Solutions:

1. It is easily verified by induction that

$$f(n) = \begin{cases} n^2/4 & \text{when } n \text{ is even.} \\ (n^2 - 1)/4 & \text{when } n \text{ is odd.} \end{cases}$$

Therefore, since $x + y$ and $x - y$ always have the same parity, in any case we must have

$$f(x + y) - f(x - y) = \frac{(x + y)^2 - (x - y)^2}{4} = xy.$$

2. Let $P_1, P_2, \ldots, P_n$ be the verticies of P. Let $P_1', P_2', \ldots, P_n'$ be the projections of $P_1, P_2, \ldots, P_n$ upon one of the sides of the squares, and let $P_1'', P_2'', \ldots, P_n''$ be the projections of $P_1, P_2, \ldots, P_n$ upon a side that is orthogonal to the previous one. Since P is convex, the first side will be covered at most twice by the segments $\overline{P_1' P_2'}, \ldots \overline{P_{n-1}' P_n'}, \overline{P_n' P_1'}$. We thus deduce the inequality $\overline{P_1' P_2'}^2 + \ldots + \overline{P_1' P_2'}^2 \le 2$. Similarly $\overline{P_1'' P_2''}^2 + \ldots + \overline{P_n'' P_1''}^2 \le 2$. Adding these two inequalities and using the Pythagorean theorem the assertion follows.

3. Any common factor of two of such numbers would have to be divisible by 2,3,5 or 7. So it is sufficient to prove that among any ten consecutive integers there is at least one that is not divisible by 2,3,5 or 7. We get such an integer by elimination as follows. Strike out those divisible by 3. There may be either 3 or 4 of them. Among these there is either at least one or two respectively that are divisible also by 2. Thus if we strike off also those that are divisible by 2 we will have eliminated at most seven of the integers. Note that by so doing we have stricken off at least one number divisible by five. Thus we are left with three integers only two of which can be divisible by 5 or 7.

4. $$|a_1 + 2a_2 + \dots + na_n| = |f'(0)| = \lim_{x\to 0}\left|\frac{f(x) - f(0)}{x}\right|$$
$$= \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \cdot \left|\frac{\sin x}{x}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \le 1.$$

5. The standard approach, from elementary calculus, applies. By division, rewrite the integrand as a polynomial plus a rational function with numerator of degree less than 2. The solution follows easily.

6. The easy solution is obtained by selecting a rational number p with $0 < p < \epsilon$ and solving the linear system

$$ar + bs = e + p$$
$$cr + ds = f + p.$$

The solution for r and s exist, since $ad \neq bc$, and are rational numbers which satisfy the given inequalities.

7. The proof is by induction. For a singleton set $\{1\}$ the list is $\emptyset, \{1\}$. Thus the result is true for singleton sets. Suppose the result is true for all sets with $n-1$ members. Let $S = \{1,2,3,\dots,n\}$ and $T = \{1,2,3,\dots,n-1\}$. Let $T_o T_1,\dots,T_t (t = 2^{n-1} - 1)$ be the list of subsets of T satisfying the requirements. Then the desired list of subsets of S are $S_o, S_1,\dots,S_s (s = 2^{n-1})$ where $S_i = T_i$, for $0 \le i < t$, and $S_t = T_t \cup \{n\}$, $S_{t+1} = T_{t-1} \cup \{n\},\dots,S_s = \{n\}$. Comments: This problem is equivalent to finding a Hamiltonian circuit on an n-cube.

8. Denote the four events $x^o = 70^o$, $y^o = 70^o$, $\max(x^o, y^o) = 70^o$, $\min(x^o, y^o) = 70^o$ by A, B, C, D, respectively. Then $A \cup B = C \cup D$, and $A \cap B = C \cap D$. Hence $P(A) + P(B) = P(A \cup B) + P(A \cap B) = P(C \cup D) + P(C \cap D) = P(C) + P(D)$ and $P(\min(x^o, y^o) = 70^o) = P(x^o = 70^o) + P(y^o = 70^o) - P(\max(x^o, y^o) = 70^o)$.

9. Let g be any element of G. The set $\{ga^{-1} | a \in A\}$ has the same number of elements as A. If these two sets are disjoint, their union would contain more elements than G. Thus there exist $a_1, a_2 \in A$ such that $a_1 = ga_2^{-1}$ and $g = a_1a_2$.

<u>Alternate Solution</u>: Let G have n elements, A have m elements, and consider the multiplication table of G. An element g in G must appear exactly once in each row and column of the multiplication table. It appears at most $2(n - m)$ times outside the table for A and n times in the table for G. Thus it appears at least $n - 2(n - m) = 2m - n$ times in the table for A, and we are given that $2m > n$.

10. If $a = 0$ then $d = 1$, and if $a = 1$ then $d = 0$. In either case $bc = 0$ and b or c is 0, while the other is arbitrary. There are $2p - 1$ distinct solutions to $bc = 0$ and thus the case $a = 0$ or $a = 1$ accounts for a total of $4p - 2$ solutions. If $a \neq 0$ or 1, then d is uniquely determined and $bc \equiv ad \neq 0 \pmod p$ implies that for each $b \neq 0$, there is a unique c, since the integers mod p form a field. Hence for each a in this case, there are $p - 1$ solutions. The total number of solutions is $4p - 2 + (p - 2)(p - 1) = p^2 + p$.

11. Subtract the first column from every other column. Then add the first row to every other row. The last row now has all zeros except for $(n - 1)$ in the first column. D_n is $(-1)^{n-1}(n - 1)$ times the minor formed by deleting the first column and last row from the transformed determinant. This minor has only zeros below the main diagonal and thus is equal to the product of its diagonal elements. Hence the minor has value 2^{n-2} and $D_n = (-1)^{n-1}(n - 1)2^{n-2}$.

<u>Alternate Solution</u>: From the bottom row of D_{n+1}, subtract $1/(n - 1)$ times the first row and $n/(n - 1)$ times the nth row. This shows that $D_{n+1} = -[2n/(n - 1)]D_n$, for $n > 1$. The result follows easily by iteration and the observation that $D_2 = -1$.

