NONLINEAR DAMPING AND AVERAGING

1. An energy balance method for limit cycles

Consider the damped system

$$\ddot{x} + h(x, \dot{x}) + g(x) = 0.$$

The kinetic energy is $T = (1/2)\dot{x}^2$ and the potential energy is

$$V(x) = \int g(x)dx.$$

For the undamped system with $h = 0$, the total energy $E = T + V$ is conserved.

$$\frac{1}{2}y^2 + V(x) = C, \quad y = \dot{x}.$$

The phase portrait can be obtained by

$$y = \pm\sqrt{2C - 2V(x)}.$$

The graph of $2C - 2V$ can be read off from the graph of V and the range of x must be chosen so that $2C - 2V$ is non-negative. See Figure 1.

For the damped system with $h \neq 0$, consider how E changes along a trajectory.

$$\begin{aligned}\frac{dE}{dt} &= \frac{dT}{dt} + \frac{dV}{dt} \\ &= \dot{x}\ddot{x} + V'(x)\dot{x} \\ &= -\dot{x}h(x, \dot{x}).\end{aligned}$$

Integrate from $t = \tau_0$ to $t = \tau$,

$$E(\tau) - E(\tau_0) = -\int_{\tau_0}^{\tau} \dot{x}h(x, \dot{x})dt.$$

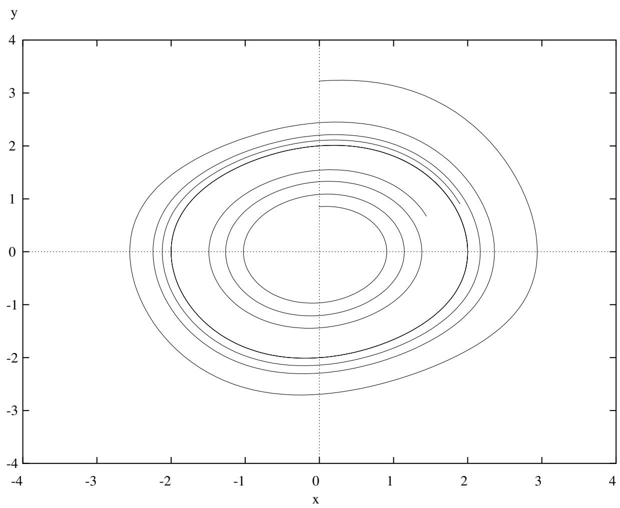

Figure 1. The graph of $2V(x) - 2C$ and the phase portrait

If in the region where the trajectory lies,

(1) $\dot{x}h(x,\dot{x}) = yh(x,y) > 0$, then $E(\tau) < E(\tau_0)$; that is the energy decreases as h has the damping effect, causing the decrease in amplitude; or

(2) $\dot{x}h(x,\dot{x}) < 0$, then $E(\tau) > E(\tau_0)$; the effect is a negative damping causing the increase in amplitude.

(3) $\dot{x}h(x,\dot{x})$ can be both positive and negative along the trajectory. In this case, it is possible that the positive and negative dampings reach balance so that the amplitude of oscillation is preserved.

The third case is of great interest in engineering and other applications. Examples are abundant. We will consider the famous van der Pol's equation. Consider the family of equations of the form

$$\ddot{x} + \epsilon h(x,\dot{x}) + x = 0.$$

The small damping coefficient ϵ reflects the fact in many mechanical system the damping is weak. Assume that $h(0,0) = 0$ so that the origin is an equilibrium point.

