ALGEBRAIC DICHOTOMIES WITH AN APPLICATION TO THE STABILITY OF RIEMANN SOLUTIONS OF CONSERVATION LAWS

XIAO-BIAO LIN

Abstract. Recently, there has been some interest on the stability of waves where the functions involved grow or decay at an algebraic rate $|x|^m$. In this paper we define the so called algebraic dichotomy that may aid in treating such problems. We discuss the basic properties of the algebraic dichotomy, methods of detecting it, and calculating the power of the weight function.

We present several examples: (1) The Bessel equation. (2) The n-degree Fisher type equation. (3) Hyperbolic conservation laws in similarity coordinates. (4) A system of conservation laws with a Dafermos type viscous regularization. We show that the linearized system generates an analytic semigroup in the space of algebraic decay functions. This example motivates our work on algebraic dichotomies.

1. Algebraic dichotomy

Dichotomies, ordinary or weighted, are essential in many applications of dynamical systems theory. A definition of the ordinary dichotomy can be found in [4], where the decay or growth rate of the flow on stable or unstable space is unspecified. We are interested in the weighted dichotomies where the decay or growth rate of the flow is controlled by monotone weight functions $w(x)$ that approach zero or infinity as $x \to c$ or d where $I = (c, d)$ is the domain of interest (allowing $c = -\infty$ and/or $d = \infty$). Although the most important case is the exponential dichotomy where $w(x) = e^{\mu x}$, in applications, we may find systems with non-exponential growth or decay solutions. For example, to solve some PDEs written in polar or spherical coordinates, we encounter Bessel's equations and their variations of which the solutions decay to 0 or grow to ∞ algebraically as $x \to \infty$. For some PDEs with one dimensional spatial domain, to study the stability of solutions, it is useful to limit solutions to spaces of functions that decay to zero with certain given rate [30, 14, 38]. Method used in this paper may be useful in those problems.

This paper is motivated by the study of the stability of conservation laws. In [24, 26], conservation laws consisting of Lax shocks written in similarity coordinates were considered. It is found that the linearized system has an algebraic dichotomy and eigenfunctions to such system decay to zero algebraically. In this paper we will define general spaces that are suitable to study systems with algebraic dichotomies. We will apply our theory to Riemann problems of hyperbolic conservation laws and

Date: May 12, 2009.

Key words and phrases. algebraic dichotomy, conservation laws, Dafermos regularization, stability of Riemann solutions, analytic semigroup, singular perturbations.

This work was supported in part by the National Science Foundation under grant DMS-0708386.

their viscous regularizations. By carefully choosing viscosity terms, we show that the regularized systems can be posed in spaces of algebraic decay functions just like the original hyperbolic system. And the linearized equation around viscous Riemann solutions is sectorial, thus generates an analytic semigroup.

We assume that the weight function satisfies the following conditions:

(1) $w(x)$ is positive and monotone in the domain (c, d) where the equation is well-posed and the dichotomy exists. We allow $c = -\infty$ and/or $d = \infty$.

(2) At $x = c$ and/or $x = d$, $w(x)$ may have a zero or a pole, i.e., where $w(x) = 0$ or ∞. The zero or pole is said to be finite or infinite if c or d is finite or infinite.

Examples of scalar equations, weight functions and their zeros and poles are:

(a) $u' = \frac{\mu}{x}u,\ x \in I = (0, \infty)$ with $\beta = \Re\mu \neq 0$.

In this case the weight function is $w(x) = x^\beta$. If $\beta > 0$ then w has a zero at 0 and a pole at ∞. If $\beta < 0$ then w has a pole at 0 and a zero at ∞.

(b) $u' = \frac{\mu}{x(1-x)}u,\ x \in I = (0, 1)$ with $\beta = \Re\mu \neq 0$.

The weight function is $w(x) = (x/(1-x))^\beta$. If $\beta > 0$ then w has a zero at $x = 0$ and a pole at $x = 1$. If $\beta < 0$ then w has a zero at $x = 1$ and a pole at $x = 0$.

(c) $u' = \pm\frac{\mu}{\sqrt{1+x^2}}u,\ x \in I = (-\infty, \infty)$ with $\beta = \Re\mu \neq 0$.

The weight function is $w = (x + \sqrt{1+x^2})^\beta$. If $\beta > 0$ then w has a zero at $-\infty$ and a pole at ∞. If $\beta < 0$ then w has a zero at ∞ and a pole at $-\infty$.

(d) In §7 we will discuss an example from a Laplace transformed PDE in dual variable s. Although in some finite interval (x^i, x^{i+1}), none of the end points is a zero or a pole of the weight function, the algebraic dichotomy is still important because the power β is related to the dual variable s which is in a region unbounded to the right.

We will focus on systems defined on $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$ where the weight function has a zero or a pole at $\pm\infty$. The growth or decay rate is measured by

$$a(x) = x + \sqrt{1+x^2}, \quad a(x)^{-1} = a(-x) = -x + \sqrt{1+x^2}.$$

Both functions are positive, strictly monotone and asymptotically satisfy:

$$a(x) \sim \begin{cases} 2x, & x \to \infty, \\ 1/(2|x|), & x \to -\infty. \end{cases}, \quad a(x)^{-1} \sim \begin{cases} 1/(2x), & x \to \infty, \\ 2|x|, & x \to -\infty. \end{cases}$$

The weight function $a(x)^\mu$ has properties similar to that of $e^{\mu x}$:

$$\begin{aligned} &\frac{dx}{\sqrt{1+x^2}} = \frac{da(x)}{a(x)}, \\ &\int a(x)^\mu \frac{dx}{\sqrt{1+x^2}} = \frac{1}{\mu}a(x)^\mu + C, \quad \mu \neq 0. \end{aligned} \tag{1.1}$$

The weight function $a(x)^\mu$ is flexible to adapt to some other situations. For example, the weight functions used in [38] and [14] are equivalent to $1 + a(x)$ and $a(x) + a(x)^{-1}$ respectively.

Norms in $\mathbb{C}^n$ will be denoted by $|\cdot|$ while norms in infinite dimensional Banach spaces or norms of operators will be denoted by $\|\cdot\|$.

Definition 1.1. A continuous function $u : \mathbb{R} \to \mathbb{C}^n$ is said to grow (or decay) algebraically with the power $\mu > 0$ (or $\mu < 0$) as $x \to \infty$ if there exists a constant C such that

$$|u(x)| \leq Ca(x)^\mu.$$

Define the Banach space E_μ of continuous functions $u(x)$, with the following norm being finite:

$$\|u\|_{E_\mu} = \sup_{x\in\mathbb{R}}\{|u(x)| \cdot a(x)^{-\mu}\}.$$

Similarly, we can define the space $E_\mu(J)$ for $x \in J$ where J is an unbounded or bounded interval in $\mathbb{R}$.

Define the Banach space $\widehat{E}_\mu$ of continuous functions $u(x)$ such that $u \in E_\mu(\mathbb{R}^+) \cap E_{-\mu}(\mathbb{R}^-)$ with the norm

$$\|u\|_{\widehat{E}_\mu} = \max\{\|u\|_{\mu(R^+)}, \|u\|_{-\mu(R^-)}\}.$$

Remark 1.1. Assume that $\mu < \nu$. Then we have $E_\mu(\mathbb{R}^+) \subset E_\nu(\mathbb{R}^+)$, $E_\nu(\mathbb{R}^-) \subset E_\mu(\mathbb{R}^-)$ and $\widehat{E}_\mu \subset \widehat{E}_\nu$. However, we do not have $E_\mu(\mathbb{R}) \subset E_\nu(\mathbb{R})$ or $E_\nu(\mathbb{R}) \subset E_\mu(\mathbb{R})$.

A continuous function $u : \mathbb{R} \to \mathbb{C}^n$ is said to grow (or decay) algebraically with the power $\mu > 0$ (or $\mu < 0$) as $|x| \to \infty$ if there exists a constant C such that

$$|u(x)| \leq C(\sqrt{1+x^2})^\mu.$$

If $u \in \widehat{E}_\mu$, then it is straight forward to verify that $u(x)$ grow (or decay) algebraically with the power $\mu > 0$ (or $\mu < 0$) as $|x| \to \infty$. An alternative norm on $\widehat{E}_\mu$ can be defined as:

$$|||u|||_{\widehat{E}_\mu} = \sup_{x\in\mathbb{R}}\{|f(x)| \cdot (\sqrt{1+x^2})^{-\mu}\}.$$

See [14]. Notice that

$$\begin{aligned} a(x) + a(x)^{-1} &= 2\sqrt{1+x^2}, \\ (\sqrt{1+x^2})^\mu &< a(x)^\mu + a(x)^{-\mu}, \quad \text{for } \mu \in \mathbb{R}, \\ a(x)^\mu + a(x)^{-\mu} &< 2(2\sqrt{1+x^2})^\mu, \qquad \text{for } \mu > 0. \end{aligned} \tag{1.2}$$

If $x \geq 0$, then

$$\begin{aligned} a(x)^\mu \leq \sqrt{1+x^2}^\mu \leq 2^{-\mu}a(x)^\mu, \quad \mu \leq 0, \\ 2^{-\mu}a(x)^\mu \leq \sqrt{1+x^2}^\mu \leq a(x)^\mu, \quad \mu \geq 0. \end{aligned} \tag{1.3}$$

Similar estimates hold for $x \leq 0$. Thus we have

$$\begin{aligned} \|u\|_{\widehat{E}_\mu} \leq |||u|||_{\widehat{E}_\mu} \leq 2^\mu\|u\|_{\widehat{E}_\mu}, \quad \mu \geq 0, \\ 2^\mu\|u\|_{\widehat{E}_\mu} \leq |||u|||_{\widehat{E}_\mu} \leq \|u\|_{\widehat{E}_\mu}, \quad \mu \leq 0. \end{aligned}$$

Let $T(x, y)$, with $T(y, y) = I$, be the principal matrix solution of the homogeneous part of the system in $\mathbb{C}^n$:

$$u'(x) = A(x)u(x) + g(x). \tag{1.4}$$

Definition 1.2. *Algebraic dichotomy: The principal matrix solution $T(x,y)$ of the homogeneous part of* (1.4) *is said to have an algebraic dichotomy on $J=(-\infty,\infty)$ with a power $\beta>0$, constants $K_1,K_2>0$ and projections $P(x)+Q(x)=I$ if $P(x)$ is continuous and the following properties hold:*

$$\begin{aligned}
&T(x,y)P(y)=P(x)T(x,y),\\
&\|T(x,y)P(y)\|\le K_1\left(a(x)/a(y)\right)^{-\beta}, \ \text{if } y\le x,\\
&\|T(x,y)Q(y)\|\le K_2\left(a(y)/a(x)\right)^{-\beta}, \ \text{if } x\le y.
\end{aligned}$$

The ranges of $P(y)$ and $Q(y)$ are called the stable and unstable subspaces for $T(x,y)$ at $y\in\mathbb{R}$, and are invariant under $T(x,y)$.

If instead, there exist $\gamma>\delta$ such that,

$$\begin{aligned}
&\|T(x,y)P(y)\|\le K_1(a(x)/a(y))^{\delta}, && y\le x\in J,\\
&\|T(x,y)Q(y)\|\le K_2(a(x)/a(y))^{\gamma}, && x\le y\in J,
\end{aligned}$$

then $T(x,y)$ is said to have a pseudo algebraic dichotomy, or an asymmetric dichotomy if $\gamma+\delta\neq 0$. A dichotomy is symmetric if $\gamma+\delta=0$.

We sometimes use $K=\max\{K_1,K_2\}$ as the constant of the dichotomy for simplicity. We say (1.4) *has an algebraic dichotomy if $T(x,y)$ has an algebraic dichotomy.*

If $u(x)$ satisfies a linear system $u'=A(x)u$, then $w(x)=e^{\mu x}u(x)$ satisfies $w'=(A+\mu I)w$. A similar property holds if the exponential function is replaced by $a(x)^{\mu}$.

Lemma 1.1. *If $u(x)$ satisfies* (1.4)*, then $w(x)=a(x)^{\mu}u(x)$ satisfies*

$$w'(x)=(A(x)+\frac{\mu}{\sqrt{1+x^2}}I)w(x)+a(x)^{\mu}g(x). \tag{1.5}$$

The principal matrix solution becomes $\widetilde{T}(x,y)=(a(x)/a(y))^{\mu}T(x,y)$. Moreover, if (1.4) *has a pseudo algebraic dichotomoy with powers $\gamma>\delta$, then* (1.5) *has a pseudo algebraic dichotomy with the powers $\tilde\gamma=\gamma+\mu$, $\tilde\delta=\delta+\mu$, the projections $\widetilde{P}(x)=P(x)$, $\widetilde{Q}(x)=Q(x)$ and constants $\widetilde{K}_1=K_1$, $\widetilde{K}_2=K_2$ remain the same.*

Remark 1.2. Definition 1.2 works on finite intervals where there is no finite zero or pole to the weight function of the dichotomy. For a scalar equation $u'=\mu(x)u$, the solution is $u(x)=e^{\int^x\mu(t)dt}u(y)/e^{\int^y\mu(t)dt}$, and the weight function is precisely $e^{\int^x\mu(t)dt}$. This also suggest that the growth or decay should be measured by the quotient of two functions at x and y respectively. In an interval (c,d), in order to have a zero at c and a pole at d, it is necessary to have $\mu(x)\sim k/(x-c)$ as $x\to c$ and $\mu(x)\sim h/(d-x)$ as $x\to d$, $\mu(x)\sim k/|x|$ if $c=-\infty$, or $d=\infty$).

It is straightforward to verify that if $y\le x$, and $\alpha,\ \beta>0$, then

$$e^{-\alpha(x-y)}\le (a(x)/a(y))^{-\beta}.$$

Therefore, if a linear system has an exponential dichotomy with the exponent $\alpha>0$, then it also has an algebraic dichotomy with any $\beta>0$.

By reversing the time $x\to -x$ in (1.4), we have

$$\tilde{u}'(x)=\widetilde{A}(x)\tilde{u}(x)+\tilde{g}(x), \tag{1.6}$$

where $\widetilde{A}(x) = -A(-x), \tilde{g}(x) = -g(-x)$. The following simple facts can be checked easily (the L^2 type spaces H_μ and $\widehat{H}_\mu$ will be defined in §6).

Lemma 1.2. *The principal matrix solution for* (1.6) *is related to* (1.4) *by* $\widetilde{T}(x,y) = T(-x,-y)$. *If* (1.4) *has an algebraic dichotomy with the power* β, *constants* K_1, K_2 *and projections* $P(x), Q(x)$, *then* (1.6) *has an algebraic dichotomy with the same power* β, *constants* K_2, K_1. *The projections to the stable and unstable subspaces are* $\widetilde{P}(x) = Q(-x), \widetilde{Q}(x) = P(-x)$. *If* $g \in E_\mu$ *or* H_μ *for some* $\mu \in \mathbb{R}$, *then* $\tilde{g} \in E_{-\mu}$ *or* $H_{-\mu}$. *If* $g \in \widehat{H}_\mu$ *then* $\tilde{g} \in \widehat{H}_\mu$.

Sections 2 and 6 are devoted to estimates of the solution u of (1.4) if the forcing term g is given in some function spaces with specified growth/decay rates. §2 deals with spaces of continuous functions. §6 deals with spaces of L^2 type functions. An important property of the algebraic dichotomies is the roughness under perturbations, which is also treated in §2.

In §3 we introduce the method of asymptotic factorization which is related to the method of frozen coefficients and the WKB method. Our method works on some problems typically treated by the WKB method [1]. The difference is after finding the time-dependent exponents, we go back to solve the original equation rather than solve the equations of the exponents.

The simple example on Bessel functions in §4 shows what can go wrong with the method of frozen coefficients if not used correctly. In §5 we discuss an example treated by Wu, Xing and Ye [38]. In §7 we study an example from Riemann solutions of conservation laws. In §8, we discuss a singularly perturbed hyperbolic system in similarity coordinates. We show it generalizes an analytic semigroup in spaces of algebraic decay functions. And there exists $\eta > 0$ such that if $\Re s > -\eta$, then s is either a normal eigenvalue or a resolvent point of the associated linear variational system.

2. Algebraic dichotomy in spaces of continuous functions

2.1. **Estimates of solutions.** If a system has a dichotomy on $\mathbb{R}$, then it has dichotomies on $\mathbb{R}^\pm$. By Lemma 1.2, it suffices to consider dichotomies on $\mathbb{R}^+$. We also find if the forcing terms is expressed in the form $f(x)/\sqrt{1+x^2}$, then results of this paper become much simpler. Without loss of generality we will assume that the forcing term is of the form $f(x)/\sqrt{1+x^2}$ for the rest of the paper. Consider

$$u'(x) = A(x)u(x) + f(x)/\sqrt{1+x^2}, \quad x \in [0,\infty). \tag{2.1}$$

Theorem 2.1. *Assume that* (2.1) *has an algebraic dichotomy with the power* β *and constants* K_1, K_2. *If* $f \in E_\mu(\mathbb{R}^+)$ *where* $\beta > |\mu|$ *and* $P(0)u(0)$ *is given, then there exists a unique solution* $u \in E_\mu(\mathbb{R}^+)$. *Moreover,*

$$\|u(x)\|_{E_\mu(R^+)} \le \left(\frac{K_1}{\beta+\mu} + \frac{K_2}{\beta-\mu}\right)\|f\|_{E_\mu(R^+)} + K_1 a(x)^{-(\beta+\mu)}|P(0)u(0)|. \tag{2.2}$$

$$\|u(x)\|_{E_\mu(R^-)} \le \left(\frac{K_1}{\beta+\mu} + \frac{K_2}{\beta-\mu}\right)\|f\|_{E_\mu(R^-)} + K_2 a(x)^{\beta-\mu}|Q(0)u(0)|. \tag{2.3}$$

Proof. We first show the existence of the solution. Let $u(x) = P(x)u(x) + Q(x)u(x)$, where

$$P(x)u(x) = \int_0^x T(x,y)P(y)f(y)\frac{dy}{\sqrt{1+y^2}} + T(x,0)P(0)u(0), \tag{2.4}$$

$$Q(x)u(x) = \int_\infty^x T(x,y)Q(y)f(y)\frac{dy}{\sqrt{1+y^2}}. \tag{2.5}$$

It is easy to see that if both integrals are convergent then u as defined is a solution.