12. The condition $24 \mid n+1$ is equivalent to $n \equiv -1 \pmod 3$ and $n \equiv -1 \pmod 8$. Let d be a divisor of n, then $d \equiv 1$ or $2 \pmod 3$ and $d \equiv 1, 3, 5$ or $7 \pmod 8$. Since $d(n/d) = n \equiv -1$ (mod 3) or (mod 8), the only possibilities are:

$d \equiv 1, \quad n/d \equiv 2 \pmod 3$ or vice versa

$d \equiv 1, \quad n/d \equiv 7 \pmod 8$ or vice versa

$d \equiv 3, \quad n/d \equiv 5 \pmod 8$ or vice versa.

In every case, $d + n/d \equiv 0$ (mod 3) and (mod 8). Thus $d + n/d$ is a multiple of 24. Note that $d \neq n/d$ and thus no divisor is used twice in the pairing, so the sum of all the divisors is a multiple of 24.

13. Let $P(t) = at^3 + bt^2 + ct + d$. The equation

$$\frac{1}{2T}\int_{-T}^{T} P(t)dt = \frac{1}{2}\{P(t_1) + P(t_2)\}$$

is satisfied for all values of a, b, c, and d if and only if $t_2 = -t_1 = \pm T/\sqrt{3}$. If $T = 3$ hrs, $T/\sqrt{3} \approx 1$ hr, 43.92 min. Therefore, in the case considered, the critical times are 1 hr 44 min. each side of noon.

14. Using the given laws we have

$$x \circ y = (x \circ y) \circ (x \circ y) = [(x \circ y) \circ x] \circ y = [(y \circ x) \circ x] \circ y$$

$$= [(x \circ y) \circ y] \circ y = (x \circ y) \circ y = (y \circ y) \circ x = y \circ x.$$

From this commutative law we obtain

$$(x \circ y) \circ z = (y \circ z) \circ x = x \circ (y \circ z).$$

15. In the given functional equation

$$(1) \qquad F(x) + F\left[\frac{x-1}{x}\right] = 1 + x$$

we substitute $\frac{x-1}{x}$ for x, obtaining

(2) $$F\left[\frac{x-1}{x}\right] + F\left[\frac{-1}{x-1}\right] = \frac{2x-1}{x}.$$

Also in (1), we substitute $\frac{-1}{x-1}$ for x and obtain

(3) $$F\left[\frac{-1}{x-1}\right] + F(x) = \frac{x-2}{x-1}.$$

Adding (1) and (3) and subtracting (2) gives

(4) $$2F(x) = 1 + x + \frac{x-2}{x-1} - \frac{2x-2}{x} = \frac{x^3 - x^2 - 1}{x(x-1)}.$$
$$F(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}.$$

That F(x), defined in (4), does satisfy the given functional equation is easily verified.

Therefore (4) is the only solution of the problem.

16. Label the given laws (1) and (2), respectively.

I. We first show that

(3) $$(x * y) * x = y.$$

This follows from $(x * y) * x = (x * y)[(x * y) * y] = y$. (First apply (2) with x and y interchanged; then apply (1) with x replaced by $x * y$.)

We now obtain

(4) $$y * x = [(x * y) * x] * x = x * y.$$

(First apply (3); then apply (2) with y replaced by $x * y$.) This proves that * is commutative.

II. Let S be the set of all integers. Define $x * y = - x - y$. Then

(5) $$x * (y * z) = - x + y + z; (x * y) * z = x + y - z.$$

It follows from (5) that, in the first place, (1) and (2) hold and, secondly, * fails to be associative: simply choose $x \neq z$ in (5).

<u>Alternative Solution</u>, <u>Part I</u> (suggested by Martin Davis):

Write the equation $x * y = z$ as P(x,y,z). Then law (1) may be written "If $x * y = z$ then $x * z = y$" or

(6) $P(x,y,z)$ implies $P(x,y,z)$.

Similarly, the law (2) may be written

(7) $P(y,x,z)$ implies $P(z,x,y)$.

These two implications, (6) and (7), show that the permutations (23) and (13) on the location of the variables in $P(x,y,z)$ are permitted. Since (13,(23) generate the symmetric group S_3, we find (12) is also permitted.

Thus, $P(x,y,z)$ implies $P(y,x,z)$ or $x * y = z$ implies $y * x = z$, which means $x * y = y * x$.

17. We take v_o as positive (see Comment) and consider the graph of v as a function of t (see Figure 1). From the given data we know that the curve starts at the origin and is concave downward since the acceleration $a = dv/dt$ does not increase.

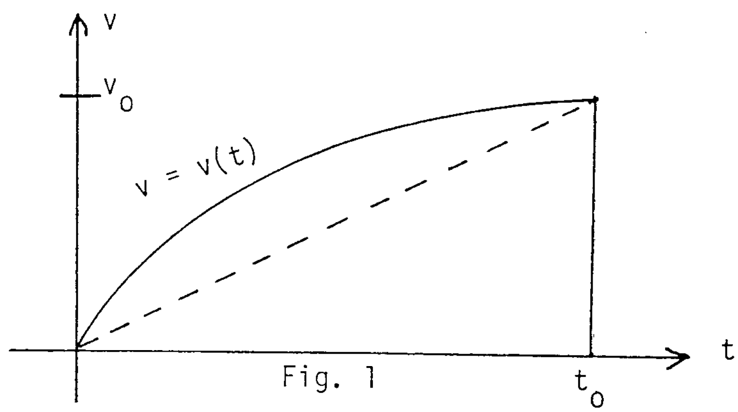

Fig. 1

Let t_o be the time of the traverse. Then $v(t_o) = v_o$. The distance s_o is represented by the area bounded by the curve $v = v(t)$, the t–axis, and the line $t = t_o$. The area area of the right triangle with vertices at $(0,0),(t_o,0)$ and (t_o,v_o) has area less than or equal to s. Thus $1/2\, v_o t_o \le s_o$ or

$$t_o \le \frac{2s_o}{v_o}.$$

Equality is possible and gives the maximum value of t_o (for given s_o and v_o) when the graph of $v(t)$ is the straight line $v(t) = (v_o/t_o)t = (v_o^2/2s_o)t$.