When $\epsilon = 0$, the equation $\ddot{x} + x = 0$ has 2π periodic solutions $x(t) = a\cos(t+\alpha)$. With out the loss of generality, assume $a > 0,\ \alpha = 0$. When ϵ is small but non-zero, we look for a periodic solution that is close to one of the unperturbed solution $x(t) = a\cos t$. The constant a has to be determined. Then $\dot{x}(t) = y(t) = -a\sin t$. The change in energy over a cycle is approximately given by

$$E(2\pi) - E(0) = -\epsilon\int_0^{2\pi}(-a\sin t)h(a\cos t, -a\sin t)dt.$$

For any periodic solution which is close to the linear oscillation, we must have

$$\int_0^{2\pi}(-a\sin t)h(a\cos t, -a\sin t)dt = 0.$$

In principle, this equation determines a if the periodic solution is a limit cycle, and is identically zero around a center (with infinitely many periodic solutions).

Example: Find the limit cycle for the van der Pol's equation

$$\ddot{x} + \epsilon(x^2-1)\dot{x} + x.$$

Assuming $x \approx a\cos t$, the energy equation gives

$$\int_0^{2\pi}(a^2\cos^2 t - 1)\sin t\sin t dt = 0.$$

Thus, $\frac{1}{4}a^2 - 1 = 0$ and $a = 2$. Numerical solutions of the limit cycles for $\epsilon = 0.1$ and 0.5 are shown in Figure 2.

We study the stability of the limit cycle. The solutions close to the limit cycle can also be approximately given by $x = a\cos t, y = -a\sin t$. On a trajectory correspooonding to $0 \le t \le 2\pi$ the change of energy is approximately

$$E(2\pi) - E(0) = \epsilon a\int_0^{2\pi}h(a\cos t, -a\sin t)\sin t dt = G(a).$$

If $G(a_0) = 0,\ G'(a_0) < 0$, then

$$\begin{aligned} G(a) > 0 \quad &\text{when } a_0 - \delta < a < a_0, \\ G(a) < 0 \quad &\text{when } a_0 < a < a_0 + \delta. \end{aligned}$$

On the interior of the limit cycle energy is gained and on exterior energy is lost, so the limit cycle is stable.

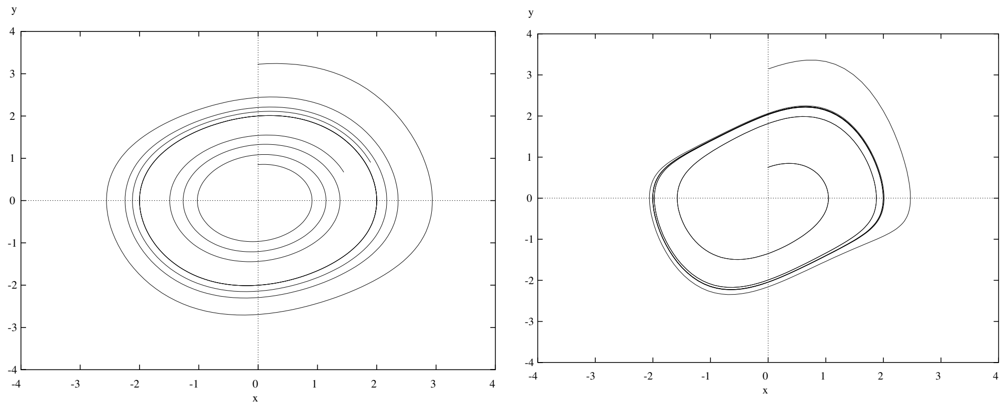

FIGURE 2. The limit cycle and solution that approach the cycle

Because $E = \frac{1}{2}(y^2 + x^2)$, the argument can be carried by using the polar coordinates and the change in distance from the origin can be used to show the stability of the limit cycle.