To show that the integrals are convergent and to obtain estimates of solutions, observe the integral term I_1 in (2.4) satisfies

$$\begin{aligned}|I_1| &\leq \int_0^x K_1 \left(a(x)/a(y)\right)^{-\beta} \|f\|_\mu a(y)^\mu \frac{dy}{\sqrt{1+y^2}} \\ &= K_1\|f\|_\mu a(x)^{-\beta} \int_0^x a(y)^{\beta+\mu} \frac{dy}{\sqrt{1+y^2}} \\ &\leq \frac{K_1\|f\|_\mu}{\beta+\mu} a(x)^\mu, \quad \text{if } \beta+\mu > 0.\end{aligned}$$

Therefore,

$$|P(x)u(x)| \leq \frac{K_1}{\beta+\mu}\|f\|_\mu a(x)^\mu + K_1 a(x)^{-\beta}|P(0)u(0)|, \quad \text{if } \beta+\mu>0. \tag{2.6}$$

Similarly,

$$\begin{aligned}|Q(x)u(x)| &\leq \int_x^\infty K_2((a(x)/a(y))^\beta \|f\|_\mu a(y)^\mu \frac{dy}{\sqrt{1+y^2}} \\ &\leq \frac{K_2}{\beta-\mu}\|f\|_\mu a(x)^\mu, \quad \text{if } \beta-\mu>0.\end{aligned} \tag{2.7}$$

To show that the solution is unique, let u be a corresponding solution of (2.1) in E_μ with $f = 0$ and $P(0)u(0) = 0$.

From the variation of constant formula, $f = 0$ and $P(0)u(0) = 0$ clearly imply that $P(x)u(x) = 0$ for all $x \geq 0$.

For any $0 \leq x \leq x_0$,

$$\begin{aligned}|Q(x)u(x)| &= |T(x,x_0)Q(x_0)u(x_0)| \\ &\leq K \left(a(x_0)/a(x)\right)^{-\beta} \|u\|_\mu a(x_0)^\mu \\ &\leq K_2\|u\|_\mu a(x)^\beta a(x_0)^{-\beta+\mu}.\end{aligned}$$

Our assumption $\beta > |\mu|$ implies that the last expression goes to 0 as $x_0 \to \infty$. This proves that $Q(x)u(x) = 0$.

Since $f = 0$ and $P(0)u(0) = 0$ imply $u = 0$, the solution $u \in E_\mu$ is unique.

□

In applications, we often look for solutions that decay to zero as $x \to \pm\infty$, i.e., $u \in \widehat{E}_{\mu(\mathbb{R}}$ with $\mu < 0$.

Theorem 2.2. *If the homogeneous part of* (2.1) *has an algebraic dichotomy of power* β *in* $\mathbb{R}$ *and if* $f(x) \in \widehat{E}_\mu(\mathbb{R})$, $\mu \le 0$, *with* $\beta > |\mu|$, *then* (2.1) *has a unique solution* $u \in E_{\hat{\mu}}(\mathbb{R})$. *Moreover,*

$$\|u\|_{\widehat{E}_\mu} \le \frac{(K_1 + K_2)}{\beta - |\mu|} \|f\|_{\widehat{E}_\mu}.$$

Proof. We starts from the integral expression

$$P(x)u(x) = \int_{-\infty}^{x} T(x,y)P(y)f(y) \frac{dy}{\sqrt{1+y^2}}.$$

If $x \le 0$, then

$$\begin{aligned} |P(x)u(x)| &\le K_1 \|f\| \int_{-\infty}^{x} (a(x)/a(y))^{-\beta} a(y)^{-\mu} \frac{dy}{\sqrt{1+y^2}} \\ &\le \frac{K_1}{\beta - \mu} \|f\| a(x)^{-\mu}. \end{aligned}$$

If $x \ge 0$, then

$$P(x)u(x) = \int_{-\infty}^{0} + \int_{0}^{x} T(x,y)P(y)f(y) \frac{dy}{\sqrt{1+y^2}} = I_1(x) + I_2(x),$$

$$\begin{aligned} \text{where } |I_1(x)| &\le (a(x)/a(0))^{-\beta} |P(0)u(0)| \le \frac{K_1}{\beta - \mu} \|f\| a(x)^{-\beta}, \\ |I_2(x)| &\le K_1 \|f\| \int_0^x (a(x)/a(y))^{-\beta} a(y)^{\mu} \frac{dy}{\sqrt{1+y^2}} \\ &\le K_1 \|f\| a(x)^{-\beta} \frac{a(x)^{\beta+\mu} - 1}{\beta + \mu}. \end{aligned}$$

Since $\mu < 0$, using the fact $1/(\beta+\mu) > 1/(\beta-\mu)$, we have for $x \ge 0$,

$$|P(x)u(x)| \le |I_1(x) + I_2(x)| \le K_1 \|f\|_{\widehat{E}_\mu} \frac{a(x)^{\mu}}{\beta + \mu}.$$

Combined with the estimate for $x \le 0$, we have

$$\|P(\cdot)u(\cdot)\|_{\widehat{E}_\mu} \le \frac{K_1}{\beta - |\mu|} \|f\|_{\widehat{E}_\mu}.$$

Similarly, from (2.5) we can show

$$\|Q(\cdot)u(\cdot)\|_{\widehat{E}_\mu} \le \frac{K_2}{\beta - |\mu|} \|f\|_{\widehat{E}_\mu}.$$

The desired result follows from $\|u\| \le \|Pu\| + \|Qu\|$.

$\square$

2.2. **Roughness of algebraic dichotomies.** An important property of the algebraic dichotomy is its *roughness under perturbations*. Let $T_B(x,y)$ be the principal matrix solution for the following linear system.

$$u'(x) = (A(x)+B(x))u(x), \qquad u \in \mathbb{R}^n,\ x \in I. \tag{2.8}$$

Assume that the system $u'(x) = A(x)u(x)$ has an algebraic dichotomy on I. The interval we consider could be finite, infinite or semi-infinite. In the following theorem we consider the case $I = (0,\infty)$ where 0 is neither a zero or a pole of the dichotomy. The other cases can be treated similarly.

Theorem 2.3. *Assume that $u' = A(x)u$ has an algebraic dichotomy with the power β and constant $K = \max\{K_1, K_2\}$ on $I = [0,\infty)$ where 0 is not a pole or zero of the dichotomy. Let $B(x)$ be piecewise continuous and there exists a constant $\zeta > 0$ such that*

$$|B(x)| \le \zeta/\sqrt{1+x^2}, \quad x \ge 0.$$

Let $-\beta < \mu < 0$ and $C_1 = \frac{2K}{\beta-|\mu|}$. Assume that ζ is sufficiently small so that

$$C_1\zeta < 1, \text{ and } C_2\zeta < 1 \text{ where } C_2 = \frac{2K^2}{(\beta-\mu)(1-C_1\zeta)}.$$

Then (2.8) also has an algebraic dichotomy on I with projections $\widetilde{P}(x)$ and $\widetilde{Q}(x)$, the power $\tilde{\beta}$ and the constant $\widetilde{K}$. Moreover, we have $\tilde{\beta} = |\mu|$,
$\widetilde{K} = K(1-C_1\zeta)^{-1}(1-C_2\zeta)^{-1}$ and

$$\|T_B(x,y)\widetilde{P}(y)\| \le \widetilde{K}(a(x)/a(y))^{-\tilde{\beta}}, \quad y \le x, \tag{2.9}$$

$$\|T_B(y,x)\widetilde{Q}(x)\| \le \widetilde{K}(a(x)/a(y))^{-\tilde{\beta}}, \quad y \le x, \tag{2.10}$$

$$\|\widetilde{P}(x) - P(x)\| \le \frac{C_2\zeta}{1-C_2\zeta}. \tag{2.11}$$

Proof. The proof is adapted from a proof in [22] where exponential dichotomies are considered. For any $\phi \in \mathbb{R}^n, y \in I$, we first show that the function $\mathcal{F}(y)$ maps $E_\mu(y,\infty)$ into itself:

$$\begin{aligned}(\mathcal{F}(y)u)(x) = T(x,y)P(y)\phi &+ \int_y^x T(x,\xi)P(\xi)B(\xi)u(\xi)d\xi \\ &+ \int_\infty^x T(x,\xi)Q(\xi)B(\xi)u(\xi)d\xi, \quad y \le x < \infty.\end{aligned} \tag{2.12}$$

Using the algebraic dichotomy on I, we have,

$$|T(x,y)P(y)\phi| \le K(a(x)/a(y))^{-\beta}|\phi| \le K(a(x)/a(y))^{\mu}|\phi|, \tag{2.13}$$

$$(2.14)\qquad |\int_y^x T(x,\xi)P(\xi)B(\xi)u(\xi)d\xi| \le \int_y^x K(a(x)/a(\xi))^{-\beta}\|u\|_\mu a(\xi)^\mu \frac{\zeta d\xi}{\sqrt{1+\xi^2}} \le \frac{\zeta K}{\beta+\mu}a(x)^\mu \|u\|_\mu,$$

$$(2.15)\qquad |\int_\infty^x T(x,\xi)Q(\xi)B(\xi)u(\xi)d\xi| \le \int_x^\infty K(a(\xi)/a(x))^{-\beta}\|u\|_\mu a(\xi)^\mu \frac{\zeta d\xi}{\sqrt{1+\xi^2}} \le \frac{\zeta K}{\beta-\mu}a(x)^\mu \|u\|_\mu.$$

Therefore

$$\|\mathcal{F}(y)u\|_\mu \le K|\phi|a(y)^{-\mu} + \frac{2\zeta K}{\beta-|\mu|}\|u\|_\mu.$$

Since $\frac{2\zeta K}{\beta-|\mu|} = C_1\zeta < 1$, $\mathcal{F}(y)$ is a contraction mapping in $E_\mu(y,\infty)$. Let the unique fixed point of $\mathcal{F}(y)$ be $u(x,y,\phi)$. Then it is a solution for (2.8) with

$$(2.16)\qquad |u(x,y,\phi)| \le \frac{K}{1-C_1\zeta}(a(x)/a(y))^\mu|\phi|, \quad \text{where } C_1 = \frac{2K}{\beta-|\mu|}.$$

Conversely, any solution $u(x)$ of (2.8) satisfying $u \in E_\mu(y,\infty)$ and $P(y)\phi = P(y)u(y)$ is a fixed point for $\mathcal{F}(y)$.

Similarly, for any $\phi \in \mathbb{R}^n, y \in I$, the function $\mathcal{H}(y)$ maps $E_{-\mu}(0,y)$ into itself:

$$(2.17)\qquad (\mathcal{H}(y)v)(x) = T(x,y)Q(y)\phi + \int_y^x T(x,\xi)Q(\xi)B(\xi)v(\xi)d\xi + \int_0^x T(x,\xi)P(\xi)B(\xi)v(\xi)d\xi, \quad 0 \le x \le y.$$

Observe that,

$$(2.18)\qquad |T(x,y)Q(y)\phi| \le K(a(y)/a(x))^{-\beta}|\phi| \le K(a(x)/a(y))^{-\mu}|\phi|,$$

$$(2.19)\qquad |\int_y^x T(x,\xi)Q(\xi)B(\xi)v(\xi)d\xi| \le \int_x^y K(a(\xi)/a(x))^{-\beta}\|v\|_{-\mu}a(\xi)^{-\mu}\frac{\zeta d\xi}{\sqrt{1+\xi^2}} \le \frac{K\zeta}{\beta+\mu}\|v\|_{-\mu}a(x)^{-\mu},$$

$$(2.20)\qquad |\int_0^x T(x,\xi)P(\xi)B(\xi)v(\xi)d\xi| \le \int_0^x K(a(x)/a(\xi))^{-\beta}\|v\|_{-\mu}a(\xi)^{-\mu}\frac{\zeta d\xi}{\sqrt{1+\xi^2}} \le \frac{K\zeta}{\beta-\mu}\|v\|_{-\mu}a(x)^{-\mu}.$$

Therefore

$$\|\mathcal{H}(y)v\|_{-\mu} \le K|\phi|a(y)^\mu + \frac{2\zeta K}{\beta-|\mu|}\|v\|_{-\mu}.$$

Since $\frac{2\zeta K}{\beta-|\mu|} < 1$, $\mathcal{H}(y)$ is a contraction mapping in $E_{-\mu}(0,y)$. Let the unique fixed point for $\mathcal{H}(y)$ be $v(x,y,\phi)$. The it is a solution for (2.8) with

$$|v(x,y,\phi)| \leq \frac{K}{1-C_1\zeta}(a(y)/a(x))^{\mu}|\phi|, \quad \text{where } C_1 = \frac{2K}{\beta-|\mu|}. \tag{2.21}$$

Conversely, any solution $v(x)$ of (2.8) satisfying $v \in E_{-\mu}(0,y)$, and $Q(y)v(y) = Q(y)\phi, P(0)v(0) = 0$ is a fixed point for $\mathcal{H}(y)$.

Let

$$W^s(y) := \{u(y,y,\phi) : \phi \in \mathbb{R}^n\}, \quad W^u(y) := \{v(y,y,\phi) : \phi \in \mathbb{R}^n\}.$$

Then It is obvious that

$$\begin{aligned} W^s(y) &= \{\phi_s : T_B(\cdot,y)\phi_s \in E_\mu(y,\infty)\}, \\ W^u(y) &= \{\phi_u : T_B(0,y)\phi_u \in \mathcal{R}Q(0)\}, \quad -\beta < \mu < 0. \end{aligned} \tag{2.22}$$

Moreover, $\phi \to u(y,y,\phi)$ and $\phi \to v(y,y,\phi)$ are homeomorphisms from $\mathcal{R}P(y)$ to $W^s(y)$ and from $\mathcal{R}Q(y)$ to $W^u(y)$ respectively.

We now show that

$$W^s(y) + W^u(y) = \mathbb{R}^n.$$

For $\phi \in \mathbb{R}^n$, consider

$$\begin{aligned} w &= u(y,y,\phi) + v(y,y,\phi) \\ &= \phi + \int_0^y T(y,\xi)P(\xi)B(\xi)v(\xi,y,\phi)d\xi + \int_\infty^y T(y,\xi)Q(\xi)B(\xi)u(\xi,y,\phi)d\xi \\ &= \phi + I_1 + I_2. \end{aligned}$$

From (2.15), (2.16) and (2.20), (2.21),

$$|I_1| \leq \frac{\zeta K}{\beta-\mu} \cdot \frac{K|\phi|}{1-C_1\zeta}, \quad |I_2| \leq \frac{\zeta K}{\beta-\mu} \cdot \frac{K|\phi|}{1-C_1\zeta}. \tag{2.23}$$

Therefore,

$$|w-\phi| \leq C_2\zeta|\phi|, \quad C_2 = \frac{2K^2}{(\beta-\mu)(1-C_1\zeta)}. \tag{2.24}$$

Since $C_2\zeta < 1$, $u(y,y,\phi) + v(y,y,\phi) : \phi \to w$ is a homeomorphism in $\mathbb{R}^n$ and we denote the inverse by $\phi = \Phi(y,w)$. We have

$$\|\Phi(y,w)\| \leq (1-C_2\zeta)^{-1}|w|. \tag{2.25}$$

We now show that $W^s(y) \cap W^u(y) = \{0\}$. Let (ϕ_1,ϕ_2) satisfy

$$u(y,y,\phi_1) = v(y,y,\phi_2).$$

Let $\phi = P(y)\phi_1 - Q(y)\phi_2$. It can be verified that

$$w = 0 = u(y,y,\phi) + v(y,y,\phi).$$

Thus $\phi = \Phi(y,0) = 0$. Consequently, $P(y)\phi_1 = 0$ and $Q(y)\phi_2 = 0$. Therefore $W^s(y) \cap W^u(y) = \{0\}$.

Define

$$\widetilde{P}(y)w = u(y, y, \Phi(y, w)),$$
$$\widetilde{Q}(y)w = v(y, y, \Phi(y, w)).$$

We have shown that $\widetilde{P}(y)$ and $\widetilde{Q}(y)$ are projections associated to the splitting $W^s(y) \oplus W^u(y) = \mathbb{R}^n$. The property

$$T_B(x, y)\widetilde{P}(y) = \widetilde{P}(x)T_B(x, y)$$

can be proved by the invariance of W^s and W^u. See (2.22).

To show (2.11), observe that

$$|(\widetilde{P}(y) - P(y))w| \leq |u(y, y, \Phi(y, w)) - P(y)\Phi(y, w)| + |P(y)\Phi(y, w) - P(y)w|$$
$$\leq |\int_y^\infty T(y, \xi)Q(\xi)B(\xi)u(\xi, y, \Phi(y, w))d\xi| + |\int_0^y T(y, \xi)P(\xi)B(\xi)v(\xi, y, \Phi(y, w))d\xi|$$

The above is bounded by $|I_1| + |I_2|$ as in (2.23). Using (2.24) and (2.25), we have

$$|(\widetilde{P}(y) - P(y))w| \leq C_2\zeta|\Phi| \leq \frac{C_2\zeta}{1 - C_2\zeta}|w|.$$

This proves (2.11).

