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Chaotic Traveling Wave Solutions in Coupled Chua's Circuits

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Abstract Coupled arrays of Chua's circuits have been studied for many years. The existence of traveling wave solutions for such system was shown numerically in Perez-Munuzuri et al. (Traveling wave front and its failure in a one-dimensional array of Chua's circuit. Chua's circuit : a paradigm for Chaos. World Scientific, Singapore, pp 336–350, 1993). The existence of periodic traveling wave solutions has been proved recently (Chow et al. in J Appl Anal Comput 3:213–237, 2013). The purpose of this paper is to prove the existence of chaotic traveling wave solutions for such system. Using the method of singular perturbations, we show that the ODE system for the traveling waves can have a heteroclinic loop consisting of two traveling waves moving at the same speed. Moreover, at the equilibrium points of the heteroclinic loop, the dominant eigenvalues of the system are a pair of complex numbers with negative real parts. By a generalization of Shilnikov's theorem of symbolic dynamics, the system can have chaotic behavior near the traveling heteroclinic orbits.

Keywords Coupled Chua's circuits · Traveling waves · Heteroclinic orbits · Shilnikov's chaos · Melnikov integral

1 Introduction

1.1 Derivation of the Traveling Wave Equations

Chua's circuit is a nonlinear circuit that has assumed a paradigmatic role in mathematical, physical and experimental demonstrations of chaos. The advantages of Chua's circuitare that

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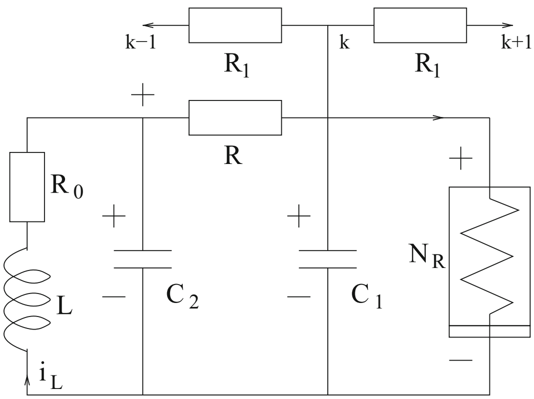
Fig. 1 The kth unit of the coupled Chua's circuits



the equations for the circuit can be derived accurately and the outcome of the experiment can be measured precisely. See the use of Chua's circuits in teaching high school students about chaos [2,14]. In [22], the theory of circuits is used to illustrate the dynamics of arterial systems. See [30] for more references. We consider the coupled Chua's circuits depicted in Fig. 1. Using k as the index for the kth circuit, the model for many coupled Chua's circuits can be written as the following system of ODEs,

$$\begin{aligned} C_1\frac{dV_{C_1}^k}{dt} &= \frac{1}{R}\left(V_{C_2}^k - V_{C_1}^k\right) - G\left(V_{C_1}^k\right) + \frac{1}{R_1}\left(V_{C_1}^{k-1} - 2V_{C_1}^k + V_{C_1}^{k+1}\right), \\ C_2\frac{dV_{C_2}^k}{dt} &= \frac{1}{R}\left(V_{C_1}^k - V_{C_2}^k\right) + i_L^k, \\ L\frac{di_L^k}{dt} + R_0\, i_L^k &= -V_{C_2}^k. \end{aligned}$$

Here $V_{C_1},\ V_{C_2}$ are voltages across the two capacitors, i_L is the current through the inductor L, G is the conductance of Chua's diode, and R_0 is the resistance of the inductor. The resistor R connects the LC oscillator to the diode, and R_1 determines the coupleing between adjacent circuits. By combining $V_{C_1}^k + RG(V_{C_1}^k)$ into $h(u)$, the above system can be transformed into the dimensionless form:

$$\begin{aligned} \dot{u}_k &= \alpha(y_k - h(u_k)) + \bar{D}(u_{k-1} - 2u_k + u_{k+1}), \\ \dot{y}_k &= u_k - y_k + z_k, \\ \dot{z}_k &= -\beta y_k - \gamma z_k, \qquad k \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

In this paper we assume that h is an N-shaped function with odd symmetry, e.g. $h(u) = mu(u+c)(u-c),\ c > 0,\ m > 0.$

Let $\epsilon = 1/\alpha$, $\Delta x = \sqrt{\epsilon}$, $u_k(t) = u(t, k\Delta x)$. If Δx is small, the last three terms can be approximated by ϵu_{xx}. Following the works of [30,33], (1.1) can be approximated by the following singularly perturbed reaction-diffusion PDE coupled with two ODEs,

$$\begin{aligned} \epsilon u_t &= (y - h(u)) + \epsilon^2 D u_{xx}, \quad 0 < \epsilon \ll 1, \\ y_t &= u - y + z, \\ z_t &= -\beta y - \gamma z. \end{aligned} \tag{1.2}$$

The constants D, β, γ are all positive. This system is a generalization of the FitzHugh–Nagumo equations where the PDE is coupled with one ODE.

In this paper, we study (1.2) which is a PDE approximation of the coupled Chua's circuits. We look for traveling wave solutions of (1.2) with undetermined wave speed s. In the traveling coordinates $\xi = x - st$, we have the following slow system of 4 ODEs, where " $\cdot$ " denotes $d/d\xi$,

$$\begin{aligned} \epsilon \dot{u} &= v/D, & s\dot{y} &= y - u - z, \\ \epsilon \dot{v} &= h(u) - sv/D - y, & s\dot{z} &= \beta y + \gamma z. \end{aligned} \tag{1.3}$$

The singular limit of (1.3) is obtained by setting $\epsilon = 0$:

$$\begin{aligned} 0 &= v, & s\dot{y} &= y - u - z, \\ 0 &= h(u) - y, & s\dot{z} &= \beta y + \gamma z. \end{aligned} \tag{1.4}$$

Introducing the fast time scale $\eta = \xi/\epsilon$ and $' = d/d\eta$, we have the fast system

$$\begin{aligned} u' &= v/D, & sy' &= \epsilon(y - u - z), \\ v' &= h(u) - sv/D - y, & sz' &= \epsilon(\beta y + \gamma z). \end{aligned} \tag{1.5}$$

Its singular limit system is:

$$\begin{aligned} u' &= v/D, & y' &= 0 \quad \text{if } s \neq 0, \\ v' &= h(u) - sv/D - y, & z' &= 0, \quad \text{if } s \neq 0. \end{aligned} \tag{1.6}$$

Let $y_m = h(u_m)$ (and $y_M = h(u_M)$) be the local minimum (and maximum) of the function $y = h(u)$. Then $y_m < y_M$ and $u_M < u_m$. The inverse of $y = h(u)$ has three branches, denoted by h_-^{-1}, h_0^{-1} and h_+^{-1} defined on $y < y_M$, $y_m < y < y_M$ and $y > y_m$ respectively. The solutions of the two algebraic equations in (1.4), $v = 0$, $u = h^{-1}(y)$, form three slow manifolds of (u, v, y, z) in $\mathbb{R}^4$:

$$\begin{aligned} S^- &:= \{v = 0, y < y_M, u = h_-^{-1}(y), z \in \mathbb{R}\}, \\ S^0 &:= \{v = 0, y_m < y < y_M, u = h_0^{-1}(y), z \in \mathbb{R}\}, \\ S^+ &:= \{v = 0, y > y_m, u = h_+^{-1}(y), z \in \mathbb{R}\}. \end{aligned}$$

And these three slow manifolds consist of the equilibrium points of the limiting fast system (1.6).

From (1.6), with $y = \bar{y}$ as a parameter, the equations for (u, v) become

$$u' = v/D, \quad v' = h(u) - sv/D - \bar{y}. \tag{1.7}$$

System (1.7) has been studied by Fife in 1974, cf. [12]. Following his approach, we look for traveling waves that connect saddle to saddle equilibrium points of (1.7), which requires that $dh/du > 0$ at the equilibrium points under consideration. Therefore, the manifold S^0 will not be of interest in this paper. For the slow system on $S^\pm$, there are two equilibrium points $P_\pm$, determined by $y - h^{-1}(y) - z = 0$ and $\beta y + \gamma z = 0$. Assume that γ/β is sufficiently small so $P_\pm$ are between the local minimum and maximum of the curve $\dot{y} = 0$, see Fig. 2.

The case $\gamma = 0$ has been considered in [3], where the last equation of (1.3) becomes $s\dot{z} = \beta y$, and the equilibria $P_\pm$ satisfy $y_\pm = 0$. Due to the symmetry of the function $h(u)$, if there is a heteroclinic orbit for the (u, v) system with $\bar{y} = y_\pm = 0$, then $s = 0$. See [12] for details. In this case the derivatives $(\dot{y}, \dot{z})$ in (1.3) simply drop out. Such "degenerate" case will not be considered in this paper.

Since $dh/du(u) > 0$ for $u = h_\pm^{-1}(y_\pm)$, the equilibrium points for (1.7) on $S^\pm$ are hyperbolic with one stable and one unstable real eigenvalues. From [12], we have the following results.

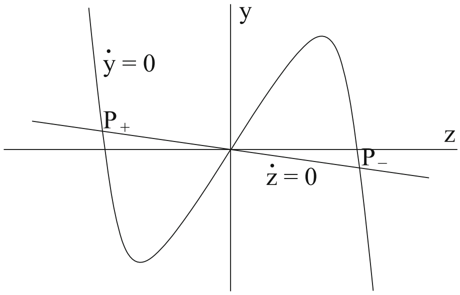


Fig. 2 Equilibrium points $P_{\pm} \in S^{\pm}$ for the slow system

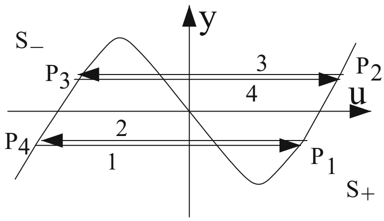


Fig. 3 The orbits (*1*) and (*3*) correspond to $s > 0$, while the orbits (*2*) and (*4*) correspond to $s < 0$

Lemma 1.1 *There are* 4 *kinds of singular heteroclinic orbits for* (1.7). (1) *For an increasing solution* $u_0(\tau)$*, if* $\bar{y} > 0$*, then* $s < 0$*; while if* $\bar{y} < 0$*, then* $s > 0$. (2) *For a decreasing solution* $u_0(\tau)$*, if* $\bar{y} > 0$*, then* $s > 0$*; while if* $\bar{y} < 0$ *then* $s < 0$.

In Fig. 3, the equilibrium point P_4 (or P_2) corresponds to P_- (or P_+) in our paper. We shall only consider the case $s > 0$ in the rest of the paper, for the case $s < 0$ can be treated similarly.

Consider the slow system on $S^{\pm}$

$$s\dot{y} = y - h_{\pm}^{-1}(y) - z, \quad s\dot{z} = \beta y + \gamma z. \tag{1.8}$$

Assume that on $S^{\pm}$, the slopes of $z = y - h_{\pm}^{-1}(y)$ are negative between the local minimum and maximum of $y = h(u)$, see Fig. 2. We have the following lemma on the eigenvalues of $P_{\pm} = (y_{\pm}, z_{\pm})$ in (1.8).

Lemma 1.2 *Assume that* $s > 0$ *and* $(d/dy)[y_{\pm} - h_{\pm}^{-1}(y_{\pm})] < 0$ *respectively. Then if* γ *is sufficiently small and* β *is sufficiently large, both* $P_{\pm} \in S^{\pm}$ *are stable equilibrium points of* (1.8)*, with a pair of complex eigenvalues.*

Proof Let $A_{\pm}$ be the Jacobian matrix of (1.8) at $P_{\pm}$. If $\gamma > 0$ is sufficiently small, then $tr(A_{\pm}) < 0$. Fix that γ and let β be sufficiently large, we have $det(A_{\pm}) > 0$. For such (β, γ), $A_{\pm}$ has two stable eigenvalues. Further increase β if necessary, then the two stable eigenvalues at $P_{\pm}$ are complex with negative real parts. □

1.2 Outline of Our Method

Our strategy of proving chaos in (1.2) can be divided into three steps.