<u>Comment</u>: If v_o is zero or negative, there is no maximum time t_o for the traverse. In the case $v_o = 0$ the equation of motion

$$S = s_o[3(t/t_o)^2 - 3(t/t_o)^3], \quad 0 \leq t \leq t_o$$

satisfies the conditions of the problem for any $t_o > 0$.

18. Let $n = [(a^2 + a)/2] + [(b^2 + b)/2]$, with a and b integers. Then

$$4n + 1 = 2a^2 + 2a + 2b^2 + 2b + 1 = (a + b + 1)^2 + (a = b)^2.$$

Conversely, let $4n + 1 = x^2 + y^2$, with x and y integers. Then exactly one of x and y is odd and so $a = (x + y - 1)/2$ and $b = (x - y - 1)/2$ are integers. One easily verifies that

$$[(a^2 + a)/2] + [(b^2 + b)/2] = (x^2 + y^2 - 1)/4 = n.$$

19. The desired region is the inside of the triangle with vertices $(0, -1),(2,1),(-2,1)$. The boundary seqments lie on the lines

$$L_1{:}c = 1, L_2{:}c - b + 1 = 0, L_3{:}c + b + 1 = 0.$$

To see this, we let $f(z) = z^2 + bz + c$ and denote its zeros by r and s. Then $-b = r + s$ and $c = rs$. Also

$$(r + 1)(s + 1) = rs + r + s + 1 = c - b + 1 = f(-1).$$

$$(r - 1)(s - 1) = rs - r - s + 1 = c + b + 1 = f(1).$$

On or below L_2, at least one zero is real and not greater than -1; this follows either from $(r + 1)\cdot(s + 1) \leq 0$ or from $f(-1) \leq 0$ and the fact that the graph of $y = f(x)$, for x real, is an upward opening parabola. Similarly, on or below L_3 one zero is real and at least 1. On or above L_1, at least one zero has absolute value greater than or equal to 1. Hence the desired points (b,c) must be inside the described triangle.

Conversely, if (b,c) is inside the triangle, $|c| < 1$ and so $|r| < 1$ or $|s| < 1$ or both. If the zeros are complex, they are conjugates and $|r| = |s|$; then $|r| = |s| < 1$

follows from $|c| < 1$. If the zeros are real, $|c| < 1$ implies that at least one zero is in $(-1,1)$. Then $(r + 1)(s + 1) = f(-1) > 0$ and $(r - 1)\cdot(s - 1) = f(1) > 0$ imply that the other zero is also in $(-1,1)$.

For full credit, the region had to be depicted.

20. No. Since the mapping with $(x,y) \longrightarrow (-x,-y)$ is a homeomorphism of the unit circle on itself, the complement $-B$ of such a subset B would also be closed. Thus the existence of such a B would make C the union $-B \cup B$ of disjoint nonempty closed subsets; this would contradict the fact that C is connected.

21. A line meeting the graph in four points has an equation $y = mx + b$. Then the x_i are the roots of

$$2x^4 + 7x^3 + (3 - m)x - (5 + b) = 0,$$

their sum is $-7/2$, and their arithmetic mean $(\Sigma x_i)/4$ is $-7/8$, which is independent of the line.

22. We show that there are an infinite number of expressions for $u(x)$ in terms of f and g; some of the simpler ones are:

$$\begin{aligned} u(x) &= g(x) - f(x+3) + f(x+1) + f(x-1) - f(x-3) \\ &= -g(x+2) + f(x+5) - f(x+3) + f(x+1) + f(x-1) \\ &= g(x+4) - f(x+7) + f(x+5) - f(x+3) + f(x+1). \end{aligned}$$

Let E be the shift operator on functions A defined by $EA(x) = A(x+1)$. Then $(E+E^{-1})u(x) = 2f(x)$ and $(E^4+E^{-4})u(x) = 2g(x)$ are given. Thus $(E^2+1)u(x) = 2Ef(x)$ and $(E^8+1)u(x) = 2E^4g(x)$. Motivated by the fact that E^2+1 and E^8+1 are relatively prime polynomials in E, one finds that

$$1 = \tfrac{1}{2}(E^8+1) - \tfrac{1}{2}(E^6-E^4+E^2-1)(E^2+1),$$

$$u(x) = \tfrac{1}{2}(E^8+1)u(x) - \tfrac{1}{2}(E^6-E^4+E^2-1)(E^2+1)u(x),$$

$$u(x) = E^4g(x) - (E^6-E^4+E^2-1)Ef(x),$$

$$u(x) = E^4g(x) + (-E^7+E^5-E^3+E)f(x),$$

$$u(x) = g(x+4) - f(x+7) + f(x+5) - f(x+3) + f(x+1).$$

Other expressions are obtained using

$$g(y) = -g(y-2) + f(y+3) + f(y-5)$$
$$= -g(y+2) + f(y+5) + f(y-3).$$

23. Each of the twenty integers of A must be in one of the eighteen disjoint sets

$$\{1\}, \{52\}, \{4,100\}, \{7,97\}, \{10,94\},\ldots,\{49,55\}.$$

Hence some (at least two) of the pairs $\{4,100\},\ldots,\{49,55\}$ must have two integers from A. But the sum for each of these pairs is 104.

