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Abstract We study the existence of traveling wave solutions for a diffusive predator–prey system. The system considered in this paper is governed by a Sigmoidal response function which in some applications is more realistic than the Holling type I, II responses, and more general than a simplified form of the Holling type III response considered before. Our method is an improvement to the original method introduced in the work of Dunbar (J Math Biol 17:11–32, 1983; SIAM J Appl Math 46:1057–1078, 1986). A bounded Wazewski set is used in this work while unbounded Wazewski sets were used in Dunbar (1983, 1986). The existence of traveling wave solutions connecting two equilibria is established by using the original Wazewski's theorem which is much simpler than the extended version in Dunbar's work.

Keywords Traveling wave solution · Shooting method · Wazewski set · Egress set · LaSalle's invariance principle

Mathematics Subject Classification (2000) 34C37 · 35K57 · 92D25

1 Introduction

Predator–prey models are important tools that help us to understand the bio and ecosystems surrounding us [12]. An important aspect of the model is how the predator interacts with the

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prey [17], which can be described by a functional response that specifies the rate of feeding of predator upon prey as a function of the prey density.

The existence of traveling wave solutions in population dynamics plays an important role in understanding the long time asymptotic property of such systems. In his pioneering works [2,4,3], Dunbar obtained the existence of several kinds of traveling wave solutions for diffusive predator–prey systems with type I, II functional responses. He considered the existence of small amplitude periodic traveling waves, and "heteroclinic traveling waves" that correspond to heteroclinic orbits connecting equilibrium-to-equilibrium or equilibrium-to-periodic orbits. The methods used by Dunbar include the invariant manifold theory, the shooting method, Hopf bifurcation analysis, and LaSalle's invariance principle. Huang et al. [13] extended the work in [4] to $\mathbb{R}^4$ by using Dunbar's method in [3]. An interesting question is whether those results can be extended to a system with type III functional response. Recently, Li et al. [15] proved the existence of traveling waves in a diffusive predator–prey system with a simplified type III functional response $\varphi(u) = Bu^2/(1 + Eu^2)$ by employing the method similar to that used in [2,4]. In this paper we will consider a type III response function which is more general than in [15].

Besides the response function, the method in Dunbar's work can also be improved in several directions. First, the shooting method used by Dunbar is based on a variant of Wazewski's theorem [2,4,3]. The Wazewski set $\mathbb{W}$ constructed by Dunbar has the property that there is an orbit that starts from the unstable manifold of an equilibrium and stays in $\mathbb{W}$ afterwards. However, the Wazewski set $\mathbb{W}$ constructed there was unbounded. In order to ensure the boundedness of the orbit, several additional Lemmas (see Lemma 7 in [2], Lemma 10 in [4], Lemma 15 in [3]) were proved to rule out the possibility that the constructed orbit might escape to infinity. In this paper we construct a simple, bounded Wazewski set $\mathbb{W}$ and use the original Wazewski's theorem [10] to simplify the proof of the existence of heteroclinic traveling waves related to various predator–prey systems.

Secondly, to establish the existence of type I traveling wave, Dunbar first showed that the wave existed for $c > c_*$, he then applied a limiting argument to the case $c = c_*$ via long analysis. In this paper, we combine the two cases $c > c_*$ and $c = c_*$ by using a geometric method, which makes the argument shorter and easier to follow.

Finally, Dunbar only gave a lower estimate for the threshold value which determined the long time property of traveling waves. In this paper, by letting $\gamma = \frac{e}{r}$, which is the quotient of the predator natural death rate verses the prey growth rate, we obtain an exact threshold value γ^*. If $\gamma > \gamma^*$, the traveling waves are oscillatory. Biologically, if γ is big, then a larger death rate of the predator must compensate the abundance of food in the process of approaching the interior equilibrium. This causes the greater rate of exchange of bio-mass between the predator and prey, therefore causes the oscillation. While if $0 < \gamma \leq \gamma^*$, the traveling waves are non-oscillatory. It means the predators possess enough food to consume and have low death rate. This leads to the population density to increase monotonically.

Consider the following diffusive predator–prey system

$$\begin{aligned}\frac{\partial N_1}{\partial t} &= D_1\frac{\partial^2 N_1}{\partial x^2} + rN_1\left(1 - \frac{N_1}{K}\right) - \frac{N_1^2}{a_1 + b_1N_1 + N_1^2}N_2\\ \frac{\partial N_2}{\partial t} &= D_2\frac{\partial^2 N_2}{\partial x^2} + N_2\left(\frac{\alpha N_1^2}{a_1 + b_1N_1 + N_1^2} - e\right),\end{aligned} \tag{1.1}$$

where N_1 and N_2 are the population densities of the prey and predator respectively. To the best of our knowledge, no rigorous work has been done on the existence of traveling wave solutions for system (1.1).

The shooting technique has been an important method in proving the existence of traveling waves solutions for predator–prey type system, e.g., see [2,4,6,7,13,15] and the references therein. The method used in this paper is motivated by the techniques used there. However, as mentioned above, we shall construct a bounded set $\mathbb{W}$ to replace the unbounded Wazewski set introduced in [2,4,13,15]. Assume that our system has three equilibria E, E_1 and E^*. Instead of the extended Wazewski's theorem used in [2,4], we shall use the original Wazewski's topological principle to prove the existence of an orbit connecting E_1 and E^* in $\mathbb{W}$. The ideas are as follows: We first exam the flow on the surfaces of $\mathbb{W}$ to determine its egress sets. Then we construct a curve $\mathcal{C}$ on the two dimensional unstable manifold with two endpoints in two disjoint egress sets. We show that there is a point on $\mathcal{C} \cap \mathbb{W}$ such that the orbit starting from this point will remain in $\mathbb{W}$ by using Theorem 2.1 in Hartman [10, pp. 279]. A Lyapunov function is constructed to show that any orbit that remains in $\mathbb{W}$ shall approach an invariant set in $\mathbb{W}$ while the maximal invariant set in $\mathbb{W}$ is E^*. Thus the solution that stays in $\mathbb{W}$ will converge to the positive equilibrium E^*. In other words, it is a heteroclinic orbit connecting the unstable equilibrium E_1 to E^*. We also show the existence of traveling wave solution connecting E to E^*. Our method is more straightforward than the shooting technique used in [2,4,13,15].

In this paper we assume $D_1 = 0$. It is the limiting case where the prey species diffuses much slower than the predator species such as the case a plant species being consumed by a relatively mobile herbivore. Although a special case, it has strong bearing on the balance of ecological environment. For example, in 1980, the eruption of Mount St Helens caused complete extermination of all plant and animal species in a large area known as the Pumice Plains. One year later, a plant species began to recolonize the Pumice Plains region. In the mid 1980s, the first herbivore species invaded, and in 1990 these herbivores first reached the plant colony. Experiments at the site showed that the invasion of herbivores induced a decrease of the plant species [5]. If the plant species is fit for the predation, then as time goes, two species can achieve a coexistence state. The case D_1 is nonzero and small can be handled by the geometric singular perturbation method and will appear in a separate paper.

To write system (1.1) in a non-dimensional form, we rescale the variables

$$A = \frac{a_1}{K^2},\ B = \frac{b_1}{K},\ U = \frac{N_1}{K},\ W = \frac{N_2}{rK},\ t' = rt,\ x' = \sqrt{\frac{r}{D_2}}x.$$

By dropping the primes on t, x for notational convenience, (1.1) becomes

$$\begin{aligned} \frac{\partial U}{\partial t} &= U(1-U) - \frac{U^2 W}{A + BU + U^2}, \\ \frac{\partial W}{\partial t} &= \frac{\partial^2 W}{\partial x^2} + \frac{W}{r}\left(\frac{\alpha U^2}{A + BU + U^2} - e\right), \end{aligned} \tag{1.2}$$

where $r, e, \alpha > 0$, $A, B \geq 0$. Let $\beta = \frac{\alpha}{e}$. We require that $\beta > 1 + A + B > 1$, which ensures (1.2) has a positive equilibrium corresponding to the coexistence of the two species. It is easy to see that system (1.2) has three spatially constant equilibria given by

$$E(0,0), \qquad E_1(1,0) \qquad \text{and} \qquad E^*(u^*, w^*),$$
$$\text{where} \quad u^* = \frac{B + \sqrt{B^2 + 4A(\beta - 1)}}{2(\beta - 1)}, \quad w^* = \frac{(1-u^*)\left[A + Bu^* + (u^*)^2\right]}{u^*}. \tag{1.3}$$

By using the techniques described above, we shall establish the existence of traveling wave solutions of system (1.2) connecting the equilibria E_1 and E^* (type I wave), which is called the "waves of invasion" (see Chow and Tam [1], Shigesada and Kawasaki [18]). This is of ecological interest since it corresponds to a situation where an environment is initially inhabited only by the prey species at its carrying capacity, the small invasion of the predators drives the system to a new stable state of coexistence of both species. We also establish the existence of traveling wave solutions of system (1.2) connecting E and E^* (type II wave). This describes new species invade the region where all species went to extinct. The establishment of type II waves is more difficult than type I waves technically since one of eigenvectors corresponding to the positive eigenvalues is in the u-axis. Moreover, a threshold property is given for these two types of traveling waves to be oscillatory or non-oscillatory. See Theorem 2.1 for details.