(I) First, for $\epsilon = 0$, we construct a formal heteroclinic loop which is a concatenation of orbits in regular and singular layers. We show that under some general assumptions, a singular heteroclinic orbit can be constructed as follows. It starts as a traveling wave from $P_- \in S^-$ to an equilibrium point $P_m \in S^+$, in fast time scale τ with $u'(\tau) > 0$, then followed

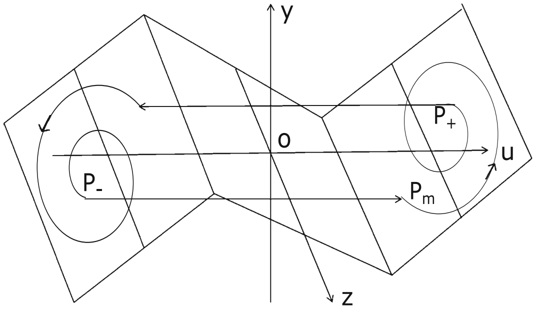


Fig. 4 Traveling wave solution with stable spirals on $\mathcal{S}^+$

by a slow motion on S^+ that spirals from P_m to P_+ in slow time scale t. By symmetry, with the same wave speed, there exists a traveling wave that starts from $P_+ \to S^-$ with $u'(\tau) < 0$, then spirals slowly on S^- to P_-. See Fig. 4. To construct such solutions, we use analytic methods on singular perturbation and heteroclinic bifurcations, see [6,20,21,25].

(II) Next, for $0 < \epsilon \ll 1$, we prove the existence of exact heteroclinic orbits near the singular heteroclinic orbits. The singular orbits obtained in (I) are approximations to the exact orbits, with small residual and jump errors, and small error in the wave speed s. In singularly perturbed systems, corrections of the wave speeds s are usually done in the internal layers which are short intervals in time scale t, but are long intervals in time scale τ, cf. [7,8,25,31]. An unexpected difficulty in this paper is that the singular heteroclinic orbit from P_- to P_m is defined on $(-\infty, b]$ in time t, which is unbounded to the left no matter how we choose b. Therefore the correction of wave speed s cannot be separated from the correction of other variables. The problem is solved by dividing the heteroclinic orbit from P_- to P_m into two parts. In the first part, the orbit simply stays at P_-, and in the second part the orbit jumps from P_- to P_m. This seems to go against any common sense at the first look, but is guided by the classical singular perturbation theory, and therefore works nicely. See the constructions in Sect. 4 for details.

After correctly dividing the heteroclinic orbits into outer and inner layers, using the theory of shadowing lemma for continuous flows, cf. [23,32], we can eliminate the residual and jump errors of the formal heteroclinic orbits and the error of wave speed to obtain a pair of exact heteroclinic orbits. To deal with the residual and jump errors, we introduce exponential dichotomies for the variational equations around the approximations. The residual and jump errors are eliminated by an iteration method, which appears to be the most technical part of the paper.

(III) We show that at the equilibrium points $P_\pm$, the system has a pair of stable complex eigenvalues. Using the symbolic dynamics near a homoclinic orbit discovered by Shilnikov [36,37], and slightly extended in [24] to treat solutions near a heteroclinic loop, we show that there exist complicated traveling wave solutions to the singularly perturbed PDE (1.2). Near the heteroclinic loop, there exists a countably infinite set of periodic traveling waves, and an uncountable set of aperiodic traveling waves. And each traveling wave near the heteroclinic loop can be associated to a unique time sequence $\{\omega_i\}_{i\in\mathbb{Z}}$, where each ω_i describes the time the solutions take turns to stay near the equilibrium P_- or P_+ then move to near another equilibrium. The time counting is achieved by using Poincare sections that are transverse to the heteroclinic orbits from P_- to P_+, and from P_- to P_+. See Theorem 5.2 for details.

In the geometric singular perturbation theory, $\dot{s} = 0$ becomes another slow equation. The heteroclinic orbit from P_- to P_+ is a transverse intersection of the center unstable manifold $W^{cu}(P_-)$ to the center stable manifold $W^{cs}(P_+)$. See [9–11,17–19]. Since we only look for a particular orbit, not the exact invariant manifolds nearby, we only use the singular approximations of $W^{cu}(P_-)$ and $W^{cs}(P_+)$, obtained when $\epsilon = 0$. This observation guides us to construct approximations and correction terms of traveling waves in Sect. 4. Moreover, the singular perturbation method allows us to quickly reduce the intersection problem in the 4D system (1.5) to the 2D system (1.7). Finally, for the original system in 4D, the Melnikov integral used in Sect. 4 rigorously shows that the intersection of $W^{cu}(P_-)$ to $W^{cs}(P_+)$ is transverse. Our method combines the geometric idea with the analytic approach and is based on many previous research such as [1,29,32].

The rest of the paper is organized as follows. In Sect. 2, we recast the traveling wave equations as a general system (2.1) and introduce some assumptions to be used in the rest of the paper. In Sect. 3, we define the exponential dichotomies and introduce some basic lemmas that are useful when we study the linear variational systems around the heteroclinic orbits. In Sect. 4, we establish the existence of heteroclinic solutions when $\epsilon \neq 0$. To this end, we have to deal with the residual and jump errors of the approximations by the singular heteroclinic orbits. The approximations and corrections are introduced in Sects. 4.1 and 4.2. We then treat the residual errors in Sect. 4.3, and the jump errors in Sect. 4.4. The exact heteroclinic solutions are obtained in Sect. 4.5, by the contraction mapping principle. Our main results are proved in Sect. 5. In Sect. 5.1, we study the eigenvalue problems when $\epsilon > 0$ and small. In the slow time t, we show that the dominant, slow eigenvalues are complex conjugate as $(-\alpha \pm i\beta) + O(\epsilon)$, and the fast eigenvalues are real and of the form $\lambda_f(\epsilon)/\epsilon$. The existence of countably many periodic orbits and uncountably many aperiodic orbits are proved in Sect. 5.2. In Remark 5.3, we explain how our results on (1.2) can be used to prove the chaotic traveling wave solutions of (1.1).

Without further specification, the norm of a continuous function defined on an interval will be the supremum norm. For example, in the later part of this paper, $\|X_i\| = \sup\{|X_i(t)| \,|t \in [\alpha_i, \beta_i]\}, \quad i = 1, 2, 3.$

2 Basic Assumptions on the Singular Heteroclinic Orbits When $\epsilon = 0$

Consider traveling wave solutions to the following system of a reaction-diffusion PDE coupled with a system of 2 ODEs:

$$\begin{aligned} \epsilon U_t &= \epsilon^2 U_{xx} - F(U, Y), \\ Y_t &= -G(U, Y). \end{aligned}$$

Using the traveling coordinates $\xi = x - st$ where $s \neq 0$ is the wave speed, we have a singularly perturbed second order fast ODE coupled with 2 first order slow ODEs

$$\begin{aligned} \epsilon^2 U_{\xi\xi} + \epsilon s U_\xi - F(U, Y) &= 0, \\ Y_\xi &= G(U, Y)/s, \end{aligned} \tag{2.1}$$

where $U \in \mathbb{R}$, $Y \in \mathbb{R}^2$; F and G are C^2 functions with bounded derivatives; $\epsilon \geq 0$ is a small parameter.

For definiteness, we assume $s > 0$. The second order system for U is equivalent to a first order system of two equations. Switching ξ to t for easy typing, we consider the singularly perturbed system

$$\epsilon \dot{U} = V, \quad \epsilon \dot{V} = F(U, Y) - sV, \quad \dot{Y} = G(U, Y)/s, \tag{2.2}$$

Switching to the fast time $\tau = \frac{t}{\epsilon}$ and $' = \frac{d}{d\tau}$, (2.2) becomes

$$U' = V, \quad V' = F(U,Y) - sV, \quad Y' = \epsilon G(U,Y)/s. \tag{2.3}$$

We call (2.2) the slow system and (2.3) the fast system, where (U, V) are the fast variables and Y is the slow variable. Let $\epsilon = 0$. Then (2.2) and (2.3) reduce to their singular limits

$$0 = V, \quad 0 = F(U,Y) - sV, \quad \dot{Y} = G(U,Y)/s, \tag{2.4}$$

$$U' = V, \quad V' = F(U,Y) - sV, \quad Y' = 0. \tag{2.5}$$

Assume that F, G satisfy the following hypotheses:

($\mathbf{H_0}$) (F, G) are odd functions: $F(-U, -Y) = -F(U, Y),\ G(-U, -Y) = -G(U, Y)$.
($\mathbf{H_1}$) For $\epsilon = 0$, the equation $F(U, Y) = 0$ has at least two branches of solutions $U = H^{\pm}(Y)$ where $H^- = -H^+$, $H^{\pm} \in C^2(O_2, \mathbb{R})$ and O_2 is an open subset in $\mathbb{R}^2$.

Moreover, assume that $F_U(U, Y) > 0$ for (U, Y) in each of the two branches.

Define the slow manifolds

$$S^{\pm} = \{(U, V, Y) | U = H^{\pm}(Y),\ V = 0,\ Y \in O_2\}.$$

If $U = H^{\pm}(Y)$, then the last equation of system (2.4) yields an equation on $S^{\pm}$:

$$\dot{Y} = G(H^{\pm}(Y), Y)/s, \tag{2.6}$$

Let $Y_{\pm} \in O_2$ be equilibrium points for (2.6) where Y_- or Y_+ corresponds to H^- or H^+ respectively. Then $(U, V, Y) = (H^{\pm}(Y_{\pm}), 0, Y_{\pm})$ are equilibrium points for (2.4), denoted by $P_{\pm}$. Notice that $DH^{\pm}(Y) = -F_U^{-1}(H^{\pm}(Y), Y)F_Y(H^{\pm}(Y), Y)$.

($\mathbf{H_2}$) Let $U_{\pm} = H^{\pm}(Y_{\pm})$. We assume that

$$DG(H^{\pm}(Y_{\pm}), Y_{\pm}) = G_Y(U_{\pm}, Y_{\pm}) - G_U(U_{\pm}, Y_{\pm})F_U^{-1}(U_{\pm}, Y_{\pm})F_Y(U_{\pm}, Y_{\pm})$$

has two eigenvalues $\lambda_{1,2} = -\alpha \pm i\beta$ with $-\alpha < 0,\ \beta > 0$.

We now look for heteroclinic orbits of (2.5) joining $P_- \to P_+$ (or $P_+ \to P_-$). To satisfy (2.5), Y must be a constant. The eigenvalues of the (U, V) equations satisfy $\lambda^2 + s\lambda - F_U(U, Y) = 0$. From ($\mathbf{H_1}$), it has two real roots of opposite signs. They are called the fast eigenvalues for future reference, and are denoted by

$$\lambda_1^f < 0 < \lambda_2^f. \tag{2.7}$$

Thus the slow manifolds $S^{\pm}$ are normally hyperbolic. Setting $Y = Y_{\pm}$ on $S^{\pm}$ respectively, then (2.5) becomes

$$U' = V, \quad V' = F(U, Y_{\pm}) - sV. \tag{2.8}$$

To construct heteroclinic orbits of (2.8) joining $P_{\pm}$, we assume that:

($\mathbf{H_3}$) For $Y = Y_{\pm}$, there exists a unique $s = s_0 > 0$ such that the fast system (2.8) has a heteroclinic orbit $(U^0, V^0)(\tau)$ from $(H^-(Y_-), 0)$ to $(H^+(Y_-), 0)$ (or from $(H^+(Y_+), 0)$ to $(H^-(Y_+), 0)$). The heteroclinic orbits break transversely if s is perturbed away from s_0.

