JOURNAL OF DIFFERENTIAL EQUATIONS 76, 339–352 (1988)

A Multiplicity Theorem for Hyperbolic Systems

ALOISIO FREIRIA NEVES*

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

AND

XIAO-BIAO LIN†

Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14853

Received May 15, 1986

In this paper we show, for a class of hyperbolic systems, that the dimension of the range of the spectral projection corresponding to a single characteristic root λ_0, is equal to the multiplicity of the spectral point λ_0, as root of the characteristic equation.

I. INTRODUCTION AND STATEMENT OF MAIN RESULT

Let $\dot{x} = Ax$ be a system of linear ODE in $\mathbb{R}^n$ and let λ_0 be a root of the characteristic polynomial with multiplicity m. As a consequence of the Jordan canonical form we know that there exist q such that $\mathbb{R}^n$ can be decomposed as the direct sum of the two complementary subspaces $N(A-\lambda_0 I)^q \oplus R(A-\lambda_0 I)^q$ where N and R denote the kernel and the range, respectively; moreover, the dimension of $N(A-\lambda_0 I)^q$ is m. Similarly for an abstract autonomous evolution equation $\dot{x} = Ax$ in a Banach space X it is important to know the dimension of the spectral projection associated to an isolated eigenvalue of A; for instance, in problems involving the center manifold, the knowledge of that dimension is essential.

For special cases in which the spectrum $\sigma(A)$ of the operator A is given by zeros of an entire function $h(\lambda)$, the problem is to know whether the

* Affiliated with the Universidade Estadual de Campinas, Brazil and presently at the Lefschetz Center for Dynamical Systems, Brown University. Research was supported in part by FAPESP, Sao Paulo, Brazil.

† Research partially supported by the Center for Applied Mathematics and Department of Theoretical and Applied Mechanics, Cornell University.

0022-0396/88 $3.00

dimension of the range of the spectral projection is equal to the multiplicity of λ_0 as a root of $h(\lambda)$. This question has been answered affirmatively by B. W. Levinger for retarded functional differential equations (FDEs) (see [3]). In this paper, we do the same thing for a class of hyperbolic systems which is a slight generalization of the hyperbolic systems studied in [4]. The generalized systems will include retarded FDEs and some of the neutral FDEs as special cases. Therefore, the resemblance in the theory and method of FDEs and hyperbolic systems is clear. We would like to thank Professor Lopes for calling our attention to this problem and Professor Hale for suggesting a collaboration on this paper.

We consider the class of hyperbolic systems

$$\frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \qquad \begin{matrix} 0<x<l \\ t>0 \end{matrix}$$

$$\frac{d}{dt}[v(t,l) - Du(t,l)] = Fu(t,\cdot) + Gv(t,\cdot)$$

with the boundary condition

$$u(t,0) = Ev(t,0),$$

where D and E are real matrices of appropriate dimension,

$$K(x) = \operatorname{diag}(k_i(x))_{i=1,\dots,n}, \qquad k_i \in C^1([0,l],\mathbb{R}),$$
$$k_i > 0 \text{ for } i=1,\dots,N \text{ and } k_j < 0 \text{ for } j = N+1,\dots,n$$
$$C(x) = (c_{ij}(x))_{i,j=1,\dots,n}, \qquad c_{ij} \in C([0,l],\mathbb{R})$$
$$u(t,x) \in \mathbb{R}^N \qquad (\text{or } \mathbb{C}^N)$$
$$v(t,x) \in \mathbb{R}^{n-N} \qquad (\text{or } \mathbb{C}^{n-N}),$$

and

$$F\colon (W^{1,p}[0,l])^N \to \mathbb{R}^{n-N} \,(\text{or } \mathbb{C}^{n-N})$$
$$G\colon (W^{1,p}[0,l])^{n-N} \to \mathbb{R}^{n-N} \,(\text{or } \mathbb{C}^{n-N})$$

are linear continuous operators. Furthermore if

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

are column vectors in $(W^{1,p}[0,l])^n$, then we define

$$(F,G)\begin{bmatrix} u_1 & \cdots & u_n \\ v_1 & & v_n \end{bmatrix}_{n\times n}$$

