_{t \to 0} \left\{ \int_0^1 [bx + a(1 - x)]^t dx \right\}^{1/t}$$

[The final answer should not involve any operations other than addition, subtraction, multiplication, division, and exponentation.]

56. Prove or disprove: there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.
[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

57. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in the simplest form.)

58. Two distinct squares of the 8 by 8 chessboard C are said to be adjacent if they have a vertex or side in common. Also, g is called a C−gap if for every numbering of the squares of C with all the integers 1,2,...,64 there exist two adjacent squares whose numbers differ by at least g. Determine the largest C−gap g.

59. Find

$$\lim_{n\to\infty}\left[\frac{1}{n^5}\sum_{h=1}^{n}\sum_{k=1}^{n}(5h^4 - 18h^2k^2 + 5k^4)\right].$$

60. Let p_n be the probability that $c + d$ is a perfect square when the integers c and d are selected independently at random from the set (1,2,3,...,n). Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists and express this limit in the form $r(\sqrt{s} - t)$, where s and t are integers and r is a rational number.