We remark that the last condition will be re-formulated analytically by the Melnikov method.

From ($\mathbf{H_2}$), the eigenvalues of (2.6) evaluated at $Y_{\pm}$ are complex with negative real parts. We further assume that:

($\mathbf{H_4}$) For (2.6), the point $Y = H^-(Y_+)$ (or $Y = H^+(Y_-)$) is contained in the attraction domain of the equilibrium P_- on S^- (or P_+ on S^+). And the orbits through which will approach P_- (or P_+) in such a way that is tangent to the 2D eigenspace of the eigenvalues $\lambda_{1,2} = -\alpha \pm i\beta$.

Remark 2.1 The hypothesis (**H$_4$**) may look complicated. In the example of Chua's circuits, it can be satisfied by applying certain conditions on the circuits, [3].

Based on (**H$_3$**) and (**H$_4$**), for $\epsilon = 0$, define a singular heteroclinic orbit connecting P_- to P_+ as follows. Let $X_i^0(t) = (U_i^0(t), V_i^0(t), Y_i^0(t))$, $i = 1, 3$, be solutions of (2.4) on the slow manifold S^- if $i = 1$ (or on S^+ if $i = 3$). Among them, $X_1^0(t) = P_-$ is the constant solution in slow time t, and $X_3^0(t)$ is the solution on S^+ that connects P_m to P_+. Let $X_2^0(\tau) = (U_2^0(\tau), V_2^0(\tau), Y_2^0(\tau))$ be a heteroclinic solution of (2.5) connecting S^- to S^+. with the wave speed s_0.

The domains for X_i^0, $i = 1, 2, 3$ are defined as

$$t \in [-\infty, 0] \text{ for } X_1^0, \quad \tau \in [-\infty, \infty] \text{ for } X_2^0, \quad t \in [0, \infty] \text{ for } X_3^0.$$

Notice that the following matching conditions are satisfied:

$$\lim_{t\to-\infty} X_1^0(t) = P_-, \quad \lim_{\tau\to-\infty} X_2^0(\tau) = X_1^0(0), \quad \lim_{\tau\to\infty} X_2^0(\tau) = X_3^0(0), \quad \lim_{t\to\infty} X_3^0(t) = P_+. \tag{2.9}$$

Then a formal heteroclinic orbit connecting P_- to P_+ can be constructed by the concatenation of X_1^0, X_2^0 and X_3^0, see Fig. 4. By symmetry, a formal heteroclinic orbit connecting P_+ to P_- can be constructed.

Remark 2.2 As mentioned in Sect. 1.2, defining the approximation X_1^0 as a constant is an important step in constructing the approximations. See Sect. 4.1 where the correction terms are constructed.

3 Exponential Dichotomies and Linear Nonhomogeneous Systems with Boundary Conditions

We introduce some basic concept of exponential dichotomies, see [5,24].

Definition 3.1 Consider the linear system $\dot{x} = A(t)x$, $x \in \mathbb{R}^m$, where $A(t)$ is a continuous matrix defined on a finite or infinite interval $I \subset \mathbb{R}$. Let $\Phi(t, s)$ be the principal matrix solution of $\dot{x} = A(t)x$. We say that $\dot{x} = A(t)x$ has an exponential dichotomy on I if there exist constants $K, \zeta > 0$, and projections to the stable and unstable subspaces, $P_s(t) + P_u(t) = I_d$ for $t \in I$, such that

$$\begin{aligned} &\text{(i)} \quad \Phi(t, s)P_s(s) = P_s(t)\Phi(t, s), \ s, t \in I, \\ &\text{(ii)} \quad |\Phi(t, s)P_s(s)| \le Ke^{-\zeta(t-s)}, \ s \le t, \\ &\text{(iii)} \quad |\Phi(t, s)P_u(s)| \le Ke^{-\zeta(s-t)}, \ t \le s. \end{aligned}$$

With the principal matrix solution $\Phi^*(s, t) := (\Phi(t, s)^{-1})^*$, the adjoint equation

$$\frac{dy}{ds} + A^*(s)y = 0.$$

is solved from the initial time t to the moving time s. If $x(t)$ is the solution of the original equation and $x^*(t)$ is a solution of the adjoint equation, then

$$< x^*(t), \Phi(t, s)x(s) >=< \Phi^*(s, t)x^*(t), x(s) > .$$

If $\Phi(t, s)$ has an exponential dichotomy, then $\Phi^*(s, t)$ has an exponential dichotomy with the projections to the stable and unstable subspaces, $P_s^*(t)$ and $P_u^*(t)$, being adjoint operators of

$P_s(t)$ and $P_u(t)$ respectively. The name of the stable and unstable subspaces for the adjoint equation are meaningful if it is solved backward in time:

$$\Phi^*(s,t)P_s^*(t) = P_s^*(s)\Phi^*(s,t),$$
$$|\Phi^*(s,t)P_s^*(t)| \le Ke^{-\gamma(t-s)}, \quad s \le t,$$
$$|\Phi^*(s,t)P_u^*(t)| \le Ke^{-\gamma(s-t)}, \quad t \le s.$$

We now consider the linear nonhomogeneous system

$$\dot{x} - A(t)x = f(t), \ x \in \mathbb{R}^m, t \in I, \tag{3.1}$$

where the interval I is sufficiently large so the term $Ke^{-\gamma|t-s|}$ can be sufficiently small. We consider the following two cases.
∼ Case I. System (3.1) has an exponential dichotomy on $[a, b]$, possibly $a = -\infty$ and/or $b = \infty$.

Denote the range of an projection operator P by $\mathcal{R}P$ in the rest of this paper. We have the following result for Case I.

Lemma 3.1 *For a given $f \in C[a,b]$ and $(\phi_s, \phi_u) \in (\mathcal{R}P_s(a), \mathcal{R}P_u(b))$, consider the nonhomogeneous boundary value problem:*

$$\begin{aligned} &\dot{x} - A(t)x = f(t), \quad a \le t \le b, \\ &P_s(a)x(a) = \phi_s, \quad P_u(b)x(b) = \phi_u. \end{aligned} \tag{3.2}$$

The system has a unique C^1 solution $x(t)$, $a \le t \le b$ given by

$$x(t) = \Phi(t,a)\phi_s + \int_a^t \Phi(t,s)P_s(s)f(s)ds + \Phi(t,b)\phi_u + \int_b^t \Phi(t,s)P_u(s)f(s)ds.$$

And the solution satisfies

$$|x(t)| \le C(\|f\| + e^{-\gamma(t-a)}|\phi_s| + e^{-\gamma(b-t)}|\phi_u|). \tag{3.3}$$

Moreover, we allow $a = -\infty$ and/or $b = \infty$. If $a = -\infty$ (and/or $b = \infty$) then the term ϕ_s (and/or ϕ_u) should be dropped in (3.2) and (3.3).

∼ Case II: System (3.1) has exponential dichotomies on $[-L, 0]$ and $[0, M]$.

In this case let $\phi(t) = \Phi(t,0)\phi(0)$ be a solution to the homogeneous problem with $\phi(0) \in \mathcal{R}P_u(0-) \cap \mathcal{R}P_s(0+)$. Also assume that

$$\dim \mathcal{R}P_u(0-) = \dim \mathcal{R}P_u(0+) = d^+.$$
$$\mathcal{R}P_u(0-) \cap \mathcal{R}P_s(0+) = \text{span}\{\phi(0)\}.$$

Note that $\mathcal{R}P_u(0-) + \mathcal{R}P_s(0+)$ is of codimension one. Let $\psi(t)$ be a solution to the adjoint system $dy/ds + A^*(s)y = 0$ such that $\psi(0) \perp (\mathcal{R}P_u(0-) + \mathcal{R}P_s(0+))$. Then $\psi(0) \in \mathcal{R}P_s^*(0-) \cap \mathcal{R}P_u^*(0+)$. Let $[E(0-)]^c$ and $[E(0+)]^c$ be orthogonal complementary to $\phi(0)$ in $\mathcal{R}P_u(0-)$ and $\mathcal{R}P_s(0+)$ respectively. Then

$$\text{span}\{\psi(0)\} \oplus \text{span}\{\phi(0)\} \oplus [E(0-)]^c \oplus [E(0+)]^c = \mathbb{R}^m.$$

Lemma 3.2 *Let $L, M > 0$. For a given $f \in C[-L, M]$ and $(\phi_s, \phi_u) \in (\mathcal{R}P_s(-L), \mathcal{R}P_u(M))$, consider the nonhomogeneous boundary value problem*

$$\begin{aligned} &\dot{x} - A(t)x = f(t), \quad -L \le t \le M, \\ &P_s(-L)x(-L) = \phi_s, \quad P_u(M)x(M) = \phi_u. \end{aligned} \tag{3.4}$$

The system has a unique C^1 solution $x(t)$ with $x(0) \perp \phi(0)$ if and only if

$$\int_{-L}^{M} < \psi(t), f(t) > dt + < \psi(-L), \phi_s > - < \psi(M), \phi_u >= 0. \tag{3.5}$$

Moreover, if conditions (3.5)*and $x(0) \perp \phi(0)$ are satisfied, then*

$$|x(t)| \le C(\|f\| + e^{-\gamma(t+L)}|\phi_s| + e^{-\gamma(M-t)}|\phi_u|). \tag{3.6}$$

Proof The general solutions of (3.4) can be written piecewise as follows.

For $-L \le t \le 0$, with undetermined $x_u^c(0) \in [E(0-)]^c$, $\zeta_1 \in \mathbb{R}$,

$$\begin{aligned} x(t) = \Phi(t, -L)\phi_s + \int_{-L}^{t} \Phi(t, s)P_s(s)f(s)ds \\ + \Phi(t, 0)x_u^c(0) + \zeta_1\phi(t) + \int_{0}^{t} \Phi(t, s)P_u(s)f(s)ds. \end{aligned}$$

For $0 \le t \le M$, with undetermined $x_s^c(0) \in [E(0+)]^c$, $\zeta_2 \in \mathbb{R}$,

$$\begin{aligned} x(t) = \Phi(t, 0)x_s^c(0) + \zeta_2\phi(t) + \int_{0}^{t} \Phi(t, s)P_s(s)f(s)ds \\ + \Phi(t, M)\phi_u + \int_{M}^{t} \Phi(t, s)P_u(s)f(s)ds. \end{aligned}$$

Being orthogonal to $\phi(0)$, $x(0+) - x(0-)$ is determined by the unique triple: $(x_u^c(0-),\ x_s^c(0+),\ G\psi(0))$:

$$\begin{aligned} x_s^c(0+) - x_u^c(0-) - G\ \psi(0) = \Phi(0, -L)\phi_s + \int_{-L}^{0} \Phi(0, s)P_s(s)f(s)ds \\ - \Phi(0, M)\phi_u + \int_{0}^{M} \Phi(0, s)P_u(s)f(s)ds + (\zeta_1 - \zeta_2)\phi(0). \end{aligned} \tag{3.7}$$

In (3.7), $G\psi(0)$ represents the projection of the gap $x(0+) - x(0-)$ to the direction spanned by $\psi(0)$, and $\zeta_1 - \zeta_2$ is undetermined. The expression for G can be obtained by taking the inner product of $\psi(0)$ to (3.7) and integrating by parts. Using $P_s^*(s)\Phi^*(s, 0)\psi(0) = \psi(s)$, $s \le 0$ and $P_u^*(s)\Phi^*(s, 0)\psi(0) = \psi(s)$, $s \ge 0$, in order to have $G = 0$, we obtain the condition (3.5). Finally, if (3.5) is satisfied, then we can uniquely determine $x_s^c(0+)$ and $x_u^c(0-)$ in $[E(0-)]^c \oplus [E(0+)]^c$ from (3.7), and then find the unique ζ_1 and ζ_2 such that $x(0\pm) \perp \phi(0)$ are satisfied. □

Remark 3.3 Lemma 3.2 was proved in [24] for functional differential equations. We give a simple proof for systems of ODEs for easy understanding.