as being the $(n-N)\times n$ complex matrix

$$[Fu_1+Gv_1 \cdots Fu_n+Gv_n].$$

This system may be rewritten as an abstract equation

$$\dot{w}=Aw \qquad \text{in } X_p=(L_p[0,l])^n\times\mathbb{C}^{n-N},\ 1\leqslant p<\infty,$$

where $w=(u,v,d)$,

$$A: D(A)\subset X_p\to X_p$$

$$A(u,v,d)=\left(-K(x)\frac{d}{dx}\begin{pmatrix}u\\v\end{pmatrix}-C(x)\begin{pmatrix}u\\v\end{pmatrix}; Fu(\cdot)+Gv(\cdot)\right)$$

with

$$D(A)=\{(u,v,d)\in X_p: (u,v)\in(W^{1,p}[0,l])^n, u(0)=Ev(0), d=v(l)-Du(l)\}.$$

A special case where

$$Fu(t,\cdot)=\mathrm{F}u(t,l)$$
$$Gv(t,\cdot)=\mathrm{G}v(t,l)$$

has been studied in [4], where F is an $(n-N)\times N$ matrix and G is an $(n-N)\times(n-N)$ matrix.

For this system we have that

$$\sigma(A)=p\sigma(A)=\{\lambda\in\mathbb{C}: h(\lambda)=0\},$$

where

$$h(\lambda)=\det H(\lambda)$$
$$H(\lambda)=-(\lambda D\delta_l+F, G-\lambda I\delta_l)\, X(\cdot,0,\lambda)\begin{pmatrix}E\\I\end{pmatrix}$$

with $X(x,y,\lambda)$ denoting the fundamental matrix of the system

$$\frac{d}{dx}\begin{pmatrix}u\\v\end{pmatrix}=-K^{-1}(x)(\lambda I+C(x))\begin{pmatrix}u\\v\end{pmatrix} \tag{1}$$

and $\delta_l: W^{1,p}\to\mathbb{C}$ is the δ function, i.e., $\delta\phi(\cdot)=\phi(l)$. Also, for any λ such that $h(\lambda)\neq 0$, the resolvent operator $R(\lambda:A)$ is given by

$$\begin{aligned}
R(\lambda:A)(f, g, b) &= (\lambda I - A)^{-1}(f, g, b) \\
&= \left(X(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H(\lambda)^{-1} \beta(\lambda)(f, g, b) \right. \\
&\quad + \int_0^{\cdot} X(\cdot, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy; \\
&\quad (-D, I) \left[X(l, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H(\lambda)^{-1} \beta(\lambda)(f, g, b) \right. \\
&\quad \left. \left. + \int_0^{l} X(l, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \right] \right), \qquad (2)
\end{aligned}$$

where

$$\beta(\lambda): X_p \to \mathbb{C}^{n-N} \text{ is given by}$$

$$\beta(\lambda)(f, g, b) = b + (\lambda D\delta_l + F, G - \lambda I \delta_l) \int_0^{\cdot} X(\cdot, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy. \quad (3)$$

Several examples of hyperbolic systems with $F = \mathrm{F} \cdot \delta_l$ and $G = \mathrm{G}\delta_l$ are given in [4]. Hyperbolic systems in Lebesgue spaces have been used by many authors in the study of FDEs. We mention the work of Krasovskii [9], Borisovic and Turbabin [6], Banks and Burns [7], and Marcus and Mizel [8]. However the system given in this paper is more general. Here we show how the hyperbolic systems contain FDEs [2].

EXAMPLE 1. Let $N = 0$ and $n - N = n$. The variable u is not present and the boundary condition $u(t, 0) = Ev(t, 0)$ is empty. Consider

$$\frac{\partial}{\partial t} v(t, x) = \frac{\partial}{\partial x} v(t, x), \qquad -1 < x < 0, t > 0 \qquad (4)$$

and

$$\frac{d}{dt} v(t, 0) = \int_{-1}^{0} d\eta(\theta)\, v(t, \theta), \qquad (5)$$

where $\eta(\theta)$ is a matrix valued function of bounded variation which vanishes at $\theta = 0$ and is left continuous in $(-1, 0)$. The right-hand side of (5) defines a continuous operator $G: C[-1, 0]^n \to \mathbb{C}^n$. Since $(W^{1,p}[-1, 0])^n \subset (C[-1, 0])^n$, G can be considered as a continuous operator defined on $(W^{1,p}[-1, 0])^n$.

