NOTES ON LINEAR NON-AUTONOMOUS SYSTEMS

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1. General Linear Systems

Consider the linear nonhomogeneous system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{g}(t) \tag{1.1}$$

where $A(t)$ and $\mathbf{g}(t)$ are continuous on an interval I.

Theorem 1.1. *If $A(t)$, $\mathbf{g}(t)$ are continuous on some interval $a \le t \le b$, if $a \le t_0 \le b$, and if $\eta \in \mathbb{R}^n$, then the system* (1.1) *has a unique solution $\phi(t)$ satisfying the initial condition $\phi(t_0) = \eta$ and existing on the interval $a \le t \le b$.*

1.1. Linear Homogeneous Systems.

Consider the liner homogeneous system associated with (1.1)

$$\mathbf{y}' = A(t)\mathbf{y}. \tag{1.2}$$

For a homogeneous system, $\phi(t) = 0$ is the only solution that satisfies $\phi(t_0) = 0$. Moreover, if ϕ_1 and ϕ_2 are any solutions of (1.2) on an interval I, and c_1 and c_2 are any constants, then $c_1\phi_1 + c_2\phi_2$ is again a solution of (1.2).

Definition 1.1. A set of vectors $v_1, v_2, \ldots, v_k$ is linearly dependent if there exist scalers $c_1, c_2, \ldots, c_k$, not all zero, such that the linear combination

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0.$$

A set of vectors $v_1, v_2, \ldots, v_k$ is linearly independent if it is not linearly dependent.

A set S of vectors is said to form a basis of a vector space V if it is linearly independent and if every vector in V can be expressed as a linear combination of vectors in S.

We can define the dimension of a particular vector space V to be the number of elements in any basis of V. A vector space is called finite-dimensional if is has a finite basis.

Theorem 1.2. *if the complex $n \times n$ matrix $A(t)$ is continuous on an interval I, then the solutions of the system* (1.2) *on I form a vector space of dimension n over the complex numbers.*

We say that the linearly independent solutions $\phi_1, \phi_2, \ldots, \phi_n$ form a fundamental set of solutions. There are infinitely many different fundamental sets of solutions of (1.2).

A matrix of n rows whose columns are solutions (1.2) is called a solution matrix. An $n \times n$ solution matrix whose columns form a fundamental set of solutions is called a fundamental matrix for (1.2) on I. Denote the fundamental matrix formed from the solutions $\phi_1, \ldots, \phi_n$ by Φ. The statement that every solution ϕ of (1.2) is the linear combination of $\phi_1, \ldots, \phi_n$ for some unique choice of the constants $c_1, \ldots, c_n$ is simply that

$$\phi(t) = \Phi(t)\mathbf{c} \tag{1.3}$$

where Φ is the fundamental matrix solution and $\mathbf{c}$ is the column vector with the components $c_1, \ldots, c_n$.

Theorem 1.3. *If Φ is a solution matrix of (1.2) on I and if t_0 is any point of I, then*

$$(det\Phi)' = (\sum_{j=1}^{n} a_{jj}(t))det\Phi,$$
$$det\Phi(t) = det\Phi(t_0)exp[\int_{t_0}^{t}\sum_{j=1}^{n} a_{jj}(s)ds], \quad \textit{for every } t \textit{ in } I.$$

It follows that either $det\Phi(t) \neq 0$ for each $t \in I$ or $det\Phi(t) = 0$ for every $t \in I$.

Proof. Because the column vectors of Φ is a solution of (1.2), we have

$$\phi'_{ij} = \sum_{k=1}^{n} a_{ik}\phi_{kj}, \quad i, j = 1, \ldots, n. \tag{1.4}$$

Therefore,

$$(det\Phi)' = \begin{vmatrix} \phi'_{11} & \phi'_{12} & \cdots & \phi'_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$
$$+ \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi'_{21} & \phi'_{22} & \cdots & \phi'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \cdots$$
$$+ \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi'_{n1} & \phi'_{n2} & \cdots & \phi'_{nn} \end{vmatrix}.$$

Using (1.4), we have

$$(det\Phi)' = \begin{vmatrix} \sum_{k=1}^{n} a_{1k}\phi_{k1} & \sum_{k=1}^{n} a_{1k}\phi_{k2} & \dots & \sum_{k=1}^{n} a_{1k}\phi_{kn} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \sum_{k=1}^{n} a_{2k}\phi_{k1} & \sum_{k=1}^{n} a_{2k}\phi_{k2} & \dots & \sum_{k=1}^{n} a_{2k}\phi_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^{n} a_{nk}\phi_{k1} & \sum_{k=1}^{n} a_{nk}\phi_{k2} & \dots & \sum_{k=1}^{n} a_{nk}\phi_{kn} \end{vmatrix}.$$

Using elementary row elimination, we find that the first row of the first determinant simplifies to

$$a_{11}\phi_{11}\ a_{11}\phi_{12} \dots\ a_{11}\phi_{1n}.$$

Similarly for the ith row of the ith determinant. Thus

$$(det\Phi)' = a_{11}det\Phi + a_{22}det\Phi + \cdots + a_{nn}det\Phi$$

for every $t \in I$. This proves the first part of the theorem.