We now present some properties of the 2nd order equation

$$u'' + s_0 u' - F(u, Y_\pm) = 0. \tag{3.8}$$

From $(\mathbf{H_1})$, we have $F_u(U_\pm, Y_\pm) > 0$. We assume that the equation has a heteroclinic solution $u_0(\tau)$ connecting two saddle equilibrium points $U_\pm$. Then the linear variational equation around u_0,

$$u''(\tau) + s_0 u'(\tau) - F_U(u_0(\tau))u(\tau) = 0,$$

has a unique bounded solution $u_0'(\tau)$ up to constant multiples. The adjoint equation

$$\psi''(\tau) - s_0\psi' - F_U(u_0)\psi = 0,$$

has a unique bounded solution $\psi_0(\tau) = e^{s_0\tau}u_0'(\tau)$ up to constant multiples. Obviously

$$\int_{-\infty}^{\infty} \psi_0(\tau)u_0'(\tau)d\tau \neq 0. \tag{3.9}$$

To convert the properties on the second order equation to those on a first order system, let $w = (U, V)^T$, $\Xi_0 = (0, u_0')^T$. Define the matrix $A(\tau) = \begin{pmatrix} 0 & I \\ F_U & -s_0 \end{pmatrix}$. The linear system $w' = A(\tau)w$ has exponential dichotomies for $\tau \in \mathbb{R}^{\pm}$ respectively. It has a unique bounded solution $\Phi_0(\tau) = (u_0', u_0'')^T$ and the adjoint system $\Psi' + A^T(\tau)\Psi = 0$ has a unique bounded solution $\Psi_0(\tau) = e^{s\tau}(-u_0'', u_0')^T$. Based on (3.9), we assume that the following condition is satisfied:

(H$_5$) $\int_{-\infty}^{\infty} < \Psi_0, \Xi_0 > d\tau \neq 0$.

If $L, M > 0$ are sufficiently large, then $w' = A(\tau)w$ has exponential dichotomies on $[-L, 0]$ and $[0, M]$ respectively, and the constants (K, ζ) in Definition 3.1 satisfy $Ke^{-\zeta L} \ll 1$, $Ke^{-\zeta M} \ll 1$. Let $\phi_1 \in \mathcal{R}P_s(-L)$, $\phi_2 \in \mathcal{R}P_u(M)$ be two given vectors, and $g \in C[-L, M]$. Consider the linear differential equation on $[-L, M]$ with boundary conditions:

$$\begin{aligned} &w' - A(\tau)w = s\Xi_0(\tau) + g(\tau), \\ &P_s(-L)w(-L) = \phi^1,\ \ P_u(M)w(M) = \phi^2. \end{aligned} \tag{3.10}$$

From Lemma 3.2, we have the following well-known result:

Lemma 3.4 *Assume the function $\Xi_0 = (0, u_0')^T$ where $u_0(\tau)$ is the unique bounded solution to (3.8), Ψ_0 is the unique bounded solution to the adjoint equation $\Psi' + A^T(\tau)\Psi = 0$, and condition* (**H$_5$**) *is satisfied. Then there exists a unique $s \in \mathbb{R}$ such that (3.10) has a solution $w \in C^1[-L, M]$. If we assume the phase condition $w(0) \perp \Phi_0(0)$, then the solution w is unique and the following estimates holds with C being independent of L, M.*

$$|s| \leq C(e^{-\zeta L}|\phi_1| + e^{-\zeta M}|\phi_2| + \|g\|), \tag{3.11}$$

$$|w| \leq C(|\phi_1| + |\phi_2| + \|g\|). \tag{3.12}$$

Moreover, at the two boundaries of the interval $[-L, M]$,

$$|P_u(-L)w(-L)| + |P_s(M)w(M)| \leq C(e^{-\zeta L}|\phi_1| + e^{-\zeta M}|\phi_2| + \|g\|). \tag{3.13}$$

Proof Combine the r.h.s. $s\Xi_0(\tau) + g(\tau)$ of (3.10) into $f(t)$ in (3.4). Then from Lemma 3.2, and using (**H$_5$**), we can uniquely determine $s \in \mathbb{R}$, so that (3.5) in Lemma 3.2 is satisfied. The estimates (3.11)–(3.13) also come from Lemma 3.2. □

4 Existence of Exact Heteroclinic Solutions when $\epsilon \neq 0$

In this section, we show that there exist heteroclinic orbits for $0 < \epsilon \ll 1$, that are near the singular orbits defined for $\epsilon = 0$ at the end of Sect. 2. First, we construct an approximation $X^{ap}(\tau)$, using the fast time τ, as follows.

4.1 Construction of an Approximated Solution for $0 < \epsilon \ll 1$

In the rest of this paper, let $L = \epsilon^{-0.5}$ be an intermediate time scale such that $1 \ll L \ll \epsilon^{-1}$. Divide the entire domain as $\mathbb{R} = I_1 \cup I_2 \cup I_3$, where $I_1 = (-\infty, -L]$, $I_2 = [-L, L]$ and $I_3 = [L, \infty)$. The solutions for $\epsilon = 0$ now become approximations for $\epsilon > 0$:

$$X^{ap}(\tau) = (U_i^{ap}, V_i^{ap}, Y_i^{ap})^T(\tau) := \begin{cases} X_i^0(\epsilon\tau), & \tau \in I_1 = (-\infty, -L], \quad i = 1, \\ X_i^0(\tau), & \tau \in I_2 = [-L, L], \quad i = 2, \\ X_i^0(\epsilon\tau), & \tau \in I_3 = [L, \infty), \quad i = 3. \end{cases}$$

As in Sect. 2, $X_1^0(t)$ is an orbit for (2.4) that stays at P_-, $X_2^0(\tau)$ is an orbit for (2.5) that connects S^- to S^+, and $X_3^0(t)$ is an orbit for (2.4), that spirals towards P_+ on S^+.

It is easy to see that X^{ap} does not satisfy (2.2) or (2.3) for $\epsilon \neq 0$. Let the small residual errors (r_i, p_i, q_i) be defined by

$$\begin{aligned} U'_{ap} = V_{ap} + r_i(\tau), \quad V'_{ap}(t) = F(U_{ap}, Y_{ap}) - s_0 V_{ap}(\tau) + p_i(\tau), \\ Y'_{ap}(t) = \epsilon G(U_{ap}, Y_{ap})/s_0 + \epsilon q_i(\tau), \qquad i = 1, 2, 3. \end{aligned} \tag{4.1}$$

Observe that the two terms in the r.h.s. of Y'_{ap} are not small even they have the factor ϵ, since the left hand side, if we write $Y'_{ap}(t) = Y'_{ap}(\epsilon\tau)$, also has a factor ϵ. Since $(U_i^{ap}, V_i^{ap}, Y_i^{ap})$ satisfy (2.4) and (2.5), not (2.2) or (2.3), it is straight forward to check that the residual errors satisfy:

$$|r_i| + p_i| + |q_i| = O(\epsilon), \quad i = 1, 2, 3,$$

in slow (or fast) layers using the time t (or τ).

When $\epsilon = 0$ there is no jump error between the singular and regular layers due to the matching condition (2.9). However, when $\epsilon \neq 0$, there will be jump errors at the junction points $\tau = \pm L$.

Definition 4.1 For a piecewise continuous function $w(\tau)$ with $\bar{\tau}$ in its domain, define the jump at $\bar{\tau}$ as

$$\Delta[w](\bar{\tau}) := w(\bar{\tau}+) - w(\bar{\tau}-).$$

For $X = (U, V, Y)$, the jump errors of approximation for $\epsilon \neq 0$ satisfy

$$J_{12}^{ap} := \Delta[X^{ap}](-L) = O(\epsilon^{0.5}), \quad J_{23}^{ap} := \Delta[X^{ap}](L) = O(\epsilon^{0.5}). \tag{4.2}$$

The $O(\epsilon^{0.5})$ estimates in (4.2) can be proved by the asymptotic matching conditions in (2.9) and the choice of the intermediate variable $L = \epsilon^{-1/2}$. For example, using $(e^{-\lambda_2^f L}) < 1/(\lambda_2^f L)$ if $\lambda_2^f L > 1$, we have

$$|\Delta[X^{ap}](L)| \leq |X_3^0(\epsilon L) - X_3^0(0)| + |X_3^0(0) - X_2^0(L)| \leq C\epsilon L + Ce^{-\lambda_2^f L} \leq C\epsilon^{0.5}.$$

4.2 The Correction Functions

Let X^{ex} be the exact traveling wave solution with the wave speed s^{ex}, then the correction terms are $X^{cr} = X^{ex} - X^{ap}$, $s^{cr} = s^{ex} - s_0$. We shall use fast time τ for corrections in both the fast layer and slow layers, namely,

$$X_i^{cr}(\tau) = (U_i(\tau), V_i(\tau), Y_i(\tau)),$$

$$\tau \in [-L, L],\ i = 2, \quad \tau \in (-\infty, -L],\ i = 1, \quad \tau \in [L, \infty),\ i = 3.$$

Here we rewrite X_i^{cr} as (U_i, V_i, Y_i), and s^{cr} as s for simplicity. Notice that $1/(s_0 + s) \approx 1/s_0 - s/s_0^2$. From (2.2) and (2.3), linearizing $F(U_i^0 + U_i, Y_i^0 + Y_i)$ and $G(U_i^0 + U_i, Y_i^0 + Y_i)$ around (U_i^0, Y_i^0), we find that the correction terms satisfy:

$$\begin{aligned} &U_i'(\tau) = V_i(\tau) - r_i, \quad V_i' = F_u U_i + F_Y Y_i - s_0 V_i - s V_i^0 - p_i(\tau) + h.o.t., \\ &Y_i'(\tau) = \epsilon(G_U U_i + G_Y Y_i)/s_0 - \epsilon s G(U_i^0, Y_i^0)/s_0^2 - \epsilon q_i(\tau) + h.o.t., \quad i = 1, 2, 3. \end{aligned} \tag{4.3}$$

Notice that r_i, p_i, q_i are given functions of τ; while $h.o.t. = O(|U_i|^2 + |Y_i|^2 + |V_i|^2 + |s|^2)$ are remainders after linearization. We do not include terms like $|U_i||Y_i|$ or $|s||Y_i|$ in the order estimate, sine they are bounded by $O(|U_i|^2 + |Y_i|^2)$ or $O(|Y_i|^2 + |s|^2)$ already.

After linearization, (4.3) is still too complicated to solve directly. We now use two important methods from the theory of singular perturbations to further simply (4.3). To this end, we have to treat $i = 2$ and $i = 1, 3$, differently.