From (4), $v(t, x) = w(t+x)$, $-1 < x < 0$, $t > 0$. Substituting into (5), one has

$$\frac{d}{dt} w(t) = \int_{-1}^{0} d\eta(\theta)\, w(t+\theta)$$

which is the usual linear retarded FDE.

EXAMPLE 2. We replace (5) in Example 1 by

$$\frac{d}{dt} v(t, 0) = \int_{-1}^{0} d\eta(\theta)\, v(t, \theta) + \int_{-1}^{0} \psi(\theta) \frac{\partial}{\partial \theta} v(t, \theta)\, d\theta, \tag{5'}$$

where η is as before, $\psi \in L_q[-1, 0]^{n \times n}$, and q is the dual number to p. Again $v(t, x) = w(t+x)$ and one recognizes it as a neutral FDE.

EXAMPLE 3. Another possible generalization is replacing (5) in Example 1 by

$$v(t, 0) = \sum_{k=1}^{\infty} A_k v(t, -w_k), \qquad t \geqslant 0, \tag{5''}$$

where $0 < w_k \leqslant 1$, $\sum |A_k| < \infty$, and $\sum_{w_k < \varepsilon} |A_k| \to 0$ as $\varepsilon \to 0_+$. The class of hyperbolic systems, thus defined, is a difference equation.

If P denotes the spectral projection of A corresponding to a single characteristic root λ_0, then we can state the following:

THEOREM. *If λ_0 is a root of $h(\lambda)$ of mutiplicity m, then we have:*

(i) $X_p = N(\lambda_0 I - A)^m \oplus R(\lambda_0 I - A)^m$,

(ii) $N(\lambda_0 I - A)^m = P(X_p)$ *where*

$$P = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \delta} R(\lambda : A)\, dy \qquad \text{and} \qquad \delta > 0$$

is such that $\sigma(A) \cap \{z \in \mathbb{C} : |z - \lambda_0| \leqslant \delta\} = \{\lambda_0\}$,

(iii) *the dimension of* $N(\lambda_0 I - A)^m$ *is* m.

Remark. The least number for $X_p = N(\lambda_0 I - A)^J \oplus R(\lambda_0 I - A)^J$ is $J = q$, which is also the least number for $P(X_p) = N(\lambda_0 I - A)^J$, and q may be obtained from Theorem 4 constructively.

II. Preliminary Results

Lemma 1. *If $B(y)$ is a $k \times n$ matrix whose entries are continuous functions on $[0, l]$ such that the rows are linearly independent elements of $C([0, l], \mathbb{C}^n)$, then*

$$T\colon (L_p[0, l])^n \to \mathbb{C}^k \text{ given by } Tf = \int_0^l B(y)\, f(y)\, dy$$

is surjective for every $k \geqslant 1$.

Proof. If α is a k-rowed vector and

$$\int_0^l \alpha B(y)\, f(y)\, dy = 0 \qquad \text{for every} \quad f \in (L_p[0, l])^n,$$

then we have that $\alpha B(y) = 0$ for y in $[0, l]$. Then $\alpha = 0$ because the rows of $B(y)$ are linearly independent and the lemma is proved.

Lemma 2. *If $X(x, y, \lambda)$ is the fundamental matrix of* (1), *then, for each $j = 1, 2, 3, \ldots$, there exists a C^1-matrix $F_j(x, y, \lambda)$ such that*:

$$\text{(i)} \quad \frac{\partial^j}{\partial \lambda^j} X(x, y, \lambda) = F_j(x, y, \lambda) \cdot X^{-1}(y, 0, \lambda)$$

$$\text{(ii)} \quad \frac{\partial}{\partial y} F_j(x, y, \lambda) = j \left[\frac{\partial^{j-1}}{\partial \lambda^{j-1}} X(x, y, \lambda) \right] K^{-1}(y)\, X(y, 0, \lambda).$$