The system studied in this paper is a simplification of the full system by assuming $D_1 = 0$. In order to study the original system with $D_1 \neq 0$, we may use the more advanced topological method like Conley index which is suitable for analyzing heteroclinic connections in higher than three dimensional spaces. See [8,9,16] for references and related works. We hope our experience in constructing the Wazewski set may be useful in constructing the isolated invariant set in the Conley index approach.

The organization of the paper is as follows. In Sect. 2, we state our main results on the existence of traveling wave solutions. Sections 3 and 4 are devoted to proving the existence or non-existence for type I and type II waves, respectively. The construction of the Wazewski set $\mathbb{W}$ will also be given in Sects. 3 and 4. In Sect. 5, we discuss the threshold property for the oscillation of the traveling waves in terms of γ, the quotient of the predator natural death rate verses the prey growth rate.

2 Main Results

A traveling wave solution of (1.2) is a solution of the special form $U(t,x) = u(x+ct) = u(s)$ and $W(t,x) = w(x+ct) = w(s)$, where $s = x+ct$ and $c > 0$ is the wave speed. Substituting this solution into (1.2), we have the wave system

$$\begin{aligned} cu'(s) &= u(1-u) - \frac{u^2 w}{A + Bu + u^2}, \\ cw'(s) &= w''(s) + \frac{w}{r}\left(\frac{\alpha u^2}{A + Bu + u^2} - e\right). \end{aligned} \tag{2.1}$$

Note that (2.1) also has three equilibria E, E_1 and E^*, where E corresponds to the absence of both species, E_1 corresponds to the prey at the environment carrying capacity in the absence of the predator, and E^* corresponds to the coexistence of both species. Rewrite system (2.1) as an equivalent system in $\mathbb{R}^3$

$$
\begin{aligned}
u'(s) &= \frac{1}{c} u \left(1 - u - \frac{uw}{A + Bu + u^2}\right),\\
w'(s) &= z,\\
z'(s) &= cz + \gamma w \left(1 - \frac{\beta u^2}{A + Bu + u^2}\right),
\end{aligned} \tag{2.2}
$$

where $\beta = \frac{\alpha}{e}$, $\gamma = \frac{e}{r}$. Then E, E_1 and E^* correspond to the critical points $(0, 0, 0)$, $(1, 0, 0)$ and $(u^*, w^*, 0)$ of (2.2), respectively. In what follows, we shall still use E, E_1 and E^* to denote $(0, 0, 0)$, $(1, 0, 0)$ and $(u^*, w^*, 0)$. The Jacobian matrix of (2.2) takes the form

$$
\begin{pmatrix}
\frac{1}{c}(1-2u) - \frac{(2A+Bu)uw}{c(A+Bu+u^2)^2} & -\frac{u^2}{c(A+Bu+u^2)} & 0\\
0 & 0 & 1\\
-\frac{\beta\gamma(2A+Bu)uw}{(A+Bu+u^2)^2} & \gamma\left(1 - \frac{\beta u^2}{A+Bu+u^2}\right) & c
\end{pmatrix}. \tag{2.3}
$$

We call the nonnegative solutions of (2.2) satisfying

$$
\lim_{s\to-\infty}(u(s), w(s)) = (1, 0), \quad \lim_{s\to+\infty}(u(s), w(s)) = (u^*, w^*) \tag{2.4}
$$

type I waves, and the nonnegative solutions of (2.2) satisfying

$$
\lim_{s\to-\infty}(u(s), w(s)) = (0, 0), \quad \lim_{s\to+\infty}(u(s), w(s)) = (u^*, w^*) \tag{2.5}
$$

type II waves.

Consider the prey isocline of (2.1)

$$
w = h(u) := \frac{(1-u)(A + Bu + u^2)}{u}, \quad u \in (0, +\infty).
$$

It is easy to check that $h'(u) = \frac{k(u)}{u^2}$, where

$$
k(u) := -2u^3 + (1 - B)u^2 - A, \quad u \in (0, +\infty).
$$

If $B \geq 1$, then $h(u)$ is monotone decreasing. Next, we consider the other case $0 \leq B < 1$. Obviously, there is a negative real root to $k(u) = 0$. Let $P := (1 - B)^2$, $Q := -18A$, $R := 3A(1 - B)$, then $\triangle := Q^2 - 4PR = 12A[27A - (1 - B)^3]$. (i) If $27A > (1 - B)^3$, there is a complex conjugate pair of roots with positive real part to $k(u) = 0$. Hence, $k(u) < 0$ for all $u \in (0, +\infty)$. (ii) If $27A = (1 - B)^3$, there is a double positive root $u = \frac{1-B}{3}$ to $k(u) = 0$. It follows that $\frac{1-B}{3}$ is a local maximum point of $k(u) = 0$ and thus $k(u) \leq 0$ for all $u \in (0, +\infty)$. (i) and (ii) imply that $h(u)$ is monotone decreasing. (iii) If $27A < (1 - B)^3$, there is three distinct real roots to $k(u) = 0$. Note that $k'(u) = 2u(1 - B - 3u)$, then $k(u) = 0$ has two positive roots α_1, α_2 satisfying $0 < \alpha_1 < \frac{1-B}{3} < \alpha_2 < 1$ (see Fig. 1). We now state the following result.

Lemma 2.1 *If* $0 \leq B < 1$ *and* $27A < (1 - B)^3$, *then* $k(u) = 0$ *has two positive roots* α_1, α_2 *satisfying* $0 < \alpha_1 < \frac{1-B}{3} < \alpha_2 < 1$, *and there exist a unique* $u_0 \in (0, \alpha_1)$ *satisfying*

$$
h(u_0) = h(\alpha_2) \tag{2.6}
$$

and a unique $u_1 \in (\alpha_2, 1)$ *satisfying*

$$
h(u_1) = h(\alpha_1). \tag{2.7}
$$

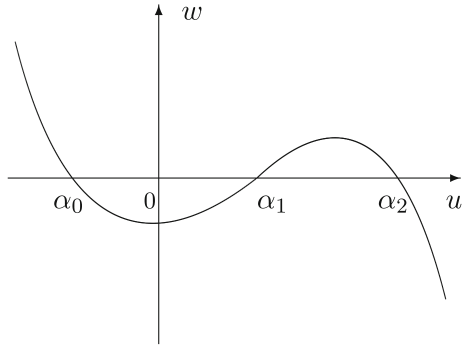

Fig. 1 The graph of $k(u)$ for $0 \le B < 1$ and $27A < (1-B)^3$

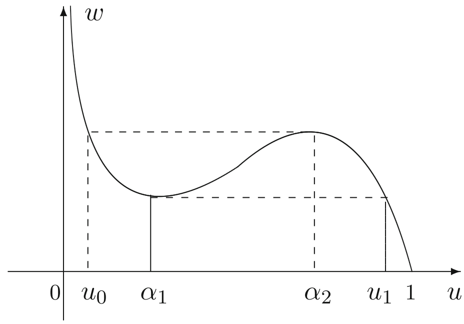

Fig. 2 The prey isocline $w = h(u)$ for $0 < B < 1$ and $27A < (1-B)^3$

Proof Note that $h(1) = 0$, $\lim_{u\to 0^+} h(u) = +\infty$, $h'(u) > 0$ for $u \in (\alpha_1, \alpha_2)$ and $h'(u) < 0$ for $u \in (0, \alpha_1) \cup (\alpha_2, 1)$. Here $\alpha_1 < \alpha_2$ are the roots for $h'(u) = \frac{k(u)}{u^2} = 0$. Then $w = h(u)$ has a local minimum at α_1 and a local maximum at α_2. Moreover, there exist a unique $u_0 \in (0, \alpha_1)$ satisfying $h(u_0) = h(\alpha_2)$ and a unique $u_1 \in (\alpha_2, 1)$ satisfying $h(u_1) = h(\alpha_1)$ (see Fig. 2). □

Lemma 2.2 *Under one of the following cases:*

(i) $B \ge 1$,
(ii) $0 \le B < 1$, $27A \ge (1-B)^3$,
(iii) $0 \le B < 1$, $27A < (1-B)^3$, $u^* \le u_0$ *or* $u^* \ge u_1$, *where* u_0 *is given by* (2.6), *and* u_1 *by* (2.7),

we have $(u - u^*)[h(u) - h(u^*)] \le 0$ *for* $0 < u < 1$, *where* $(u^*, w^*) = (u^*, h(u^*))$ *is the positive equilibrium of* (2.1).