From (2.16) and (2.25), we have for $x \geq y$,

$$|T_B(x, y)\widetilde{P}(y)w| = |u(x, y, \Phi(y, w)|$$
$$\leq \frac{K}{1 - C_1\zeta}\left(\frac{a(x)}{a(y)}\right)^\mu \frac{1}{1 - C_2\zeta}|w|.$$

This proves (2.9) with the constant $\widetilde{K}$. The estimate (2.10) can be proved similarly. □

Assume that the coefficients $A(x, s), B(x, s)$ depend on a parameter s and is analytic in s. Then it is well known that the solution matrices $T(x, y, s)$ and $T_B(x, y, s)$ for systems

(2.26) $$u'(x) = A(x, s)u(x),$$
(2.27) $$u'(x) = (A(x, s) + B(x, s))u(x), \qquad u \in \mathbb{R}^n,\ x \in I,$$

are analytic functions of s.

Corollary 2.4. *If system* (2.26) *has an algebraic dichotomy with projections analytic in the parameter s, then the perturbed system* (2.27) *has an algebraic dichotomy with projections $\widetilde{P}(x, s), \widetilde{Q}(x, s)$ analytic in s.*

Remark 2.1. The unstable subspace $\mathcal{R}\widetilde{Q})$ in $\mathbb{R}^+$ is not uniquely defined. The projections are analytic in s if the unstable subspace is chosen to satisfy

$$W^u(y, s) = \{\phi_u : T_B(0, y, s)\phi_u \in \mathcal{R}Q(0, s)\}.$$

However, this additional condition is not unique. Other choice of $W^u(y, s)$ is possible if it is transverse to the range of the stable subspace (which is unique).

Proof. The proof is based on careful examination of the proof of Lemma 2.3, details as follows.

Based on the analyticity of $T(x,y,s), P(x,s), Q(x,s)$ and $B(x,s)$ in s, from (2.12), the contraction mapping $\mathcal{F}(y,s)u$ is analytic in s. Thus the fixed point $u(x,y,s,\phi)$ of $\mathcal{F}(y,s)$ is analytic is s. More specifically, $u(x,y,s,\phi)$ can be obtained by an iteration method of which each step we have an analytic function and the uniformly convergent sequence of analytic functions converges to an analytic function. Similarly, the fixed point $v(x,y,s,\phi)$ is analytic in s. The mapping $\phi \to w$ defined as

$$w = u(y,y,s,\phi) + v(y,y,s,\phi),$$

is analytic in s, so is the inverse $\Phi(y,s,w)$. Finally, as the composition of analytic functions, the projections

$$\widetilde{P}(y,s)w = u(y,y,s,\Phi(y,s,w)), \quad \widetilde{Q}(y,s)w = v(y,y,s,\Phi(y,s,w)),$$

are clearly analytic functions of s. □

If $B(x)$ decays to zero faster than that assumed in Theorem 2.3, e.g., there exists $b, C_1 > 0$ such that

$$|B(x)| \le C_1/\sqrt{1+x^2}^{(1+b)}, \tag{2.28}$$

then $|B(x)| \le \zeta/\sqrt{1+x^2}$ with an arbitrarily small ζ on an interval $[N,\infty)$ if N is sufficiently large. We conclude from Theorem 2.3 that $T_B(x,y)$ has an algebraic dichotomy on $[N,\infty)$ with the power $\tilde{\beta}$ close to β. In the next theorem, we show under (2.28) we can choose the power of the dichotomy $\tilde{\beta} = \beta$ on $I = [N,\infty)$. Observe that (2.28) implies that $|B(x)| \le \frac{2^b C_1}{\sqrt{1+x^2}\cdot a(x)^b}$. In the following theorem we replace $2^b C_1$ by C for simplicity.

Theorem 2.5. *Assume that $u' = A(x)u$ has an algebraic dichotomy with the power β on $I = [0,\infty)$. Assume that $B(x)$ is piecewise continuous and there exit $C, b > 0$ with $b < \beta$ such that*

$$|B(x)| \le \frac{C}{\sqrt{1+x^2}\cdot a(x)^b}. \tag{2.29}$$

Let the constant N be sufficiently large such that $C_3 := 2CK/(\beta\ a(N)^b) < 1$. then (2.8) has an algebraic dichotomy with the power $\tilde{\beta} = \beta$ on $[N,\infty)$, and the new constant $\widetilde{K} = \frac{K}{1-C_3}$.

Remark 2.2. The dichotomy obtained in this theorem can be extend from $[N,\infty)$ to the original interval $[0,\infty)$ by flowing the flow backwards in time. The power of the dichotomy will not change, but the constant in the larger interval is not the same $\widetilde{K}$.

Proof. The main steps of the proof are exactly as in the proof of Theorem 2.3. Define $\mathcal{F}$ and $\mathcal{H}$ as before. The main difference is that we now have $\mu = -\beta < 0$.

(2.13) is still valid with $\mu = -\beta$.

The L.H.S. of (2.14) is bounded by

$$\int_y^x K(a(x)/a(\xi))^{-\beta}\|u\|_\mu \frac{a(\xi)^\mu}{a(\xi)^b}\frac{Cd\xi}{\sqrt{1+\xi^2}} \le \frac{CK}{b}a(y)^{-b}\|u\|_\mu a(x)^\mu.$$

The L.H.S. of (2.15) is bounded by

$$\int_x^\infty K(a(\xi)/a(x))^{-\beta}\|u\|_\mu \frac{a(\xi)^\mu}{a(\xi)^b}\frac{Cd\xi}{\sqrt{1+\xi^2}} \le \frac{CK}{2\beta+b}a(y)^{-b}\|u\|_\mu a(x)^\mu.$$

Therefore

$$\|\mathcal{F}(y)u\|_\mu \le K|\phi|a(y)^\beta + \frac{2CK}{\beta\ a(y)^b}\|u\|_\mu.$$

If $N>0$ is a constant such that $2CK/(ba(N)^b)<1$, then $\mathcal{F}(y)$ is a contraction mapping in $E_\mu(y,\infty)$ for $N\le y<\infty$. The unique fixed point for which is denoted by $u(x,y,\phi)$ and is a solution for (2.8) with

$$|u(x,y,\phi)| \le \frac{K}{1-C_3}(a(x)/a(y))^{-\beta}|\phi|, \text{ where } C_3 = 2CK/(ba(N)^b).$$

Similarly, (2.18) is still valid with $-\mu=\beta$.

The L.H.S. of (2.19) is bounded by

$$\int_x^y K(a(\xi)/a(x))^{-\beta}\|v\|_{-\mu}\frac{a(\xi)^{-\mu}}{a(\xi)^b}\frac{Cd\xi}{\sqrt{1+\xi^2}} \le \frac{KC}{b}a(x)^{-b}\|v\|_{-\mu}a(x)^\beta.$$

The L.H.S. of (2.20) is bounded by

$$\int_0^x K(a(x)/a(\xi))^{-\beta}\|v\|_{-\mu}\frac{a(\xi)^{-\mu}}{a(\xi)^b}\frac{Cd\xi}{\sqrt{1+\xi^2}} \le \frac{KC}{2\beta-b}a(x)^{-b}\|v\|_{-\mu}a(x)^\beta.$$

Using $b<\beta$, we have

$$\|\mathcal{H}(y)v\|_{-\mu} \le K|\phi|a(y)^\mu + \frac{2CK}{\beta\ a(x)^b}\|v\|_{-\mu}.$$

If $N>0$ is sufficiently large such that $2CK/(\beta\ a(N)^b)<1$, then $\mathcal{H}(y)$ is a contraction mapping in $E_\mu(N,y)$ for $N\le x\le y$. The unique fixed point for which is denoted by $v(x,y,\phi)$ and is a solution for (2.8) with

$$|v(x,y,\phi)| \le \frac{K}{1-C_3}(a(y)/a(x))^{-\beta}|\phi|, \text{ where } C_3 = 2CK/(\beta\ a(N)^b).$$

It Follows from the rest of the proof of Theorem 2.3 that $T_B(x,y)$ has an algebraic dichotomy on $[N,\infty)$ with the power $\tilde{\beta}=-\mu=\beta$. □

Corollary 2.6. *Assume that $u'=A(x)u$ has a pseudo algebraic dichotomy with the powers $\gamma>\delta$ and constant $K\ge1$ on $I=[0,\infty)$.*

(1) Let $B(x)$ be piecewise continuous with

$$|B(x)| \le \zeta/\sqrt{1+x^2},\ x\ge 0.$$

Assume that

$$\gamma-\delta > 8K^2\zeta.$$

Then the perturbed systems has a pseudo algebraic dichotomy with the powers $\tilde{\delta}<\tilde{\gamma}$ such that

$$\delta+4K^2\zeta \le \tilde{\delta} < \tilde{\gamma} \le \gamma-4K^2\zeta.$$

Let $d = \min\{\gamma - \tilde{\gamma}, \tilde{\delta} - \delta\}$*. Then the projections and the constant satisfy*

$$\|\widetilde{P}(x) - P(x)\| \le \frac{4K^2\zeta}{d},$$
$$\widetilde{K} = \frac{K}{(1 - \frac{2K\zeta}{d})(1 - \frac{4K^2\zeta}{d})}.$$

In particular, as $\zeta \to 0$*,*

$$\widetilde{K} \to K, \quad \widetilde{P} \to P, \quad \widetilde{Q} \to Q.$$

(2) If there exists $b, C > 0$ *such that*

$$|B(x)| \le C/\sqrt{1+x^2}^{(1+b)},$$

then (2.8) *has a pseudo algebraic dichotomy with the same powers* $\tilde{\gamma} = \gamma$ *and* $\tilde{\delta} = \delta$ *on* I*.*

Proof. Let $\mu = -(\gamma + \delta)/2$, $w(x) = u(x)a(x)^{\mu}$. Then from (1.5),

$$w'(x) = (A(x) + \frac{\mu}{\sqrt{1+x^2}} I + B(x))w. \tag{2.30}$$

By Lemma 1.1, with $B = 0$, (2.30) has a symmetric algebraic dichotomy with the power $\beta = (\gamma - \delta)/2$. From Lemmas 2.3 and 2.5, if $|B|$ is sufficiently small, (2.30) with $B \neq 0$ has a symmetric algebraic dichotomy. The results of the lemma follow after changing the variable back from w to u.

□

3. Asymptotic factorization and the Riccati equation

For a second order equation

$$u'' + p(x)u' + q(x)u = f(x),$$

if the coefficients $p(x), q(x)$ are slow varying, we expect that the dynamics to be governed by the eigenvalues $\lambda_j(x)$ calculated at each fixed x. This is called the method of frozen coefficients. We will introduce a method that extends the method of the frozen coefficients.

The system can be written as

$$(D_x - \nu(x))(D_x - \mu(x))u = f(x), \tag{3.1}$$

where

$$\nu + \mu = -p(x), \quad \nu\mu - D_x\mu = q(x).$$

The function $\mu(x)$ satisfies the Riccati equation [1]:

$$\mu' + \mu^2 + p(x)\mu + q(x) = 0.$$

The exact solution for the Riccati equation may be hard to find. Using the method of frozen coefficients as the first approximation, which is a quadratic equation with $\mu' = 0$, two asymptotic series solutions (μ_j, ν_j), $j = 1, 2$ can be computed to arbitrarily small error terms e_j, which are included in the equation:

$$\mu' + \mu^2 + p(x)\mu + q(x) = e_j(x), \quad j = 1, 2. \tag{3.2}$$

Let $u_1 = (D_x - \mu_1(x))u$, $u_2 = (D_x - \mu_2(x))u$ and $\nu_j = -p(x) - \mu_j(x)$. Then $u = (u_1 - u_2)/(\mu_2(x) - \mu_1(x))$. The second order equation is equivalent to the first order system

$$\begin{aligned} u_1' - \nu_1(x)u_1 &= f(x) + \frac{e_1(x)}{\mu_1(x) - \mu_2(x)}(u_1 - u_2), \\ u_2' - \nu_2(x)u_2 &= f(x) + \frac{e_2(x)}{\mu_1(x) - \mu_2(x)}(u_1 - u_2). \end{aligned} \tag{3.3}$$

Assume that the decoupled linear homogeneous system

$$u_1' - \nu_1(x)u_1 = 0, \quad u_2' - \nu_2(x)u_2 = 0$$

has an algebraic dichotomy on $\mathbb{R}^+$. Then

Theorem 3.1. *(1) If the remainder term* $\frac{|e_j(x)|}{|\mu_1(x)-\mu_2(x)|} \le \delta/\sqrt{1+x^2}$ *with a sufficiently small* $\delta > 0$*, then the system* (3.3) *has an algebraic dichotomy on* $\mathbb{R}^+$ *with a power close to the one determined by the decoupled system.*

(2) If $\frac{|e_j(x)|}{|\mu_1(x)-\mu_2(x)|} \le C/\sqrt{1+x^2}^{(1+b)}$ *for some* $C, b > 0$*, then* (3.3) *has an algebraic dichotomy with the power equal to the one determined by the decoupled system.*

If the remainder term is not small enough, we can use asymptotic expansions to extract higher order terms from the Riccati equation until $|e_j(x)|$ is sufficiently small so that Theorem 2.3 or Theorem 2.5 may apply.

The second order equation for u is often written as a first order system

$$u' = v, \quad v' = -q(x)u - p(x)v + f(x).$$

If the system for (u_1, u_2) has a dichotomy with stable and unstable subspaces spanned by $(u_1, 0)$ and $(0, u_2)$ respectively, then the system for (u, v) has an algebraic dichotomy with projections close to the splitting $(u, v) \to (u_1, u_2)$ where $u_1 = v - \mu_1(x)u, u_2 = v - \mu_2(x)u$.

If the coefficients $(p(x), q(x))$ are slow-varying, we often approximate (μ_1, μ_2) by the eigenvalues $(\lambda_1(x), \lambda_2(x))$. Then from (3.2), $e_j(x) = D_x\mu_j$. In the literature, this is called the method of frozen coefficients, which works if $D_x\mu_j/(\mu_1(x) - \mu_2(x))$ is small as required by Theorem 2.3 or Theorem 2.5.

The method of factorization can be used to convert a second order singularly perturbed equation to a first order system of which the fast and slow variables naturally split [4, 37, 16, 25].

4. Bessel Functions

The Bessel equation of order α contains a parameter $\alpha \ge 0$:

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \alpha^2)u = 0. \tag{4.1}$$

It has two linearly independent solutions – the first and second kind Bessel functions, $J_\alpha(x)$ and $Y_\alpha(x)$. It is known that,

$$J_\alpha(x) = O(x^\alpha), \qquad Y_\alpha(x) = O(x^{-\alpha}), \qquad x \to 0, \quad \text{if } \alpha \neq 0,$$
$$J_\alpha(x) = O(x^{-1/2}), \qquad Y_\alpha(x) = O(x^{-1/2}), \qquad x \to \infty.$$

We will check these asymptotic rates by the method of asymptotic factorization.

First consider $x \to \infty$. The system

$$u'' + \frac{1}{x}u' + (1 - \frac{\alpha^2}{x^2})u = 0$$

can be written as $(D - \nu_j(x))(D - \mu_j(x))u = 0$ if

$$D\mu_j + \mu_j^2 + \mu_j/x + 1 - \alpha^2/x^2 = 0, \quad \nu_j = -\mu_j - x^{-1}.$$

Freezing the coefficients and setting $D\mu_j = 0$, we approximate μ_j by the eigenvalues:

$$\mu_{1,2} \approx -\frac{1}{2x} \pm i\beta, \quad \beta = \sqrt{1 + \frac{1/4 - \alpha^2}{x^2}}, \nu_j \approx -\frac{1}{2x} \mp i\beta.$$

The error term satisfies

$$D\mu_{1,2}(x)/|\mu_1(x) - \mu_2(x)| = O(x^{-2} + |1/2 - \alpha^2|x^{-3})$$

The system $(D - \nu_j(x))u_j = 0, j = 1, 2$ that approximates (4.1) has an algebraic dichotomy with the power $\beta = 1/2$. Based on the discussion in §3, the Bessel equation has two linearly independent solutions of $O(x^{-1/2}), x \to \infty$, consistent with the properties of $J_\alpha(x), Y_\alpha(x)$.

To use our method on the asymptotic rates as $x \to 0$, let $\xi = 1/x, u = u(\xi) = u(1/x)$. Bessel's equation in the new variable becomes

$$u_{\xi\xi} + \xi^{-1}u_\xi + (\xi^{-4} - \alpha^2\xi^{-2})u = 0. \tag{4.2}$$

We wish to convert this to $(D-\nu_j(\xi))(D-\mu_j(\xi))u = 0$. The corresponding Riccati equation is

$$\mu_j' + \mu_j^2 + \xi^{-1}\mu_j + \xi^{-4} - \alpha^2\xi^{-2} = 0.$$

Ignoring the μ_j' term renders an incorrect answer. To improve the accuracy, we set $\mu_j(\xi) = \sum_1^\infty c_j\xi^{-j}$ in the Riccati equation and find two solutions:

$$\mu_1(x) = \alpha\xi^{-1} + \frac{1}{2(1-\alpha)}\xi^{-3} + \cdots, \quad \mu_2(x) = -\alpha\xi^{-1} + \frac{1}{2(1+\alpha)}\xi^{-3} + \cdots.$$

By Theorem 2.5, it suffices to keep $\mu_j(\xi) \approx \pm\alpha\xi^{-1}$ in order to obtain the leading powers of the dichotomy. Since $\nu_j = -\mu_j - \xi^{-1}$, the first order approximating system

$$(D - \nu_j)u_j = 0, \quad j = 1, 2,$$

has two solutions

$$u_1(\xi) \sim C_1\xi^{-\alpha-1}, \quad u_2(\xi) \sim C_2\xi^{\alpha-1},$$

Recall that $u = (u_1 - u_2)/(\mu_2(x) - \mu_1(x))$. Thus, system (4.2) has two solutions $u \sim \xi^{\pm\alpha}$. Changing back to $x = \xi^{-1}$, the Bessel equation of order α has two solutions $u(x) \sim C_1x^{-\alpha}$, $u(x) \sim C_2x^\alpha$ as $x \to 0$, consistent to the asymptotics of $J_\alpha(x)$ and $Y_\alpha(x)$.