Proof. From (1), for each $j = 1, 2, \ldots$, we have

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial^j}{\partial \lambda^j} X(x, y, \lambda) \right) = {} & -K^{-1}(x)(\lambda I + C(x)) \frac{\partial^j}{\partial \lambda^j} X(x, y, \lambda) \\ & - jK^{-1}(x) \frac{\partial^{j-1}}{\partial \lambda^{j-1}} X(x, y, \lambda) \end{aligned}$$

with

$$\frac{\partial^j}{\partial \lambda^j} X(y, y, \lambda) = 0$$

and then

$$\frac{\partial^j}{\partial \lambda^j} X(x, y, \lambda) = -j \int_y^x X(x, s, \lambda)\, K^{-1}(s) \frac{\partial^{j-1}}{\partial \lambda^{j-1}} X(s, y, \lambda)\, ds; \tag{6}$$

in particular, for $j=1$, we have

$$\frac{\partial}{\partial\lambda} X(x, y, \lambda) = -\int_y^x X(x, s, \lambda)\, K^{-1}(s)\, X(s, 0, \lambda)\, ds \cdot X^{-1}(y, 0, \lambda).$$

Therefore, it is sufficient to take

$$F_1(x, y, \lambda) = -\int_y^x X(x, s, \lambda)\, K^{-1}(s)\, X(s, 0, \lambda)\, ds.$$

Now, if the lemma is true for $j-1$, we have from (6) that

$$\frac{\partial^j}{\partial\lambda^j} X(x, y, \lambda) = -j\int_y^x X(x, s, \lambda)\, K^{-1}(y)\, F_{j-1}(s, y, \lambda)\, X^{-1}(y, 0, \lambda)\, ds$$

and, therefore, taking

$$F_j(x, y, \lambda) = -j\int_y^x X(x, s, \lambda)\, K^{-1}(s)\, F_{j-1}(s, y, \lambda)\, ds$$

we complete, by induction, the proof of the lemma.

LEMMA 3. *Let $M_i(\lambda)$, $i=1, ..., r$, be $k\times k$ matrices functions, analytic in λ. Let $T_l(M_i(\lambda))$ be defined as*

$$T_l(M_i(\lambda)) = \begin{pmatrix} M_i(\lambda) & M_i'(\lambda) & \frac{1}{2}M_i''(\lambda) & \cdots & \frac{1}{l!} M_i^{(l)}(\lambda) \\ & M_i(\lambda) & M_i'(\lambda) & \cdots & \frac{1}{(l-1)!} M_i^{(l-1)}(\lambda) \\ & & M_i(\lambda) & & \\ & & & \ddots & \vdots \\ 0 & & & & M_i(\lambda) \end{pmatrix},$$

where $l \geqslant 1$ is any positive integer. Then we have

$$T_l(M_1(\lambda)\cdot M_2(\lambda)\cdots M_r(\lambda)) = T_l(M_1(\lambda))\cdots T_l(M_r(\lambda)).$$

Proof. For $k=2$, the proof is a direct computation. The general case follows by induction.

We shall use $p(\lambda)$ to denote any analytic function with $p(0)\neq 0$ if the specific feature of this function is of no importance. Let $M(\lambda)=\sum_{i=0}^{\infty} \lambda^i M_i$ be a $k\times k$ analytic matrix. Assume

$$\det M(\lambda) = \lambda^m p(\lambda), \qquad p(0)\neq 0.$$

Define the truncation of $M(\lambda)$ as $\tilde{M}(\lambda)=\sum_{i=0}^{m}\lambda^i M_i$. Polynomial matrices $A(\lambda)$ and $B(\lambda)$ can be found such that $\det A(\lambda)=\det B(\lambda)=1$, and

$$\tilde{D}(\lambda) \overset{\text{def}}{=} A(\lambda)\,\tilde{M}(\lambda)\,B(\lambda)=\operatorname{diag}\{d_1(\lambda),\ldots,d_k(\lambda)\}$$

is the Smith canonical form for $\tilde{M}(\lambda)$ with

$$\frac{d_2}{d_1},\frac{d_3}{d_2},\ldots,\frac{d_k(\lambda)}{d_{k-1}(\lambda)}$$

being polynomials [1]. The following theorem is known from Levinger [3]. We give a shorter proof based on a simpler idea than the original one.