2. Amplitude and frequency estimates

As expected, a periodic solution of the equation

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0,$$

or the equivalent system

$$\dot{x} = y, \quad \dot{y} = -\epsilon h(x, y) - x,$$

with $h(0, 0) = 0$ and $\epsilon << 1$ is a small distortion of one of the circular orbit of the linearized equation $\ddot{x} + x = 0$. Using the polar coordinates

$$x = a \cos\theta, \quad y = a \sin\theta,$$

we can calculate the perturbation of the amplitude and frequency of the periodic solution.

$$\begin{aligned}
\dot{a} &= (x\dot{x} + y\dot{y})/a = -\epsilon y h(x, y)/a \\
&= -\epsilon h(a\cos\theta, a\sin\theta)\sin\theta, \\
\dot{\theta} &= (x\dot{y} - y\dot{x})/a^2 \\
&= -1 - \frac{\epsilon}{a} h(a\cos\theta, a\sin\theta)\cos\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{da}{d\theta} &= \frac{-\epsilon h(a\cos\theta, a\sin\theta)\sin\theta}{-1 - \frac{\epsilon}{a} h(a\cos\theta, a\sin\theta)\cos\theta} \\
&= \epsilon h(a\cos\theta, a\sin\theta)\sin\theta + O(\epsilon^2).
\end{aligned}$$

Since $da/d\theta = O(\epsilon)$, over one period

$$a(\theta) = a_0 + O(\epsilon),$$

where a_0 is a constant obtained before. To find the period T, we have

$$\begin{aligned} T &= \int_0^T dt = \int_{2\pi}^0 (\frac{d\theta}{dt})^{-1} d\theta \\ &= \int_0^{2\pi} \frac{d\theta}{1 + \frac{\epsilon}{a} h(a\cos\theta, a\sin\theta)\cos\theta} \\ &\approx 2\pi - (\epsilon/a_0) \int_0^{2\pi} h(a_0\cos\theta, a_0\sin\theta)\cos\theta d\theta. \end{aligned}$$

The frequency ω is given to order ϵ by

$$\omega = 2\pi/T \approx 1 + \frac{\epsilon}{2\pi a_0} \int_0^{2\pi} h(a_0\cos\theta, a_0\sin\theta)\cos\theta d\theta.$$

3. Gaps for the linear nonhomogeneous system

Consider the damped system

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0. \tag{1}$$

We show for small ϵ there is a unique periodic solution bifurcates from $x(t) = 2\cos t$.

First consider a linear system

$$\begin{aligned} \dot{x} &= y + f_1(t), \\ \dot{y} &= -x + f_2(t). \end{aligned}$$

It has two linearly independent period solutions

$$\phi_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \phi_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

The homogenous system is self-adjoint. The periodic solutions of the adjoint systems are the same as for the linear system:

$$\psi_1(t) = \phi_1(t), \quad \psi_2(t) = \phi_2(t).$$

The general solutions for the nonhomogeneous system are:

$$\Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-s)\mathbf{f}(s)ds.$$

Denote the convolution by $\mathcal{K}\mathbf{f}$, then the solution is

$$\Phi(t)\mathbf{x}(0) + \mathcal{K}\mathbf{f}.$$

Since $\Phi(2\pi) = I$, the gap from $t = 0$ to $t = 2\pi$ is

$$\mathbf{x}(2\pi) - \mathbf{x}(0) = \mathcal{K}\mathbf{f}(2\pi).$$

Let $w_1 = (1, 0)$, $w_2 = (0, 1)$. The x-gap g_1 and y-gap g_2 are the gaps along the direction of x and y axes.

$$\begin{aligned} g_1 &= w_1 \mathcal{K}\mathbf{f} = \int_0^{2\pi} \psi_1(s)\mathbf{f}(s)ds \\ g_2 &= w_2 \mathcal{K}\mathbf{f} = \int_0^{2\pi} \psi_2(s)\mathbf{f}(s)ds. \end{aligned}$$

Intuitively the x-gap g_1 is related to the energy gain $E(2\pi) - E(0)$ and the y-gap is related to the change of period.