By computing the eigenvalues at E_1, E and E^* in Sections 3, 4 and 5, we find that E, E_1, E^* are all saddle points in the parameter ranges considered in this paper. The local unstable manifolds for E and E_1 are two dimensional and the local stable manifold for E^* is two dimensional. Generically, both type I and type II waves are transversal heteroclinic orbits. Therefore, we expect that both waves should exist for an open set of parameters, plus maybe some of its boundary points.

We now state our main results.

Theorem 2.1 *Let* $\beta > A + B + 1 > 1$, $c_* := \sqrt{4\gamma\left(\frac{\beta}{A+B+1} - 1\right)}$ *and* (u^*, w^*) *be as in* (1.3).

(1) *If* $0 < c < c_*$, *then type II waves do exist while type I waves do not.*

(2) (*i*) *If* $c \geq c_*$, A, B *satisfy one of the three conditions in Lemma* 2.2, *then type I waves do exit while type II waves do not.*

(*ii*) *If* $c \geq c_*, 0 \leq B < 1$, $27A < (1-B)^3$ *and* $u_0 < u^* < u_1$, *then there is a traveling wave* $\phi(s)$ *with* $\lim_{s\to-\infty} \phi(s) = (1,0)$ *and* $\phi(s) \in \mathbb{W}_{(u,w)} = \{(u,w)|0 \leq w \leq \beta(1+cd)(1-u), 0 \leq u \leq 1\}$ *for* $s \geq 0$ *with a parameter* $d > \frac{c}{2}$, *while type II waves do not exist.*

(3) *Let* (u,w) *be a type I or type II traveling wave. Then there exists a value* $\gamma^* = \gamma^*(A,B,\beta,c)$ *such that if* $0 < \gamma \leq \gamma^*$, (u,w) *is non-oscillatory and approaches* (u^*,w^*) *monotonically if* s *is sufficiently large, while if* $\gamma > \gamma^*$, (u,w) *have exponentially damped oscillations about* (u^*,w^*) *as* $s \to \infty$.
Furthermore, we have

$$\gamma^* = \frac{c}{27\delta_2 q(u^*)}[(2\omega_c^2 + 6\delta_1)\sqrt{\omega_c^2 + 3\delta_1} - (2\omega_c^3 + 9\delta_1\omega_c)],$$

where $\omega_c := \frac{-\delta_1}{c} + c$, $\delta_1 := -k(u^*)q(u^*)$, $\delta_2 := (2A + Bu^*)(1-u^*)$ *and*

$$q(u) := \frac{1}{A + Bu + u^2}, \quad k(u) := -2u^3 + (1-B)u^2 - A.$$

Remark 2.1 The results in [15, Theorem 2.2] is a special case in (2) of Theorem 2.1, i.e. $B = 0$. Our contributions to this special case are as follows: (I) The existence of type I waves for cases $c = c_*$ and $c > c_*$, $u^* = u_1$, $u^* \leq u_0$ was not in that paper. (II) Type II waves and the bounded waves in our case $B = 0$, $27A < (1-B)^3$, $u_0 < u^* < u_1$ were not discussed in [15]. In fact, the latter can only be obtained by a bounded Wazewski set, not the unbounded one as in their work. (III) Non-monotone traveling waves were not discussed in [15].

The proof of Theorem 2.1 will be given in the following three sections. In Sections 3 and 4, we prove the assertion (1) and (2) respectively. The procedure of proofs will be divided into several steps with the main argument stated as several Lemmas for easy understanding.

Consider the differential equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \tag{2.8}$$

where $\mathbf{f}(t,\mathbf{y})$ is a continuous function defined on an open $(t,\mathbf{y})$-set Ω. Let Ω_0 be an open subset of Ω, $\partial\Omega_0$ be the boundary and $\bar{\Omega}_0$ be the closure of Ω_0.

Definition 2.1 [10, pp. 278] A point $(t_0, \mathbf{y}_0) \in \Omega \cap \partial\Omega_0$ is called an egress point of Ω_0, with respect to (2.8), if for every solution $\mathbf{y} = \mathbf{y}(t)$ of (2.8) satisfying

$$\mathbf{y}(t_0) = \mathbf{y}_0, \tag{2.9}$$

there is an $\epsilon > 0$ such that $(t, \mathbf{y}(t)) \in \Omega_0$ for $t_0 - \epsilon \leq t < t_0$. An egress point $(t_0, \mathbf{y}_0)$ of Ω_0 is called a strict egress point of Ω_0 if $(t, \mathbf{y}(t)) \notin \bar{\Omega}_0$ for $t_0 < t < t_0 + \epsilon$ with a small $\epsilon > 0$. The set of egress points of Ω_0 will be denoted by Ω_0^e and the set of strict egress points by Ω_0^{se}. As a sufficient condition, the set of egress points Ω_0^e can be determined by verifying $\overrightarrow{f} \cdot \overrightarrow{n} > 0$, where $\overrightarrow{f}$ is the vector field, and $\overrightarrow{n}$ is the outward normal vector of $\partial\Omega_0$.

If U is a topological space and V a subset of U, a continuous mapping $\pi : U \to V$ defined on all of U is called a retraction of U onto V if the restriction $\pi|_V$ of V to V is the identity; i.e., $\pi(u) \in V$ for all $u \in U$ and $\pi(v) = v$ for all $v \in V$. When there exists a retraction of U onto V, V is called a retract of U.

Lemma 2.3 *[10, pp. 279] Let $f(t, \mathbf{y})$ be continuous on an open $(t, \mathbf{y})$-set Ω with the property that an initial value determines a unique solution of (2.8). Let Ω_0 be an open subset of Ω satisfying $\Omega_0^e = \Omega_0^{se}$. Let S be a nonempty subset of $\Omega_0 \cup \Omega_0^e$ such that $S \cap \Omega_0^e$ is not a retract of S but is a retract of Ω_0^e. Then there exists at least one point $(t_0, \mathbf{y}_0) \in S \cap \Omega_0$ such that the solution arc $(t, \mathbf{y}(t))$ of (2.8)- (2.9) is contained in Ω_0 on its right maximal interval of existence.*

3 The Existence of Type I Traveling Waves

At the equilibrium (1, 0, 0), (2.3) becomes

$$J_{(1,0,0)} = \begin{pmatrix} -\frac{1}{c} & -\frac{1}{c(A+B+1)} & 0 \\ 0 & 0 & 1 \\ 0 & \gamma\left(1 - \frac{\beta}{A+B+1}\right) & c \end{pmatrix}. \tag{3.1}$$

Then by (3.1), we see that the eigenvalues of (2.2) at (1, 0, 0) are

$$\mu_1 = -\frac{1}{c}, \quad \mu_2 = \frac{c - \sqrt{c^2 - 4\gamma\left(\frac{\beta}{A+B+1} - 1\right)}}{2}, \quad \mu_3 = \frac{c + \sqrt{c^2 - 4\gamma\left(\frac{\beta}{A+B+1} - 1\right)}}{2}.$$

If $\mu_2 \neq \mu_3$, then the eigenvectors corresponding to μ_2, μ_3 are

$$\begin{aligned} \mathbf{X}_2 &= \left(\frac{-1}{A+B+1}, 1 + c\mu_2, \mu_2(1 + c\mu_2)\right)^T, \\ \mathbf{X}_3 &= \left(\frac{-1}{A+B+1}, 1 + c\mu_3, \mu_3(1 + c\mu_3)\right)^T. \end{aligned}$$

Proof of the non-existence of the type I waves for $0 < c < c_*$. If $0 < c < c_*$, then μ_2 and μ_3 are complex. We shall show that the heteroclinic orbit Γ of (2.2) would oscillate around $w = 0$ for large negative s. The heteroclinic solution Γ satisfies:

$$w'(s) = z, \quad z'(s) = cz - \gamma w\left(\frac{\beta u^2}{A + Bu + u^2} - 1\right). \tag{3.2}$$

Let $\ell := \gamma(\frac{\beta}{A+B+1} - 1)$. Since $\beta > A+B+1$, we know that $\ell > 0$. In a small neighborhood of (1, 0, 0), using the polar coordinates $w = \rho\sin(\theta),\ z = \rho\cos(\theta)$, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{z^2 - cwz + \gamma w^2(\frac{\beta u^2}{A+Bu+u^2} - 1)}{\rho^2}.$$

Note $c_* = \sqrt{4\ell}$. For any given $c \in (0, c_*)$, let ε be a positive number such that $0 < \varepsilon < \ell - \frac{c^2}{4}$. Since $\lim_{s\to-\infty} u(s) = 1$, one can choose large $s_c > 0$ such that

$$\ell - \varepsilon < \gamma(\frac{\beta u^2(s)}{A + Bu(s) + u^2(s)} - 1) < \ell + \varepsilon \text{for} s < -s_c.$$