THEOREM 4. *Suppose that* $\tilde{M}(\lambda)^{-1}$ *has* $\lambda=0$ *as a pole of order* q, *then* $M(\lambda)^{-1}$ *also has* $\lambda=0$ *as a pole of order* q. *Moreover, for each* $J\geqslant q-1$, $T_J(M(0))$ *has* rank $k(J+1)-m$ *and* $k(J+1)-\operatorname{rank} T_J(M(0))$ *is a strictly increasing function of* J *if* $0\leqslant J\leqslant q-1$, $q\geqslant 2$.

Proof. Obviously $\det \tilde{M}(\lambda)=\lambda^m p(\lambda)$, $p(0)\neq 0$. Thus, $q\leqslant m$. Write

$$M(\lambda)=\tilde{M}(\lambda)+\lambda^{m+1}Q_1(\lambda)=\tilde{M}(\lambda)[I+\lambda^{m+1}\tilde{M}(\lambda)^{-1}Q_1(\lambda)].$$

Since $q\leqslant m$, $Q_1(\lambda)$ and $\lambda^m\tilde{M}(\lambda)^{-1}$ are analytic. Thus, $M(\lambda)^{-1}$ has $\lambda=0$ as a pole of order q.

Since $\pi d_i=\det\tilde{D}(\lambda)=\det\tilde{M}(\lambda)=\lambda^m p(\lambda)$, we can write $d_i(\lambda)=\lambda^{j_i}p(\lambda)$, $i=1,\ldots,k$, where $j_1+\cdots+j_k=m$ and $0\leqslant j_1\leqslant\cdots\leqslant j_k=q$. Let $D(\lambda)=^{\text{def}}A(\lambda)\,M(\lambda)\,B(\lambda)=\tilde{D}(\lambda)+\lambda^{m+1}C(\lambda)$. Consider the ith diagonal elements in $D(\lambda)$, $D'(\lambda)$, ..., $D^{(j_i)}(\lambda)$, respectively. They have the form of $\lambda^{j_i}p(\lambda)$, $\lambda^{j_i-1}p(\lambda)$, ..., $\lambda p(\lambda)$, $p(\lambda)$, since $0\leqslant j_i\leqslant m$. So, the ith diagonal term is zero in $D(0)$, $D'(0)$, ..., $D^{(j_i-1)}(0)$, but nonzero in $D^{(j_i)}(0)$. It is easy to see that all the other elements in the ith rows of $D(0)$, $D'(0)$, ..., $D^{(j_i)}(0)$ are zeros.

Consider $T_J(A(0))\,T_J(M(0))\,T_J(B(0))=T_J(D(0))$, $J\geqslant q-1$. It is a $k(J+1)\times k(J+1)$ matrix. The first nonzero elements in the ith, $(i+k)$th, $(i+2k)$th, ... rows are located at the $i+k\cdot j_i$, $i+k(j_i+1)$, ... th columns (Fig. 1), until the column number exceeds $k(J+1)$. Obviously we have $J+1-j_i$ such rows when considering the ith, $(i+k)$th, ... $(i+Jk)$th rows in $T_J(D(0))$, and all the other rows among them are identically zeros. Now letting i run through $1,\ldots,k$, we have $(J+1)\,k-j_1-j_2-\cdots-j_k=(J+1)\,k-m$ such rows and they are linearly independent. The remainder of the proof is straightforward.

$T_J(D(0)) =$

$D(0)$	$D'(0)$	$\cdots$	*		
	$D(0)$	$D'(0)$	$\cdots$	*	
		$D(0)$	$D'(0)$	$\cdots$	*
$0\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots 0$
$0\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots 0$
$0\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots 0$

FIG. 1. "*" indicates the first nonzero element in the ith, $(i+k)$th, ..., rows located in $D^{(j_i)}(0)$, i.e., in the $i+kj_i$, $i+k(j_i+1)$, ..., th columns.

III. PROOF OF THE MAIN RESULT

We will prove assertion (iii) only, because the first and the second assertions follow from the general spectral theory and the compactness of $R(\lambda : A)$. See [5].

In order to calculate the dimension of the range of the spectral projection P, we start by making more explicit the several terms appearing in its expression. From (2), we have

$$P(f, g, b)(x) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\delta}\left(X(x, 0, \lambda)\begin{pmatrix}E\\ I\end{pmatrix} H(\lambda)^{-1}\beta(\lambda)(f, g, b);\right.$$
$$\left.(-D, I)\, X(l, 0, \lambda)\begin{pmatrix}E\\ I\end{pmatrix} H(\lambda)^{-1}\beta(\lambda)(f, g, b)\right) d\lambda,$$

where $\beta(\lambda)(f, g, b)$ is given in (3). The other terms of $R(\lambda : A)(f, g, b)$ are entire functions of λ and their integrals are zero.