24. The area is the same as for an octagon inscribed in a circle and with sides alternately 3 units and 2 units in length. For such an octagon, all angles measure $3\pi/4$ and one can augment the octagon into a square with sides of length $3+2\sqrt{2}$ by properly placing a $\sqrt{2}, \sqrt{2}, 2$ isosceles right triangle on each of the sides of length 2. Hence the desired area is

$$(3+2\sqrt{2})^2 - (4\sqrt{2}\cdot\sqrt{2}/2) = 13 + 12\sqrt{2}.$$

A second solution follows. Let r be the radius of the circle and let α and β be half of the central angles for the chords of lengths 3 and 2, respectively. Then $8\alpha + 8B = 2\pi$ and so $\beta = (\pi/4) - \alpha$. Also

$$\frac{3}{2r} = \sin\alpha,\ \frac{1}{r} = \sin\beta = \sin(\tfrac{\pi}{4} - \alpha) = \frac{\cos\alpha - \sin\alpha}{\sqrt{2}},$$

$$\frac{2}{3} = \frac{2r}{3}\cdot\frac{1}{r} = \frac{\cos\alpha - \sin\alpha}{\sqrt{2}\sin\alpha} = \frac{\cot\alpha - 1}{\sqrt{2}}.$$

Now $\cot \alpha = (3 + 2\sqrt{2})/3 = [(3 + 2\sqrt{2})/2]/(3/2)$ and hence the distance from the center of the circle to a chord of length 3 is $h_3 = (3 + 2\sqrt{2})/2$. Similarly the distance to a chord of length 2 is $h_2 = (2 + 3\sqrt{2})/2$. Finally, the desired area is

$$4(3h_3) + 2h_2)/2 = (9 + 6\sqrt{2}) + (4 + 6\sqrt{2}) = 13 + 12\sqrt{2}\,.$$

25. We see that $n = 660$ and that all but one of the a_i equal 3 and the exceptional a_i is a 2 as follows. No a_i can be greater than 4 since one could increase the product by replacing 5 by $2 \cdot 3$, 6 by $3 \cdot 3$, 7 by $3 \cdot 4$, etc. There cannot be both a 2 and a 4 or three 2's among the a_i since $2 \cdot 4 < 3 \cdot 3$ and $2 \cdot 2 \cdot 2 < 3.3$. Also, there cannot be two 4's since $4 \cdot 4 < 2 \cdot 3 \cdot 3$. Clearly, no a_i is a 1. Hence the a_i are 3's except possibly for a 4 or for a 2 or for two 2's. Since $1979 = 3 \cdot 659 + 2$, the only exception is a 2 and $n = 660$.

26. We show that the number is $(1 + 4r + 6r^2)(1 + 4s + 6s^2)$. Each of a,b,c,d must be of the form $3^m 7^n$ with m in $\{0,1,...,r\}$ and n in $\{0,1,...,s\}$. Also m must be r for at least two of the four numbers, and n must be s for at least two of the four numbers. There is one way to have $m = r$ for all four numbers, 4r ways to have one m in $\{0,1,...,r-1\}$ and the other three equal to r, and $\binom{4}{2}r^2 = 6r^2$ ways to have two of the m's in $\{0,1,...,r-1\}$ and the other two equal to r. Thus there are $1 + 4r + 6r^2$ choices of allowable m's and, similarly, $1 + 4s + 6s^2$ choices of allowable n's.

27. Let T consist of the points inside or on the triangle with vertices at (0,3),(−1,4),(1.4) and let U be the set of points inside or on the triangle with vertices at (0,0),(−4,4),(4,4). Then T and V overlap only on boundary points and their union is U. The centroids of T and U are (0,11/3) and (0,8/3), respectively. The areas of T, V, and U are 1, 15, and 16, respectively. Using weighted averages with the areas as weights, one has

$$1 \cdot 0 + 15\bar{x} = 16 \cdot 0, \qquad 1 \cdot \frac{11}{3} + 15\bar{y} = 16 \cdot \frac{8}{3}.$$

It follows that $\bar{x} = 0$, $\bar{y} = 13/5$.

28. The set B can be partitioned into the following sets:

(i) A itself, of volume abc;

(ii) two $a \times b \times 1$ bricks, two $a \times c \times 1$ bricks, and two $b \times c \times 1$ bricks, of total volume $2ab + 2ac + 2bc$;

(iii) four quarter-cylinders of length a and radius 1, four quarter-cylinders of length b and radius 1, and four quarter-cylinders of length c and radius 1, of total volume $(a + b + c)\pi$;

(iv) eight spherical sectors, each consisting of one-eight of a sphere of radius 1, of total volume $4\pi/3$.

Hence the volume of B is

$$abc + 2(ab + ac + bc) + \pi(a + b + c) + \frac{4\pi}{3}.$$

29. We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)[f(n+1) - f(n)].$$

It follows that we can take $P(x) = x + 3$ and $Q(x) = -x - 2$.

30. For d and m in $Z^+ = \{1,2,3,...\}$, let $d|m$ denote that d is an integral divisor of m. For m in Z^+, let r(m) be the number of d in Z^+ such that $d|m$. The number of n in Z^+ such that $n|a$ or $n|b$ is

$$r(a) + r(b) - r(\gcd(a,b)).$$

Also $r(p^s q^t) = (s + 1)(t + 1)$ for p,q,w,t in Z^+ with p and q distinct primes. Thus the desired count is

$$r(2^{40} \cdot 5^{40}) + r(2^{60} \cdot 5^{30}) - r(2^{40} \cdot 5^{30}) = 41^2 + 61 \cdot 31 - 41 \cdot 31$$
$$= 1681 + 620 = 2301.$$

31. The diameter of S must be 4 and S must be centered at the center of C. The set of points inside C nearer to v than to another vertex w is the part of that half-space, bounded by the perpendicular bisector of the segment uw, containing v which lies within C. The intersection of these sets is a cube C' bounded by the three facial planes of C through v and the three planes which are perpendicular bisectors of the edges of C at v. These last 3 planes are planes of symmetry for C and S. Hence R is one of 8 disjoint congruent regions whose union is the set of points between S and C, excepting those on the 3 planes of symmetry. Therefore

$$8\ \mathrm{vol}(R) = \mathrm{vol}(C) - \mathrm{vol}(S) = 4^3 - \frac{4\pi}{3}\cdot 2^3,$$

$$\mathrm{vol}(R) = 8 - \frac{4\pi}{3}.$$

32. Let these points be denoted by $P_1, P_2, \ldots, P_n$. To every permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1,2,3,\ldots,n)$ we associate a closed polygon, namely $P_{\sigma_1}P_{\sigma_2}\ldots P_{\sigma_n}P_{\sigma_1}$. This way we obtain $(n-1)!$ distinct closed polygons some of which may have selfintersections. we claim that anyone of these polygons whose length is the shortest possible is simple. By the hypothesis that no three P_i's are on the same line, a selfintersection occurs if and only if two segments say $\overline{P_{\sigma_1}P_{\sigma_2}}$ and $\overline{P_{\sigma_m}P_{\sigma_{m+1}}}$ cross each other. However, then the closed polygon $P_{\sigma_2}\ldots P_{\sigma_{m-1}}P_{\sigma_m}P_{\sigma_n}P_{\sigma_{n-1}}\ldots P_{\sigma_{m+1}}P_{\sigma_2}$ would have shorter length. Thus there can't be a cross if the length of $P_{\sigma_1}\ldots P_{\sigma_n}P_{\sigma_1}$ is shortest possible.