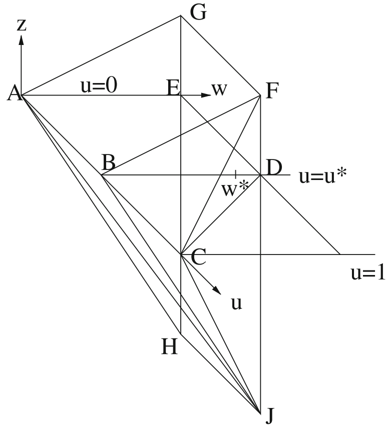

Fig. 3 The graph of set $\mathbb{W}$

Therefore, for $s < -s_c$, we have

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}s} &> \cos^2(\theta) - c\sin(\theta)\cos(\theta) + (\ell - \varepsilon)\sin^2(\theta)\\ &= [\cos(\theta) - \frac{c}{2}\sin(\theta)]^2 + (\ell - \frac{c^2}{4} - \epsilon)\sin^2(\theta). \end{aligned} \tag{3.3}$$

The last expression is a periodic function of θ and is nonzero, which must be bounded below by a constant $c_0 > 0$. Therefore if $s < -s_c$, we have $\frac{\mathrm{d}\theta}{\mathrm{d}s} > c_0 > 0$, which can leads to $\lim_{s\to-\infty} \theta(s) = -\infty$. From $w = \rho\sin(\theta)$, $w(s)$ would be negative for some $s < -s_c$. This violates the requirement that the traveling wave solution considered should be nonnegative. The proof is complete. □

We now assume that $c \geq c_*$. Counting multiplicity, then there are three real eigenvalues satisfying $\mu_1 < 0 < \mu_2 \leq \mu_3$. If $c > c_*$, then $\mu_2 < \mu_3$, there are two eigenvectors $(\mathbf{X}_2, \mathbf{X}_3)$ corresponding to (μ_2, μ_3). If $c = c_*$, then $\mu_2 = \mu_3$ is a double eigenvalue. There are two generalized eigenvectors $(\mathbf{X}_2, \mathbf{X}_3)$ corresponding to $\mu_2 = \mu_3$. By Theorems 6.1 and 6.2 in [10, pp. 242-244], there exists a two dimensional local unstable manifold $W^u_{loc}(E_1)$ tangent to the span of $\mathbf{X}_2, \mathbf{X}_3$. The points on $W^u_{loc}(E_1)$ can be represented by the local coordinates $\Phi_2 : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\Phi_2(m, n) = (1, 0, 0)^T + m \cdot \mathbf{X}_2 + n \cdot \mathbf{X}_3 + o(|m| + |n|).$$

Consider a prism shaped solid $\mathbb{W}$ (see Fig. 3) in (u, w, z) space bounded by the following five surfaces:

(1) The top surface $\mathbb{F}_t := \left\{(u, w, z) \mid z = \frac{c}{2}w, 0 < u < 1, 0 < w < w_m\right\}$, where

$$w_m := K^*(1 - u^*), \quad K^* := \frac{(1 + cd)[A + Bu^* + (u^*)^2]}{(u^*)^2} = (1 + cd)\beta, \quad d > \frac{c}{2}.$$

Namely, the quadrilateral $ABCFG$.

(2) The bottom surface $\mathbb{F}_b := \left\{(u, w, z) \mid z = -\frac{\gamma}{c}w, 0 < u < 1, 0 < w < w_m\right\}$. Namely, the quadrilateral $ABCJH$.

(3) The front surface $\mathbb{F}_f := \left\{(u, w, z) \mid w = K^*(1 - u), u^* < u < 1, -\frac{\gamma}{c}w < z < \frac{c}{2}w\right\}$. Namely, the triangle $CFDJ$.

(4) The right vertical surface $\mathbb{F}_r := \{(u, w, z) \mid w = w_m, 0 < u < u^*, -\frac{\gamma}{c} w_m < z < \frac{c}{2} w_m\}$. Namely, the quadrilateral $GFDJHE$, where one part $GFDE$ is above the plane $z = 0$, and the other part $EDJH$ is below the plane $z = 0$.
(5) The back surface $\mathbb{F}_k := \{(u, w, z) \mid u = 0, 0 < w < w_m, -\frac{\gamma}{c} w < z < \frac{c}{2} w\}$. Namely, the triangle AHG.

Note that $u^* \in (0, 1)$, thus we obtain

$$w_m = K^*(1 - u^*) = (1 - u^*) \cdot \frac{(1 + cd)[A + Bu^* + (u^*)^2]}{(u^*)^2} = \frac{1 + cd}{u^*} w^* > w^*.$$

Next, we will prove the following lemma.

Lemma 3.1 *If an initial point $P_0 = (u(0), w(0), z(0))$ is picking from the interior of $\mathbb{W}$, then the flow $\phi(s, P_0)(s \geq 0)$ of (2.2) can only exit $\mathbb{W}$ from $\mathbb{F}_t, \mathbb{F}_b, \mathbb{F}_r \cap \{z > 0\}$ or the line segment GF.*

Proof (1) The outward normal vector of $\mathbb{F}_t$ is $\overrightarrow{n_t} = (0, -\frac{c}{2}, 1)$, and the vector field is

$$\overrightarrow{f} = \left(\frac{u}{c}\left(1 - u - \frac{uw}{A + Bu + u^2}\right), z, cz + \gamma w\left(1 - \frac{\beta u^2}{A + Bu + u^2}\right)\right).$$

Then it follows that

$$\begin{aligned}
\overrightarrow{n_t} \cdot \overrightarrow{f} &= -\frac{c}{2} z + cz + \gamma w\left(1 - \frac{\beta u^2}{A + Bu + u^2}\right) \\
&= \frac{c}{2} z + \gamma w\left(1 - \frac{\beta u^2}{A + Bu + u^2}\right) \\
&= \frac{c^2}{4} w + \gamma w\left(1 - \frac{\beta u^2}{A + Bu + u^2}\right) \\
&> \frac{c^2}{4} w + \gamma w\left(1 - \frac{\beta}{A + B + 1}\right) \\
&= w\left[\frac{c^2}{4} + \gamma\left(1 - \frac{\beta}{A + B + 1}\right)\right] \geq 0
\end{aligned} \tag{3.4}$$

by the assumption $c \geq c_*$. Therefore, the top surface $\mathbb{F}_t$ belongs to the egress set.
(2) The outward normal vector of $\mathbb{F}_b$ is $\overrightarrow{n_b} = (0, -\frac{\gamma}{c}, -1)$. Then it follows that

$$\begin{aligned}
\overrightarrow{n_b} \cdot \overrightarrow{f} &= -\frac{\gamma}{c} z - cz - \gamma w\left(1 - \frac{\beta u^2}{A + Bu + u^2}\right) \\
&= \left(\frac{\gamma}{c}\right)^2 w + \gamma w - \gamma w + \gamma w \frac{\beta u^2}{A + Bu + u^2} \\
&= \left(\frac{\gamma}{c}\right)^2 w + \gamma w \frac{\beta u^2}{A + Bu + u^2} > 0,
\end{aligned}$$

and this implies that the bottom surface $\mathbb{F}_b$ belongs to the egress set.

(3) The outward normal vector of $\mathbb{F}_f$ is $\overrightarrow{n_f} = (K^*, 1, 0)$. Then

$$\begin{aligned}
\overrightarrow{n_f} \cdot \overrightarrow{f} &= \frac{K^*}{c} u \left(1 - u - \frac{uw}{A + Bu + u^2}\right) + z \\
&< \frac{K^*}{c} u \left(1 - u - \frac{uw}{A + Bu + u^2}\right) + \frac{c}{2} w \\
&= \frac{K^*}{c} u \left[1 - u - \frac{uK^*(1-u)}{A + Bu + u^2}\right] + \frac{c}{2} K^*(1-u) \\
&= K^*(1-u) \left[\frac{u}{c} - \frac{K^*}{c} \frac{u^2}{A + Bu + u^2} + \frac{c}{2}\right] \\
&< K^*(1-u) \left[\frac{1}{c} - \frac{K^*}{c} \frac{(u^*)^2}{A + Bu^* + (u^*)^2} + \frac{c}{2}\right] \\
&= K^*(1-u) \left[\frac{1}{c} - \frac{1 + cd}{c} + \frac{c}{2}\right] = K^*(1-u) \left(\frac{c}{2} - d\right) < 0
\end{aligned}$$

by the assumption $d > \frac{c}{2}$. Therefore, the front surface $\mathbb{F}_f$ belongs to the ingress set.

(4) The right vertical surface $\mathbb{F}_r$ is further divided into $\mathbb{F}_r = \mathbb{F}_{r_1} \cup \mathbb{F}_{r_2} \cup \mathbb{F}_{r_3}$, with $\mathbb{F}_{r_1} := \mathbb{F}_r \cap \{z > 0\}$, $\mathbb{F}_{r_2} := \mathbb{F}_r \cap \{z < 0\}$, $\mathbb{F}_{r_3} := \mathbb{F}_r \cap \{z = 0\}$. The outward normal vector of $\mathbb{F}_r$ is $\overrightarrow{n_r} = (0, 1, 0)$. Then $\overrightarrow{n_r} \cdot \overrightarrow{f} = z$ and this implies that $\mathbb{F}_{r_1}$ belongs to the egress set while $\mathbb{F}_{r_2}$ belongs to the ingress set.