5. The n-degree Fisher type equation

This example is based on the work of Leach, Needham, Kay [17] and Wu, Xing, Ye [38]. Consider the n-degree Fisher type equation:

$$u_t = u_{xx} + u^n(1-u), \quad n \in \mathbb{N},\ n > 1. \tag{5.1}$$

For each $n > 1$ there exists $c^*(n) > 0$ such that (5.1) has a traveling wave solution $\phi(x - ct)$ connecting $u = 1$ to $u = 0$ if and only if $c \geq c^*(n)$. The equation for ϕ is

$$\phi'' + c\phi' + \phi^n(1-\phi) = 0. \tag{5.2}$$

If $c > c^*(n)$, the equilibrium point $\phi = 1$ is a hyperbolic saddle while $\phi = 0$ is non-hyperbolic with an one dimensional center manifold. It is known that $\phi(z)$ decays to zero algebraically with the power of $1/(n-1)$.

$$\phi(z) \sim \left(\frac{c}{(n-1)z}\right)^{1/(n-1)}, \quad \text{as } z \to \infty. \tag{5.3}$$

The linear variational system is $U''+cU'+(n\phi^{n-1}-(n+1)\phi^n)U = g(z)$. Without loss of generality, we drop the smaller term $(n+1)\phi^n$ and study the algebraic dichotomy for a simplified system,

$$U'' + cU' + n\phi^{n-1}U = g(z). \tag{5.4}$$

We show that it has an algebraic dichotomy on $\mathbb{R}^+$. For each fixed z, the characteristic values for (5.4) are

$$\begin{aligned} \mu^\pm(z) &= -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - n\phi^{n-1}} \\ &\sim -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - \frac{nc}{(n-1)z}}. \\ \mu^- \sim -c < 0,& \quad \mu^+ \sim -\frac{n}{(n-1)z}. \end{aligned} \tag{5.5}$$

Although $\mu^+(z) < 0$, it approaches zero as $z \to \infty$.

Let $V = U'$, and make the change of variable $(U, V) \to (U_1, U_2)$:

$$\begin{aligned} &U_1 = V - \mu^- U, \quad U_2 = V - \mu^+ U. \\ &U_1' - \mu^+ U_1 = g(z) - \frac{\partial_z \mu^-(z)}{\mu^+ - \mu^-} \cdot (U_1 - U_2), \\ &U_2' - \mu^- U_2 = g(z) - \frac{\partial_z \mu^+(z)}{\mu^+ - \mu^-} \cdot (U_1 - U_2). \end{aligned}$$

By studying the flow on the center manifold, we can show that $\phi' \sim -\frac{1}{c}\phi^n$. From (5.5),

$$\partial_z \mu^\pm(z)/(\mu^+ - \mu^-) = O(z^{-2}), \text{ for large } z.$$

By Theorem 2.5 and Corollary 2.6, without affecting the powers of the dichotomy, system (5.4) is approximately

$$U_1' - \mu^+ U_1 = g(z), \quad U_2' - \mu^- U_2 = g(z). \tag{5.6}$$

For sufficiently large (z, z_0), the solutions to the homogeneous part of (5.6) are asymptotically

$$U_1(z) = U_1(z_0)\left(\frac{a(z)}{a(z_0)}\right)^{-n/(n-1)}, \quad U_2(z) = U_2(z_0)e^{-c(z-z_0)}.$$

Since the exponential decay implies the algebraic decay, we conclude that the system has a pseudo-dichotomy where the decay rate on the first subspace is slower than the decay rate on the second subspace.

Remark 5.1. Wu, Xing and Ye [38] proved the existence of an algebraic dichotomy for the linearized system using the method from Coppel [4]. They also studied stability of the traveling wave solutions in the space of functions that decay to zero algebraically.

6. Algebraic dichotomies in L^2 spaces

To use algebraic dichotomies on the Laplace transformed equations, we will extend the results of Section 2 to the L^2 type Hilbert spaces.

Definition 6.1. Let H_μ be the Hilbert space of locally L^2 functions with the following weighted norm being finite.

$$\|u\|_\mu = \left(\int_{\mathbb{R}} a(x)^{-2\mu}|u(x)|^2 \frac{dx}{\sqrt{1+x^2}}\right)^{1/2}.$$

Similarly we can define $H_\mu(J)$ if J is an interval of $\mathbb{R}$.

Let $\widehat{H}_\mu$ be the Hilbert space of locally L^2 functions with the following weighted norm being finite.

$$\|u\|_{\widehat{H}_\mu} = \left(\int_{-\infty}^0 a(x)^{2\mu}|u(x)|^2 \frac{dx}{\sqrt{1+x^2}} + \int_0^\infty a(x)^{-2\mu}|u(x)|^2 \frac{dx}{\sqrt{1+x^2}}\right)^{1/2}.$$

For any $\epsilon > 0$, $\widehat{E}_{\mu-\epsilon} \subset \widehat{H}_\mu$. However $\widehat{E}_\mu \subset \widehat{H}_\mu$ is not true. For example, the function $(\sqrt{1+x^2})^{\mu-\epsilon} \in \widehat{H}_\mu$ for any $\epsilon > 0$, but not for $\epsilon = 0$.

An equivalent norm for $u \in \widehat{H}_\mu$ can be defined as:

$$|||u|||_{\widehat{H}_\mu} = \left(\int_{-\infty}^\infty (1+x^2)^{-\mu}|u(x)|^2 \frac{dx}{\sqrt{1+x^2}}\right)^{1/2}.$$

Based on (1.3) for $x \geq 0$ and similar estimates for $x \leq 0$, we have

$$\|u\|_{\widehat{H}_\mu} \leq |||u|||_{\widehat{H}_\mu} \leq 2^\mu \|u\|_{\widehat{H}_\mu}, \quad \mu \geq 0,$$
$$2^\mu \|u\|_{\widehat{H}_\mu} \leq |||u|||_{\widehat{H}_\mu} \leq \|u\|_{\widehat{H}_\mu}, \quad \mu \leq 0.$$

Theorem 6.1. *Assume that* (2.1) *has an algebraic dichotomy on* $\mathbb{R}^+$ *with the power* β *and constants* K_1, K_2. *Assume that* $f \in H_\mu(R^+)$, $\beta > |\mu|$ *and* $P(0)u(0)$ *is given. Then*

(1) there exists a unique solution $u \in E_\mu(\mathbb{R}^+)$ *with*

$$\|u\|_{E_\mu(R^+)} \leq \left(\frac{K_1}{\sqrt{2(\beta+\mu)}} + \frac{K_2}{\sqrt{2(\beta-\mu)}}\right)\|f\|_{H_\mu(R^+)} + K_1|P(0)u(0)|. \tag{6.1}$$

(2) There exists a unique solution $u \in H_\mu(\mathbb{R}^+)$ with

$$\|u\|_{H_\mu(R^+)} \le (\frac{K_1}{\beta+\mu} + \frac{K_2}{\beta-\mu})\|f\|_{H_\mu(R^+)} + \frac{K_1}{\sqrt{2(\beta+\mu)}}|P(0)u(0)|. \tag{6.2}$$

Proof. The proof of the uniqueness is similar to that of Theorem 2.1 and shall be omitted. We will prove that $u = Pu + Qu$ is a solution by proving the convergence of

$$P(x)u(x) = \int_0^x T(x,y)P(y)f(y)\frac{dy}{\sqrt{1+y^2}} + T(x,0)P(0)u(0), \tag{6.3}$$

$$Q(x)u(x) = \int_\infty^x T(x,y)Q(y)f(y)\frac{dy}{\sqrt{1+y^2}}, \tag{6.4}$$

under suitable norms.

Proof of part (1): The integral term I_1 in (6.3) satisfies

$$\begin{aligned}
|I_1(x)|^2 &\le \left(\int_0^x K_1(a(x)/a(y))^{-\beta}|f(y)|\frac{dy}{\sqrt{1+y^2}}\right)^2 \\
&\le K_1^2\left(\int_{-\infty}^x a(x)^{-2\beta}a(y)^{2\beta}a(y)^{2\mu}\frac{dy}{\sqrt{1+y^2}}\right)\left(\int_{-\infty}^x a(y)^{-2\mu}|f(y)|^2\frac{dy}{\sqrt{1+y^2}}\right) \\
&\le \frac{K_1^2}{2\beta+2\mu}a(x)^{2\mu}\|f\|_\mu^2, \quad \text{if } \beta+\mu>0.
\end{aligned}$$

Therefore

$$|P(x)u(x)| \le \frac{K_1 a(x)^\mu}{\sqrt{2(\beta+\mu)}}\|f\|_\mu + K_1 a(x)^\mu |P(0)u(0)|.$$

Similarly, from (6.4),

$$|Q(x)u(x)| \le \frac{K_2 a(x)^\mu}{\sqrt{2(\beta-\mu)}}\|f\|_\mu, \quad \text{if } \beta-\mu>0.$$

The desired estimate for $|u(x)|$ in (6.1) follows from those of $|P(x)u(x)|$ and $|Q(x)u(x)|$.

Proof of part (2): Define a new variable z for every pair of $y \le x$ by

$$a(z) = \frac{a(x)}{a(y)}, \quad y \le x \Rightarrow a(z) \ge 1 \text{ and } z \ge 0. \tag{6.5}$$

Then for each fixed x,

$$\begin{aligned}
a(y) &= a(x)/a(z), \\
\frac{dy}{\sqrt{1+y^2}} &= -\frac{da(z)}{a(z)}.
\end{aligned} \tag{6.6}$$

Define $f^* : \mathbb{R}^+ \to \mathbb{R}^n$ as $f(y) = f^*(a(y))$. Then

$$\int a(y)^{2\mu}|f(y)|^2\frac{dy}{\sqrt{1+y^2}} = \int_{a=1}^\infty a^{2\mu}|f^*(a)|^2\,\frac{da}{a}. \tag{6.7}$$

The integral term I_1 in (6.3) satisfies

$$|I_1(x)| \leq \int_{-\infty}^{x} K_1 \left(\frac{a(x)}{a(y)}\right)^{-\beta} |f(y)| \frac{dy}{\sqrt{1+y^2}} \tag{6.8}$$
$$= K_1 \int_{z=0}^{\infty} a(z)^{-\beta} |f^*(a(x)/a(z))| \frac{da(z)}{a(z)}$$

From a generalized Minkovsky's inequality,

$$\|I_1\|_\mu \leq K_1 \int_{z=0}^{\infty} a(z)^{-\beta} \|f^*(\frac{a(\cdot)}{a(z)})\|_\mu \frac{da(z)}{a(z)}, \tag{6.9}$$

where the H_μ norms above are taken on functions of x for each fixed z.

As a function of x, we have $dx/\sqrt{1+x^2} = da(x)/a(x)$, and by the definition of H_μ norm, as function of x,

$$\|f^*(\frac{a(x)}{a(z)})\|_\mu^2 = \int a(x)^{-2\mu} |f^*(\frac{a(x)}{a(z)})|^2 \frac{dx}{\sqrt{1+x^2}} \tag{6.10}$$
$$= a(z)^{-2\mu} \int (a(x)/a(z))^{-2\mu} |f^*(a(x)/a(z))|^2 \frac{d(a(x)/a(z))}{a(x)/a(z)}$$
$$= a(z)^{-2\mu} \|f\|_\mu^2.$$

Substituting into (6.9), we have

$$\|I_1\|_\mu \leq K_1 \int_{z=0}^{\infty} a(z)^{-\beta-\mu} \|f\|_\mu (da(z)/a(z))$$
$$= K_1 \|f\|_\mu \int_{a=1}^{\infty} a^{-\beta-\mu-1} da$$
$$= \frac{K_1}{\beta+\mu} \|f\|_\mu, \quad \text{if } \beta+\mu > 0.$$

Therefore

$$\|Pu\|_\mu \leq \frac{K_1}{\beta+\mu} \|f\|_\mu + \frac{K_1}{\sqrt{2(\beta+\mu)}} |P(0)u(0)|.$$

Similarly, we have

$$\|Qu\|_\mu \leq \frac{K_2}{\beta-\mu} \|f\|_\mu, \quad \text{if } \beta-\mu > 0.$$

Estimate in (6.2) follows by combining those of $\|Pu\|_\mu$ and $\|Qu\|_\mu$.

□

Corollary 6.2. *Assume that* (2.1) *has an algebraic dichotomy on* $\mathbb{R}^-$ *with the power* β *and constants* K_1, K_2. *Assume that* $f \in H_\mu(\mathbb{R}^-)$, $\beta > |\mu|$ *and* $Q(0)u(0)$ *is given. Then*

(1) there exists a unique solution $u \in E_\mu(\mathbb{R}^-)$ *with*

$$\|u\|_{E_\mu(R^-)} \leq (\frac{K_1}{\sqrt{2(\beta+\mu)}} + \frac{K_2}{\sqrt{2(\beta-\mu)}}) \|f\|_{H_\mu(R^-)} + K_2 |Q(0)u(0)|.$$

(2) There exists a unique solution $u \in H_\mu(\mathbb{R}^-)$ *with*

$$\|u\|_{H_\mu(R^-)} \leq (\frac{K_1}{\beta+\mu} + \frac{K_2}{\beta-\mu})\|f\|_{H_\mu(R^-)} + \frac{K_2}{\sqrt{2(\beta+\mu)}}|Q(0)u(0)|.$$

Theorem 6.3. *Assume that* (2.1) *has an algebraic dichotomy in* $\mathbb{R}$ *with the power* $\beta > 0$ *and assume that* $f \in \widehat{H}_\mu(\mathbb{R})$ *with* $\mu < 0,\ \beta > |\mu|$. *Then*

(1) there exists a unique solution $u \in \widehat{E}_\mu(\mathbb{R})$ *such that*

$$\|u\|_{\widehat{E}_\mu} \leq \frac{(K_1+K_2)}{\sqrt{\beta-|\mu|}}\|f\|_{\widehat{H}_\mu}. \tag{6.11}$$

(2) There exists a unique solution $u \in \widehat{H}_\mu(\mathbb{R})$ *such that*

$$\|u\|_{\widehat{H}_\mu} \leq \frac{2(K_1+K_2)}{\beta-|\mu|}\|f\|_{\widehat{H}_\mu}. \tag{6.12}$$

Proof. **Proof of Part (1):** Notice that

$$\widehat{H}_\mu(\mathbb{R}) = H_{-\mu(\mathbb{R}^-)} \cap H_{\mu(\mathbb{R}^+)}.$$

Consider $x \leq 0$ first. By

$$P(x)u(x) = \int_{-\infty}^x T(x,y)P(y)f(y)\frac{dy}{\sqrt{1+y^2}},$$

$$\begin{aligned}
|P(x)u(x)|^2 &\leq K_1^2\left(\int_{-\infty}^x \frac{a(y)^\beta}{a(x)^\beta}|f(y)|\frac{dy}{\sqrt{1+y^2}}\right)^2 \\
&\leq K_1^2\left(\int_{-\infty}^x \frac{a(y)^{2\beta}}{a(x)^{2\beta}}a(y)^{-2\mu}\frac{dy}{\sqrt{1+y^2}}\right)\left(\int_{-\infty}^x a(y)^{2\mu}\frac{|f(y)|^2dy}{\sqrt{1+y^2}}\right) \\
&\leq K_1^2\frac{a(x)^{-2\mu}}{2\beta-2\mu}\|f\|_{\widehat{H}_\mu}^2.
\end{aligned}$$

Therefore

$$|P(x)u(x)| \leq \frac{K_1\|f\|_{\widehat{H}_\mu}}{\sqrt{2(\beta-\mu)}}a(x)^{-\mu}, \quad x \leq 0.$$

In particular,

$$|P(0)u(0)| \leq \frac{K_1\|f\|_{\widehat{H}_\mu}}{\sqrt{2(\beta-\mu)}}. \tag{6.13}$$

Next, consider $x > 0$. The integral term $I_1(x)$ in (6.3) satisfies

$$\begin{aligned}
|I_1(x)|^2 &\le K_1^2 \left(\int_0^x \frac{a(y)^\beta}{a(x)^\beta} |f(y)| \frac{dy}{\sqrt{1+y^2}} \right)^2 \\
&\le K_1^2 \left(\int_0^x \frac{a(y)^{2\beta}}{a(x)^{2\beta}} a(y)^{2\mu} \frac{dy}{\sqrt{1+y^2}} \right) \left(\int_0^x a(y)^{-2\mu} \frac{|f(y)|^2 dy}{\sqrt{1+y^2}} \right) \\
&\le K_1^2 \frac{a(x)^{2\mu} - a(x)^{-2\beta}}{2\beta + 2\mu} \|f\|^2_{\widehat{H}_\mu}.
\end{aligned}$$

One can verify that

$$\begin{aligned}
|T(x,0)P(0)u(0)|^2 &\le K_1^2 a(x)^{-2\beta} |P(0)u(0)|^2 \\
&\le \frac{K_1^2 a(x)^{-2\beta} \|f\|^2_{\widehat{H}_\mu}}{2(\beta - \mu)}.
\end{aligned}$$

Using $A + B \le \sqrt{2(A^2 + B^2)}$ for any positive numbers A and B, we have

$$|I_1(x)| + |T(x,0)P(0)u(0)| \le \frac{K_1 a(x)^\mu \|f\|_{\widehat{H}_\mu}}{\sqrt{\beta - |\mu|}}, \quad x > 0.$$

Therefore

$$\|Pu\|_{\widehat{E}_\mu} \le \frac{K_1 \|f\|_{\widehat{H}_\mu}}{\sqrt{\beta - |\mu|}}.$$

We can similarly prove that

$$\|Qu\|_{\widehat{E}_\mu} \le \frac{K_2 \|f\|_{\widehat{H}_\mu}}{\sqrt{\beta - |\mu|}}.$$

Estimate (6.11) follows from those of $\|Pu\|$ and $\|Qu\|$.