Alternate solution: Take two points P_1 and P_2 such that all the other points $P_3, P_4 \ldots, P_n$ are on the same side of the line connecting P_1 and P_2. Each point P_i, $i > 2$, determines an angle θ_i between P_1P_2 and P_1P_i, with $0 < \theta_i < \pi$. By hypothesis, $\theta_i \neq \theta_j$ if $i \neq j$. Let $(i_3, i_4, \ldots, i_n)$ be the permutation of $(3,4,\ldots,n)$ such that $\theta_{i_3} < \theta_{i_4} < \ldots < \theta_{i_n}$. Then $P_1P_2P_{i_3}P_{i_4}\ldots P_{i_n}P_1$ is a closed simple polygon.

33. Let $f(x) = ax^2 - bx + c = a(x-r)(x-s)$. Then $f(0)\cdot f(1) = a^2 r(r-1)\cdot s(s-1)$. The graph of $r(r-1)$ shows that $0 < r < 1$ implies $0 < r(r-1) \le \frac{1}{4}$, with equality if and only if $r = \frac{1}{2}$. Similarly, $0 < s(s-1) \le \frac{1}{4}$. Since $r \ne s$, $r(r-1)s(s-1) < 1/16$ and $0 < f(0)\cdot f(1) < a^2/16$. The coefficients a,b,c are integers and thus $1 \le f(0)\cdot f(1)$. Consequently $a^2 > 16$, i.e. $a \ge 5$.

The discriminant $b^2 - 4ac$ shows that the minimum possible value for b is 5. Furthermore, $5x^2 - 5x + 1$ has two distinct roots between 0 and 1.

34. Locker m, $1 \le m \le n$, will be unlocked after the n operations are performed if and only if m has an odd number of positive divisors. If $m = p^{\alpha}q^{\beta}\ldots r^{\gamma}$, then the number of divisors of m is $(\alpha+1)(\beta+1)\ldots(\gamma+1)$, which is odd if and only if $\alpha,\beta,\ldots,\gamma$, are all even. This is equivalent to the condition that m is a perfect square.

For part (b), the set of numbers of the form $2m^2$ are obtained by having T_k change lockers whose numbers are multiples of 2k. The set $m^2 + 1$ results from T_k changing locker i if $i - 1$ is a multiple of k, with the stipulation that locker number 1 is changed only by T_1.

35. Let $\{K_n\}$ be any sequence of compact sets of rational numbers. For each n, there is a rational $r_n \notin K_n$, with $0 \le r_n < 1/n$. Otherwise, it would be that K_n contained all rationals in $[0,1/n]$, and hence some irrationals (since K_n is closed). Let $S = \{0, r_1, r_2, \ldots\}$. Then S is compact and not included in any K_n.

36. The first relation implies that

$$T_n = \frac{(n-1)(n-3)\ldots3}{(n-2)(n-4)\ldots2}\cdot\frac{1}{T_1} \quad \text{if } n \text{ is even,}$$

$$T_n = \frac{(n-1)(n-3)\ldots2}{(n-2)(n-4)\ldots1}\cdot T_1 \quad \text{if } n \text{ is odd.}$$

If n is odd,

$$\frac{T_n}{T_{n+1}} = (T_1)^2 \cdot \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\cdots\frac{(n-1)}{(n-2)}\frac{(n-1)}{n}.$$

The Wallis product is $\pi/2 = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\ldots$. After an even number of factors the partial product is less than $\pi/2$ and after an odd number of factors the partial product is greater than $\pi/2$. Thus for the case when n is odd, $T_n/T_{n+1} < \frac{1}{2}\pi T_1^2$. Since the limit of $T_n/T_{n+1} = 1$, 1 is less than or equal to both $\frac{1}{2}\pi T_1^2$ and its reciprocal. This implies that $\pi T_1^2 = 2$.

37. Note that $e^{ax}\cos bx$ is the real part of $e^{(a+ib)x}$. Thus the power series is

$$e^{ax}\cos bx = \sum_{n=0}^{\infty} \operatorname{Re}\{(a+ib)^n\}\frac{x^n}{n!}.$$

In this form, it is easily seen that if x^n has a zero coefficient, then x^{kn} has a zero coefficient for every odd value of k.

38. For $\epsilon > 0$, let N be sufficiently large so that $|x_n - x_{n-2}| < \epsilon$ for all $n \geq N$. Note that for any $n > N$,

$$x_n - x_{n-1} = (x_n - x_{n-2}) - (x_{n-1} - x_{n-3}) + (x_{n-2} - x_{n-3}) - \ldots$$
$$\pm (x_{N+1} - x_{N-1}) \mp (x_N - x_{N-1}).$$

Thus $|x_n - x_{n-1}| \leq (n-N)\epsilon + |x_N - x_{N-1}|$ and $\lim_{n\to\infty}(x_n - x_{n-1})/n = 0$.

39. Let $y_n \to y$ with $(x_n,y_n) \in S$ for all n. The Bolzano-Weierstrass Theorem implies that a subsequence $x_{k(n)} \to x$. Then $y_{k(n)} \to y$ and since S is closed, $(x,y) \in S$. Thus y is in the projection of S on the y-axis.