(5) It is easy to see that the flow of (2.2) cannot enter or exit $\mathbb{W}$ through $\mathbb{F}_k$ since $u = 0$ is invariant under the system (2.2).

In order to check all the intersection part of two surfaces for the solid $\mathbb{W}$, we make the following observation:

If M and N are two surfaces that intersect along a line ℓ, and if the vector field is transverse to these two surfaces, then the line ℓ is on the egress set if and only if the vector field points outward to both surfaces M and N.

From this observation, we see that the line segments GF is on the egress set. It is easy to check that $\mathbb{F}_{r_3} = \mathbb{F}_r \cap \{z = 0\}$ is not on the egress set. Furthermore, notice that AC is a part of u axis which is an invariant set of (2.2), which is not a part of the egress set.

Thus, we have shown that $\mathbb{F}_t, \mathbb{F}_b, \mathbb{F}_{r_1} = \mathbb{F}_r \cap \{z > 0\}$ and the line segment GF belong to the egress set, and no more. The proof of the lemma is complete. □

Definition 3.1 Consider a two dimensional surface $\bar{S} = \text{span}\{\mathbf{u_1}, \mathbf{u_2}\}$ in $\mathbb{R}^3$ and let $\vec{n}$ be a normal vector to $\bar{S}$. Any vector $\mathbf{w} \notin \bar{S}$ is on the positive (or negative) side of $\bar{S}$ with respect to the normal $\vec{n}$ if $\vec{n} \cdot \mathbf{w} > 0$ (or < 0).

We say the two points $\mathbf{w_1}$ and $\mathbf{w_2}$ are on the same side of $\bar{S}$ (or on the opposite side of $\bar{S}$) if the product $(\vec{n} \cdot \mathbf{w_1})(\vec{n} \cdot \mathbf{w_2}) > 0$ (or < 0).

In particular, if we choose $\vec{n} = \mathbf{u_1} \times \mathbf{u_2}$, then $\vec{n} \cdot \mathbf{w} = \det(\mathbf{u_1}, \mathbf{u_2}, \mathbf{w})$.

Lemma 3.2 *There exists an arc $\mathcal{C} = \overline{C_1C_2} \subset \mathbb{W} \cap W^u_{loc}(E_1)$ of which one end point $C_1 \in \mathbb{F}_t$, and the other end point $C_2 \in \mathbb{F}_b$.*

Proof First, assume that $c > c_*$. Then E_1 has two distinct positive eigenvalues. The two dimensional unstable manifold is tangent to $S = \text{span}(\mathbf{X}_2, \mathbf{X}_3) = \text{span}(\mathbf{X}_2, -\mathbf{X}_3)$. Recall that $\overrightarrow{CF} = (u^* - 1, w_m, \frac{c}{2} w_m)^T$ and $\overrightarrow{CJ} = (u^* - 1, w_m, -\frac{\gamma}{c} w_m)^T$ are the vectors corresponding to the rays CF and CJ, and $\vec{u} = (1, 0, 0)^T$ be the unit vector on the $u-$axis with a start point $C = (1, 0, 0)$.

Note that $\beta > A + B + 1$, $c > c_*$ and $d > \frac{c}{2}$, we obtain

$$\begin{aligned}
\det(\mathbf{X}_2, \mathbf{X}_3, \overrightarrow{CF}) &= \frac{w_m}{A+B+1}\left(1+\frac{c^2}{2}\right) \\
&\quad \times(\mu_3-\mu_2) - (1-u^*)(1+c\mu_2)(1+c\mu_3)(\mu_3-\mu_2) \\
&= (1-u^*)(\mu_3-\mu_2)\left[\frac{K^*}{A+B+1}\left(1+\frac{c^2}{2}\right) - (1+c\mu_2)(1+c\mu_3)\right] \\
&= (1-u^*)(\mu_3-\mu_2)\left[\frac{(1+cd)\beta}{A+B+1}\left(1+\frac{c^2}{2}\right) - 1 - c^2 - c^2\cdot\frac{c_*^2}{4}\right] \\
&> (1-u^*)(\mu_3-\mu_2)\left[\left(1+c\cdot\frac{c}{2}\right)\left(1+\frac{c^2}{2}\right) - 1 - c^2 - \frac{c^4}{4}\right] = 0,
\end{aligned}$$

$\det(\mathbf{X}_2, \mathbf{X}_3, \vec{u}) = (\mu_3-\mu_2)(1+c\mu_2)(1+c\mu_3) > 0,$

$$\begin{aligned}
\det(\mathbf{X}_2, -\mathbf{X}_3, \overrightarrow{CJ}) &= -(1-u^*)(\mu_3-\mu_2) \\
&\quad \times\left[\frac{K^*}{A+B+1}\left(1+\gamma+c^2\right) - (1+c\mu_2)(1+c\mu_3)\right] < 0,
\end{aligned}$$

$\det(\mathbf{X}_2, -\mathbf{X}_3, \vec{u}) = -\det(\mathbf{X}_2, \mathbf{X}_3, \vec{u}) < 0.$

This implies that $\vec{u}$ and $\overrightarrow{CF}$ are on the same side of the tangent plane S. Also, $\vec{u}$ and $\overrightarrow{CJ}$ are on the same side of S.

Next assume that $c = c_*$. Then E_1 has a double real eigenvalue $\mu_2 = \mu_3 = \frac{c}{2}$. The two dimensional unstable manifold is tangent to $S = \mathrm{span}(\mathbf{X}_2, \mathbf{X}_3)$, where $\mathbf{X}_2$ and $\mathbf{X}_3$ are generalized eigenvectors corresponding to μ_2. Denote the first row of the matrix

$$\left(\frac{c}{2}I - J_{(1,0,0)}\right)^2 = \begin{pmatrix} \left(\frac{c}{2}+\frac{1}{c}\right)^2 & \frac{\frac{c}{2}+\frac{1}{c}}{c(A+B+1)} + \frac{1}{2(A+B+1)} & \frac{-1}{c(A+B+1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

by $\mathbf{r}_1$. Since any generalized eigenvector $\mathbf{X}$ satisfies $(\frac{c}{2}I - J_{(1,0,0)})^2\mathbf{X} = 0$, we see that $\mathbf{r}_1$ is a normal vector to S. Then for any $\vec{v} \in \mathbb{R}^3$, whether $\vec{v}$ is on which side of S can be determined by the sign of the inner product $\mathbf{r}_1 \cdot \vec{v}$. Recall that $\vec{u} = (1,0,0)^T$ and $\frac{\overrightarrow{CF}}{w_m} = (-\frac{1}{K^*}, 1, \frac{c}{2})^T$, it is easily checked

$$\mathbf{r}_1 \cdot \vec{u} = \left(\frac{c}{2}+\frac{1}{c}\right)^2 > 0,$$

$$\begin{aligned}
&\mathbf{r}_1 \cdot \frac{\overrightarrow{CF}}{w_m} \\
&= \left[\frac{\frac{c}{2}+\frac{1}{c}}{c(A+B+1)} + \frac{1}{2(A+B+1)}\right] - \frac{(\frac{c}{2}+\frac{1}{c})^2}{K^*} - \frac{1}{2(A+B+1)} \\
&= \left(\frac{c}{2}+\frac{1}{c}\right)\cdot\frac{K^* - (A+B+1)(1+\frac{c^2}{2})}{c(A+B+1)K^*}.
\end{aligned}$$

Since $K^* = (1+cd)(\frac{A}{(u^*)^2} + \frac{B}{u^*} + 1) > (1+cd)(A+B+1) \ge (1+\frac{c^2}{2})(A+B+1)$, we have $\mathbf{r}_1 \cdot \overrightarrow{CF} > 0$.

Next recall that $\frac{\overrightarrow{CJ}}{w_m} = (-\frac{1}{K^*}, 1, -\frac{\gamma}{c})^T$. It is easily verified

$$\mathbf{r}_1 \cdot \frac{\overrightarrow{CJ}}{w_m} > \mathbf{r}_1 \cdot \frac{\overrightarrow{CF}}{w_m} > 0.$$

In conclusion, $\vec{u}$, $\overrightarrow{CF}$ and $\overrightarrow{CJ}$ are on the same side of S just like the case $c > c_*$. Denote this side as the "positive-u side of S".

In geometry, same notations are often used for vectors and rays. In the rest of the section, we assume $(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$ are three rays with vertex C. Let $\text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$ be the convex hull generated by the rays $\overrightarrow{CF}$, $\overrightarrow{CJ}$ and the ray pointing to the positive $u-$axis with the starting point C. This is an infinite cone of triangular cross section and with the vertex C. Then except for the point C, $\text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$ is on the "positive-u side" of the tangent plane S.