If

$$G(\lambda) = (\lambda-\lambda_0)^m H(\lambda)^{-1} \tag{7}$$

we have that $G(\lambda)$ is analytic in $\{\lambda \in \mathbb{C}/|\lambda-\lambda_0| \leqslant \delta\}$ and

$$P(f, g, b)(x) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial\lambda^{m-1}}\left[\left(X(x, 0, \lambda)\begin{pmatrix}E\\ I\end{pmatrix} G(\lambda)\,\beta(\lambda)(f, g, b);\right.\right.$$
$$\left.\left.(-D, I)\, X(l, 0, \lambda)\begin{pmatrix}E\\ I\end{pmatrix} G(\lambda)\,\beta(\lambda)(f, g, b)\right)\right]_{\lambda=\lambda_0}.$$

A short computation shows that the first component of the derivative above is given by

$$\frac{\partial^{m-1}}{\partial\lambda^{m-1}}\left[X(x,0,\lambda)\begin{pmatrix}E\\ I\end{pmatrix}G(\lambda)\,\beta(\lambda)(f,g,b)\right]_{\lambda=\lambda_0}$$
$$=\mathbf{X}(x,\lambda_0)\,\mathrm{E}S_{m-1}(G(\lambda_0))\,B(f,g,b),$$

where

$\mathbf{X}(x,\lambda_0)$ is the $n\times mn$ matrix

$$\left(X(x,0,\lambda_0),\binom{m-1}{1}\frac{\partial}{\partial\lambda}X(x,0,\lambda)_{\lambda=\lambda_0},\ldots,\binom{m-1}{m-1}\frac{\partial^{m-1}}{\partial\lambda^{m-1}}X(x,0,\lambda)_{\lambda=\lambda_0}\right),$$

E is the $mn\times m(n-N)$ diagonal block matrix whose $n\times(n-N)$ blocks are $\binom{E}{I}$,

$S_{m-1}(G(\lambda_0))$ is the $m(n-N)\times m(n-N)$ matrix

$$\begin{bmatrix}
G(\lambda_0) & \binom{m-1}{1}G'(\lambda_0) & \binom{m-1}{2}G''(\lambda_0) & \cdots & \binom{m-1}{m-1}G^{(m-1)}(\lambda_0)\\
 & G(\lambda_0) & \binom{m-2}{1}G'(\lambda_0) & \cdots & \binom{m-2}{m-2}G^{(m-2)}(\lambda_0)\\
 & & G(\lambda_0) & \cdots & \binom{m-3}{m-3}G^{(m-3)}(\lambda_0)\\
 & & & \ddots & \vdots\\
 & 0 & & & G(\lambda_0)
\end{bmatrix},$$

and $B: X_p\to\mathbb{C}^{m(n-N)}$ is given by

$$B(f,g,b)=\begin{pmatrix}\beta^{(m-1)}(\lambda_0)\\ \vdots\\ \beta'(\lambda_0)\\ \beta(\lambda_0)\end{pmatrix}(f,g,b).$$

It is easy to see, for $j\geqslant 1$, that

$$\beta^{(j)}(\lambda_0)(f,g,b)=(\lambda_0 D\delta_l+F,\,G-\lambda_0 I\delta_l,\,jD\delta_l,\,-jI\delta_l)$$
$$\times\int_0^{\cdot}\begin{bmatrix}\dfrac{\partial^j}{\partial\lambda^j}X(\cdot,y,\lambda)_{\lambda=\lambda_0}\\[2mm] \dfrac{\partial^{j-1}}{\partial\lambda^{j-1}}X(\cdot,y,\lambda)_{\lambda=\lambda_0}\end{bmatrix}K^{-1}(y)\begin{pmatrix}f(y)\\ g(y)\end{pmatrix}dy.$$

The second component of the derivative has a similar expression.

The proof will be carried out in three steps.