For $i = 2$, $[-L, L]$ is so called the internal layer where $\int_{-L}^{L} \epsilon(G_u U_i + G_Y Y_i) d\tau = O(\sqrt{\epsilon}(|U_i| + |Y_i|))$. Therefore $\epsilon(G_u U_i + G_Y Y_i) = h.o.t.$ Similarly $\int_{-L}^{L} \epsilon s G(U_i^0, Y_i^0) d\tau = O(\sqrt{\epsilon} s) = h.o.t.$. Here $h.o.t.$s are not defined as the same as in (4.3) for all $i = 1, 2, 3$. Finally $Y_2'(\tau) = \epsilon q_2 + h.o.t.$ and for $i = 2$, (4.3) simplifies to

$$\begin{aligned} &Y_2(\tau) = Y_2(-L) - \int_{-L}^{\tau} \epsilon q_2(\eta) d\eta + h.o.t., \\ &U_2'(\tau) = V_2(\tau) - r_2,\ V_2' = F_u U_2 + F_Y Y_2 - s_0 V_2 - s V_2^0 - p_2(\tau) + h.o.t. \end{aligned} \tag{4.4}$$

However, for $i = 1, 3$, the domains for Y_i are unbounded so the r.h.s. of (4.3) cannot be simplified like $i = 2$. We shall take the advantage that $S^{\pm}$ are normally hyperbolic and the flows on which are tangent to $S^{\pm}$. Notice that $U = H^{\pm}(Y)$ on $S^{\pm}$. Define the distance from (U, V, Y) to $S^{\pm}$, along the U axis, by

$$Z = U - H^{\pm}(Y). \tag{4.5}$$

In the new coordinates (Z, V, Y), the vector field satisfies

$$Z'(\tau) := U'(\tau) + F_U^{-1} F_Y Y'(\tau).$$

On the slow manifolds, $Z'(\tau) = 0$, so the vector $(U, Y) = (-F_U^{-1} F_Y Y_i, Y_i),\ i = 1, 3$, is tangent to $S^{\pm}$. Let $Z_i = U_i + F_U^{-1} F_Y Y_i$, Then $Z_i'(\tau) = V_i(\tau) - r_i + \frac{d}{d\tau}(F_U^{-1} F_Y Y_i),\ i = 1, 3$, where $\frac{d}{d\tau}(F_U^{-1} F_Y Y_i) = O(\epsilon)$, which is a h.o.t. The system for $i = 1, 3$ then becomes:

$$\begin{aligned} &Z_i'(\tau) = V_i(\tau) - r_i(\tau) + h.o.t., \\ &V_i'(\tau) = F_U Z_i - s_0 V_i - s V_i^0 - p_i(\tau) + h.o.t., \\ &Y_i(\tau)' = \epsilon(G_Y - G_U F_U^{-1} F_Y) Y_i / s_0 + \epsilon G_U Z_i / s_0 \\ &\qquad - \epsilon s G(U_i^0, Y_i^0)/s_0^2 - \epsilon q_i(t) + h.o.t., \quad i = 1, 3. \end{aligned} \tag{4.6}$$

Here the $h.o.t.$s for (V_i, Y_i) equations are inherited from those of (4.3).

Motivated by (4.4), (4.6), we consider the following linear system. For $i = 2$,

$$\begin{aligned} &Y_2(\tau) = Y_2(-L) + \int_{-L}^{\tau} \epsilon h_2(\eta) d\eta, \\ &U_2'(\tau) = V_2(\tau) + f_2(\tau), \quad V_2'(\tau) = F_u U_2 + F_Y Y_2 - s_0 V_2 - s V_2^0 + g_2(\tau). \end{aligned} \tag{4.7}$$

For $i = 1, 3$,

$$\begin{aligned} &Z_i'(\tau) = V_i(\tau) + f_i(\tau), \quad V_i'(\tau) = F_U Z_i - s_0 V_i - s V_i^0 + g_i(\tau), \\ &Y_i'(\tau) = \epsilon(G_Y - G_U F_U^{-1} F_Y) Y_i / s_0 + \epsilon G_U Z_i / s_0 - \epsilon s G(U_i^0, Y_i^0)/s_0^2 + \epsilon h_i(\tau). \end{aligned} \tag{4.8}$$

Based on (4.2), the following jump conditions must be satisfied by $X = (U, V, Y)$

$$\Delta[X](-L) = -J_{12}^{ap}, \quad \Delta[X](L) = -J_{23}^{ap}, \tag{4.9}$$

where J_{12}^{ap}, J_{23}^{ap} are defined as before. It is easy to check that (4.7) and (4.8) satisfy the following properties.

Lemma 4.1 (1) *If* $(\mathbf{H_1})$ *is satisfied, then the equations for* (U_2, V_2) *in* (4.7)*have exponential dichotomies on* $[-L, 0]$ *and* $[0, L]$ *respectively. Let the projections associated to the dichotomies be* $P_u(\tau) + P_s(\tau) = Id$*, which exist for* $\tau \in [-L, 0]$ *or* $\tau \in [0, L]$ *respectively.*

(2) *If* $(\mathbf{H_2})$ *is satisfied, then the equation for* Y_i, $i = 1, 3$ *in* (4.8)*, is weakly exponentially stable with the exponents* $\epsilon(-\tilde{\alpha} \pm i\tilde{\beta})$ *where* $\tilde{\alpha} = \alpha/s_0$, $\tilde{\beta} = \beta/s_0$*. The system for* (Z_i, V_i), $i = 1, 3$ *has exponential dichotomies on* $(-\infty, -L]$ *and* $[L, \infty)$*, with the projections* $P_u(\tau) + P_s(\tau) = Id$*. Moreover, the projections* $(P_s(\pm L), P_u(\pm L))$ *associated to the dichotomies for* (Z_i, V_i) *at* $\pm L$ *are exactly the same as those for* (U_2, V_2) *in Part (1) of this lemma.*

Remark 4.2 To simplify the notation we shall assume $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$, which can be achieved by rescaling the time so that $s_0 = 1$.

The rest of this section is devoted to solving (4.7), (4.8), with jump conditions (4.9).

4.3 Solving the Non-homogeneous Systems Without Jump Conditions

For $i = 2$, we set $Y_2(-L) = 0$ and solve $Y_2(\tau)$ from (4.7) first. Since $L = \epsilon^{-0.5}$, the solution satisfies

$$|Y_2| \le C\sqrt{\epsilon}|h_2|.$$

We then plug Y_2 into the equations for (U_2, V_2). The system

$$U_2' - V_2 = f_2(\tau), \quad V_2' - F_u U_2 + s_0 V_2 = -s V_2^0 + F_Y Y_2 + g_2(\tau), \tag{4.10}$$

is exactly like (3.10) where $g(\tau)$ now becomes $(f_2(\tau), F_Y Y_2 + g_2(\tau))^T$. We can apply Lemma 3.4 to (4.10) with $(\phi_1, \phi_2) = (0, 0)$ and $\Xi_0 = (0, V_2^0)^T$. Based on $(\mathbf{H_5})$, there exists a unique $s = \bar{s} = O(|h_2| + |f_2| + \|g_2\|)$ so that (4.10) has a solution $(U_2, V_2) = O(|h_2| + |f_2| + \|g_2\|)$, and the constants in the estimates are independent of L as $L \to \infty$.

For $i = 1, 3$, using $s = \bar{s}$ obtained in the previous step when looking for (U_2, V_2), the system for (Z_i, V_i) becomes

$$\begin{aligned} &Z_i'(\tau) - V_i(\tau) = f_i(\tau), \\ &V_i'(\tau) - F_U Z_i + s_0 V_i = -\bar{s} V_i^0 + g_i(\tau), \quad \text{where } i = 1, 3. \end{aligned} \tag{4.11}$$

It has exponential dichotomies on $(-\infty, -L]$ and $[L, \infty)$. Using Lemma 3.1 and by setting $P_u(-L)(Z_1, V_1)(-L) = 0$ and $P_s(L)(Z_3, V_3)(L) = 0$, we can find a unique (Z_i, V_i), $i = 1, 3$. The solutions satisfy

$$(Z_i, V_i)(\tau) \le C(|f_i| + |\bar{s}| + |g_i|),$$

where the constant C is independent of L. We then plug (Z_i, V_i) into (4.8) for Y_i, $i = 1, 3$. Notice the homogeneous part of the system for Y_i is weakly exponentially stable with the exponents $-\epsilon\alpha$. However, the r.h.s. of Y_i' in (4.8) has a factor of ϵ, so the solutions are $O(|\bar{s}| + |f_i| + |g_i| + |h_i|)$, $i = 1, 3$, with constants uniformly valid as $L \to \infty$.

Denote the solutions of this subsection by $s = \bar{s}$, $\bar{X} = (\bar{U}, \bar{V}, \bar{Y})$, or $(\bar{Z}, \bar{V}, \bar{Y})$ if the change of variables near $S^{\pm}$ has been made.

4.4 Solving (4.7), (4.8), (4.9) Without Forcing Terms

In this subsection we look for $s = \hat{s}$, $\hat{X} = (\hat{U}, \hat{V}, \hat{Y})$, or $(\hat{Z}, \hat{V}, \hat{Y})$ if the change of variables near $S^{\pm}$ has been made. They satisfy (4.7), (4.8) with $(f_i, g_i, h_i) = (0, 0, 0)$, $i = 1, 2, 3$. Moreover, the jump conditions for $\hat{X}$ must be specified so $X^{ap} + \bar{X} + \hat{X}$ is a smooth function defined for $\tau \in \mathbb{R}$. Let the jumps for $\bar{X}$ and $\hat{X}$ be

$$\bar{J}_{12} := \Delta[\bar{X}](-L), \quad \bar{J}_{23} := \Delta[\bar{X}](L),$$
$$\hat{J}_{12} := \Delta[\hat{X}](-L), \quad \hat{J}_{23} := \Delta[\hat{X}](L).$$

Then the jump conditions for $\hat{X}$ are given by

$$\hat{J}_{12} = -(J_{12}^{ap} + \bar{J}_{12}), \quad \hat{J}_{23} = -(J_{23}^{ap} + \bar{J}_{23}).$$

Let $(\hat{J}_{12}^{U,V}, \hat{J}_{23}^{U,V})$ be the (U, V) component of the jumps $\hat{J}_{12}$ and $\hat{J}_{23}$ and let $(\hat{J}_{12}^{Y}, \hat{J}_{23}^{Y})$ be the Y component of the jumps $\hat{J}_{12}$ and $\hat{J}_{23}$.

The jump conditions expressed in (U, V) are good for solutions in the fast layer I_2. In slow layers I_1 and I_3, the system shall be written in the variables (Z, V, Y). In order to obtain boundary conditions for the slow layers where $i = 1, 3$ on $S^{\pm}$, we will rewrite the jump condition in terms of (Z, V, Y).

To describe the dynamics near $S^{\pm}$, (Z, V, Y) will only be used in a neighborhood of $S^{\pm}$, i.e. either $U < u_M$ or $U > u_m$. If ϵ is sufficiently small and hence L is sufficiently large, then the fast solutions $(U_2(\pm L), V_2(\pm L))$ are close to $S^{\pm}$ due to the asymptotic matching conditions. For such large L, the junction points of J_{12} and J_{23} are near $S^{\pm}$, where $H^{\pm}(Y)$ will be used to define the variable Z.

Recall that the distance from (U, V, Y) to $S^{\pm}$, along the U axis, was defined in (4.5) as $Z = U - H^{\pm}(Y)$. We finally obtain the jump conditions for (Z, V) as follows:

$$\Delta[(Z, V)](\pm L) = \Delta[(U, V)](\pm L) - \Delta[H^{\pm}(Y), 0)](\pm L). \tag{4.12}$$

Denote the jump for $(Z, V)_i$, $i = 1, 2, 3$ by $\hat{J}_{12}^{Z,V}$, $\hat{J}_{23}^{Z,V}$,

$$\Delta[(Z, V)](-L) = \hat{J}_{12}^{Z,V}, \quad \Delta[(Z, V)](L) = \hat{J}_{23}^{Z,V}.$$

Then using the jumps for Y_2 at $\pm L$, we have

$$\begin{aligned} \hat{J}_{12}^{Z,V} &= \hat{J}_{12}^{U,V} - \Delta[(H^{\pm}(Y), 0)](-L), \\ \hat{J}_{23}^{Z,V} &= \hat{J}_{23}^{U,V} - \Delta[(H^{\pm}(Y), 0)](+L). \end{aligned} \tag{4.13}$$

In this subsection, we consider the jumps at $\tau = \pm L$ as given conditions. Depending on whether (U, V) or (Z, V) are used as fast variables, they are

$$(\hat{J}_{12}^{U,V}, \hat{J}_{12}^{Y}) \text{ or } (\hat{J}_{12}^{Z,V}, \hat{J}_{12}^{Y}), \text{ at } \tau = -L, \tag{4.14}$$
$$(\hat{J}_{23}^{U,V}, \hat{J}_{23}^{Y}) \text{ or } (\hat{J}_{23}^{Z,V}, \hat{J}_{23}^{Y}), \text{ at } \tau = L. \tag{4.15}$$

Assuming L is sufficiently large, then from (3.13) in Lemma 3.4, the influences of the two jumps at $-L$ and L on each other are very small. As the first approximation, we shall decompose the jumps into the stable and unstable subspaces of the exponential dichotomies. Then the solutions can be solved as boundary value problems on each $I_i, i = 1, 2, 3$ separately. The real jump will be satisfied if we use the iterations described in Part A and Part B below.