Define

$$S_1 := \left\{(u, w, z) \mid z = \frac{c}{2}w\right\}, \quad S_2 := \left\{(u, w, z) \mid z = -\frac{\gamma}{c}w\right\}.$$

Note that $\mathbb{F}_t$ ($\mathbb{F}_b$) is a part of S_1 (S_2).

Note the plane S intersects with S_1 transversely and S is the tangent plane of $W^u_{loc}(E_1)$ at E_1, the unstable manifold $W^u_{loc}(E_1)$ intersects with S_1 on a smooth line segment ℓ_1. If we make ℓ_1 sufficiently short, then ℓ_1 does not enter $\text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$. Denote by ℓ_t the part of ℓ_1 where $z > 0$ and which is on $\mathbb{F}_t$. We now select C_1 on ℓ_t. It is clear that $C_1 \in W^u_{loc}(E_1) \cap \mathbb{F}_t$ but not in $\text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$.

Similarly, by using the property of S_2, we can construct a short line segment ℓ_b which is on $W^u_{loc}(E_1) \cap \mathbb{F}_b$ but not in $\text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$. Select a point $C_2 \in \ell_b$. Using the local coordinates $u = \tilde{u}(w, z)$ for $W^u_{loc}(E_1)$, we can construct a curve segment $\mathcal{C} = \overline{C_1C_2}$ on $W^u_{loc}(E_1)$, connecting C_1 and C_2 and is between the two surfaces $\mathbb{F}_t$ and $\mathbb{F}_b$.

Let $d(P_1, P_2)$ be the distance function in $\mathbb{R}^3$. Define the minimum distance

$$\eta := \min_{P_1, P_2} \left\{d(P_1, P_2) \mid \overrightarrow{CP_1} \in \text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u}), \|CP_1\| = 1, P_2 \in S\right\}.$$

Then $0 < \eta \le 1$. If $\overrightarrow{CQ}$ is a nonzero vector in $\text{co}(\overrightarrow{CF}, \overrightarrow{CJ}, \vec{u})$, then the distance $d(Q, S) \ge \eta\rho$, where $\rho = \|CQ\|$. On the other hand, since $W^u_{loc}(E_1)$ is tangent to S, the distance of any point $Q \in \overline{C_1C_2}$ to S is $O(\rho^2)$. If ρ is sufficiently small, $O(\rho^2) < \eta\rho$. Based on this, we find that the entire curve $\overline{C_1C_2}$ is inside $\mathbb{W}$ if its distance to the vertex C is sufficiently small. □

Lemma 3.3 *There is a point $P_0 = (u_0, w_0, z_0) \in \mathcal{C} \cap \mathbb{W}$ such that the flow $\phi(s, P_0)$ will remain in $\mathbb{W}$ for all $s \ge 0$.*

Proof Note that $A + Bu + u^2 = 0$ has roots $\frac{-B\pm\sqrt{B^2-4A}}{2}$. Set

$$\hat{u} = \begin{cases} \frac{-B+\sqrt{B^2-4A}}{2}, & B^2 - 4A \ge 0, \\ -\infty, & B^2 - 4A < 0. \end{cases}$$

Define $\mathcal{D} = (\hat{u}, \infty)$, then u', z' has no singularity on $\mathcal{D}$. Let $\Omega = \mathcal{D} \times \mathbb{R}^2$ and $\Omega_0 = \mathbb{W}$. Then $\Omega_0^e = \Omega_0^{se} = \mathbb{F}_t \cup (\mathbb{F}_r \cap \{z > 0\}) \cup GF \cup \mathbb{F}_b$. Also, $C_1 \cup C_2$ is not a retract of $\mathcal{C}$ but a retract of Ω_0^e. By Lemma 2.3, there is a point $P_0 \in \mathcal{C} \cap \mathbb{W}$ such that the flow $\phi(s, P_0)$ will remain in $\mathbb{W}$ for all $s \ge 0$. □

Lemma 3.4 *Let P_0 be defined as in Lemma 3.3. Then $\phi(s, P_0) \to (u^*, w^*, 0)^T$ as $s \to +\infty$.*

Proof Let u^- be the negative root of the equation $1-\frac{\beta u^2}{A+Bu+u^2}=0$, then the two roots of $1-\frac{\beta u^2}{A+Bu+u^2}=0$ are u^* and u^-. Define a Lyapunov function

$$V(u,w,z)=c\gamma\int_{u^*}^{u}\frac{(\xi-u^*)(\xi-u^-)}{\xi^2}\mathrm{d}\xi+[cw-w^*-z]+w^*\left[\frac{z}{w}-c\log\frac{w}{w^*}\right].$$

It is easy to check that $V(u,w,z)$ is continuously differentiable and bounded below on the compact set $\mathbb{W}$. Moreover,

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}s}&=\frac{\partial V}{\partial u}u'+\frac{\partial V}{\partial w}w'+\frac{\partial V}{\partial z}z'\\
&=\frac{c\gamma(u-u^*)(u-u^-)}{u^2}\cdot\frac{u}{c}\left(1-u-\frac{uw}{A+Bu+u^2}\right)+\left(c-\frac{w^*z}{w^2}-\frac{cw^*}{w}\right)z\\
&\quad+\left(\frac{w^*}{w}-1\right)\left[cz+\gamma w\left(1-\frac{\beta u^2}{A+Bu+u^2}\right)\right]\\
&=\frac{\gamma(u-u^*)(u-u^-)}{A+Bu+u^2}[h(u)-w]+\gamma\left[\frac{\beta u^2}{A+Bu+u^2}-1\right](w-w^*)-\frac{w^*z^2}{w^2}\\
&=\frac{\gamma(u-u^*)(u-u^-)}{A+Bu+u^2}[h(u)-w]+\frac{\gamma(u-u^*)(u-u^-)}{A+Bu+u^2}(w-w^*)-\frac{w^*z^2}{w^2}\\
&=\frac{\gamma(u-u^*)(u-u^-)}{A+Bu+u^2}[h(u)-w^*]-\frac{w^*z^2}{w^2}\\
&=\frac{\gamma(u-u^-)}{A+Bu+u^2}(u-u^*)[h(u)-w^*]-\frac{w^*z^2}{w^2},
\end{aligned}$$

where $h(u)$ is defined in Sect. 2 and $w^*=h(u^*)$. By Lemma 2.2, it immediately follows that $(u-u^*)[h(u)-w^*]\le 0$ for $0<u<1$. Furthermore, $\frac{\mathrm{d}V}{\mathrm{d}s}=0$ if and only if $\{u=u^*, 0<w<w_m, z=0\}$. The largest invariant subset of this line segment in $\mathbb{W}$ is the positive equilibrium $(u^*,w^*,0)$. By the LaSalle's invariance principle [14], it follows that $\phi(s,P_0)\to(u^*,w^*,0)^T$ as $s\to+\infty$. □

Proof of existence of traveling waves for $c\ge c_*$**.** Assume that one of the conditions in Lemma 2.2 holds. Choose a point P_0 in $\mathcal{C}\cap\mathbb{W}$ as defined by Lemma 3.3. By Lemmas 3.3–3.4, we see that $\phi(s,P_0)$ will remain in $\mathbb{W}$ and further approach the positive equilibrium $(u^*,w^*,0)^T$. Also, $\phi(s,P_0)\to(1,0,0)^T$ as $s\to-\infty$ since $P_0\in W^u_{loc}(E_1)$. Thus, a type I traveling wave solution has been constructed for $c\ge c_*$.

If $0\le B<1$, $27A<(1-B)^3$, $u_0<u^*<u_1$, then by Lemma 3.3, there is a traveling wave $\phi(s,P_0)$ with $\lim_{s\to-\infty}\phi(s,P_0)=(1,0,0)$ and $\phi(s,P_0)\in\mathbb{W}$ for $s\ge 0$. Note that the projection of $\mathbb{W}$ on (u,w) plane is $\mathbb{W}_{(u,w)}:=\{(u,w)|0\le w\le K^*(1-u), 0\le u\le 1\}$, where $K^*=\beta(1+cd)$. The proof has been completed. □

4 The Existence of Type II Traveling Waves

At the equilibrium $E = (0, 0, 0)$, (2.3) becomes

$$J_{(0,0,0)} = \begin{pmatrix} \frac{1}{c} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \gamma & c \end{pmatrix}. \tag{4.1}$$

Then the eigenvalues of (2.2) at E are

$$\nu_1 = \frac{c - \sqrt{c^2 + 4\gamma}}{2} < 0, \quad \nu_2 = \frac{1}{c} > 0, \quad \nu_3 = \frac{c + \sqrt{c^2 + 4\gamma}}{2} > 0.$$

Thus there is a two dimensional local unstable manifold $W^u_{loc}(E)$ based at E. The eigenvectors corresponding to ν_2, ν_3 are respectively $\mathbf{Y}_2 = (1, 0, 0)^T, \mathbf{Y}_3 = (0, 1, \nu_3)^T$. Let $L = (0, 1, 0)^T$ be a vector that is complementary to the plane $S = \text{span}\{\mathbf{Y}_2, \mathbf{Y}_3\}$. In a small neighborhood of E, the points on $W^u_{loc}(E)$ can be expressed as:

$$(u, w, z)^T = (0, 0, 0)^T + m \cdot \mathbf{Y}_2 + n \cdot \mathbf{Y}_3 + \ell^*(m, n)L,$$

where $\ell^*(m, n) = O(m^2 + n^2)$ is a smooth function of (m, n). Since the u-axis is invariant under the flow of (2.2), if m is small, then $m\mathbf{Y}_2 \in W^u_{loc}(E)$, that is, $\ell^*(m, 0) = 0$.