Proof of Part (2): We first consider the $x \le 0$. We starts from

$$|P(x)u(x)| \le K_1 \int_{-\infty}^x \left(\frac{a(x)}{a(y)} \right)^{-\beta} \frac{|f(y)|}{\sqrt{1+y^2}} dy, \quad x \le 0. \tag{6.14}$$

Define the new variable z by (6.5). Then for each fixed x, (6.6) is satisfied.

As a function of x, we have $dx/\sqrt{1+x^2} = da(x)/a(x)$. Let $f^*(a(x)) = f(x)$. Then using (6.7),

$$\begin{aligned}
&\|f^*(a(\cdot)/a(z))\|^2_{H_{-\mu}(R^-)} \\
&= \int a(x)^{2\mu} |f^*(a(x)/a(z))|^2 \frac{dx}{\sqrt{1+x^2}} \\
&= a(z)^{2\mu} \int (a(x)/a(z))^{2\mu} |f^*(a(x)/a(z))|^2 \frac{d(a(x)/a(z))}{a(x)/a(z)} = a(z)^{2\mu} \|f\|^2_{H_{-\mu}(R^-)}.
\end{aligned}$$

From (6.14) and a generalized Minkovski inequality,

$$\|Pu\|_{H_{-\mu}(R^-)} \le \int_{z=\infty}^{0} K_1 a(z)^{-\beta} \|f^*(a(\cdot)/a(z))\|_{H_{-\mu}(R^-)}(-da(z)/a(z))$$
$$\le K_1 \int_{a=1}^{\infty} a^{-\beta+\mu-1} \|f\|_{H_{-\mu}(R^-)} da \le \frac{K_1}{\beta-\mu} \|f\|_{H_{-\mu}(R^-)}.$$

Next consider $x > 0$. The integral term $I_1(x)$ in (6.3) satisfies

$$\|I_1\|_{H_\mu(R^+)} \le \frac{K_1}{\beta+\mu} \|f\|_{H_\mu(R^+)},$$

as in the proof of part (2) of Theorem 6.1. Using (6.13), we can show that

$$\|T(\cdot,0)P(0)u(0)\|_{H_\mu(R^+)} \le K_1 |P(0)u(0)| \int_0^\infty a(x)^{-2\beta-2\mu} \frac{dx}{\sqrt{1+x^2}}$$
$$\le \frac{K_1}{\sqrt{2(\beta+\mu)}} |P(0)u(0)|$$
$$\le \frac{K_1 \|f\|_{\widehat{H}_\mu}}{\sqrt{2(\beta+\mu)}\sqrt{2(\beta-\mu)}}$$
$$\le \frac{K_1}{2(\beta+\mu)} \|f\|_{\widehat{H}_\mu}.$$

Therefore

$$\|Pu\|_{\widehat{H}_\mu(R)} \le \frac{2K_1}{\beta-|\mu|} \|f\|_{\widehat{H}_\mu}.$$

We can similarly show that

$$\|Qu\|_{\widehat{H}_\mu(R)} \le \frac{2K_2}{\beta-|\mu|} \|f\|_{\widehat{H}_\mu}.$$

Estimate (6.12) follows from $\|u\| \le \|Pu\| + \|Qu\|$.

□

7. Riemann solutions of conservation laws in similarity coordinates

In this section, we present an example from Riemann solutions of hyperbolic conservation laws where algebraic dichotomies naturally occur.

A Riemann solution: $u = \hat{u}(X/T)$, $\hat{u} : \mathbb{R} \to \mathbb{R}^n, X \in \mathbb{R}, T \ge 0$, is a solution to the Riemann problem of the conservation laws

$$u_T + f(u)_X = 0, \quad u(X,0) = \begin{cases} u^\ell, & \text{if } X < 0, \\ u^r, & \text{if } X > 0. \end{cases}$$

Consider the Riemann solution that satisfies the following hypotheses:

(H1) The system is strictly hyperbolic on an open set Ω that contains the Riemann solution $\hat{u}$, i.e. the eignvalues of $Df(u)$ are real and distinct if $u \in \Omega$.

(H2) The Riemann solution has n consecutive Lax i-shocks: Λ^i, $i = 1, \ldots, n$, with the speed $\bar{s}^i$. Let $\bar{s}^0 = -\infty$ and $\bar{s}^{n+1} = \infty$, then $\bar{u}(X,T) = \bar{u}^i$ if $\bar{s}^i < X/T < \bar{s}^{i+1}$. By definition, if Λ^i is a lax i-shock with the speed $\bar{s}^i$, and then the eigenvalues

$\nu_1^{i-1} < \cdots < \nu_n^{i-1}$ of $Df(\bar{u}^{i-1})$ and the eigenvalues $\nu_1^i < \cdots < \nu_n^i$ of $Df(\bar{u}^i)$ are real, distinct and satisfy

$$\nu_{i-1}^{i-1} < \bar{s}^i < \nu_i^{i-1}, \quad \nu_i^i < \bar{s}^i < \nu_{i+1}^i.$$

(H3) The Rankine-Hugoniot jump condition is satisfied at each shock Λ^i, i.e.:

$$f(\bar{u}^i) - f(\bar{u}^{i-1}) = \bar{s}^i(\bar{u}^i - \bar{u}^{i-1}).$$

(H4) Majda's stability condition is satisfied at each Λ^i. By definition, this means that if the eigenvectors corresponding to ν_j^i are $\mathbf{r}_j^i$, then the following n vectors in $\mathbb{R}^n$ are linearly independent:

$$\mathbf{r}_1^{i-1}, \ldots, \mathbf{r}_{i-1}^{i-1}, \bar{u}^i - \bar{u}^{i-1}, \mathbf{r}_{i+1}^i, \ldots, \mathbf{r}_n^i.$$

By the change of variables $x = X/T, t = \log T$, the conservation laws become:

$$u_t + (Df(u) - xI)u_x = 0, \quad u : \mathbb{R} \to \mathbb{R}^n, \ x, t \in \mathbb{R}. \tag{7.1}$$

The Riemann solution becomes stationary to (7.1). We can study its eigenvalues and we can use the spectral method to find sufficient and necessary conditions for the stability of Riemann solutions.

Assume that the location of the shock in (x, t) coordinates is at x^i for the shock Λ^i. It means in the (X, T) coordinates the speed of the ith shock is $X/T = \bar{s}^i = x^i$. As a convention, $x^0 = -\infty, x^{n+1} = \infty$.

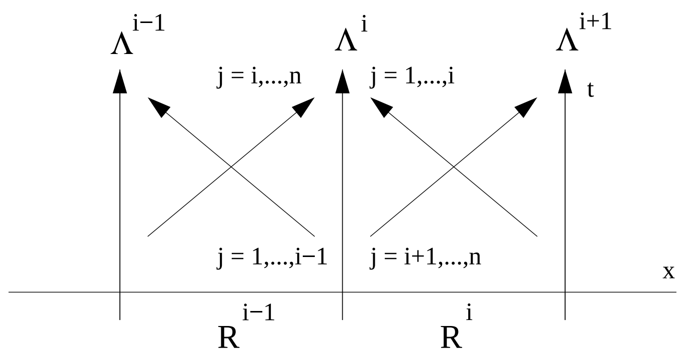

FIGURE 7.1. The left and right going characteristics in R^{i-1} and R^i.

In $R^i = (x^i, x^{i+1})$, where $u = \bar{u}^i$, condition (H2) implies that there are i characteristics moving to the left and $n-i$ moving to the right. Let $\nu_j(\bar{u}^i)$ be the jth eigenvalue then $\nu_j(\bar{u}^i) \notin R^i$, and

$$\nu_\ell(\bar{u}^i) < x^i < x^{i+1} < \nu_r(\bar{u}^i), \quad \ell = 1, \ldots, i; \ r = i+1, \ldots, n. \tag{7.2}$$

Consider the linear variational system around the Riemann solution:

$$U_t + (Df(\bar{u}^i) - xI)U_x = g(x, t). \tag{7.3}$$

At the shock Λ^i, let $\Delta u^i = \bar{u}^i - \bar{u}^{i-1}$,and let $X^i(t)$ be the variation of the shock speed x^i. Linearize around the Rankine Hugoniot condition: $f(u(x^i+, t)) - f(u(x^i-, t)) = (\dot{x}^i(t) + x^i(t))(u(x^i+, t) - u(x^i-, t))$, we have

$$[Df(\bar{u}^i) - x^iI]U(x^i+, t) - [Df(\bar{u}^{i-1}) - x^iI]U(x^i-, t) = (\dot{X}^i(t) + X^i(t))\Delta u^i. \tag{7.4}$$

Instead of study the evolution of $(U, \{X^i\}_{i=1}^n)$, we write the jump condition as

$$[Df(\bar{u}^i) - x^i I]U(x^i+, t) - [Df(\bar{u}^{i-1}) - x^i I]U(x^i-, t) = 0, \qquad \text{mod } (\Delta u^i). \tag{7.5}$$

This together with (7.3) determine the evolution of $U(\cdot, t)$ without having to know $X^i(t), i = 1, \ldots, n$.

Applying the Laplace transform to the linear system (7.3) and (7.5), we obtain the system in dual variable s:

$$s\hat{U} + (Df(\bar{u}^i) - xI)\hat{U}_x = \hat{g}(x, s) + h(x), \tag{7.6}$$

$$[Df(\bar{u}^i) - x^i I]\hat{U}(x^i+, s) - [Df(\bar{u}^{i-1}) - x^i I]\hat{U}(x^i-, s) = 0, \qquad \text{mod } (\Delta u^i). \tag{7.7}$$

Condition (H1) implies that in (x^i, x^{i+1}), $Df(\bar{u}^i)$ has n real eigenvalues/eigenvectors $\nu_j(\bar{u}^i)$ and $\mathbf{r}_j(\bar{u}^i)$. Let

$$U^i(x) = \sum_j u_j^i(x)\mathbf{r}_j(\bar{u}_i), \text{ where } U^i(x) := U(x) \text{ for } x \in R^i.$$

On each R^i, (7.6) reduces to a system on $u_j^i(x), j = 1, \ldots, n$ which can be solved by integration factors. Using (7.2), it is easy to show that the homogeneous part of the reduced system has an algebraic dichotomy in each R^i with the power $\Re s$. See [24, 26].

We look for $\hat{U}(x, s)$ in the weighted L^2 space L^2_η, $\eta \in \mathbb{R}$, defined as follows:

$$\|U\|_{L^2_\eta} := \left(\sum_{i=0}^n \sum_{j=1}^n \|u_j^i\|^2 \right)^{1/2}, \text{ where}$$

$$\|u_j^i\| := \left(\int_{R^i} \left| \left(x - \nu_j(\bar{u}^i)\right)^\eta u_j^i(x) \right|^2 \frac{dx}{|x - \nu_j(\bar{u}^i)|} \right)^{1/2}.$$

Assume $\eta > 0$. Then as $x \to \pm\infty$, $u_j^0(x)$ and $u_j^n(x)$ decay to 0 algebraically of order $|x - \nu_j|^{-\eta}$.

The next is to fit the solutions u_j^i on R^i according to the jump conditions (7.7). It has been shown that the system for $\hat{U}(x, s)$ can be solved if we assume that the determinant $d(s)$ of a characteristic matrix is nonzero. On the other hand the zeros of the characteristic equation $d(s) = 0$ correspond to the eigenvalues of the linear system [21, 24, 26].

Sufficient conditions for the stability of the Riemann solutions have been studied by many authors [34, 19, 20, 21] in BV and L^1 norms. We look for conditions that are sufficient and necessary for the stability of Riemann solutions. By using the algebraic dichotomy, it is shown in [26] that

(1) in the space L^2_η, the linear variational system generates a C_0 semigroup $T(t), t \geq 0$. If $U(0) = U_0$, then the solution of the initial value problem can be expressed as:

$$U(\cdot, t) = T(t)U_0 + \int_0^t T(t - s)g(\cdot, s)ds.$$

(2) We say σ_0 is the coordinate of a resonance line if

$$\inf_\omega |d(\sigma_0 + i\omega)| = 0.$$

The largest real parts of the eigenvalues and the coordinates of the resonance lines, denoted by Γ, is finite.

(3) If $\gamma > \Gamma$ and $g(t) = O(e^{\gamma t})$ then $U(t) = O(e^{\gamma t})$.

The concept of resonance lines, defined in [26], is related to the pseudo-eigenvalues and the Gearhart-Prüss Theorem on the growth rate of C^0 semigroups on Hilbert spaces. The growth rate in (3) is optimal, cf. [36, 5].

8. Conservation laws with Dafermos type viscous regularization

Consider the Riemann solutions in similarity coordinates as in (7.1):

$$u_t + f(u)_x - xu_x = 0, \quad u : \mathbb{R} \to \mathbb{R}^n,\ x, t \in \mathbb{R}.$$

Assume that $f \in C^3(\mathbb{R}^n)$ with bounded derivatives. If a diffusion term ϵu_{xx} is added to the system,

$$u_t + f(u)_x - xu_x = \epsilon u_{xx}, \tag{8.1}$$

the resulting system is equivalent to the Dafermos regularization in the original variable (X, T),

$$u_T + f(u)_X = \epsilon T u_{XX}.$$

For (8.1), Schecter and Szmolyan proved that for small ϵ, structurally-stable, classical Riemann solutions which consist of n rarefaction and Lax shock waves have Riemann-Dafermos solutions $u(x, \epsilon)$ nearby [35]. The proof of these results uses geometric singular perturbation theory [13]. It is known that linearized system around $u(x, \epsilon)$ of (8.1) generates an analytic semigroup in the space of super-fast decay functions, $u = O(e^{-\alpha x^2})$, cf. [23]. However, such function space excludes solutions that decay to zero algebraically in x. As seen in §7, such solutions naturally occur when studying linearized system near a Riemann solution in similarity coordinates.

In this section a stronger regularization term will be added to (7.1),

$$u_t + f(u)_x - xu_x = \epsilon(1 + x^2)u_{xx}, \tag{8.2}$$

so that its the linear variational system can generate an analytic semigroup in the space of algebraic decay functions.

System (8.2) has some similar properties to (8.1). Suppose $\bar{u}(x)$ is a structurally-stable [31], classical Riemann solution to (7.1) which consists of rarefaction and Lax shock waves. Then for small ϵ, system (8.2) has regularized smooth solution $u^\epsilon(x)$ nearby. The method of proof is similar to that used by schecter [32] that treats Dafermos regularization of Riemann solutions of conservation laws consisting of shocks, and will not been given here. Assuptions on u^ϵ will be specified shortly. Equation (8.2) will be called a Dafermos type regularization since the stationary solutions of which correspond to similarity solutions $u = u(X/T)$ in the original coordinates (X, T).

The purpose of introducing u^ϵ is to provide a tool for studying the nonlinear stability of the Riemann solutions of the conservation laws. While the linear stability is completely solved by the spectrum method, the semigroup $T_0(t)$ obtained in [26] only maps L^2_η to L^2_η. It is not smooth enough to handle nonlinear terms. For example variation of constant formulas cannot be used to write integral equations for the solutions of the nonlinear conservation laws. On the other hand, if the semigroup

$T_\epsilon(t)$ generated by the linear variational system of (8.2) is analytic, it has enough smoothness to handle nonlinear problems. In particular, the variation of constant formula works and many dynamical systems tool can apply.

For any $0 < \theta_0 < \pi$ and $a \in \mathbb{R}$, let

$$\Sigma(\theta_0) = \{\lambda \in \mathbb{C} \ : \ |\arg(\lambda)| \le \theta_0\},$$
$$\Sigma(\theta_0) + a = \{\lambda \in \mathbb{C} \ : \ |\arg(\lambda - a)| \le \theta_0\}.$$

We call a linear operator A in a Banach space X a sectorial operator if it is a closed, densely defined operator, and there exist some $M \ge 1$ and $a \in \mathbb{R}$ such that

(i) for some $\pi/2 < \theta_0 < \pi$ and $a \in \mathbb{R}$, $\{\Sigma(\theta_0) + a\} \cap \{\lambda \ne a\} \subset \rho(A)$;

(ii) there exists a constant M such that

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - a|} \quad \text{for } \lambda \in \Sigma(\theta_0) + a, \ \lambda \ne a.$$

A necessary and sufficient condition for T_ϵ to be an analytic semigroup is that its infinitesimal generator is a sectorial operator in appropriate function spaces. See [8, 11, 29] for reference of the sectorial operators and related analytic semigroups.

According to Hale and Lunel [10], page 278, an eigenvalue of a closed linear operator T on a Banach space is called a normal eigenvalue if it is an isolated point of the spectrum of T, and the corresponding generalized eigenspace is finite dimensional. A normal point of T is either a normal eigenvalue or a resolvent point of T.