40. Clearly u_n is continuous. So, if F is continuous, then $u_n \circ F$ is the composition of continuous functions and hence is continuous. Conversely, suppose $u_n \circ F$ is continuous for all n. To prove F is continuous it is enough to show $F^{-1}[(a,b)]$ is open for every bounded interval (a,b). Let $n > \max(|a|,|b|)$. Then $u_n^{-1}[(a,b)] = (a,b)$ so

$$F^{-1}[(a,b)] = F^{-1}[u_n^{-1}\{(a,b)\}] = (u_n \circ F)^{-1}[(a,b)],$$

which is an open set by the continuity of $u_n \circ F$.

41. The set of all lattice points can be divided into eight classes according to the parities of the coordinates, namely, (odd, odd, odd), (odd, odd, even), etc. With nine lattice points some two, say P and Q, belong to the same class. The midpoint of the segment PQ is a lattice point.

42. $P(0) = 0$, $P(1) = [P(0)]^2 + 1 = 1$, $P(2) = [P(1)]^2 + 1 = 2$, $P(5) = [P(2)]^2 + 1 = 5$, $P(5^2+1) = [P(5)]^2 + 1 = 26$, etc. Thus the polynomial $P(x)$ agrees with x for more values than the degree of $P(x)$, so $P(x) \equiv x$.

43. For a given n and r, in order for the first three binomial coefficients to be in arithmetic progression, we must have

$$(1) \qquad 2\binom{n}{r+1} = \binom{n}{r} + \binom{n}{r+2}$$

or equivalently

$$(2) \qquad 2 = \frac{r+1}{n-r} + \frac{n-r-1}{r+2}.$$

The condition that the last three given binomial coefficients are in arithmetic progression is found from (1) by replacing r by $r+1$. Consequently both r and $r+1$ must satisfy equation (2) if all four terms are in arithmetic progression.

Note that the two terms in equation (2) are interchanged if r is replaced by $n - r - 2$. Thus the quadratic equation (2) has roots

$r, r+1; n-r-3,\ n-r-2.$

Since (2) can have only two roots, $r = n - r - 3$ and $n = 2r + 3$. The four binomial coefficients must be

$$\binom{2r+3}{r},\ \binom{2r+3}{r+1},\ \binom{2r+3}{r+1},\ \binom{2r+3}{r+3}$$

which are the four middle terms. They cannot be in arithmetic progression since binomial coefficients increase to the middle term(s) and then decrease.

44. From $ABA = BA^2B = BA^{-1}B$, we have

$$AB^2 = ABA \cdot A^{-1}B = BA^{-1}BA^{-1}B = BA^{-1} \cdot ABA = B^2A.$$

By induction, $AB^{2r} = B^{2r}A$ so that $AB = AB^{2n} = B^{2n}A = BA$. Since A and B commute, $ABA = BA^2B$ implies $A^2B = A^2B^2$, or $B = B^2$, or $b = 1$.

Alternate Solution: It can be shown that A and B commute by expressing each as powers of the same group element. Because $A^3 = 1$ it is tempting to multiply $ABA = BA^2B$ on the right by A^2 and then on the left by BA^2 to get $B^2 = (BA^2)^3$. Set $X = BA^2$ and use $B^{2n} = B$ to obtain

(1) $\qquad B = X^{3n}.$

From $X = BA^2$, we get $XA = B$, $A = X^{-1}B$, or

(2) $\qquad A = X^{3n-1}.$

The conclusion that $B = 1$ is as before.

45. Let $x = t^n, y = t^{n+1}, z = t + t^{n+2}$. We construct a polynomial $P(x,y,z)$ with integral coefficients such that $P(x,y,z) = t$. We have

$$\begin{aligned} z &= t + t^{n+2}, \\ zy &= t^{n+1} + t^{2n+3}, \\ zy^2 &= t^{2n+3} + t^{3n+4}, \\ &\cdots\cdots\cdots \\ z^{n-2} &= t^{n^2-n-1} + t^{n^2}. \end{aligned}$$

Multiply the above equations alternately by +1 and −1 and add:

$$z[1 - y + y^2 - ... + (-1)^{n-2}y^{n-2}] = t + (-1)^{n-2}t^{n^2} = t + (-1)^n x^n.$$

Hence, if we define

$$P(x,y,x) = z\left[\sum_{t=0}^{n-2} (-1)^i y^i\right] + (-1)^{n-1}x^n,$$

Then $P(t^n, t^{n+1}, t + t^{n+2}) = t$.

46. A conspiratorial subset (CS) of {1,2,...,16} has at most two numbers from the pairwise relatively prime set {1,2,3,5,7,11,13} and so has at most 16 − (7−2) = 11 numbers. But

{2,3,4,6,8,9,10,12,14,15,16}

is a CS with 11 elements; hence the answer is ll.

47. If $p \equiv 1 (\mathrm{mod}\ 4)$, either (A): $p \equiv 1(\mathrm{mod}\ 8)$ or (B): $p \equiv 5(\mathrm{mod}\ 8)$. We show that (A) and (B) are necessary and sufficient for (a) and (b), respectively. If $p = m^2 + n^2$ and p is odd, one can let m be odd and n be even. Then $p = m^2 + 4v^2$ with $m^2 \equiv 1(\mathrm{mod}\ 8)$. With (A), v is even and $p = m^2 + 16w^2$. Conversely, $p = m^2 + 16w^2$ implies $p \equiv m^2 \equiv 1(\mathrm{mod}\ 8)$. With (B), v is odd, $m = 2u + v$ for some integer u, and $p = (2u + v)^2 + 4v^2$ $= 4u^2 + 4uv + 5v^2$. Conversely, $p = 4u^2 + 4uv + 5v^2$ with p odd implies $p = (2u + v)^2 + 4v^2$ with v odd and hence $p \equiv 5(\mathrm{mod}\ 8)$.

48. Let F be the fixed focus, M be the moving focus, and T be the (varying) point of mutual tangency. The reflecting property of parabolas tells us that the tanget line at T makes equal angles with FT and with a vertical line. This and congruence of the two parabolas imply that MT is vertical and that the segments $\overline{FT}$ and $\overline{MT}$ are equal. Now M must be on the horizontal fixed directrix $y = 1/4$ by the focus-directrix definition of a parabola.