The rest of the proof follows by solving the scalar equation for $det\Phi$. □

Theorem 1.4. *A solution matrix* Φ *of* (1.2) *on an interval* I *is a fundamental matrix of* (1.2) *if and only if* $det\Phi(t) \neq 0$ *for every* $t \in I$.

Theorem 1.5. *If* Φ *is a fundamental matrix for* (1.2) *on* I *and* C *is a nonsingular constant matrix, then* ΦC *is also a fundamental matrix for* (1.2). *Every fundamental matrix of* (1.2) *is of this form for some nonsingular matrix* C.

1.2. **Linear Nonhomogeneous Systems.** Consider the linear nonhomogeneous system as in (1.1)

$$y' = A(t)y + g(t). \tag{1.1}$$

Suppose ϕ_1 and ϕ_2 are any two solutions of (1.1) on I. Then $\phi_1 - \phi_2$ is a solution of the associated homogeneous systems (1.2) on I. By the remark following Theorem 1.2, there exists a constant vector $\mathbf{c}$ such that

$$\phi_1 - \phi_2 = \Phi\mathbf{c}.$$

The general solutions for (1.1) are

$$\psi = \Phi\mathbf{c} + \phi_0, \tag{1.5}$$

where Φ is a fundamental matrix solution of (1.2), $\mathbf{c}$ is an arbitrary constant vector and ϕ_0 is a particular solution of (1.1). If the initial condition $\psi(t_0) = \eta$ is given for $t_0 \in I$, then, the constant vector c can be solved from the given vector η. Note that the matrix $\Phi(t_0)$ is nonsingular.

Theorem 1.6. *If Φ is a fundamental matrix of* (1.2) *on I, then the function*

$$\psi(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) g(s) ds$$

is the (unique) solution of (1.1) *valid on I and satisfying the initial condition*

$$\psi(t_0) = 0.$$

If the initial condition $\psi(t_0) = \eta$ is given, then

$$\phi(t) = \phi_h(t) + \psi(t),$$

where ψ is the solution given in Theorem 1.6, and ψ_h is the solution of the homogeneous system (1.2) satisfying the initial condition $\phi_h(t_0) = \eta$, the same η as the initial condition for ϕ.

$$\phi_h(t) = \Phi(t)\Phi^{-1}(t_0)\eta.$$

We have

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)\eta + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) g(s) ds.$$

Let $T(t, s) = \Phi(t)\Phi^{-1}(s)$. Then $T(t, s)$ is a special fundamental matrix solution that satisfies $T(s, s) = I$, and is called the principal matrix solution. We can write the solution to (1.1) as

$$\phi(t) = T(t, t_0)\eta + \int_{t_0}^{t} T(t, s) g(s) ds.$$

2. Linear Systems with Periodic Coefficients

2.1. **Homogeneous systems with Periodic Coefficients.** Consider the homogeneous system

$$y' = A(t)y, \tag{1.2}$$

where A(t) is a continuous periodic $n \times n$ matrix of period ω, i.e., $A(t + \omega) = A(t), t \in \mathbb{R}$. If $\Phi(t)$ is a fundamental matrix for (1.2), then $\Phi(t + \omega)$ is also a fundamental matrix of (1.2). Therefore there exists a nonsingular constant matrix C such that

$$\Phi(t + \omega) = \Phi(t)C, \quad -\infty < t < \infty.$$

In the rest of this paper, we assume that $\Phi(t)$ satisfies the initial condition $\Phi(0) = I$. Then we have

$$\Phi(t + \omega) = \Phi(t)\Phi(\omega). \text{ proof?}$$

The nonsingular matrix $C = \Phi(\omega)$ is called the monodromy matrix. Since C is nonsingular, there exists a matrix R such that $C = exp(\omega R)$. Note that

$$\Phi(\omega) = exp(\omega R). \tag{2.1}$$

Theorem 2.1. *Let $A(t)$ be a continuous periodic matrix of period ω and let $\Phi(t)$ be any fundamental matrix of* (1.2)*. Then there exists a periodic nonsingular matrix $P(t)$ of period ω and a constant matrix R such that*

$$\Phi(t) = P(t) exp(tR).$$

Remark 2.1. (1) Note that $\Phi(t)$ may not be periodic.

(2) If $A(t)$ is a constant matrix, hence periodic of any period ω, then $P(t) = I$ and $R = A$.