4. Linear variational system for the perturbed nonlinear system

Rewrite (1) as a system

$$\frac{d}{dt}x = y,$$
$$\frac{d}{dt}y = -x - \epsilon h(x, y).$$

Assume that the perturbed system has a solution of period $2\mu\pi$ with $\mu \approx 1$. Let $t = \mu\tau,\ u(\tau) = x(\mu\tau), v(\tau) = y(\mu\tau)$. We look for period 2π solutions for the system

$$\frac{d}{d\tau}u = \mu v,$$
$$\frac{d}{d\tau}v = -\mu u - \epsilon\mu h(u, v).$$

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix}' - A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & (\mu-1) \\ -(\mu-1) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\epsilon\mu h(u, v) \end{pmatrix}.$$

Let E be the projection to the range of $L(u, v)^T = \frac{d}{d\tau}(u, v)^T - A(u, v)^T$. Using the Lyapunov-Schimidt reduction, the system split into two system

$$\begin{pmatrix} u \\ v \end{pmatrix}' - A\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} 0 & (\mu-1) \\ -(\mu-1) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\epsilon\mu h(u, v) \end{pmatrix},$$
$$0 = (I - E)\begin{pmatrix} 0 & (\mu-1) \\ -(\mu-1) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\epsilon\mu h(u, v) \end{pmatrix}.$$

Since the system is autonomous, we can impose that $(u, v) \perp (\sin\tau, \cos\tau)$. The first equation can be solved for $(u, v) = (a\cos\tau + u^*(\tau, a, \epsilon, \mu), -a\sin\tau + v^*(\tau, a, \epsilon, \mu))$ where $(a\cos\tau, -a\sin\tau) \in \ker L$ and $(u^*, v^*) \perp \ker L$. Then $(u^*(\tau, a, 0, 1), v^*(\tau, a, 0, 1)) = (0, 0)$. Substitue (u, v) into the second equation, we have the bifurcation equations

$$G_j(a, \epsilon, \mu) = \int_0^{2\pi} \psi_j(\tau)\begin{pmatrix} 0 & (\mu-1) \\ -(\mu-1) & 0 \end{pmatrix}\begin{pmatrix} a\cos\tau + u^* \\ -a\sin\tau + v^* \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ -\epsilon\mu h(a\cos\tau + u^*, -a\sin\tau + v^*) \end{pmatrix} d\tau = 0, \quad j = 1, 2.$$

We have

$$\int_0^{2\pi} \psi_1(\tau) \cdot ((\mu-1)(-\sin\tau), -(\mu-1)(\cos\tau))^T d\tau = 0,$$
$$\int_0^{2\pi} \psi_2(\tau) \cdot ((\mu-1)(-\sin\tau), -(\mu-1)(\cos\tau))^T d\tau$$
$$= \int_0^{2\pi} (\mu-1)(-\sin^2\tau - \cos^2\tau) d\tau = -2\pi(\mu-1).$$

Therefore,

$$G_1(a,\epsilon,\mu) = \epsilon\mu \int_0^{2\pi} (\sin\tau) h(a\cos\tau + u^*, -a\sin\tau + v^*) d\tau,$$

$$G_2(a,\epsilon,\mu) = -2a\pi(\mu - 1) - \epsilon\mu \int_0^{2\pi} (\cos\tau) h(a\cos\tau + u^*, -a\sin\tau + v^*) d\tau.$$

If we look solutions with $\epsilon\mu \neq 0$, then from $G_1 = 0$, we have

$$g_1(a,\epsilon,\mu) := \int_0^{2\pi} (\sin\tau) h(a\cos\tau + u^*, -a\sin\tau + v^*) d\tau = 0.$$

Assume that $g_1(a,0,1) = \int_0^{2\pi} (\sin\tau) h(a\cos\tau, -a\sin\tau) d\tau$ has a simple zero at $a = a_0$, i.e.

$$\int_0^{2\pi} (\sin\tau) h(a_0\cos\tau, -a_0\sin\tau) d\tau = 0,$$

$$\frac{\partial}{\partial a} \int_0^{2\pi} (\sin\tau) h(a_0\cos\tau, -a_0\sin\tau) d\tau \neq 0.$$