49. Let $\Sigma d_i = d$ and let S be a sphere of radius $r > d/2$. The area of S contained in slab S_i is at most $2\pi d_i$. It follows that the area of S contained in the union of the slabs S_i is at most $2\pi d < 2\pi r$ = (area of S). Hence there are points of S that are not in any of the slabs.

The problem may also be done using volumes of intersection of the slabs with an appropriately chosen sphere.

50. Let μ be the angle bisector of $\angle AOB$ and λ be the perpendicular of μ through P. Then the intersections of λ with OA and OB are chosen as X and Y respectively.

This construction makes $OX = OY$ and there is a circle Γ tangent to OA at X and to OB at Y. Let $\overline{X_1Y_1}$ be any other segment containing P with X_1 on OA and Y_1 on OB. Let X_2 and Y_2 be the intersections of $\overline{X_1Y_1}$ with Γ. A theorem of Euclidean geometry states that $(PX)(PY) = (PX_2)(PY_2)$. Clearly $(PX_2)(PY_2)$ is less than $(PX_1)(PY_1)$. Hence $(PX)(PY)$ is a minimum.

One can also locate X and Y by saying that $(\pi - \angle AOB)/2$ should be chosen as the measure of $\angle OXP$ or $\angle OYP$.

51. It is shown below that $a = 2\pi$, $b = 1$, and $c = 1$. We use $I[S]$ to denote the integral of $e^{-D(x,y)}$ over a region S. Since $D(x,y) = 0$ on R, $I[R] = A$. Now let σ be a side of R, s be the length of σ, and $S(\sigma)$ be the half strip consisting of the points of the plane having a point on σ as the nearest point of R. Changing to (u,v)-coordinates with u measured parallel to σ and v measured perpendicular to σ, one finds that

$I[S(\sigma)] = \int_0^s \int_0^\infty e^{-v}\, dv\, du = s$. The sum Σ_1 of these integrals for all the sides of R is L.

If v is a vertex of R, the points with v as the nearest point R lie in the inside $T(v)$ of an angle bounded by the rays from v perpendicular to the edges meeting at v; let $\alpha = \alpha(v)$ be the measure of this angle. Using polar coordinates, one has

$$I[T(v)] = \int_0^{\alpha}\int_0^{\infty} re^{-r}\, dr d\theta = \alpha.$$

The sum Σ_2 of the $I[T(v)]$ for all vertices v of R is 2π. Now the original double integral equals $\Sigma_2 + \Sigma_1 + A = 2\pi + L + A$. Hence $a = 2\pi$ and $b = 1 = c$.

52. The answers are (a) 8; (b) $1, A^2, B, B^3, B^4, B^5, B^6$. Since $B = (B^4)^2$, $B^3 = (B^5)^2$, $B^5 = (B^6)^2$, the elements in the answer to (b) are all squares in G. They are distinct since B has order 7 and A has order 4. To show that there are no other squares, we first note that $ABA^{-1}B = 1$ implies $AB = B^{-1}A$. Then

$$AB^2 = (B^{-1}A)B = B^{-1}(AB) = B^{-1}(B^{-1}A) = B^{-2}A.$$

Similarly $AB^n = B^{-n}A$ for the other n's in $\{0,1,\ldots,6\}$ and so for all integers n. With this, one obtains

(P) $$(B^iA^j)(B^hA^k) = B^uA^v \text{ with } u = i + (-1)^j h, v = j + k.$$

Thus the set S of elements of the form B^iA^j is closed under multiplication. S is finite since i and j may be restricted to $0 \le i \le 6$ and $0 \le j \le 3$. Hence S is a group and so $S = G$. It then follows from (P) that the squares in G are the B^uA^v with $u = i[1 + (-1)^j]$ and $v = 2j$. If j is odd, $u = 0$ and $v \equiv 2 (\text{mod } 4)$. If j is even, $v \equiv 0 (\text{mod } 4)$. Thus there are no squares other than those listed above.

53. We show that w must equal one of x, y, z and that the remaining two unknowns must be negatives of each other. Let $s = x + y$ and $p = xy$. Then the given equations imply that $w - z = s$ and that

$$\frac{s}{p} = \frac{x + y}{xy} = \frac{1}{y} + \frac{1}{x} = \frac{1}{w} - \frac{1}{z} = \frac{z - w}{zw} = -\frac{s}{zw}.$$

Then $s/p = s/(-zw)$ implies that either $s = 0$ or $-zw = p$. If $s = 0$, then $y = -x$ and $w = z$. If $-zw = p = xy$, then $-z$ and w are the roots of the quadratic equation $T^2 - sT + p = 0$, which has x and y as its roots; this case thus leads to either $w = x$ and $-z = y$ or $w = y$ and $-z = x$.

Alternate solution Substitute w from the first equation into the second. The resulting expression factors into $(x + y)(x + z)(y + z) = 0$. In this form we see that two of the numbers, x, y, z must be negatives of each other, and the third number must equal w (by the first equation).

54. It is well known that $\binom{p}{i} \equiv 0 \bmod p$ for $i = 1,2,\ldots,p-1$ or equivalently that in $Z_p[x]$ one has $(1 + x)^p = 1 + x^p$, where Z_p is the field of the integers modulo p. Thus in $Z_p[x]$,

$$\sum_{k=0}^{pa} \binom{pa}{k} x^k = (1 + x)^{pa} = [(1 + x)^p]^a = \sum_{j=0}^{a} \binom{a}{j} x^{jp}.$$

Since coefficients of like powers must be congruent modulo p in the equality

$$\sum_{k=0}^{pa} \binom{pa}{k} x^k = \sum_{j=0}^{a} \binom{a}{j} x^{jp}$$

in $Z_p[x]$, one sees that

$$\binom{pa}{pb} \equiv \binom{a}{b} (\bmod p)$$

for $b = 0,1,\ldots,a$.