Part A: Approximating the jump conditions by boundary conditions on each I_i.

From Lemma 4.1, the homogeneous system $U_2' = V_2,\ V_2' = F_U U_2 - s_0 V_2$, has exponential dichotomies for $\tau \in \mathbb{R}^{\pm}$ respectively. If L is sufficiently large, the system has exponential dichotomies on $[-L, 0]$ and $[0, L]$ respectively. Denote the projections by $P_u(\tau)+P_s(\tau) = I_d$ for $\tau \in [-L, 0]$ or $[0, L]$.

We decompose the jump discontinuities into the unstable subspace of the previous interval and the stable subspace of the next interval. This yields the following boundary conditions on $[-L, L]$,

$$\phi_2^s = P_s(-L)(U_2, V_2)(-L) = -P_s(-L)\hat{J}_{12}^{U,V}, \quad \phi_2^u = P_u(L)(U_2, V_2)(L) = P_u(L)\hat{J}_{23}^{U,V}, \tag{4.16}$$

As for $i = 1, 3$, the homogeneous parts of the linear systems for (Z_i, V_i) are exactly the same as those for (U_2, V_2). Using the same projections to the stable and unstable subspaces, we have the following boundary conditions on$(-\infty, -L]$ and $[L, \infty)$,

$$\phi_1^u = P_u(-L)(Z_1, V_1)(-L) = -P_u(-L)\hat{J}_{12}^{Z,V}, \quad \phi_3^s = P_s(L)(Z_3, V_3)(L) = P_s(L)\hat{J}_{23}^{Z,V}. \tag{4.17}$$

Complementary to (4.16), (4.17), we also obtain the boundary conditions for (Z_2, V_2) at $\pm L$, which are consistent with those for (U, V), due to (4.13).

$$P_s(-L)(Z_2, V_2)(-L) = P_s(-L)\hat{J}_{12}^{Z,V}, \quad P_u(L)(Z_2, V_2)(L) = -P_u(L)\hat{J}_{23}^{Z,V}. \tag{4.18}$$

Since (Z_2, V_2) are only defined near $S^{\pm}$, such conditions shall not be used in our paper. However, by comparing (4.17) and (4.18), we can see that the decompositions of $\hat{J}_{12}^{Z,V}$ and $\hat{J}_{23}^{Z,V}$ at $\pm L$ are correctly done.

Iteration method shall be used to solve the homogeneous part of systems (4.7) and (4.8) with jump conditions. Recall that $\mathbb{R} = I_1 \cup I_2 \cup I_3$. Since the correction of wave speed s appears in all the three regions, we start from I_2 first, which allows us to find s that will be used in I_1 and I_3.

First, in I_2, we set $Y_2(-L) = \hat{J}_{12}^Y$. From (4.7) we have $Y_2(\tau) = \hat{J}_{12}^Y,\ \tau \in I_2$. The system for (U_2, V_2) now becomes

$$U_2'(\tau) = V_2(\tau), \quad V_2'(\tau) = F_u U_2 + F_Y \hat{J}_{12}^Y - s_0 V_2 - s V_2^0. \tag{4.19}$$

Notice that the forcing term $F_Y \hat{J}_{12}^Y$ can be dealt with first, that is, we can solve a nonhomogeneous system with forcing term $F_Y \hat{J}_{12}^Y$, as in Sect. 4.3. In order to focus on the jump errors in this subsection, we will consider $F_Y \hat{J}_{12}^Y$ as an residual error to (4.19) and try to cancel it in the next iteration when residual errors will be treated again.

After dropping $F_Y \hat{J}_{12}^Y$, (or combining it with $g_2(\tau)$ in (4.7)), the system simplifies to

$$U_2'(\tau) = V_2(\tau), \quad V_2'(\tau) = F_u U_2 - s_0 V_2 - s V_2^0. \tag{4.20}$$

By Lemma 3.4, there exists a unique solution (U_2, V_2) that satisfies the boundary conditions (4.16) at $\tau = -L$ and L provided the parameter $s = \hat{s}$ is uniquely determined by Lemma 3.4. The following estimates are satisfied

$$|\hat{s}| + |U_2| + |V_2| \le C(|\phi_2^s| + |\phi_2^u|), \tag{4.21}$$

$$|P_u(-L)(U_2, V_2)^T(-L)| + |P_s(L)(U_2, V_2)^T(L)| \le Ce^{-\gamma L}(|\phi_2^s| + |\phi_2^u|). \tag{4.22}$$

Next, in I_1, using $s = \hat{s}$ obtained in the first step when looking for (U_2, V_2) in I_2, we can solve for (Z_1, V_1) from

$$Z_i'(\tau) = V_i(\tau), \quad V_i'(\tau) = F_U Z_i - s_0 V_i - \hat{s} V_i^0, \quad i = 1.$$

Notice again that the forcing term $\hat{s} V_1^0$ can be considered as a residual error term and dealt with later in the next iteration. To focus on jump errors, we will drop it from the above for now. Using $\phi_1^u = -P_u(-L)\hat{J}_{12}^{Z,V}$ from (4.17), the solution $(Z, V)^T(\tau) = \Phi(\tau, -L)\phi_1^u$ satisfies $|(Z, V)^T(\tau)| \le K|\phi_1^u|e^{\zeta(\tau+L)}$.

Now we look for Y_1 in (4.8). The given forcing term $\epsilon \hat{s} G(U_1^0, Y_1^0)/s_0^2$ can be treated as an residual error in the next iteration, so the system becomes

$$Y_1'(\tau) = \epsilon(G_Y - G_U F_U^{-1} F_Y)Y_1/s_0 + \epsilon G_U Z_1/s_0.$$

The system is weakly stable in τ with exponents being $-\epsilon\alpha$, $\alpha > 0$. Since the forcing term $Z_1(\tau) = O(|\phi_1^u|e^{\zeta(\tau+L)})$, the system has a unique bounded solution that satisfies

$$|Y_1(\tau)| \le C\epsilon|\phi_1^u|, \quad -\infty < \tau \le -L. \tag{4.23}$$

Finally, in I_3, the system for (Z_3, V_3) in (4.8) has an exponential dichotomy so a bounded solution (Z_3, V_3) can be determined first. Dropping $\hat{s} V_3^0$ and considering it as a residual error again, using the boundary condition from (4.17), we find that the solution satisfies $(Z_3, V_3)^T(\tau) = \Phi(\tau, L)\phi_3^s = O(|\phi_3^s|)$. Then we plug (Z_3, V_3) into the equation for Y_3.

$$Y_3(\tau)' = \epsilon(G_Y - G_U F_U^{-1} F_Y)Y_3/s_0 + \epsilon G_U Z_3/s_0 - \epsilon \hat{s} G(U_3^0, Y_3^0)/s_0^2.$$

Again the term $\epsilon \hat{s} G(U_3^0, Y_3^0)/s_0^2$ will be treated as a residual error in the next iteration and will be dropped for now. Observe that the linear homogeneous part of the equation is weakly exponentially stable, and there is a factor ϵ in the r.h.s. of the equation. With the boundary condition $Y_3(L) = Y_2(L) + \hat{J}_{23}^Y = \hat{J}_{12}^Y + \hat{J}_{23}^Y$, the system has a bounded solution $Y_3(\tau)$, $\tau \in [L, \infty)$, which satisfies

$$|Y_3(\tau)| \le C\left(|\hat{J}_{12}^Y + \hat{J}_{23}^Y| + |\phi_3^s|\right).$$

Part B: We first calculate the jump errors for solutions (U_i, V_i, Y_i) obtained in Part A, due to approximating the jump conditions by local boundary conditions. Denote the jump by $\hat{J} := \left(\hat{J}_{12}, \hat{J}_{23}\right)$. Even though the jump conditions are not exactly satisfied, the error $\delta\hat{J}$ can be much smaller compared to the prescribed $\hat{J}$. Part of the error is from (4.23) which causes an additional term $Y_1(-L)$ at the junction point $\tau = -L$. (The boundary value $Y_2(-L) = \hat{J}_{12}^Y$ has no error only if $Y_1(-L) = 0$.) The other part of the error is from (4.22) which shows the exponentially small influences of the junction points $-L$ to L, and from L to $-L$. Denote the original jump between I_1 to I_2 (or I_2 to I_3) by $\hat{J}_{12}^0$ (or $\hat{J}_{23}^0$). If ϵ is sufficiently small and $L = \epsilon^{-0.5}$ is sufficiently large, we have

$$|\delta\hat{J}_{12}| \le \frac{1}{2}|\hat{J}_{12}^0|, \quad |\delta\hat{J}_{23}| \le \frac{1}{2}|\hat{J}_{23}^0|.$$

Decompose $-\hat{J}_{12}$ and $-\hat{J}_{23}$ into $\phi^{u,s}(\pm L)$ as in (4.16) and (4.17) and using Part A again, we obtain a sequence of approximations $(U_2^{(k)}, V_2^{(k)}, Y_2^{(k)})$, $(Z_i^{(k)}, V_i^{(k)}, Y_i^{(k)},\ i = 1, 3)$, for $k \in \mathbb{Z}^+$. Each of them satisfies the linear homogeneous system, and the jumps $\delta\hat{J}_{12}^{(k)}$ and $\delta\hat{J}_{23}^{(k)}$ decay by a factor of $1/2$ as $k \to \infty$. Finally, the converging series

$$(U_2, V_2, Y_2) = \sum_{k=0}^{\infty}\left(U_2^{(k)}, V_2^{(k)}, Y_2^{(k)}\right), \quad (Z_i, V_i, Y_i) = \sum_{k=0}^{\infty}\left(Z_i^{(k)}, V_i^{(k)}, Y_i^{(k)}\right),\ i = 1, 3$$

is the exact solution that satisfies the jump condition. A simple proof using "approximate right inverse" can be found in the Appendix of [26]. For completeness, we outline the proof as follows.

Denote the linear operator which evaluates the size of jump by $\Delta[(\hat{U}, \hat{V}, \hat{Y})] = \hat{J}$. Then in Part A, we constructed the so called "approximate right inverse operator" $\mathcal{R} \approx \Delta^{-1}$. From part B, if ϵ is sufficiently small,

$$\|\hat{J} - \Delta[\mathcal{R}\hat{J}]\| \leq (1/2)|\hat{J}|.$$

Let the desired jump be $\hat{J}_0$, then by repeating Part A and B we create a sequence

$$\hat{J}_k - \Delta[\mathcal{R}\hat{J}_k] = \hat{J}_{k+1}, \quad k = 0, 1, 2, \ldots.$$

Take the sum of the above equations for $k \in \mathbb{Z}^+$, we have

$$\hat{J}_0 - \Delta\left[\left(\sum_{k=0}^{\infty} \mathcal{R}\hat{J}_k\right)\right] = 0.$$

This shows that $(\hat{U}, \hat{V}, \hat{Y}) = \sum_{k=0}^{\infty} \mathcal{R}\hat{J}_k$ is the solution with the prescribed jump $\hat{J}_0$.