Proof. Let R be as in (2.1). Define

$$P(t) = \Phi(t)exp(-tR), \quad -\infty < t < \infty.$$

$P(t)$ is nonsingular and

$$\begin{aligned} P(t+\omega) &= \Phi(t+\omega)exp(-(t+\omega)R) \\ &= \Phi(t)Cexp(-(t+\omega)R) \\ &= \Phi(t)Cexp(-\omega R)exp(-tR) \\ &= \Phi(t)exp(-tR) = P(t). \end{aligned}$$

□

The Floquet theorem can be used to transform (1.2) to a linear system with constant coefficients. Let

$$y = P(t)u. \tag{2.2}$$

Since y is a solutioin and $\Phi(t)$ is a fundamental matrix to (1.2), there exits a constant $\mathbf{c}$ such that

$$\begin{aligned} y &= \Phi(t)\mathbf{c} \\ &= P(t)exp(tR)\mathbf{c} \\ &= P(t)u, \end{aligned}$$

it follows that

$$u = exp(tR)\mathbf{c}.$$

Therefore

$$u' = Ru. \tag{2.3}$$

Corollary 2.2. *The change of variable $y = P(t)u$ transforms the periodic system* (1.2) *to the system* (2.3) *with constant coefficients.*

Corollary 2.3. *A nontrivial solution $\phi(t)$ of* (1.2) *has the property*

$$\phi(t+\omega) = k\phi(t), \quad -\infty < t < \infty,$$

where k is a constant, if and only if k is an eigenvalue of $\Phi(\omega) = exp(\omega R)$.

The eigenvalues ρ_i of the nonsingular matrix $\Phi(\omega) = exp(\omega R)$ is called the characteristic multipliers of the system (1.2), and the eigenvalues λ_i of the matrix R is called the characteristic exponents of (1.2). One can show that

$$\rho_j = e^{\omega\lambda_j}, \quad \lambda_j = \frac{1}{\omega}\log\rho_j, \ (mod\frac{2\pi i}{\omega}), \quad j = 1, \dots, n.$$

$$\Pi_{j=1}^n \rho_j = \exp(\int_0^\omega trA(s)ds), \quad \sum_{j=1}^n \lambda_j = \frac{1}{\omega}\int_0^\omega trA(s)ds, \ (mod\frac{2\pi i}{\omega}).$$

Corollary 2.4. *If the characteristic exponents of* (1.2) *have negative real parts (or equivalently, it the multipliers* (1.2) *have magnitude strictly less than* 1*), then all solutions of* (1.2) *approach zero as $t \to \infty$.*

2.2. **Nonhomogeneous Systems with Periodic Coefficients.** We consider the nonhomogeneous system

$$y' = A(t)y + g(t), \tag{1.1}$$

where we assume that $A(t)$ and $g(t)$ are continuous and periodic in t of the same period ω.

Theorem 2.5. *A solution $\phi(t)$ of* (1.1) *is periodic of period ω if and only if $\phi(\omega) = \phi(0)$.*

Proof. The condition is clearly necessary. To prove the sufficiency, let $\psi(t) = \phi(t+\omega)$. Then ϕ and ψ are both solutions of (1.1) and $\psi(0) = \phi(\omega) = \phi(0)$. By the uniqueness theorem, $\phi(t) = \psi(t) = \phi(t+\omega)$ for all t. □

Theorem 2.6. *The system* (1.1) *has a periodic solution of period ω for any periodic forcing vector g of period ω if and only if the homogeneous system* (1.2) *has no periodic solution of period ω except for the trivial solution $y = 0$.*

Proof. Since $\Phi(t)$ be a fundamental matrix of (1.2) with $\Phi(0) = I$. A solution of (1.1) has the form

$$\psi(t) = \Phi(t)\psi(0) + \Phi(t)\int_0^t \Phi^{-1}(s)g(s)ds.$$

The solution ψ is periodic of and only if $\psi(0) = \psi(\omega)$. But

$$\psi(\omega) = \Phi(\omega)\psi(0) + \Phi(\omega)\int_0^\omega \Phi^{-1}(s)g(s)ds.$$

Therefore, $\psi(0) = \psi(\omega)$ if and only if

$$[I - \Phi(\omega)]\psi(0) = \Phi(\omega)\int_0^\omega \Phi^{-1}(s)g(s)ds.$$

The homogeneous algebraic system must be solved for every periodic forcing term g. This is possible if and only if $det(I - \Phi(\omega)) \neq 0$. On the other hand, (1.2) has a periodic solution of period ω if and only if 1 is a multiplier of $y' = A(t)y$. Thus, $det(I - \Phi(\omega)) \neq 0$ if and only if (1.2) has no nontrivial periodic solution of period ω. □

Theorem 2.7. *(Fredholm's alternative). If A and g are continuous and periodic in t with period ω, then* (1.1) *has a period ω solution if and only if*

$$\int_0^\omega z(t)g(t)dt = 0,$$

for all period ω solutions z of the adjoint equation

$$\dot{z} = -zA(t).$$

Note that $z(t)$ is a row vector in the adoint equation.