On the other hand, equation (2.2) is linear if $u = 0$. For any $n \in \mathbb{R}$, the line $n\mathbf{Y}_3$ is invariant under the flow and $n\mathbf{Y}_3 \in W^u_{loc}(E)$. This shows $\ell^*(0, n) = 0$. Based on $\ell^*(m, 0) = \ell^*(0, n) = 0$, we have a better estimate $\ell^*(m, n) = O(|mn|)$. And any point on $W^u_{loc}(E)$ can be expressed as

$$Q(m, n) = (u, w, z)^T = (0, 0, 0)^T + m \cdot \mathbf{Y}_2 + n \cdot \mathbf{Y}_3 + (O(|mn|)) \cdot L. \tag{4.2}$$

Proof of the non-existence of the type II waves for $c \geq c_*$**.** We present an indirect proof. If there is a heteroclinic solution $\left\{\mathbf{y}(s) = (u(s), w(s), z(s))^T : s \in \mathbb{R}\right\}$ connecting E to E^*, then for sufficiently large negative s_1, we have $\mathbf{y}(s_1) \in W^u_{loc}(E)$. Let $(m(s), n(s))$ be the parameter representation for $\mathbf{y}(s)$ as in (4.2). Then we claim that $n(s_1) > 0$. For from (4.2), if $n(s_1) < 0$, then $w(s_1) < 0$, which is meaningless in the biological context. If $n(s_1) = 0$, then $w(s_1) = z(s_1) = 0$, thus $\mathbf{y}(s_1)$ is on the u-axis. However, the u-axis is an invariant manifold connecting E to E_1, this is a contradiction.

Using $n(s_1) > 0$, we have $z(s_1) = \nu_3 w(s_1) + o(|n|) > \frac{c}{2} w(s_1)$. We can show that $z(s) > \frac{c}{2} w(s)$ and $w(s) > 0$ for all $s > s_1$. Here is the proof.

Consider the open set $\Lambda := \left\{0 < u < 1, w > 0, z > \frac{c}{2} w\right\}$. Let $\chi(s) = z(s) - \frac{c}{2} w(s)$. From (2.2), we calculate $\chi'(s)$ in Λ:

$$\begin{aligned} \chi'(s) &= cz(s) + \gamma w(s) \left[1 - \frac{\beta u^2(s)}{A + Bu(s) + u^2(s)}\right] - \frac{c}{2} z(s) \\ &> w(s) \left[\frac{c^2}{4} + \gamma \left(1 - \frac{\beta u^2(s)}{A + Bu(s) + u^2(s)}\right)\right] \\ &\geq w(s) \left[\frac{c_*^2}{4} + \gamma \left(1 - \frac{\beta}{A + B + 1}\right)\right] = 0. \end{aligned}$$

Now the heteroclinic solution $\mathbf{y}(s)$ satisfies $\mathbf{y}(s_1) \in \Lambda$ and $0 < u(s) < 1$ for all s. We can prove $\mathbf{y}(s) \in \Lambda$ for all $s > s_1$ by contradiction. Assume that $s_2 > s_1$ is the first time that $\mathbf{y}(s)$ hits the boundary of Λ. Then either (1) $w(s_2) = 0$ or (2) $\chi(s_2) = 0$. Case (1) is impossible since for $s_1 < s < s_2$, $w'(s) = z(s) > 0$ in Λ. Case (2) is also impossible since for $s_1 < s < s_2$, $\chi'(s) > 0$ which leads to $\chi(s_2) > \chi(s_1) > 0$.

However, if $\mathbf{y}(s) \in \Lambda$ for $s \geq s_1$, then $\lim_{s\to\infty} \mathbf{y}(s) \neq E^*$. The proof has been completed. □

The existence of the type II waves for $0 < c < c_*$**.** Define a solid $\mathbb{W}_0$ which is a modification of $\mathbb{W}$ as in §3, Fig 3. Compared to $\mathbb{W}$, the top surface is replaced by $\mathbb{F}_t := \{(u, w, z) \mid 0 < u < 1, 0 < w < w_m, z = dw\}$ where $d > \nu_3$; the right vertical surface is replaced by $\mathbb{F}_r := \left\{(u, w, z) \mid 0 < u < u^*, w = w_m, -\frac{\gamma}{c} w_m < z < dw_m\right\}$; and the front surface is replaced by $\mathbb{F}_f := \left\{(u, w, z) \mid u^* < u < 1, w = K^*(1-u), -\frac{\gamma}{c} w < z < dw\right\}$. The back surface is $\mathbb{F}_k := \left\{(u, w, z) \mid u = 0, 0 < w < w_m, -\frac{\gamma}{c} w < z < dw\right\}$.

The bottom surface is unchanged.

Lemma 4.1 *The egress sets for $\mathbb{W}_0$ are the bottom surface $\mathbb{F}_b$ and part of the right vertical surface $\mathbb{F}_r \cap \{z > 0\}$.*

Proof Notice that $\{u = 0\}$ is an invariant set thus the back surface $\mathbb{F}_k$ is not an egress set.

The set $\mathbb{F}_r \cap \{z > 0\}$ is obviously an egress set due to $w' = z > 0$ there.

The bottom surface $\mathbb{F}_b$ remains the same and hence is an egress set.

The front surface $\mathbb{F}_f$ is larger than that of $\mathbb{W}$. However its normal vector remains the same. Modifying the estimates in the proof of Lemma 3.1, we have

$$
\begin{aligned}
\overrightarrow{n_f} \cdot \overrightarrow{f} &= \frac{K^*}{c} u \left(1 - u - \frac{uw}{A + Bu + u^2}\right) + z \\
&< \frac{K^*}{c} u \left(1 - u - \frac{uw}{A + Bu + u^2}\right) + dw \\
&= \frac{K^*}{c} u \left[1 - u - \frac{uK^*(1-u)}{A + Bu + u^2}\right] + dK^*(1-u) \\
&< K^*(1-u) \left[\frac{1}{c} - \frac{1 + cd}{c} + d\right] = 0.
\end{aligned}
$$

Thus, $\mathbb{F}_f$ is still an ingress set.

Since $\nu_3 < d$, similar to the proof of Lemma 3.1, we have

$$
\begin{aligned}
\overrightarrow{n_t} \cdot \overrightarrow{f} &= -dz + cz + \gamma w \left(1 - \frac{\beta u^2}{A + Bu + u^2}\right) \\
&= (-d + c) dw + \gamma w \left(1 - \frac{\beta u^2}{A + Bu + u^2}\right) \\
&= w \left[-d^2 + cd + \gamma \left(1 - \frac{\beta u^2}{A + Bu + u^2}\right)\right] \\
&< -w(d^2 - cd - \gamma) \\
&< -w(\nu_3^2 - c\nu_3 - \gamma) = 0.
\end{aligned}
$$

Hence, $\mathbb{F}_t$ belongs to the ingress set.

Whether the edges of $\mathbb{W}_0$ is an egress set can be checked easily. In particular, the line GF as in Fig. 3 is not an egress set. □

In order to use Lemmma 2.3, we shall construct a curve $\mathcal{E} \subset W^u(E) \cap \mathbb{W}_0$, of which the two end points belong to the two disjoint egress sets of $\mathbb{W}_0$.

Lemma 4.2 *Assume that $0 < c < c_*$, then there exists an $P_1 \in W^u_{loc}(E) \cap \Theta$ such that the flow $\phi(s, P_1)$ enters $Q = \{(u, w, z)|u > u^*, w < w^*, z < 0\}$ for some finite $s = \bar{s}$, where $\Theta = \{(u, w, z)|u \geq 0, w \geq 0, z \geq 0\}$.*

Proof The proof is similar to that Dunbar in Lemma 10 [2], and shall be skipped. □

Choose a constant $\bar{w}$ such that $w(\bar{s}) < \bar{w} < w^*$. Define a solid as $\mathbb{W}_2 := \text{co}(B, C, D, J) \cap \{w < \bar{w}\}$.

Lemma 4.3 *The only egress set for $\mathbb{W}_2$ is the egress set $\mathbb{F}_b$ for $\mathbb{W}_0$.*

Proof The surface $w = \bar{w}$, $z < 0$, where $w' < 0$, cannot be an egress set for $\mathbb{W}_2$.