Let $\mu < 0$ be a constant independent of ϵ. In the space $u \in \widehat{H}_\mu(\mathbb{R})$, define $\mathcal{A}^\epsilon$ to be the linear variational operator of (8.2):

$$\mathcal{A}^\epsilon u := \epsilon(1 + x^2)u_{xx} + xu_x - Df(u^\epsilon)u_x - Df(u^\epsilon)_x u,$$

where

$$D(\mathcal{A}^\epsilon) := \{u | u \in \widehat{H}_\mu, u_x \in \widehat{H}_{\mu-1}, u_{xx} \in \widehat{H}_{\mu-2}\}.$$

We would like to show:

(1) that $\mathcal{A}^\epsilon$ is a sectorial operator in $\widehat{H}_\mu(\mathbb{R})$;
(2) in the complex plane, the region $\Re s \ge -\eta$, $\eta > 0$ consists of normal points only, i.e., if s is in that region, then either s is a normal eigenvalue or a resolvent point for the operator $\mathcal{A}^\epsilon$, ;
(3) the eigenvalues of $\mathcal{A}^\epsilon$ either come from the Riemann solutions of the conservation laws (slow eigenvalues) or from the traveling waves that approximate shocks (fast eigenvalues); and
(4) solutions of the initial value problem for the hyperbolic conservation laws are limits of the solutions of the initial value problems of (8.2) as $\epsilon \to 0$.

If we can prove (1)-(4), and if the viscous shock waves $u^\epsilon \to u^0$ as $\epsilon \to 0$, then the limit u^0 is a weak solution to the conservation laws. If the solutions u^ϵ are nonlinearly stable, so is u^0. The approach outlined above provides a new method to study nonlinear stability of the Riemann solutions based on its linear stability.

Only parts (1) and (2) will be proved in this paper. The convergence of viscous solutions to Riemann solutions has only been proved recently [3]. Much stronger assumptions will be needed to prove (3) and (4) and the proofs are likely to be difficult. But to prove (1) and (2), the assumptions used in this section are rather weak:

(H): System (8.2) has a solution $u^\epsilon \in C^3(\mathbb{R})$ that satisfies

$$|u^\epsilon| \le C_1,\ |u^\epsilon_x| \le C_2/\epsilon, \quad \epsilon \to 0.$$

(HH): There exists $N > 0$ such that the Riemann solution $\bar{u}(x)$ of (7.1) is constant if $|x| \ge N$: $\bar{u}(x) = u^\ell$, if $x \le -N$ and $\bar{u}(x) = u^r$ if $x \ge N$. Moreover, for $|x| \ge N$, the Riemann solution is strictly hyperbolic and the regularized solution $u^\epsilon(x)$ satisfies

$$|u^\epsilon - \bar{u}| \le C_1\epsilon,\ |u^\epsilon_x| \le C_2, \text{ as } \epsilon \to 0.$$

For $\sigma_0 > 0$, define the following region in the complex plane:

$$\mathcal{P}(\sigma_0) = \{\sigma + i\omega \ :\ \sigma \ge (\sigma_0) - \omega^2/(4\sigma_0)\}.$$

Lemma 8.1. *Let $\tau = \sigma + i\omega$, $z = x + iy$ be two complex numbers with $\sqrt{\tau} = z$, where $\sigma \ge 0$ and the principal value of the square root is used. Then*

(1) for any $M > 0$, if $\tau \in \mathcal{P}(M^2)$, then $\Re\sqrt{\tau} \ge M$. See Fig. 8.1.

(2) For any $0 < \theta < \pi/2$, if $z \in \Sigma(\theta)$, then $\Re(z) \ge \cos(\theta)|z|$.

(3) For any $M > 0$ if $\tau \in \mathcal{P}(M^2) \cap \Sigma(2\pi/3)$, then

$$\Re(\sqrt{\tau} - M/2) \ge \frac{1}{4}\,|\sqrt{\tau} - M/2|. \tag{8.3}$$

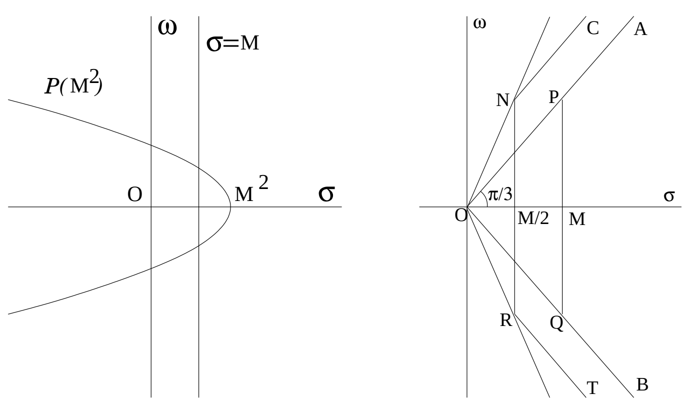

FIGURE 8.1. The first figure shows if $\tau \in \mathcal{P}(M^2)$, then $\Re\sqrt{\tau} \ge M$. The second figure illustrates the proof of (3).

(4) If $\Re s \ge -\eta$ for some $\eta > 0$, then for any $b > 0$,

$$\Re\frac{\sqrt{b^2 + \epsilon s} - b}{\epsilon} \ge \frac{-\eta}{2b} + O(\epsilon).$$

Proof. Proof of (3): Since $\sqrt{\tau} \in \Sigma(\pi/3)$, the points of $\sqrt{\tau}$ are in the region right to the path $APQB$ where $OP = M/\cos(\pi/3) = 2M$. See Fig. 8.1. After shifting to the left by $M/2$ the points are in the region to the right of the path $CNRT$ where $ON < 2M$. The sector $CNRT$ is contained in $\Sigma(\theta)$ with $\cos(\theta) > (M/2)/(2M) = 1/4$.

Proof of (4): If ϵ is sufficiently small, then $b^2 - \epsilon\eta > 0$. Since $\Re\epsilon s \geq -\epsilon\eta$, from part (1) of this lemma, we have

$$\Re\sqrt{b^2+\epsilon s} \geq \sqrt{b^2-\epsilon\eta} = b - \frac{\epsilon\eta}{2b} + O(\epsilon^2\eta^2).$$

The estimate in (4) follows from this. □

8.1. $\mathcal{A}^\epsilon$ **is sectorial in** $\widehat{H}_\mu(\mathbb{R})$**.** In this subsection, we assume that hypothesis (H) is satisfied. For the point s, we assume that

$$M \geq 1 \text{ and } \tau = \epsilon s \in \mathcal{P}(M^2) \cap \Sigma(2\pi/3).$$

Then from Lemma 8.1, $\Re\sqrt{\epsilon s} \geq M,\ |\epsilon s| \geq M$, and (8.3) hold. The resolvent equation $\mathcal{A}^\epsilon u - su = g$ becomes

$$\epsilon(1+x^2)u_{xx} + (xI - Df(u^\epsilon))u_x - (D^2 f(u^\epsilon)u^\epsilon_x)u - su = g(x), \tag{8.4}$$

$$u_{xx} - \frac{su}{\epsilon(1+x^2)} = \frac{g(x)}{\epsilon(1+x^2)} - \frac{xu_x}{\epsilon(1+x^2)} + \frac{Df(u^\epsilon)u_x + (D^2 f(u^\epsilon)u^\epsilon_x)u}{\epsilon(1+x^2)}.$$

Define the n-dimensional vector valued functions:

$$u^\pm := u_x - \lambda^\pm(x)u \in \widehat{H}_{\mu-1},$$

$$\text{where } \lambda^\pm(x) = \frac{\pm\sqrt{\epsilon s}}{\epsilon\sqrt{1+x^2}}.$$

$$\text{Then } u_x = \frac{1}{2}(u^+ + u^-),$$

$$\text{and } u = \frac{\epsilon\sqrt{1+x^2}}{2\sqrt{\epsilon s}}(u^- - u^+). \tag{8.5}$$

Observe that $\partial_x(\frac{1}{\sqrt{1+x^2}}) = \frac{-x}{(1+x^2)^{3/2}}$. Therefore

$$u^\pm_x \pm \frac{\sqrt{\epsilon s}}{\epsilon\sqrt{1+x^2}}u^\pm = u_{xx} - \frac{su}{\epsilon(1+x^2)} \pm \frac{\sqrt{\epsilon s}}{\epsilon}\frac{xu}{(1+x^2)^{3/2}}$$
$$= \frac{g(x)}{\epsilon(1+x^2)} - \frac{xu_x}{\epsilon(1+x^2)} \pm \frac{\sqrt{\epsilon s}}{\epsilon}\frac{xu}{(1+x^2)^{3/2}} + \frac{Df(u^\epsilon)u_x + (D^2 f(u^\epsilon)u^\epsilon_x)u}{\epsilon(1+x^2)}.$$

Expressing (u_x, u) by (u^+, u^-) and observe that $u^\epsilon_x = O(1/\epsilon)$, we have

$$\left|\frac{xu_x}{\epsilon(1+x^2)}\right| + \left|\frac{\sqrt{\epsilon s}}{\epsilon}\frac{xu}{(1+x^2)^{3/2}}\right| + \left|\frac{Df(u^\epsilon)u_x + (D^2 f(u^\epsilon)u^\epsilon_x)u}{\epsilon(1+x^2)}\right|$$
$$\leq \frac{C}{\epsilon\sqrt{1+x^2}}(|u^+| + |u^-|).$$

System for (u^+, u^-) can be written as a system of $2n$ equations:

$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix}' = (A(x,s) + B(x,s))\begin{pmatrix} u^+ \\ u^- \end{pmatrix} + \begin{pmatrix} f(x)/\sqrt{1+x^2} \\ f(x)/\sqrt{1+x^2} \end{pmatrix}, \tag{8.6}$$

where $(u^+, u^-) \in \widehat{H}_{\mu-1}$, both sides are evaluated in $\widehat{H}_{\mu-2}$ and

$$A(x,s) = \mathrm{diag}(-\frac{\sqrt{\epsilon s}}{\epsilon\sqrt{1+x^2}}, \dots, -\frac{\sqrt{\epsilon s}}{\epsilon\sqrt{1+x^2}}, \frac{\sqrt{\epsilon s}}{\epsilon\sqrt{1+x^2}} \dots, \frac{\sqrt{\epsilon s}}{\epsilon\sqrt{1+x^2}}),$$

$$|B(x,s)| \le \frac{k}{\epsilon\sqrt{1+x^2}}, \text{ for some } k > 0, \quad f(x) = \frac{g(x)}{\epsilon\sqrt{1+x^2}}.$$

It is easy to see that $(u^{+\prime}, u^{-\prime})^\tau = A(x,s)(u^+, u^-)^\tau$ has an algebraic dichotomy with the power

$$\beta = \frac{\Re\sqrt{\epsilon s}}{\epsilon} \ge \frac{M}{\epsilon}.$$

We have shown that $|B(x,s)| \le \delta/\sqrt{1+x^2}$ with $\delta = k/\epsilon$ for some $k > 0$. Using Theorem 2.3, if M is sufficiently large, then (8.6) has an algebraic dichotomy with the power

$$\tilde\beta = \frac{\Re\sqrt{\epsilon s}}{\epsilon} - \frac{M}{4\epsilon}.$$

More specifically, as in Theorem 2.3, let

$$C_1 = \frac{2K}{M/(4\epsilon)}, \quad C_2 = \frac{2K^2}{(M/(4\epsilon))(1 - C_1 k/\epsilon)}.$$

If we choose M to be sufficiently large so that $C_1 k/\epsilon < 1/2$. Then the condition $C_1\delta < 1$ is satisfied. Also

$$C_2\delta \le \frac{4\delta K^2}{M/(4\epsilon)} < 1,$$

if M is sufficiently large. By Theorem 2.3 has an algebraic dichotomy with the power $\tilde\beta \ge 3M/(4\epsilon)$. Moreover, we can choose $M > 0$ sufficiently large so that the projections $\widetilde{P}(x,s) - P(x,s) = O(1/M)$ can be arbitrarily small. The constant $\widetilde{K}$ is uniformly bounded with respect to large M and small ϵ.

Theorem 8.2. *Assume the hypothesis (H), then there there exists a sufficiently large $M > 0$, independent of ϵ such that if $\epsilon s \in \mathcal{P}(M^2) \cap \Sigma(2\pi/3)$, then the resolvent equation* (8.4) *has a unique solution $u \in \widehat{H}_\mu$ with*

$$\|u\|_{\widehat{H}_\mu} \le \frac{C}{|s - M^2/(4\epsilon)|}\|g\|_{\widehat{H}_\mu}, \quad \text{where } \epsilon s \in \mathcal{P}(M^2) \cap \Sigma(2\pi/3).$$

$\mathcal{A}^\epsilon$ is sectorial where the sector is $O(1/\epsilon)$ from the origin of the complex plane.

Proof. Since $g \in \widehat{H}_\mu$, then $f(x) = g(x)/\epsilon\sqrt{1+x^2} \in \widehat{H}_{\mu-1}$, with

$$\|f\|_{\widehat{H}_{\mu-1}} \le \frac{2}{\epsilon}\|g\|_{\widehat{H}_\mu}.$$

Since μ is independent of ϵ, $M/(4\epsilon) > |\mu - 1|$ if $\epsilon > 0$ is sufficiently small. Then (6.12) in Theorem 6.3 applies to this system because $\tilde\beta > |\mu - 1|$. There exists a unique solution $u^\pm \in \widehat{H}_{\mu-1}$. It follows from (8.5) that $u \in \widehat{H}_\mu$.

We now track the coefficients of the inequalities involved. By (6.12),

$$\begin{aligned} \|u^{\pm}\|_{\mu-1} &\leq \frac{2(K_1+K_2)}{\tilde{\beta}-|\mu-1|}\|f\|_{\mu-1} \\ &\leq \frac{4(K_1+K_2)}{\epsilon(\tilde{\beta}-|\mu-1|)}\|g\|_{\mu}. \end{aligned} \tag{8.7}$$

Since $\epsilon|\mu-1| \leq M/4$, then by Lemma 8.1,

$$\begin{aligned} &\frac{4(K_1+K_2)}{\epsilon(\tilde{\beta}-|\mu-1|)} \\ &\leq \frac{4(K_1+K_2)}{\Re\sqrt{\epsilon s}-M/4-M/4} \\ &\leq \frac{16(K_1+K_2)}{|\sqrt{\epsilon s}-M/2|}, \\ &\leq \frac{16(K_1+K_2)|\sqrt{\epsilon s}+M/2|}{|\epsilon s-M^2/4|}. \end{aligned}$$

Using $|\sqrt{\epsilon s}| \geq M$, we have

$$\begin{aligned} \|\frac{\epsilon\sqrt{1+x^2}}{2\sqrt{\epsilon s}}u^{\pm}\|_{\mu} &\leq \frac{C\epsilon}{|\sqrt{\epsilon s}|}\|u^{\pm}\|_{\mu-1} \\ \leq &\frac{C\epsilon|\sqrt{\epsilon s}+M/2|}{|\sqrt{\epsilon s}||\epsilon s-M^2/4|}\|g\|_{\mu} \\ \leq &\frac{C}{|s-M^2/(4\epsilon)|}\|g\|_{\mu}. \end{aligned}$$

From (8.5), we have

$$\|u\|_{\widehat{H}_\mu} \leq \frac{C}{|s-M^2/(4\epsilon)|}\|g\|_{\widehat{H}_\mu}, \quad \text{where } \epsilon s \in \mathcal{P}(M^2) \cap \Sigma(2\pi/3).$$

□

Note that s is outside a disk of radius $O(M^2/\epsilon)$. This is consistent with the fact that some eigenvalues that correspond to eigenfunctions whose support are near the shocks are of $O(1/\epsilon)$.

8.2. **The region $\Re s \geq -\eta$ consists of normal points.** In this subsection, we assume that the hypothesis (HH) is satisfied. Assume $\Re s \geq -\eta$ for a constant $\eta > 0$.