We look for solutions $(a,\mu) \approx (a_0, 1)$ for small ϵ. Observe that when $\epsilon = 0, \mu = 1$, $\partial u^*/\partial a = 0, \partial u^*/\partial\mu = 0$. Similarly for v^*. Thus the matrix at $\epsilon = 0, \mu = 1$ is nonsingular:

$$\begin{pmatrix} \frac{\partial}{\partial a} g_1 & \frac{\partial}{\partial \mu} g_1 \\ \frac{\partial}{\partial a} G_2 & \frac{\partial}{\partial \mu} G_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial a} g_1 & 0 \\ 0 & -2a\pi \end{pmatrix}.$$

Therefore, there exists a unique solution $(a,\mu) = (\tilde{a}(\epsilon), \tilde{\mu}(\epsilon))$ near $(a_0, 1)$. In particular, from $G_2 = 0$, as predicted by the method of averaging, we have

$$\tilde{\mu}(\epsilon) - 1 \approx \frac{-\epsilon}{2\pi a_0} \int_0^{2\pi} (\cos\tau) h(a_0\cos\tau, -a_0\sin\tau) d\tau,$$

$$T - 2\pi \approx \frac{-\epsilon}{a_0} \int_0^{2\pi} (\cos\tau) h(a_0\cos\tau, -a_0\sin\tau) d\tau.$$

5. VDP equation with a large damping

Consider the van der Pol's equation with a large ϵ,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x.$$

Let $\mu = 1/\epsilon$ be the small parameter and let $\tau = \mu t$. Then $\epsilon\dot{x} = (d/d\tau)x = x'$.

$$\mu^2 x'' + (x^2 - 1)x' + x = 0.$$

For the slow layer, $\mu^2 x'' \to 0$ as $\mu \to 0$, we have

$$(x^2 - 1)x' + x = 0, \text{ or } x' = \frac{x}{1 - x^2}.$$

The slow manifold in the original time scale is $y = \dot{x} = \mu x' \approx 0$ on which the flow is $\dot{x} = \frac{\mu x}{1-x^2}$. The solutions pn which move toward $x = \pm 1$ as time goes to infinity.

For the fast layer, let $t = \mu\xi$ and $Dx = dx/d\xi = \mu\dot{x}$. Then in the fast time ξ, we have

$$D^2 x + Dx(x^2 - 1) + \mu^2 x = 0.$$

Let $\mu \to 0$, then the fast motion is described by

$$D^2 x + Dx(x^2 - 1) = 0.$$

Intergrating the eqaution once, we have

$$Dx = C + x - \frac{x^3}{3}.$$

The fast motion in the original time scale is $y = \dot{x} = \frac{Dx}{\mu} = \frac{1}{\mu}(C + x - \frac{x^3}{3})$. The numerical result with $\epsilon = 10$ or $\mu = 0.1$ is included with the y-axis zoomed out by a scale of 10.

We can also write the second order equations as first order systems of equations. The original equation is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \epsilon y(x^2 - 1). \end{aligned}$$

Using $\mu = 1/\epsilon$ as the small parameter and $\tau = \epsilon t,\ \mu x' = \dot{x}$, we have

$$\begin{aligned} \mu x' &= y, \\ \mu y' &= -x - \epsilon y(x^2 - 1). \end{aligned}$$

Let $y = \mu \bar{y}$. We have the system in the slow layer:

$$\begin{aligned} x' &= \bar{y}, \\ \mu^2 \bar{y}' &= -x - \bar{y}(x^2 - 1). \end{aligned} \tag{2}$$

Let $\mu \to 0$,

$$\begin{aligned} x' &= \bar{y}, \\ 0 &= -x - \bar{y}(x^2 - 1). \\ \text{or } x' &= \frac{x}{x^2 - 1}. \end{aligned}$$