However, when solving the system with jump conditions between I_1, I_2 and I_2, I_3, we dropped some forcing terms, if they were already known. By doing so we introduced new residual errors, and those errors are bounded by the jump errors. Now the jump errors are gone, we have to go back to Sect. 4.3 to eliminate such residual errors. By an iteration process that repeatedly using the procedures in Sects. 4.3 and 4.4, we obtain the correction terms that satisfy (4.7), (4.8), with jump conditions (4.9). This can be proved as follows. Notice that in each iteration, the new residual errors used in Part A of Sect. 4.4 are controlled by the jump errors in the previous step. Therefore if the jump errors are bounded by a geometric sequence that decays to zero, then the residual errors will also be bounded by a geometric sequence that decays to zero.

4.5 Existence of Heteroclinic Solutions for the Nonlinear System (4.3)

By converting $(Z_i, V_i, Y_i),\ i = 1, 3$ to (U_i, V_i, Y_i), we have obtained the solution to the linearized system which comes from (4.3) by dropping the nonlinear h.o.t.s,

$$\begin{aligned} &U_i'(\tau) = V_i(\tau) - r_i(\tau), \quad V_i' = F_u U_i + F_Y Y_i - s_0 V_i - s V_i^0 - p_i(\tau), \\ &Y_i'(\tau) = \epsilon(G_U U_i + G_Y Y_i)/s_0 - \epsilon s G(U_i^0, Y_i^0)/s_0^2 - \epsilon q_i(\tau), \quad i = 1, 2, 3. \end{aligned}$$

Moreover the solution satisfies the boundary boundary condition (4.9). Denote the solution by

$$(s, \{U_i, V_i, Y_i\}_{i=1}^3) = \mathcal{F}\left(\{r_i, p_i, q_i\}_{i=1}^3, J_{12}^{ap}, J_{23}^{ap}\right),$$

where $\mathcal{F}$ is the bounded solution map for the linear system with boundary conditions. Denote the h.o.t.s by (Q_i, M_i, N_i), we are led to the equation

$$(s, \{U_i, V_i, Y_i\}_{i=1}^3) = \mathcal{F}\left(\{r_i + Q_i, p_i + M_i, q_i + N_i\}_{i=1}^3, J_{12}^{ap}, J_{23}^{ap}\right). \tag{4.24}$$

The h.o.t.s are functions of $(s, \{U_i, V_i, Y_i\}_{i=1}^3)$ and satisfy

$$\|\{Q_i, M_i, N_i\}_{i=1}^3\| \leq C\left(\epsilon^{0.5}\left(\|\{U_i, V_i, Y_i\}_{i=1}^3\| + |s|\right) + \|\{|U_i|^2 + |V_i|^2 + |Y_i|^2\}_{i=1}^3\| + |s|^2\right). \tag{4.25}$$

We have used $\|\{U_i, V_i, Y_i\}_{i=1}^3\| = \sup_{i=1,2,3}\{|U_i| + |V_i| + |Y_i|\}$ for the norm of functions defined on $I_1 \cup I_2 \cup I_3$. If $\|\{Q_i, M_i, N_i\}_{i=1}^3\| \leq \delta$ with a small $\delta > 0$ and if ϵ is sufficiently small, then the Lipschitz numbers of the r.h.s. of (4.24) with respect to (s, U_i, V_i, Y_i) are bounded by $1/2$. Therefore, for such small $\epsilon, \delta > 0$, $\mathcal{F}$ is a contraction mapping on $(s, \{U_i, V_i, Y_i\}_{i=1}^3)$ and (4.24) has a unique solution $(s, \{U_i, V_i, Y_i\}_{i=1}^3)$.

5 The Chaotic Solutions Near the Heteroclinic Loop

In this section, we show the existence of symbolic dynamics near the heteroclinic loop obtained in Sect. 4 for $\epsilon \neq 0$. Our main result is stated in Theorem 5.2, and the idea is similar to Shilnikov's work on symbolic dynamics near a homoclinic orbit [36]. However, since we look for solutions near a loop, we will use the result from [24] which works on an infinite chain of heteroclinic orbits. Besides having a heteroclinic loop, we must show that the dominant eigenvalues at equilibrium points $P_\pm$ are complex, for $\epsilon > 0$ and small.

5.1 The Eigenvalue Problems

In the previous sections, we studied eigenvalues for the slow system in Y and eigenvalues for the fast system in (Z, V), when $\epsilon = 0$ and the fast-slow variables are decoupled. When $\epsilon = 0$, the two eigenvalues of the fast system (denoted by $\lambda_f(0)$) are real, and the two eigenvalues of the slow system (denoted by $\lambda_s(0)$) are conjugate complex. We now study the coupled system (5.1) where $\epsilon \neq 0$, and show that (5.1) still has two fast and two slow eigenvalues, denoted by $\lambda_f(\epsilon)$ and $\lambda_s(\epsilon)$, which are the perturbations of the singular eigenvalues $\lambda_f(0)$ and $\lambda_s(0)$ respectively.

For self-completeness, we first solve the eigenvalue problem (5.1) independently, without using results from the theory of singular perturbations. In Theorem 5.1, we will link results obtained in this subsection to the notions of fast and slow eigenvalues used in the previous sections. When $\epsilon \neq 0$, (2.2) is equivalent to (2.3), so we will only study the eigenvalues of $P_\pm$ for fast system (2.3) in the fast time τ.

Denote the eigenvalues and corresponding eigenvectors for (2.3) at $P_\pm$ by $\lambda(\epsilon)$ and $(U(\epsilon), V(\epsilon), Y(\epsilon))$. Let the coefficients F_U, F_Y, G_U, G_Y be evaluated at $(U_\pm, Y_\pm)$ respectively, and the wave speed $s(\epsilon)$ be as determined in the previous sections. Then the coupled eigenvalue problem is

$$\begin{aligned} &\lambda(\epsilon)U(\epsilon) = V(\epsilon), \quad \lambda(\epsilon)V = F_U U(\epsilon) + F_Y Y(\epsilon) - s(\epsilon)V, \\ &\lambda(\epsilon)Y(\epsilon) = \epsilon(G_U U(\epsilon) + G_Y Y(\epsilon))/s(\epsilon). \end{aligned} \tag{5.1}$$

Denote the 4×4 matrix in the r.h.s. by $A(\epsilon)$. The 4th order equation $\det(\lambda I - A(\epsilon)) = 0$ has 4 eigenvalues, denoted by $\lambda^{(k)}(\epsilon),\ 1 \leq k \leq 4$. When $\epsilon = 0$, (5.1) has two simple nonzero eigenvalues and two double zero eigenvalues.

$$\lambda^{(1)}(0) < 0 < \lambda^{(2)}(0), \quad \lambda^{(3)}(0) = \lambda^{(4)}(0) = 0. \tag{5.2}$$

The two nonzero eigenvalues are given by

$$\lambda(0)U_0 = V_0, \quad \lambda(0)V_0 = F_U U_0 + F_Y Y_0 - s_0 V_0, \quad Y_0 = 0,$$

which can be reduced to $\lambda^2(0) + s(0)\lambda(0) - F_U = 0$. From (H_1), $F_U > 0$. Which proves that the two nonzero eigenvalues satisfy $\lambda^{(1)}(0) < 0 < \lambda^{(2)}(0)$. Notice that $\lambda^{(1)}(0), \lambda^{(2)})(0)$ are exactly the fast eigenvalues as in (2.7)

If $\epsilon \neq 0$ and small, then some small terms involving ϵ will be added to the second order polynomial that determines $\lambda(0)$. By the Implicit Function Theorem, simple eigenvalues change continuously with the parameter ϵ in (5.1). Therefore for $\epsilon > 0$ and small, (5.1) still has two real, simple eigenvalues

$$\lambda_1^{(1)}(\epsilon) < 0 < \lambda_2^{(2)}(\epsilon).$$

For the two zero eigenvalues in (5.2), take a contour integral around small paths surrounding $\lambda = 0$ and use the Cauchy's argument principle on $\det(\lambda I - A(\epsilon))$, then from the Hurwitz's theorem, see [4] and p. 231 of [13], we find two near zero eigenvalues if $\epsilon \neq 0$ and small. Assume that the eigenvalues and eigenfunctions for $A(\epsilon)$ have the following expansions,

$$\lambda(\epsilon) = \epsilon\lambda_1 + O(\epsilon^2). \quad U(\epsilon) = U_0 + \epsilon U_1 + O(\epsilon^2), \quad \text{similar expansions for } (V(\epsilon), Y(\epsilon)).$$

Then the $O(1)$ expansions of (5.1) satisfy

$$0 = V_0, \quad 0 = F_U U_0 + F_Y Y_0,$$

which yields

$$U_0 = -F_u^{-1} F_Y Y_0. \tag{5.3}$$

The $O(\epsilon)$ expansions of (5.1) satisfy

$$\begin{aligned} &\lambda_1 U_0 = V_1, \quad \lambda_1 U_0 = F_U U_1 + F_Y Y_1 - s_0 V_1, \\ &\lambda_1 Y_0 = (G_U U_0 + G_Y Y_0)/s_0. \end{aligned} \tag{5.4}$$

Combine (5.4) with (5.3) , we have

$$\lambda_1 Y_0 = (1/s_0)(G_Y - G_U F_U^{-1} F_Y) Y_0.$$

From ($\mathbf{H_2}$), the above has two simple eigenvalues $\lambda_1^{\pm} = -\alpha \pm i\beta$ with $\alpha > 0$. If $\epsilon \neq 0$ and small, then some small terms involving ϵ will be added to the above equation that determines λ_1 and Y_0. Since the eigenvalues $\lambda_1^{\pm}$ are simple, by the Implicit Function Theorem, for $\epsilon > 0$ and small, the two slow eigenvalues satisfy

$$\lambda^{(3)}(\epsilon), \lambda^{(4)}(\epsilon) = \epsilon(-\alpha \pm i\beta) + O(\epsilon^2).$$

Finally we remark that the factor ϵ before $-\alpha \pm i\beta$ is due to the use of fast time τ. In slow time $t = \epsilon\tau$, the leading terms of $(\lambda^{(3)}(\epsilon)/\epsilon, \lambda^{(4)}(\epsilon)/\epsilon)$ are exactly the slow eigenvalues as in (H_2).

We summarize our results in the following theorem.

Proposition 5.1 *Assume that the conditions* ($\mathbf{H_1}$) $-$ ($\mathbf{H_2}$) *hold, then system* (5.1) *has two real eigenvalues* $\lambda_f(\epsilon)$ *of order* $O(1)$, *and two complex eigenvalues of order* $O(\epsilon)$. *Moreover, they are of the form*

$$\lambda_f^{\pm}(\epsilon) = \lambda_{f0}^{\pm} + O(\epsilon), \ \textit{with}\ \lambda_{f0}^{-} < 0 < \lambda_{f0}^{+}, \quad \lambda_s^{\pm}(\epsilon) = \epsilon(-\alpha \pm i\beta) + O(\epsilon^2), \textit{with}\ \alpha > 0.$$

5.2 The Dynamics Near the Heteroclinic Loop

The proof of our main result is based on Theorem 4.8 in [24]. We will use the slow time $t = \epsilon\tau$ in this section. Denote solutions of (2.2) in slow time t by $X(t, \epsilon) = (U, V, Y)(t, \epsilon)$. Let $X_1(t, \epsilon)$, $t \in \mathbb{R}$, be the heteroclinic solution connecting P_- to P_+ and $X_2(t, \epsilon)$, $t \in \mathbb{R}$, be the heteroclinic solution connecting P_+ to P_-. First we define a sequence of heteroclinic chain $\{q_i(t)\}_{i\in\mathbb{Z}}$. For any integer $k \in \mathbb{Z}$, let $q_i = X_1, \ p_i = P_-$ if $i = 2k+1$, and let $q_i = X_2, \ p_i = P^+$ if $i = 2k$. Such sequence of heteroclinic solutions $\{q_i(t)\}_{i\in\mathbb{Z}}$ shall be called a heteroclinic chain, which is associated to the sequence of equilibria $\{p_i\}_{i\in\mathbb{Z}}$ in the way that $\lim_{t\to-\infty} q_i(t) = p_i, \quad \lim_{t\to\infty} q_i(t) = p_{i+1}$.