The surface CDJ is part of $\mathbb{F}_f$ for $\mathbb{W}_0$ thus it cannot be an egress for $\mathbb{W}_2$.

The surface $BDJ \cap \{w < \bar{w}\}$ cannot be an egress set since $u' = 0$ at $(u = u^*, w = w^*)$ from the first equation of (2.2). We now have $u = u^*$ but $w < w^*$, therefore $u' > 0$ there.

On the surface $BCD \cap \{w < \bar{w}\}$, $z = 0$. From (2.2), $z'(s) = \gamma w \left(1 - \frac{\beta u^2}{A+Bu+u^2}\right) < 0$, since it would be 0 if $u = u^*$. But now we have $u > u^*$. □

Proof of the existence of the type II waves for $0 < c < c_*$. Let P_1 be the point as in Lemma 4.2. The orbit $\phi(s, P_1)$ enters $\mathbb{W}_0$ at some $s = s_3 \geq \bar{s}$. Let $P_3 = \phi(s_3, P_1)$. The flow $\phi(s, P_3)$ cannot stay in $\mathbb{W}_2$ for all $s > s_3$ because $w'(s) < 0$, $z'(s) < 0$ there. Therefore it must exit through its egress set $\mathbb{F}_b$ as from Lemma 4.3.

On the other hand, from (4.2), for a small real $n > 0$, $Q(0, n) := n\mathbf{Y}_3$ is on $W^u_{loc}(E)$. Moreover, from (2.2), the solution $\phi(s, Q(0, n))$ hits the surface $w = w_m$ transversely in finite time. From the expression (4.2), there is a small $m > 0$ such that the corresponding $Q(m, n)$ is near $Q(0, n)$ and its u-coordinate is small and positive. Thus, $Q(m, n) \in W^u_{loc}(E) \cap \mathbb{W}_0$ and the flow $\phi(s, Q(m, n))$ hits the surface $w = w_m$ in finite time and its u-coordinate is small and positive. Let $P_2 = Q(m, n)$. Then $\phi(s, P_2)$ exits $\mathbb{W}_0$ at $\mathbb{F}_r \cap \{z > 0\}$.

With the given end points P_1 and P_2 and using the local coordinates for $W^u_{loc}(E)$, we can construct a small curve $\overline{P_1 P_2} \subset W^u_{loc}(E) \cap \mathbb{W}_0$. Let s_4 be the first time that $\phi(s, P_1)$ hits $\mathbb{F}_b$ and let s_5 be the first time that $\phi(s, P_2)$ hits $\mathbb{F}_r \cap \{z > 0\}$. Then the curve $\mathcal{E}$ can be obtained as the union of three curves

$$\mathcal{E} := \{\phi(s, P_1) : 0 \leq s \leq s_4\} \cup \overline{P_1 P_2} \cup \{\phi(s, P_2) : 0 \leq s \leq s_5\}.$$

From Lemma 2.3, there exits a point $P_0 \in \overline{P_1 P_2}$ such that the flow $\phi(s, P_0)$ remains in $\mathbb{W}_0$ for $s > 0$. By using the same Lyapunov function as in Lemma 3.4, we see that $\phi(s, P_0) \to E^*$ as $s \to \infty$. The proof has been completed. □

5 Oscillation of the Traveling Wave Solutions

Evaluating the Jacobian matrix of (2.2) at $E^* = (u^*, w^*, 0)$, we have

$$J_{(u^*,w^*,0)} = \begin{pmatrix} \frac{1}{c}k(u^*)q(u^*) & -\frac{1}{c}(u^*)^2 q(u^*) & 0 \\ 0 & 0 & 1 \\ -\beta\gamma(2A + Bu^*)(1 - u^*)q(u^*) & 0 & c \end{pmatrix}, \tag{5.1}$$

where $k(u) = -2u^3 + (1 - B)u^2 - A$ (see Lemma 2.1) and $q(u) = \frac{1}{A+Bu+u^2}$. Observe that $\beta(u^*)^2 q(u^*) = 1$. Using γ as a parameter, the characteristic polynomial of $J_{(u^*,w^*,0)}$ is

$$p(\lambda, \gamma) = \lambda \left[\lambda - \frac{k(u^*)q(u^*)}{c}\right](c - \lambda) + \frac{\gamma}{c}(2A + Bu^*)(1 - u^*)q(u^*).$$

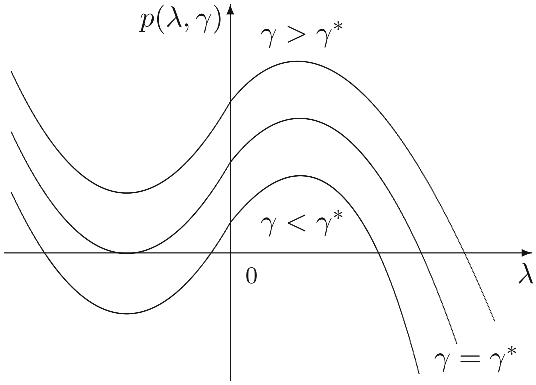

Fig. 4 The graph of $p(\lambda, \gamma)$ for $\gamma \geq 0$

Note that if $B \geq 1$, then $k(u^*) < 0$. If $0 \leq B < 1, \triangle \geq 0$, then $k(u^*) < 0$ also holds. If $0 \leq B < 1, \triangle < 0, 0 < u^* \leq u_0 < \alpha_1$ and $\alpha_2 < u_1 \leq u^* < 1$, we have $k(u^*) < 0$ (see page 6 for details) and thus $-k(u^*) > 0$. Let $\delta_1 := -k(u^*)q(u^*)$ and $\delta_2 := (2A + Bu^*)(1 - u^*)$. Then $\delta_1 > 0, \delta_2 > 0$ and

$$p(\lambda, \gamma) = -\lambda^3 + \left(\frac{-\delta_1}{c} + c\right)\lambda^2 + \delta_1\lambda + \frac{\gamma}{c}\delta_2 q(u^*).$$

Proof of (3) in Theorem 2.1. Based on

$$p'(\lambda, \gamma) = -3\lambda^2 + 2\left(\frac{-\delta_1}{c} + c\right)\lambda + \delta_1, \tag{5.2}$$

we find that $p'(0, \gamma) > 0$ and $p'(\lambda, \gamma)$ has two critical points $\lambda_- < 0 < \lambda_+$. Together with the fact that on the real line, $p(\lambda, \gamma) \to \mp\infty$ if $\lambda \to \pm\infty$, we find that when $\gamma = 0$, the graph of $p(\lambda, 0)$ is an "S" shaped function passing through the origin. When $\gamma \geq 0$, the graph is a shift up by $\frac{\gamma}{c}\delta_2 q(u^*)$ to that of $p(\lambda, 0)$ (See Fig. 4).

Therefore, if $\gamma > 0$, the equilibrium point E^* is hyperbolic, with a two dimensional $W^s_{loc}(E^*)$ and a one dimensional $W^u_{loc}(E^*)$.

From Fig. 4, it is easily seen that there is a threshold value $\gamma^* = \gamma^*(A, B, \beta, c)$. If $0 < \gamma < \gamma^*$, there are two distinct negative real eigenvalues for E^*. If $\gamma = \gamma^*$, there is a repeated negative real eigenvalue. If $\gamma > \gamma^*$, there is a complex conjugate pair of eigenvalues with negative real part.

Hence, if $0 < \gamma \leq \gamma^*$, E^* has the two real negative eigenvalues. Let $\mathbf{y}(s)$ be a solution to (2.2) that is on $W^s_{loc}(E^*)$. Then $\mathbf{y}(s)$ approaches E^* monotonically if s is sufficiently large. If $\gamma > \gamma^*$, $\mathbf{y}(s)$ approaches E^* with damped oscillations as $s \to \infty$.

The negative root λ_- of (5.2) is

$$\lambda_- = \frac{1}{3}\left(\omega_c - \sqrt{\omega_c^2 + 3\delta_1}\right)$$

with $\omega_c = \frac{-\delta_1}{c} + c$. There exists a unique γ^* such that

$$p(\lambda_-, \gamma^*) = -(\lambda_-)^3 + \omega_c(\lambda_-)^2 + \delta_1\lambda_- + \frac{\gamma^*}{c}\delta_2 q(u^*) = 0.$$

Using $p'(\lambda_-, \gamma^*) = 0$ to simplify $p(\lambda_-, \gamma^*)$ (long division), the remainder is a first order polynomial in λ_-. Some calculations show that

$$\gamma^* = \frac{2c}{27\delta_2 q(u^*)}[(\omega_c^2 + 3\delta_1)\sqrt{\omega_c^2 + 3\delta_1} - (\omega_c^3 + \frac{9}{2}\delta_1\omega_c)].$$

This completes the proof of (3) in Theorem 2.1. □

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