The resolvent equation $\mathcal{A}^\epsilon u - su = g$ can be expressed as a second order system of equations:

$$u_{xx} + \frac{xI - Df(u^\epsilon)}{\epsilon(1+x^2)}u_x - \frac{su}{\epsilon(1+x^2)} = \frac{g(x)}{\epsilon(1+x^2)} + \frac{(D^2 f(u^\epsilon)u^\epsilon_x)u}{\epsilon(1+x^2)}.$$

The equation is typically written as the first order system of equations using $(u, v = u_x)$ as the phase variables in $\mathbb{R}^{2n}$,

$$
\begin{aligned}
&u_x = v,\\
&v_x = -\frac{xI - Df(u^\epsilon)}{\epsilon(1+x^2)}v + \frac{su}{\epsilon(1+x^2)} + \frac{(D^2f(u^\epsilon)u_x^\epsilon)u}{\epsilon(1+x^2)} + \frac{g(x)}{\epsilon(1+x^2)}.
\end{aligned}
\tag{8.8}
$$

A better way to rewrite the second order equation into a first order system is to use the method of frozen coefficients. Due to the strict hyperbolicity, in $(-\infty, -N)$ and (N, ∞), $Df(\bar{u})$ has n eigenvalues $\nu_j(x)$ associated with the left/right eigenvectors $(\ell_j, \mathbf{r}_j)$, where ℓ_j is a row vector and $\mathbf{r}_j$ is a column vector. Then (u, g) can be written in coordinate forms:

$$
\begin{aligned}
&Df(\bar{u})\mathbf{r}_j = \nu_j\mathbf{r}_j,\ \ell_j Df(\bar{u}) = \nu_j\ell_j,\ \ell_j\mathbf{r}_k = \delta_{jk},\\
&u = \sum_j u_j(x)\mathbf{r}_j(x), \quad g = \sum_j g_j(x)\mathbf{r}_j(x),\\
&\ell_j u = u_j, \quad \ell_j v = u_{jx}, \quad \ell_j g = g_j.
\end{aligned}
$$

In the regions $(-\infty, x^1)$ and (x^n, ∞) where $\bar{u}(x)$, $\nu_j(x)$ and $\mathbf{r}_j(x)$ are constants, define

$$
\begin{aligned}
&b_j(x) = \frac{x - \nu_j}{2\sqrt{1+x^2}},\\
&u_j^\pm = u_{jx} - \lambda_j^\pm(x)u_j, \quad \text{where } \lambda_j^\pm = \frac{-b_j(x) \pm \sqrt{b_j^2(x) + \epsilon s}}{\epsilon\sqrt{1+x^2}},\\
&\text{then } u_{jx} = \frac{\lambda_j^+ u_j^- - \lambda_j^- u_j^+}{\lambda_j^+ - \lambda_j^-},\\
&\text{and } u_j = \frac{u_j^- - u_j^+}{\lambda_j^+ - \lambda_j^-} = \frac{\epsilon\sqrt{1+x^2}}{2\sqrt{b_j^2(x) + \epsilon s}}(u_j^- - u_j^+).
\end{aligned}
$$

We have the following system for $(u_j^+, u_j^-), j = 1, \dots, n$:

$$
\begin{aligned}
&(D_x - \lambda_j^\mp)u_j^\pm = u_{jxx} + \frac{x - \nu_j}{\epsilon(1+x^2)}u_{jx} - \frac{s}{\epsilon(1+x^2)}u_j - D_x\lambda_j^\pm(x)u_j\\
&= \frac{g_j(x)}{\epsilon(1+x^2)} + \ell_j(\frac{Df(u^\epsilon) - Df(\bar{u})}{\epsilon(1+x^2)}u_x + (\frac{(D^2f(u^\epsilon)u_x^\epsilon)}{\epsilon(1+x^2)} - D_x\lambda_j^\pm(x))u).
\end{aligned}
$$

System for $(u_j^+, u_j^-), j = 1, \dots, n$, can be rearranged as

$$
\begin{aligned}
&u_{jx}^+ = \lambda_j^- u_j^+ + \sum_k (b_{j+}^{k+}u_k^+ + b_{j+}^{k-}u_k^-) + f_j(x)/\sqrt{1+x^2},\\
&u_{jx}^- = \lambda_j^+ u_j^- + \sum_k (b_{j-}^{k+}u_k^+ + b_{j-}^{k-}u_k^-) + f_j(x)/\sqrt{1+x^2}.
\end{aligned}
\tag{8.9}
$$

where $f_j(x) = \frac{g_j(x)}{\epsilon\sqrt{1+x^2}}$, and the small terms are defined as

$$\sum_k (b_{j\pm}^{k+} u_k^+ + b_{j\pm}^{k-} u_k^-) = \ell_j(\frac{Df(u^\epsilon) - Df(\bar{u})}{\epsilon(1+x^2)} u_x + (\frac{(D^2 f(u^\epsilon) u_x^\epsilon)}{\epsilon(1+x^2)} - D_x \lambda_j^\pm(x))u),$$

where $u = \sum_j u_j \mathbf{r}_j,\ u_x = \sum_j u_{jx}\mathbf{r}_j$, and then expressing (u_j, u_{jx}) by $u_j^\pm$.

Lemma 8.3. *If $|x| \geq N$, and ϵ is sufficiently small, then*

$$b_{j\pm}^{k\pm} = O(\frac{1}{\sqrt{1+x^2}}).$$

Proof. It is straight forward to check that if $x \leq x^1 - 1$ or $x \geq x^n + 1$ then $|u_x^\epsilon|$ is uniformly bounded with respect to ϵ and x.

$$\text{Note that } |D_x b_j(x)| \leq \frac{C}{1+x^2}, \quad |D_x \lambda_j^\pm(x)| \leq \frac{C|\sqrt{b_j^2 + \epsilon s}|}{\epsilon(1+x^2)},$$

$$|\frac{(D^2 f(u^\epsilon) u_x^\epsilon)}{\epsilon(1+x^2)} - D_x \lambda_j^\pm(x)| \leq \frac{C|\sqrt{b_j^2(x) + \epsilon s}|}{\epsilon(1+x^2)},$$

$$|\sum_j u_j \mathbf{r}_j| \leq C \sum_j \frac{\epsilon\sqrt{1+x^2}}{|\sqrt{b_j^2 + \epsilon s}|}(|u_j^+| + |u_j^-|).$$

$$\text{Thus } |\ell_j(\frac{(D^2 f(u^\epsilon) u_x^\epsilon)}{\epsilon(1+x^2)} - D_x \lambda_j^\pm(x))u| \leq \frac{C}{\sqrt{1+x^2}} \sum_j (|u_j^+| + |u_j^-|).$$

$$\text{Note that } |Df(u^\epsilon) - Df(\bar{u})| \leq C\epsilon, \quad |\sum_j u_{jx}\mathbf{r}_j| \leq \sum_j (|u_j^+| + |u_j^-|).$$

$$\text{Thus } |\ell_j \frac{Df(u^\epsilon) - Df(\bar{u})}{\epsilon(1+x^2)} u_x| \leq \frac{C}{1+x^2} \sum_j (|u_j^+| + |u_j^-|).$$

The desired result follows from the above estimates. □

Define the $2n$-vector valued functions:

$$U(x) = (u_1^+(x), \ldots, u_n^+(x), u_1^-(x), \ldots, u_n^-(x))^\tau,$$
$$\mathbb{F}(x) = (f_1(x), \ldots, f_n(x), f_1(x), \ldots, f_n(x))^\tau.$$

Define the $n \times n$ matrices:

$$B_+^+ = \{b_{j+}^{k+}\}_{j,k=1}^n, B_+^- = \{b_{j+}^{k-}\}_{j,k=1}^n, B_-^+ = \{b_{j-}^{k+}\}_{j,k=1}^n, B_-^- = \{b_{j-}^{k-}\}_{j,k=1}^n.$$

Define the $2n \times 2n$ matrices:

$$\mathbb{A}(x,s)_{2n\times 2n} = \mathrm{diag}(\lambda_1^-, \ldots, \lambda_n^-, \lambda_1^+, \ldots, \lambda_n^+),$$
$$\mathbb{B}(x,s)_{2n\times 2n} = \begin{pmatrix} B_+^+ & B_+^- \\ B_-^+ & B_-^- \end{pmatrix}.$$

System (8.9) can be expressed as

$$U_x = (\mathbb{A}(x,s) + \mathbb{B}(x,s))U + \frac{1}{\sqrt{1+x^2}}\mathbb{F}(x). \tag{8.10}$$

In (8.10), both U and $\mathbb{F}$ are in $(\widehat{H}_{\mu-1})^{2n}$.

Lemma 8.4. *Assume that (HH) is satisfied, $\Re s \geq -\eta$ for a constant $\eta > 0$, and N is sufficiently large. Then system* (8.10) *has algebraic dichotomies in $(-\infty, -N]$ and $[N, \infty)$ respectively. More specifically, if $\|\mathbb{B}(x,s)\| \leq \widetilde{C}/\sqrt{1+x^2}$ (Lemma 8.3), and if K is the constant for the dichotomies of $U' = \mathbb{A}(x,s)U$, then for any constants $\tilde{\gamma}^+, \tilde{\delta}^+$ satisfying*

$$\tilde{\gamma}^+ + 1 = -\eta - 4K^2\widetilde{C}, \quad \tilde{\delta}^+ - 1 = -\frac{1}{2\epsilon} + 4K^2\widetilde{C},$$

if ϵ is sufficiently small, then system (8.9) *has an algebraic dichotomy on $[N, \infty)$ with the powers $0 > \tilde{\gamma}^+ > \tilde{\delta}^+$.*

Similarly, for any constants $\tilde{\gamma}^-, \tilde{\delta}^-$ satisfying

$$\tilde{\gamma}^- + 1 = \frac{1}{2\epsilon} - 4K^2\widetilde{C}, \quad \tilde{\delta}^- - 1 = \eta + 4K^2\widetilde{C},$$

if ϵ is sufficiently small, then system (8.9) *has an algebraic dichotomy on $(-\infty, -N]$ with the powers $0 < \tilde{\delta}^- < \tilde{\gamma}^-$.*

Proof. Let $N > 0$ satisfy $-N < x^1 - 1,\ N > x^n + 1$. When $\mathbb{B}(x,s) = 0$, it is easy to see that the uncoupled system

$$u^+_{jx} - \lambda^-_j u^+_j = 0, \quad , u^-_{jx} - \lambda^+_j u^-_j = 0,$$

has the asymmetric algebraic dichotomies on $(-\infty, -N]$ and $[N, \infty)$ respectively. We give estimates on the powers $\delta < \gamma$ of the dichotomies.

First, consider the region $x \geq N$. The solutions for $u^-_{jx} - \lambda^+_j u^-_j = 0$ satisfy

$$|u(x)/u(y)| = exp(\Re \int_y^x \frac{\sqrt{b_j^2(\xi) + \epsilon s} - b_j(x)}{\epsilon\sqrt{1+\xi^2}} d\xi), \quad N \leq y \leq x.$$

If $|x| \to \infty$, then $|b_j(x)| \to 1/2$. Thus if N is sufficiently large, $\inf_{|x|\geq N} |b_j(x)| > 0$. For a fixed N, $b_j^2 > \epsilon\eta$ if ϵ is sufficiently small. From part (4) of Lemma 8.1, we have,

$$(\Re\sqrt{b_j^2(\xi) + \epsilon s} - b_j(x))/\epsilon \geq -\eta/(2b_j) + O(\epsilon),$$
$$2b_j = 1 + O(1/\sqrt{1+x^2}), \quad \frac{-\eta}{2b_j} = -\eta + O(\frac{\eta}{\sqrt{1+x^2}}).$$

Then there exists $K \geq 1$ such that,

$$\begin{aligned}|u(x)/u(y)| &\geq exp(\int_y^x \frac{-\eta/(2b_j)+O(\epsilon)}{\sqrt{1+x^2}}) \\ &\geq exp(\int_y^x \frac{-C\eta}{1+x^2}d\xi)exp(\int_y^x \frac{-\eta+O(\epsilon)}{\sqrt{1+x^2}}d\xi) \\ &\geq K^{-1}\left(\frac{a(x)}{a(y)}\right)^{\gamma}, \quad \text{where } \gamma=-\eta+O(\epsilon).\end{aligned}$$

Similarly, the solutions for $u^+_{jx} - \lambda^-_j u^+_j = 0$ satisfy

$$|u(x)/u(y)| = exp(\Re\int_y^x \frac{-\sqrt{b_j^2(\xi)+\epsilon s}-b_j(x)}{\epsilon\sqrt{1+\xi^2}}d\xi), \quad N \leq y \leq x$$

From $2b_j(x) = 1 + O(1/\sqrt{1+x^2})$, we have

$$\begin{aligned}&-b_j(x) - \sqrt{b_j^2(x)-\epsilon\eta} = -2b_j + O(\epsilon) \\ &= -1 + O(\epsilon) + O(\frac{1}{\sqrt{1+x^2}}) \leq -\frac{1}{2},\end{aligned}$$

if ϵ is sufficiently small and N is sufficiently large. Then,

$$\begin{aligned}|u(x)/u(y)| &\leq exp(\int_y^x \frac{-1/2}{\epsilon\sqrt{1+x^2}}d\xi) \\ &\leq \left(\frac{a(x)}{a(y)}\right)^{\delta}, \quad \text{where } \delta=-1/(2\epsilon).\end{aligned}$$

We have shown that for $x \geq N$, the uncoupled system has a pseudo algebraic dichotomy with the powers

$$\gamma = -\eta + O(\epsilon), \quad \delta = -1/(2\epsilon).$$

Recall that $\|B\| \leq \widetilde{C}/\sqrt{1+x^2}$. If ϵ is sufficiently small, then

$$\gamma - \delta = \frac{1}{2\epsilon} - \eta + O(\epsilon) > 8K^2\widetilde{C}.$$

Let $\tilde{\gamma}^+, \tilde{\delta}^+$ be two constants such that

$$\tilde{\gamma}^+ + 1 = -\eta - 4K^2\widetilde{C}, \quad \tilde{\delta}^+ - 1 = -\frac{1}{2\epsilon} + 4K^2\widetilde{C}.$$

Then $\tilde{\gamma}^+ > \tilde{\delta}^+$ if ϵ is small. From Corollary 2.6, system (8.9) has an algebraic dichotomy on $[N, \infty)$ with the powers $0 > \tilde{\gamma}^+ > \tilde{\delta}^+$.

The proof for the case $x \in (-\infty, -N]$ is similar to that of $x \in [N, \infty)$. □

The variables $u^{\pm}_j(x, s)$ in $U(x, s)$ do not extend smoothly to $x \in [-N, N]$ since $b_j(x), \ell_j(x)$ and $\mathbf{r}_j(x)$ have sharp jumps across $x = x^i$, the position of a shock for the hyperbolic conservations laws.

We could extend the dichotomies to $|x| \leq N$ by using (u, v) variables. However, a careful examination of the change of variables from U to (u, v) reveals that the growth/decay power of v is one unit smaller than that of u. This actually works since the weights on u and v are different, $u \in \widehat{H}_\mu$ and $v \in \widehat{H}_{\mu-1}$. But the whole process would be too complicated to carry out since the estimates in §2 and §6 do not apply directly to (8.8).

Define the new phase variable $w = u/\sqrt{1+x^2}$ and consider the system in (w, v):

$$\begin{aligned} &(\sqrt{1+x^2}w)_x = v, \\ &v_x = -\frac{x - Df(u^\epsilon)}{\epsilon(1+x^2)}v + \frac{sw}{\epsilon\sqrt{1+x^2}} + \frac{(D^2 f(u^\epsilon)u_x^\epsilon)w}{\epsilon\sqrt{1+x^2}} + \frac{f(x)}{\sqrt{1+x^2}}, \end{aligned} \tag{8.11}$$

where $f(x) = \frac{g(x)}{\epsilon\sqrt{1+x^2}} \in (\widehat{H}_{\mu-1})^n$ and $(w, v) \in (\widehat{H}_{\mu-1})^n \times (\widehat{H}_{\mu-1})^n$.

Lemma 8.5. *System* (8.11) *has algebraic dichotomies in* $(-\infty, -N]$ *and* $[N, \infty)$*. Moreover, the powers of the dichotomies are the same as those for system* (8.10)*:*

$$\tilde{\delta}^+ < \tilde{\gamma}^+, \quad \tilde{\delta}^- < \tilde{\gamma}^-.$$

Proof. We will drop ϵ for simplicity if no confusion should arise. First, recall that $U(x) = (u_1^+, \dots, u_n^+, u_1^-, \dots, u_n^-)^\tau$ is a $2n$-vector. The variables (w, v) and U are related by

$$U = J(x,s)\begin{pmatrix} w \\ v \end{pmatrix}, \text{ where } J(x,s)_{2n\times 2n} = \begin{pmatrix} -\sqrt{1+x^2}\lambda_1^+\ell_1 & \ell_1 \\ \dots & \dots \\ -\sqrt{1+x^2}\lambda_n^+\ell_n & \ell_n \\ -\sqrt{1+x^2}\lambda_1^-\ell_1 & \ell_1 \\ \dots & \dots \\ -\sqrt{1+x^2}\lambda_n^-\ell_n & \ell_n \end{pmatrix}.$$

$$\begin{pmatrix} w \\ v \end{pmatrix} = J(x,s)^{-1}U, \quad J(x,s)^{-1} = \begin{pmatrix} \mathbf{r}_1, \dots, \mathbf{r}_n & 0_n, \dots, 0_n \\ 0_n, \dots .0_n & \mathbf{r}_1, \dots, \mathbf{r}_n \end{pmatrix}\mathbf{S},$$

where $\mathbf{S}_{2n\times 2n}$ consists of 4 block matrices each of them is diagonal:

$$\mathbf{S}_{2n\times 2n} = \begin{pmatrix} \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \frac{-1}{\sqrt{1+x^2}(\lambda_j^+ - \lambda_j^-)} & 0 & 0 & \frac{1}{\sqrt{1+x^2}(\lambda_j^+ - \lambda_j^-)} & 0 \\ 0 & 0 & \ddots & 0 & 0 & \ddots \\ \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \frac{-\lambda_j^-}{\lambda_j^+ - \lambda_j^-} & 0 & 0 & \frac{\lambda_j^+}{\lambda_j^+ - \lambda_j^-} & 0 \\ 0 & 0 & \ddots & 0 & 0 & \ddots \end{pmatrix}$$

Observe that for a fixed (s, ϵ), both $J(x, s)$ and $J(x, s)^{-1}$ are uniformly bounded for $x \in \mathbb{R}$. Therefore, the change of variables is a homeomorphism:

$$(w, v) \to U, \quad \widehat{H}_{\mu-1} \times \widehat{H}_{\mu-1} \to \widehat{H}_{\mu-1} \times \widehat{H}_{\mu-1}.$$

Let the evolution operator for (8.10) be $\mathbb{T}(x, y, s)$ and the projections to stable and unstable subspaces be $\mathbb{P}(x, s)$ and $\mathbb{Q}(x, s)$ respectively. for system (8.11), the evolution operator $T(x,y,s)$ and projections to stable and unstable subspaces, $P(x,s)$ and $Q(x,s)$, are related to those of (8.10) by:

$$\begin{aligned} T(x,y,s) &= J^{-1}(x,s)\mathbb{T}(x,y,s)J(y,s), \\ P(x,s) &= J^{-1}(x,s)\mathbb{P}(x,s)J(x,s), \\ Q(x,s) &= J^{-1}(x,s)\mathbb{Q}(x,s)J(x,s). \end{aligned}$$

This shows that system (8.11) has algebraic dichotomies in $(-\infty,-N]$ and $[N,\infty)$ respectively. □

For (8.11), we can extend the dichotomies in $|x|\geq N$ to $\mathbb{R}^{\pm}$. If $-N\leq x<0$, let

$$\begin{aligned} P(x,s) &= T(x,-N,s)P(-N,s)T(-N,x,s), \\ Q(x,s) &= T(x,-N,s)Q(-N,s)T(-N,x,s), \end{aligned}$$

while if $0<x\leq N$, let

$$\begin{aligned} P(x,s) &= T(x,N,s)P(N,s)T(N,x,s), \\ Q(x,s) &= T(x,N,s)Q(N,s)T(N,x,s). \end{aligned}$$

$P(0^{\pm},s), Q(0^{\pm},s)$ are defined as one-sided limits from $x>0$ or $x<0$ respectively. It is easy to verify that the projections defined above extend the dichotomies from $(-\infty,-N]$ and $[N,\infty)$ to $(-\infty,0]$ and $[0,\infty)$ respectively. The powers of extended dichotomies remain the same. Although the constant K depends on (s,ϵ) and can be large, it has no negative effect in this subsection.