Let Σ_i be a codimension one plane through $q_i(0)$ and orthogonal to $\dot{q}_i(0)$,

$$\Sigma_i := \{x| < \dot{q}_i(0), x - q_i(0) >= 0\}.$$

Let $x(t)$ be be a solution that lies near the heteroclinic chain and let the time spent by $x(t)$ between Σ_{i-1} to Σ_i be $2\omega_i$. We look for conditions on $\vec{\omega} = \{\omega_i\}_{i\in\mathbb{Z}}$ so that the corresponding $x(t)$ can exist. Using Lyapunov-Schmidt reduction, the existence of $x(t)$ is reduced to a system of bifurcation equations $G_i(\vec{\omega}) = 0$, $i \in \mathbb{Z}$ as follows.

We assume that the orbit of $x(t)$ is the union of those of $x_i(t)$, defined on $t \in [-\omega_i, \omega_{i+1}]$ and subject to the phase condition $x_i(0) \in \Sigma_i$. If $\{t_i\}$ is the sequence such that $x(t_i) \in \Sigma_i$, then $x_i(t) = x(t+t_i)$. Each $x_i(t)$ is near the heteroclinic segment $\Gamma_i = \{q_i(t), -\omega_i \le t \le \omega_{i+1}\}$.

Notice that the equilibrium points $p_i, \ i \in \mathbb{Z}$ are hyperbolic. Then the linearized system $\dot{x} = D_x f(q_i(t), 0)x$ has exponential dichotomies on $[-\omega_i, 0] \cup [0, \omega_{i+1}]$ if $\inf_i\{\omega_i\}$ is sufficiently large. A modified shadowing lemma for continuous systems [23] can be used to glue the end points of Γ_i and Γ_{i+1} together. Since a unified exponential dichotomy does not exists for $t \in \mathbb{R}$, to compensate this deficiency, $x_i(t)$ is allowed to have a gap at $t = 0$ along a specified direction Δ_i: $x_i(0+) - x_i(0-) = \xi_i\Delta_i$ where $\xi_i \in \mathbb{R}$ and $\Delta_i \in \mathbb{R}^4$ is a unit vector orthogonal to $TW^u(p_i) + TW^s(p_{i+1})$ at $q_i(0)$. In Theorem 3.1 of [24], it is shown that under some general conditions, there exists a unique piecewise smooth solution $x(t)$ with the specified direction of jumps $\xi_i = G_i(\vec{\omega})$.

As nonlinear systems can be approximated by linear variational systems, we assume that $x_i(t) = q_i(t) + z_i(t)$, $-\omega_i \le t \le \omega_{i+1}$, then z_i satisfies the linearized system

$$\begin{aligned} &\dot{z}_i(t) = A_i(t)z_i(t) + h_i(z_i(t), t), -\omega_i \le t \le \omega_{i+1}, \\ &z_{i-1}(\omega_i) - z_i(-\omega_i) = q_i(-\omega_i) - q_{i-1}(\omega_i), \end{aligned} \tag{5.5}$$

where $A_i(t) = D_x f(q_i(t))$ and $h_i(z,t) = f(q_i(t)+z) - f(q_i) - A_i(t)z = O(\|z\|^2)$. Applying Lemma 3.2 to system (5.5), with the phase condition $z_i(0) \perp \dot{q}_i(0)$, the jump at $t = 0$, along the direction of Δ_i written as $z_i(0-) - z_i(0+) = \xi_i\Delta_i$, satisfies,

$$\begin{aligned} \xi_i = &\int_{-\omega_i}^{\omega_{i+1}} < \psi_i(t), h_i(z_i(t), t) > ds \\ &+ < \psi_i(-\omega_i), z_i(-\omega_i) > - < \psi_i(\omega_{i+1}), z_i(\omega_{i+1}) > . \end{aligned} \tag{5.6}$$

Furthermore, due to the exponential decay rate of $\psi_i(s)$ as $s \to \pm\infty$, and $h_i(z_i(s), s) = O(\|z_i\|^2)$, one can show that

$$\int_{-\omega_i}^{\omega_{i+1}} < \psi_i(s), h_i(z_i(s), s) > ds = O(\|z_i\|^2),$$

is a small term, together with its derivatives, compared to $z_i(-\omega_i)$ and $z_i(\omega_{i+1})$, cf. [24]. Moreover, from Lemma 4.5 and Lemma 4.7 in [24], there exists $\delta > 0$ such that

$$\begin{aligned} &< \psi_i(-\omega_i), z_i(-\omega_i) > =< \psi_i(-\omega_i), q_{i-1}(\omega_i) - p_i > +o(e^{-2(\alpha+\delta)\omega_i}), \\ &< \psi_i(\omega_{i+1}), z_i(\omega_{i+1}) > =< \psi_i(\omega_{i+1}), q_{i+1}(-\omega_{i+1}) - p_i > +o(e^{-2(\alpha+\delta)\omega_{i+1}}). \end{aligned}$$

Thus

$$\xi_i =< \psi_i(-\omega_i), q_{i-1}(\omega_i) - p_i > - < \psi_i(\omega_{i+1}), q_{i+1}(-\omega_{i+1}) - p_i >$$
$$+ o(e^{-2(\alpha+\delta)\omega_i}) + o(e^{-2(\alpha+\delta)\omega_{i+1}}).$$

Here the higher order terms depend on the rates at which $q_i(t)$ approaches p_i and p_{i+1} as $t \to \mp\infty$. From the eigenvalues at p_i and p_{i+1}, as $t \to \infty$:

$$q_i(t) - p_{i+1} \sim C_1 e^{-\alpha t}(\cos\beta t, \sin\beta t)^T + C_2 e^{-\alpha t}(-\sin\beta t, \cos\beta t)^T,$$
$$\psi_i(-t) \sim D_1 e^{-\alpha t}(\cos\beta t, -\sin\beta t)^T + D_2 e^{-\alpha t}(\sin\beta t, \cos\beta t)^T,$$
$$|q_{i+1}(-t) - p_i| + |\psi_i(t)| \le C e^{-\lambda_f^+ \omega_{i+1}/\epsilon}, \quad t \to \infty.$$

Therefore,

$$\begin{aligned} < \psi_i(-\omega_i), q_{i-1}(\omega_i) - p_i >&\sim C e^{-2\alpha\omega_i}\sin(2\beta\omega_i + \theta), \\ < \psi_i(\omega_{i+1}), q_{i+1}(-\omega_{i+1}) - p_i >&\sim C e^{-2\lambda_f^+ \omega_{i+1}/\epsilon}. \end{aligned} \tag{5.7}$$

Assume that the $\vec{\omega}$ satisfies the following conditions:
($\mathbf{H_6}$) There exist $K > 1$ such that

$$\omega_i/K \le \omega_{i+1} \le K\omega_i.$$

Under the condition ($\mathbf{H_6}$), using (5.7), it becomes clear that there exists $\epsilon_0 > 0$, sufficiently small such that if $0 < \epsilon \le \epsilon_0$, then

$$\xi_i = C e^{-2\alpha\omega_i}\sin(2\beta\omega_i + \theta) + o(e^{-2\alpha\omega_i}).$$

The piecewise continuous solutions $x_i(t), i \in \mathbb{Z}$ that stay near the heteroclinic chain are uniquely determined by the sequence of times $\vec{\omega}$. To eliminate the jumps we must solve a system of bifurcation equations, which has the following asymptotic form,

$$G_i(\vec{\omega}) = C e^{-2\alpha\omega_j}\sin(2\beta\omega_j + \theta) + o(e^{-2\alpha\omega_i}) = 0, \quad i \in \mathbb{Z}. \tag{5.8}$$

We now prove the following results:

Theorem 5.2 (1) *For any $K > 1$, assume that $\vec{\omega}$ satisfies ($\mathbf{H_6}$), and there exists a sufficiently large $\hat{\omega} > 0$ such that $\inf_i\{\omega_i\} \ge \hat{\omega}$. Also assume that ω_i satisfies the asymptotic limit of (5.8), i.e. $\sin(2\beta\omega_i + \theta) = 0$ for all $i \in \mathbb{Z}$. Then*
(2) *there exists a unique traveling wave solution, for which the time spent to move from Σ_{i-1} to Σ_i is approximately $2\omega_i$. Moreover, the corresponding orbit x_i approaches q_i as $\hat{\omega} \to \infty$;*
(3) *there exists a countably infinite set of periodic traveling waves near the heteroclinic loop;*
(4) *there exists an uncountable set of aperiodic traveling waves near the heteroclinic loop.*

Proof Proof of (1): If ω_i satisfies $\sin(2\beta\omega + \theta) = 0$, then it is a simple zero. Using $\{\omega_i^{(0)}\} = \{\omega_i\}$ as the initial approximation, by an iteration method, we can obtain a sequence of approximations $\{\omega_i^{(k)}\}$ from (5.8). From the contraction mapping principle, the limit

$$\{\omega_i^{(\infty)}\} = \lim_{k\to\infty}\{\omega_i^{(k)}\}$$

is an exact solution of $G_j(\{\omega_i^{(\infty)}\}) = 0, \;\; j \in \mathbb{Z}$. Therefore the corresponding $x_i(t), i \in \mathbb{Z}$ has no jump at $t = 0$, and $q_i + z_i$ is an exact traveling wave solution for (1.2).

Proof of (2) and (3): From zeros of the equation $\sin(2\beta\omega + \theta) = 0$, choose a sequence $\{\omega_i^{(0)}\}_{i\in\mathbb{Z}}$ that is periodic (or aperiodic) in i, and satisfies all the conditions in Part (1). Using the iteration method as in Part (1) to find the limiting sequence $\{\omega_i^{(\infty)}\}$, the latter must also be periodic (or aperiodic) in i. Therefore the corresponding x_i is periodic (or aperiodic) in i. □

Remark 5.3 We have proved the existence of periodic and chaotic traveling wave solutions to (1.2), which is a PDE approximation to the coupled ODE system (1.1). For the PDE system, Perez-Munuzuri, Perez-Villar and Chua showed that the traveling wave solutions may exist, or may fail in numerical simulations [33]. We believe that the stability, or structural stability of such waves may play a role there, which still remains to be clarified by further studies.

The existence of periodic or chaotic traveling waves for (1.1) has not been proved theoretically, and it does not follow from the results of Theorem 5.2 directly. However, since systems (1.1) and (1.2) are closely related, we believe that the results obtained in this paper, and the methods used here can help to prove the existence of periodic and chaotic solutions of (1.1).

Let $w = (u, y, z)$. As a lattice differential equation, traveling waves to (1.1) are of the form $w_k(t) = \Phi(k - st)$ for some wave profile function Φ. By discretizing a periodic or chaotic traveling wave solution of (1.2), we obtain a periodic or chaotic sequence $\{w_k(t)\}_{k=-\infty}^{\infty}$, which is approximately a traveling wave solution to (1.1). If the error terms can be dealt with just like the residual and jump errors in this paper, then we will have an exact periodic or chaotic traveling wave solution to (1.1). Notice that the theory of exponential dichotomies and shadowing lemma for spatially discretized systems like (1.1) have been developed a long time ago, cf. [15,28,35]. They may prove useful when working on such systems, just like the continuous counterparts in this paper on (1.2).

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