FIRST STEP. $B\colon X_p \to C^{m(n-N)}$ is surjective.

Proof. Since

$$B(f, g, b) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \end{pmatrix} + Q \int_0^{\cdot} \begin{bmatrix} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} X(\cdot, y, \lambda)_{\lambda=\lambda_0} \\ \vdots \\ \frac{\partial}{\partial \lambda} X(\cdot, y, \lambda)_{\lambda=\lambda_0} \\ X(\cdot, y, \lambda_0) \end{bmatrix} K^{-1}(y) \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy,$$

where Q is the $m(n-N) \times mn$ matrix

$$\begin{bmatrix} (\lambda_0 D\delta_l + F, G - \lambda_0 I\delta_l, (m-1) D\delta_l, -(m-1) I\delta_l) & & & \\ & (\lambda_0 D\delta_l + F, G - \lambda_0 I\delta_l, (m-2) D\delta_l, -(m-2) I\delta_l) & & 0 \\ & & \ddots & \\ 0 & & (\lambda_0 D\delta_l + F, G - \lambda_0 I\delta_l, D\delta_l, -I\delta_l) & \\ & & & (\lambda_0 D\delta_l + F, G - \lambda_0 I\delta_l) \end{bmatrix}.$$

We only need to show that

$$\tilde{Q} \int_0^{\cdot} \begin{bmatrix} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} X(\cdot, y, \lambda)_{\lambda=\lambda_0} \\ \vdots \\ \frac{\partial}{\partial \lambda} X(\cdot, y, \lambda)_{\lambda=\lambda_0} \\ X(\cdot, y, \lambda_0) \end{bmatrix} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy$$

is surjective for $(f, g) \in (L_p[0, l])^n$, where $\tilde{Q}$ is the matrix Q without the last $(n-N)$ rows, since we know that K^{-1} is nonsingular. Notice that $\tilde{Q}$ is a lower triangular block matrix with the diaginal blocks being

$$\operatorname{diag}(-(m-1) I\delta_l, \ldots, -I\delta_l).$$

Therefore it suffices to prove that

$$\int_0^{l} \begin{bmatrix} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} X(l, y, \lambda)_{\lambda=\lambda_0} \\ \vdots \\ \frac{\partial}{\partial \lambda} X(l, y, \lambda)_{\lambda=\lambda_0} \\ X(l, y, \lambda_0) \end{bmatrix} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy$$

is surjective.

From Lemma 1, it is sufficient to prove that the rows of

$$\begin{bmatrix} \dfrac{\partial^j}{\partial \lambda^j} X(l, y, \lambda) \\ \vdots \\ X(l, y, \lambda) \end{bmatrix}$$

are linearly independent for $j \geqslant 0$. Assume, by induction, this result is true for $j-1$, and suppose

$$(a_j, a_{j-1}, ..., a_0) \begin{bmatrix} \dfrac{\partial}{\partial \lambda^j} X(l, y, \lambda) \\ \vdots \\ X(\lambda, y, \lambda) \end{bmatrix} = 0$$

with $a_i \in \mathbb{C}^n$, $i = 0, 1, ..., j$; by Lemma 2, this linear combination of the rows can be rewritten as

$$(a_j, a_{j-1}, ..., a_0) \begin{bmatrix} F_j(l, y, \lambda) \\ \vdots \\ X(l, 0, \lambda) \end{bmatrix} X^{-1}(y, 0, \lambda) = 0$$

which implies that

$$(a_j, a_{j-1}, ..., a_0) \begin{bmatrix} F_j(l, y, \lambda) \\ \vdots \\ X(l, 0, \lambda) \end{bmatrix} = 0.$$

Now, taking derivatives in y and using Lemma 2, again, we have

$$(a_j, a_{j-1}, ..., a_0) \begin{bmatrix} j \dfrac{\partial^{j-1}}{\partial \lambda^{j-1}} X(l, y, \lambda) \\ \vdots \\ X(l, y, \lambda) \\ 0 \end{bmatrix} = 0$$

and the first step is proved.

SECOND STEP. *The matrix* $S_{m-1}(G(\lambda_0))$ *has* rank m.