Now let $\mu<0$ be a fixed constant that is independent of ϵ such that

$$\mu-1<\tilde{\gamma}^{+}, \quad \tilde{\delta}^{-}<-(\mu-1).$$

If ϵ is sufficiently small, then

$$\tilde{\delta}^{+}<\mu-1<\tilde{\gamma}^{+}, \quad \tilde{\delta}^{-}<-(\mu-1)<\tilde{\gamma}^{-}.$$

We show in the space $\widehat{H}_{\mu-1}\times\widehat{H}_{\mu-1}$, if $\Re s\geq-\eta$, then s is either a normal eigenvalue or a resolvent point for $\mathcal{A}^{\epsilon}$.

For (8.11), the principal matrix solution $T(x,y;s,\epsilon)$ (previously, $T(x,y,s)$ for simplicity) depends on the parameter (s,ϵ) and is analytic in s in the region $\Re s\geq-\eta$. We will show that it is possible to choose the projections $P(x,s,\epsilon)$ and $Q(x,s,\epsilon))$ of the algebraic dichotomies on $\mathbb{R}^{\pm}$ so that they are analytic in s as well.

Let

$$\phi_1(x,s,\epsilon),\phi_2(x,s,\epsilon),\ldots,\phi_n(x,s,\epsilon)$$

be a basis for the space $\mathcal{R}Q^{-}(x,s,\epsilon)$ and let

$$\phi_{n+1}(x,s,\epsilon),\phi_{n+2}(x,s,\epsilon),\ldots,\phi_{2n}(x,s,\epsilon)$$

be a basis for the space $\mathcal{R}P^{+}(x,s,\epsilon)$.

Define the $2n\times 2n$ characteristic matrix $\Phi(s,\epsilon)$ as

$$\Phi(s,\epsilon):=$$
$$(\phi_1(0^-,s,\epsilon),\phi_2(0^-,s,\epsilon),\ldots,\phi_n(0^-,s,\epsilon),\phi_{n+1}(0^+,s,\epsilon),\phi_{n+2}(0^+,s,\epsilon),\ldots,\phi_{2n}(0^+,s,\epsilon)).$$

We call $e(s,\epsilon) := \det \Phi(s,\epsilon)$ the characteristic function, and $e(x,\epsilon) = 0$ the characteristic equation.

Theorem 8.6. *Assume the hypothesis (HH) is satisfied. There exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then it is possible to define those basis functions so that $\Phi(s,\epsilon)$ is analytic in s in the region $\Re s > -\eta$. Then for a give (s_0,ϵ) with $\Re s_0 \geq -\eta$ and $0 < \epsilon < \epsilon_0$, there are two possible cases:*

(1) The characteristic function $e(s_0,\epsilon) \neq 0$ which is equivalent to

$$\mathcal{R}Q(0^-, s_0, \epsilon) \cap \mathcal{R}P(0^+, s_0, \epsilon) = \{0\}.$$

In this case, $s_0 \in \rho(\mathcal{A}^\epsilon)$. And $(sI - \mathcal{A}^\epsilon)^{-1}$ is analytic in a neighborhood of s_0,

(2) The characteristic function $e(s_0,\epsilon) = 0$. This is equivalent to

$$\mathcal{R}Q(0^-, s_0, \epsilon) \cap \mathcal{R}P(0^+, s_0, \epsilon) \neq \{0\}.$$

In this case, s_0 is an eigenvalue of $\mathcal{A}^\epsilon$. Moreover, s_0 is an isolated singular point of $\sigma(\mathcal{A}^\epsilon)$ and is also a pole of $(sI - \mathcal{A}^\epsilon)^{-1}$ of finite order p. The order $p \leq m$ where m is the algebraic multiplicity of s_0 as a root of the characteristic equation $e(s,\epsilon) = 0$.

Proof. For $x \in (-\infty, -N]$ and $[N, \infty)$, the projections for the dichotomies of the unperturbed system $U' = \mathbb{A}(x,s)U$ is analytic in s. From Corollary 2.4, we find that the projections $\mathbb{P}(x,s,\epsilon), \mathbb{Q}(x,s,\epsilon)$ for $U' = (\mathbb{A}(x,s) + \mathbb{B}(x,s))U$ are analytic in s for $\Re s > -\eta$. It is clear that the solution operator $\mathbb{T}(x,y,s)$ (drop ϵ to simplify the notations) is analytic in s. The change of variables $J(x,s),\ J^{-1}(x,s)$ are analytic in s. Thus, for system (8.11), the solution operator $T(x,y,s)$ and projections $P(x,s), Q(x,s)$ are also analytic in s in $(-\infty,-N]$ and $[N,\infty)$ respectively. The analyticity holds even the domain is extended to $\mathbb{R}^\pm$ respectively.

We now choose vectors $(b_1, b_2, \ldots b_n)$ as a basis in $\mathcal{R}Q(0-, s, \epsilon)$ and $(b_{n+1}, b_{n+2}, \ldots, b_{2n})$ as a basis in $\mathcal{R}P(0+, s, \epsilon)$. Then let

$$\phi_1(x,s,\epsilon), \phi_2(x,s,\epsilon), \ldots, \phi_n(x,s,\epsilon) = T(x, 0-, s, \epsilon)(b_1, b_2, \ldots b_n),\ x \leq 0,$$
$$\phi_{n+1}(x,s,\epsilon), \phi_{n+2}(x,s,\epsilon), \ldots, \phi_{2n}(x,s,\epsilon) = T(x, 0+, s, \epsilon)(b_{n+1}, b_{n+2}, \ldots, b_{2n}),\ x \geq 0.$$

It should be clear that the characteristic matrix $\Phi(s,\epsilon)$ and the characteristic function $e(s,\epsilon)$ are both analytic in s, as to be proved.

Proof of case 1: If $e(s,\epsilon) \neq 0$, then the matrix $\Phi(s,\epsilon)$ is invertible. For any $g \in \widehat{H}_\mu(R),\ f(x) = g(x)/(\epsilon\sqrt{1+x^2}) \in \widehat{H}_{\mu-1}$, the solution u can be expressed as

$$\sum_{j=1}^{n} \alpha_j \phi_j(x,s,\epsilon) + \int_{-\infty}^{x} T(x,y,s)P(y,s)\begin{pmatrix} 0 \\ f(y) \end{pmatrix} \frac{dy}{\sqrt{1+y^2}}$$
$$+ \int_{0}^{x} T(x,y,s)Q(y,s)\begin{pmatrix} 0 \\ f(y) \end{pmatrix} \frac{dy}{\sqrt{1+y^2}}, \quad x \leq 0,$$

$$\sum_{j=n+1}^{2n} \alpha_j \phi_j(x,s,\epsilon) + \int_0^x T(x,y,s)P(y,s)\begin{pmatrix}0\\ f(y)\end{pmatrix}\frac{dy}{\sqrt{1+y^2}}$$
$$+ \int_\infty^x T(x,y,s)Q(y,s)\begin{pmatrix}0\\ f(y)\end{pmatrix}\frac{dy}{\sqrt{1+y^2}}, \quad x \geq 0.$$

We can determine $\alpha_j, j = 1, \ldots, 2n$, so that the two values at $0^\pm$ match.

$$(\alpha_1, \ldots, \alpha_n)^\tau = -E_n\Phi^{-1}(s,\epsilon)\left(\int_{-\infty}^0 T(0,x,s)P(x,s)\begin{pmatrix}0\\ f(x)\end{pmatrix}\frac{dx}{\sqrt{1+x^2}}\right.$$
$$\left. + \int_0^\infty T(0,x,s)Q(x,s)\begin{pmatrix}0\\ f(x)\end{pmatrix}\frac{dx}{\sqrt{1+x^2}}\right),$$

$$(\alpha_{n+1}, \ldots, \alpha_{2n})^\tau = (I - E_n)\Phi^{-1}(s,\epsilon)\left(\int_{-\infty}^0 T(0,x,s)P(x,s)\begin{pmatrix}0\\ f(x)\end{pmatrix}\frac{dx}{\sqrt{1+x^2}}\right.$$
$$\left. + \int_0^\infty T(0,x,s)Q(x,s)\begin{pmatrix}0\\ f(x)\end{pmatrix}\frac{dx}{\sqrt{1+x^2}}\right),$$

where E_n is the projection of a $2n$ vector to its first n components.

The process described above uniquely determines the bounded inverse

$$(sI - \mathcal{A}^\epsilon)^{-1} : g \to f \to (w,v) \to u,\ \widehat{H}_\mu(\mathbb{R}) \to \widehat{H}_{\mu-1}(\mathbb{R}) \to \widehat{H}_{\mu-1}(\mathbb{R}) \to \widehat{H}_\mu(\mathbb{R}).$$

Therefore, s is a resolvent point.

Proof of case 2: Let $V \in \mathcal{R}Q(0^-, s_0, \epsilon) \cap \mathcal{R}P(0^+, s_0, \epsilon)$ be a nonzero vector. Then $(w,v)(x) = T(0,x,s_0)V$ is in $\widehat{H}_{\mu-1}(\mathbb{R})$ and is a solution to (8.11) with f =0. Therefore $u = \sqrt{1+x^2}w$ is an eigenfunction in $\widehat{H}_\mu$ corresponding to the eigenvalue s_0.

We have shown that $\Phi(s,\epsilon)$ and $e(s,\epsilon)$ are analytic functions in $\Re s > -\eta$. Because the resolvent set for $\mathcal{A}^\epsilon$ is nonempty, the zeros for $e(s,\epsilon)$ are isolated points in the complex plane $\mathbb{C} \cap \{\Re s > -\eta\}$, that is, the eigenvalues are isolated points. To show that the eigenvalues are normal eigenvalues, consider a deleted neighborhood of an eigenvalue s_0: $\mathcal{O} := \{s : 0 < |s - s_0| < \delta\}$. If $\delta > 0$ is small, then $\mathcal{O}$ consists of resolvent points. From the part one of the proof, we find that formulas for $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha_{n+1}, \ldots, \alpha_{2n})$ involve $\Phi(s,\epsilon)^{-1}$ which has a pole of finite order at s_0. Assume that m is the order of s_0 as a zero of $e(s,\epsilon)$. Then $|\alpha_j| \leq C|s - s_0|^{-m}$.

□

Remark 8.1. At $x = 0$, $w(0,s) = u(0,s)/\sqrt{1+x^2} = u(0,s)$. The $2n\times 2n$ characteristic matrix $\Phi(s,\epsilon)$ and the characteristic function $e(s,\epsilon)$ can be defined by taking basis vectors from unstable subspaces at $x = 0-$ and stable subspaces at $x = 0+$ of (8.8).

Remark 8.2. It is known that for boundary value problems of many types of differential equations, an eigenvalue is a zero of the characteristic equation and the dimension of the range of the spectral projection associate to the eigenvalue is equal to the multiplicity of the root of the characteristic equation. For retarded differential equations, this result was called "A folk theorem in functional differential equations" and was proved by Levinger in 1968 [18]. The same result was proved by Lopes, Neves and Ribeiro [28], Neves and Lin [27] for systems of hyperbolic equations which, in various interpretation, include some delay, neutral and difference equations. For traveling wave solutions of reaction-diffusion equations, the characteristic function is the Evans function, cf. Evans [7]. See also Jones [12], [9, 15] and Benzoni-Gavage et al [2]. Although the characteristic function in this paper is related to "standing waves" rather than traveling waves, it can also be called as an Evanse function, and a similar "Folk Theorem" should hold with a similar proof. Details will not be discussed in this paper.

References

[1] C. M. Bender and S. A. Orszag, Advanced mathematical methods for scientists and engineers, McGraw-Hill, 1978.
[2] S. Benzoni-Gavage, D. Serre and K. Zumbrun, Alternate Evans functions and viscous shock waves, SIAM J. Math. Anal., **32** (2001), 929-962.
[3] S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Annals of Math., **161** (2005), 223-342.
[4] W. A. Coppel, Dichotomies in stability theory, Lecture Notes in Math., **629**, Springer, 1978.
[5] D. Cramer and Y. Latushkin, Gearhart-Prüss theorem in stability for wave equations: a survey. In: Evolution Equations, G. Goldstein, R. Nagel, S. Romanelli (edts), Lect. Notes Pure Appl. Math., **234**.
[6] C. M. Dafermos, Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method, Arch. Ration. Mech. Anal., **52** (1973), 1-9.
[7] J. Evans, Nerve Axon Equations: IV. The Stable and the Unstable Impulse, Indiana University Mathematics Journal, **24** (1975), 1169-1190.
[8] A. Friedman, Partial differential equations, Holt. Reinhart and Winston, New York, 1969.
[9] R. Gardner and K. Zumbrun, The gap lemma and geometric criteria for instability of viscous shock profiles, Comm. Pure Appl. Math., **51** (1998), 789-847.
[10] J. Hale, , S. V. Lunel, Introduction to functional differential equations, Springer-Verlag, 1993.
[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., **840**, Springer, Berlin, 1981.
[12] C.K.R.T. Jones, Stability of the traveling wave solution of the FitzHugh-Nagumo system. Trans. Amer. Soc., **286** (1984), 431-469.
[13] C. K. R. T. Jones, Geometric singular perturbation theory, Lecture Notes in Math., **1609**, Springer-Verlag, Berlin, 1995, 44-118
[14] T. Kapitula, On the stability of traveling waves in weighted L^∞ spaces, J. Differential Equations, **112** (1994), 179-215.
[15] T. Kapitula and B. Sandstede, Stability of bright solitary-wave solutions to perturbed nonlinear Schrodinger equations, Phys. D, **124** (1998), 58-103.
[16] Petar V. Kokotović, Hassan K. Khalil and John O'Reilly, Singular Perturbation Methods in Control: Analysis and Design, SIAM, 1999
[17] J. A. Leach, D. J. Needham and A. L. Kay, The evolution of reaction-diffusion waves in a class of scalar reaction-diffusion equations: algebraic decay rates, Physica D, **167** (2002), 153-182.
[18] B. W. Levinger, A folk theorem in functional differential equation, J. Differential equations, **4** (1968), 612-619.
[19] M. Lewicka, L^1 stability of patterns of non-interacting large shock waves, Indiana Univ. Math. J., **49** (2000), 1515-1537.
[20] M. Lewicka, Stability conditions for patterns of noninteracting large shock waves, SIAM J. Math. Anal., **32** (2001), 1094-1116.
[21] M. Lewicka and K. Zumbrun, Spectral stability conditions for shock wave patterns , Journal of Hyperbolic Equations, **4** (2007), 1-16.

[22] X.-B. Lin, Exponential dichotomies in intermediate spaces with applications to a diffusively perturbed predator-prey model, J. Differential Equations, **108** (1994), 36-63.

[23] X.-B. Lin, Analytic Semigroup Generated by the Linearization of a Riemann-Dafermos Solution, Dynamics of PDE, **1**, (2004), 193-207

[24] X.-B. Lin, L2 Semigroup and Linear Stability for Riemann Solutions of Conservation Laws, Dynamics of PDE, **2** (2005), 301-333

[25] X.-B. Lin Slow Eigenvalues of Self-similar Solutions of the Dafermos Regularization of a System of Conservation Laws: An Analytic Approach, Journal of Dynamics and Differential Equations, **18** (2006),

[26] X.-B. Lin, Gearhart-Prüss Theorem and linear stability for Riemann solutions of conservation laws, J. of Dynamics and Differential Equations, **19** (2007), 1037-1074.

[27] A. F. Neves and X.-B. Lin A multiplicity theorem for hyperbolic systems, J. Differential Equations, **76** (1988), 339-352.

[28] O. Lopes, A. Neves and H. Ribeiro, On the spectrum of evolution operators generated by hyperbolic systems, J. Funct. Anal., **67** (1986), 320-344.

[29] A. Pazy, semigroup of linear operators and applications to partial differential equations, Springer, New York, 1983.

[30] D. H. Sattinger, On the stability of waves of nonlinear parabolic systems, Adv. Math., **22** (1976), 312-355

[31] S. Schecter, D. Marchesin and B. J. Plohr (1996), Structurally stable Riemann solutions, J. Differential Equations, **126**, 303–354.

[32] S. Schecter, Undercompressive Shock Waves and the Dafermos Regularization, Nonlinearity, **15** (2002), 1361-1377.

[33] X.-B. Lin and S. Schecter, Stability of self-similar solutions of the Dafermos regularization of a system of conservation laws, SIAM J. Math. Anal., **35** (2003), 884-921.

[34] S. Schochet, Sufficient conditions for local existence via Glimn's scheme for large B.V. data, J. Differential Equations, **89** (1991), 317-354.

[35] P. Szmolyan, personal communication.

[36] Lloyd N. Trefethen and Mark Embree, Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators, Princeton University Press, 2005.

[37] R. E. Vinograd, an improved estimate in the method of freezing, Proceedings of AMS, **89** (1983), 125-129.

[38] Y. Wu, X. Xing and Q. Ye, Stability of traveling waves with algebraic decay for n-degree fisher-type equations, Discrete and Continuous Dynamical Systems, **16** (2006), 47-66.

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205