55. Let $u = bx + a(1 - x)$; then the definite integral becomes

$$I(t) = \frac{1}{b - a}\int_a^b u^t du = \frac{b^{t+1} - a^{t+1}}{(1 + t)(b - a)}.$$

Using standard calculus methods for evaluating limits of indeterminate expressions, one finds that

$$[I(t)]^{1/t} \longrightarrow e^{-1}(b^b/a^a)^{1/(b-a)} \quad \text{as } t \longrightarrow 0.$$

56. We assume that there is such a common normal and obtain a contradiction. This assumption implies

$$-\frac{a - c}{\cosh a - \sinh c} = \cosh c = \sinh a. \tag{1}$$

Since $\cosh x > 0$ for all real x and $\sinh x > 0$ only for $x > 0$, (1) implies $a > 0$. Using the fact that $\sinh x < \cosh x$ for all x and (1), one obtains

$$\sinh c < \cosh c = \sinh a < \cosh a.$$

This, $a > 0$, and the fact that $\cosh x$ increases for $x > 0$ imply that $c < a$. Thus the leftmost expression in (1) is negative and cannot equal cosh c. This contradiction shows that no common normal exists.

57. We show that $u_n > 0$ for all $n \geq 0$ if and only if $a \geq 3$. Let $\Delta u_n = u_{n+1} - u_n$. Then the recursion (i.e., difference equation) takes the form $(1 - \Delta)u_n = n^2$. Since n^2 is a polynomial, a particular solution is

$$u_n = (1 - \Delta)^{-1}n^2 = (1 + \Delta + \Delta^2 + ...)n^2 = n^2 + (2n + 1) + 2 = n^2 + 2n + 3.$$

(This is easily verified by substitution.) The complete solution is $u_n = n^2 + 2n + 3 + k\cdot 2^n$, since $u_n = k\cdot 2^n$ is the solution of the associated homogeneous difference equation $v_{n+1} - 2v_n = 0$. The desired solution with $u_o = a$ is $u_n = n^2 + 2n + 3 + (a - 3)2^n$. Since $\lim_{n\to\infty}[2^n/(n^2 + 2n + 3)] = +\infty$, u_n will be negative for large enough n if $a - 3 < 0$. Conversely, if $a - 3 \geq 0$, it is clear that each $u_n > 0$.

Alternatively, one sees that $u_o = a$ and $u_1 = 2a$ and one can prove by mathematical induction that

$$u_n = 2^n a - \sum_{k=1}^{n-1} 2^{n-1-k} k^2 \text{ for } n \geq 2.$$

Hence $u_n > 0$ for $n \geq 0$ if and only if $a > \sum_{k=1}^{n-1} 2^{-1-k} k^2$ and this holds if and only if $a \geq L$, where $L = \sum_{k=1}^{\infty} 2^{-1-k} k^2$. Let D mean d/dx. Then for $|x| < 1$,

$$(1 - x)^{-1} = \sum_{k=0}^{\infty} x^k$$

$$\therefore D(1 - x)^{-1} = (1 - x)^{-2} = \sum_{k=1}^{\infty} kx^{k-1}$$

$$D(1 - x)^{-2} = 2(1 - x)^{-3} = \sum_{k=2}^{\infty} k(k - 1)x^{k-2}.$$

Let $g(x) = 2x^3(1 - x)^{-3} + x^2(1 - x)^{-2}$. Then $L = g(1/2) = 3$ and the answer is all $a \geq 3$.

58. For any numbering, one can go from the square numbered 1 to the square numbered 64 in 7 or fewer steps, in each step going to an adjacent square; thus $(64 - 1)/7 = 9$ is a C–gap. It is the largest C–gap since with coordinates (a,b), $1 \leq a \leq 8$ and $1 \leq b \leq 8$, for the squares we can number (a,b) with $8(a - 1) + b$ and thus find that no number greater than 9 is a C–gap.

59. Let $S_k(n) = 1^k + 2^k + \ldots + n^k$. Using standard methods of calculus texts one finds that

$$S_2(n) = (n^3/3) + (n^2/2) + an$$

and

$$S_4(n) = (n^5/5) + (n^4/2) + bn^3 + cn^2 + dn,$$

with a,b,c,d constants. Then the double sum is

$$10nS_4(n) - 18[S_2(n)]^2 = (2n^6 + 5n^5 + ...) - (2n^6 + 6n^5 + ...) = -n^5 + ...$$

and the desired limit is −1.

60. Let $a(n) = [\sqrt{n+1}]$ and $b(n) = [\sqrt{2n}]$, where [x] is the greatest integer in x. For t in $\{1,2,...,a(n)\}$, there are $t^2 - 1$ ordered pairs (c,d) with c and d in $X = \{1,2,...,n\}$ and $c + d = t^2$. For t in $\{1 + a(n), 2 + a(n), ..., b(n)\}$, there are $2n + 1 - t^2$ ordered pairs (c,d) with c and d in X and $c + d = t^2$. Hence the total number F(n) of favorable (c,d) is

$$F(n) = \sum_{t=1}^{a(n)} (t^2 - 1) + \sum_{t=1+a(n)}^{b(n)} (2n + 1 - t^2)$$

$$= \left[2\sum_{t=1}^{a(n)} t^2\right] - \left[\sum_{t=1}^{b(n)} t^2\right] - a(n) + [b(n) - a(n)](2n + 1)$$

$$= \frac{2a(n)[1 + a(n)][1 + 2a(n)]}{6} - \frac{b(n)[1 + b(n)][1 + 2b(n)]}{6}$$

$$-2(n + 1)a(n) + (2n + 1)b(n).$$

Since $p_n = F(n)/n^2$,

$$\lim_{n\to\infty} (p_n\sqrt{n}) = \lim_{n\to\infty} F(n)/n^{3/2}$$

$$= \frac{2\cdot 2}{6}\lim_{n\to\infty}\left[\frac{a(n)}{\sqrt{n}}\right]^3 - \frac{2}{6}\lim_{n\to\infty}\left[\frac{b(n)}{\sqrt{n}}\right]^3 - 2\lim_{n\to\infty}\frac{a(n)}{\sqrt{n}} + 2\lim_{n\to\infty}\frac{b(n)}{\sqrt{n}}$$

$$= \frac{2}{3} - \frac{1}{3}(\sqrt{2})^3 - 2 + 2\sqrt{2} = \frac{4}{3}(\sqrt{2} - 1).$$