Proof. First of all notice that $S_{m-1}(G(\lambda_0))$ is equivalent to $T_{m-1}(G(\lambda_0))$. Next notice that as a consequence of Theorem 4, $T_{m-1}(H(\lambda_0))$ has kernel of dimension m, and from Lemma 3 and (7) we have

$$T_{m-1}(H(\lambda_0))\, T_{m-1}(G(\lambda_0)) = T_{m-1}(H(\lambda_0)\, G(\lambda_0)) = 0.$$

Then

$$\operatorname{rank} T_{m-1}(G(\lambda_0)) \leqslant m.$$

On the other hand, since $\det G(\lambda) = (\lambda - \lambda_0)^{m(n-N-1)} p(\lambda)$ with $p(\lambda_0) \neq 0$, we see that if $n - N = 1$ then $S_{m-1}(G(\lambda_0))$ is invertible and has rank m. Moreover, if $n - N \geqslant 2$ then by Theorem 4 again

$$\operatorname{rank} T_{m(n-N-1)-1} G(\lambda_0) = m(n-N-1)^2 \tag{8}$$

but

$$T_{m(n-N-1)-1}(G(\lambda_0))$$
$$= \left[\begin{array}{c|ccccc} T_{m-1}(G(\lambda_0)) & * & * & \cdots & * & \\ & G(\lambda_0) & G'(\lambda_0) & \cdots & \frac{1}{(m(n-N-2)-1)!} & G^{(m(n-N-2)-1)}(\lambda_0) \\ & & G(\lambda_0) & & & \\ 0 & 0 & & \ddots & \vdots & \\ & & & & G(\lambda_0) & \end{array}\right],$$

where the number of columns on the right side of this matrix is $m(n-N-2)(n-N)$. Then from (8), it follows that

$$\operatorname{rank} T_{m-1}(G(\lambda_0)) \geqslant m(n-N-1)^2 - m(n-N-2)(n-N) = m.$$

Therefore $\operatorname{rank} T_{m-1}(G(\lambda_0) = m$ in any case.

THIRD STEP. *The range of the spectral projection has* dimension m.

Proof. First of all using the previous steps we conclude that

$$\mathrm{E} S_{m-1}(G(\lambda_0)) B : X_p \to \mathbb{C}^{mn}$$

has an m dimensional range, because E is one-to-one. If $\{\alpha_1, ..., \alpha_m\}$ denotes any basis of this range, then

$$\{(X(x, \lambda_0)\, \alpha_1; (-D, I)\, X(l, \lambda_0)\, \alpha_1), ..., (X(x, \lambda_0)\, \alpha_m; (-D, I)\, X(l, \lambda_0)\, \alpha_m)\}$$

is a basis of the range of the spectral projection. In order to complete the proof we have to show that the above sequence is linearly independent in X_p, and for this it is sufficient to show that $(\partial^k/\partial\lambda^k)\, X(x, 0, \lambda)\alpha = 0$ implies $\alpha = 0$, for every $k \geqslant 0$; but this follows trivially from (6) and an inductive argument and the theorem is proved.

References

1. F. R. Gantmacher, "The Theory of Matrices," Vol. 1, Chelsea, New York, 1959.
2. J. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
3. B. W. Levinger, A folk theorem in functional differential equations, *J. Differential Equations* **4** (1968), 612–619.
4. O. F. Lopes, A. F. Neves, and H. S. Ribeiro, On the spectrum of evolution operators generated by hyperbolic Systems, *J. Funct. Anal.* **67** (1986), 320–344.
5. A. E. Taylor, "Functional Analysis," New York, Wiley, 1958.
6. Ju. G. Borisovic and A. S. Turbabin, On the Cauchy problem for linear inhomogeneous differential equations with retarded argument, *Dokl. Akad. Nauk. SSSR* **185** (1969), 741–744.
7. H. T. Banks and J. A. Burns, "An Abstract Framework for Approximate Solutions to Optimal Control Problems Governed by Hereditary Systems," International Conference on Differential Equations (Univ. Southern California, Los Angeles, Calif., 1974), pp. 10–25. Academic Press, New York, 1975.
8. M. Marcus and V. J. Mizel, Semilinear hereditary hyperbolic systems with nonlocal boundary conditions. II, *J. Math. Anal. Appl.* **77** (1980), 1–19.
9. N. N. Krasovskii, "Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay," Stanford University Press, Stanford, Calif., 1963.