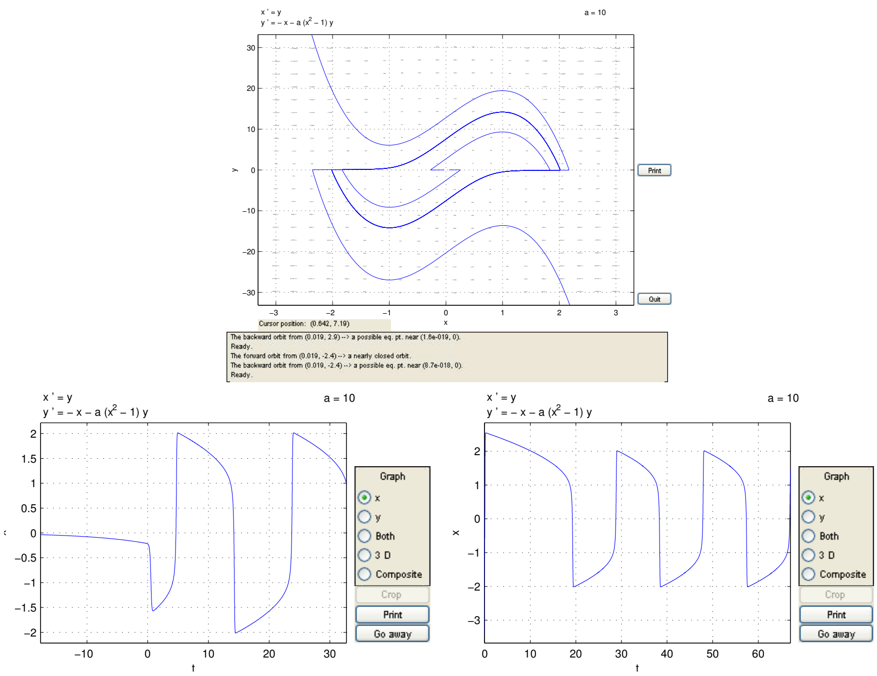

FIGURE 3. The limit cycle and solution that approach the cycle

The slow manifold $y = \mu\bar{y} \approx 0$ is normally hyperbolic if $x \neq \pm 1$, attracting if $|x| > 1$ and repelling if $|x| < 1$.

For the system in the fast layer, using $t = \mu\xi,\ Dx = \mu\dot{x}$, and letting $\bar{\bar{y}} = \mu y$ we have the system in the fast layer:

$$\begin{aligned} Dx &= \bar{\bar{y}}, \\ D\bar{\bar{y}} &= -\mu^2 x - \bar{\bar{y}}(x^2 - 1). \end{aligned}$$

Let $\mu \to 0$,

$$\begin{aligned} Dx &= \bar{\bar{y}}, \\ D\bar{\bar{y}} &= -\mu^2 x - \bar{\bar{y}}(x^2 - 1). \\ \text{or } D\bar{\bar{y}} &= -Dx(x^2 - 1). \end{aligned}$$

Integrating, we have the family of fast curves:

$$\bar{\bar{y}} = C + x - \frac{x^3}{3}.$$

The motion on the fast curve is determined by

$$Dx = C + x - \frac{x^3}{3}.$$

The slow manifold $\bar{\bar{y}} = \mu^2\bar{y} \approx 0$ is nomally hyperbolic. This can be seen from

$$\begin{aligned} Dx &= \mu^2\bar{y}, \\ D\bar{y} &= -x - \bar{y}(x^2 - 1). \end{aligned} \tag{3}$$

The fist equation says that near the slow manifold $Dx = 0$, the value of x is freezed. The second equation, with x as a parameter, describes the rate of change of $\bar{y}$ that is unstable if $|x| < 1$ and stable if $|x| > 1$.

Systems (2) and (3), for the variables $(x, \bar{y})$ are normally considered as a pair of slow and fast systems in the geometric singular perturbation theory. However, numerical results shows that $\bar{y}$ is unbounded so that $\bar{\bar{y}} = \mu^2\bar{y}$ is introduced. to show that fast motion curve.