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A SHADOWING LEMMA WITH APPLICATIONS TO SEMILINEAR PARABOLIC EQUATIONS*

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Abstract. The property of hyperbolic sets that is embodied in the Shadowing Lemma is of great importance in the theory of dynamical systems. In this paper a new proof of the lemma is presented, which applies not only to the usual case of a diffeomorphism in finite-dimensional space but also to a sequence of possibly noninvertible maps in a Banach space. The approach is via Newton's method, the main step being the verification that a certain linear operator is invertible. At the end of the paper an application to parabolic evolution equations is given.

Key words. hyperbolic, exponential dichotomy, Shadowing Lemma, pseudo-orbit, Newton's method, parabolic evolution equation

AMS(MOS) subject classifications. 34C35, 35K22, 58F15

1. Introduction. Let f be a diffeomorphism from $\mathbf{R}^k$ into itself. Given an initial point, the iterates of f and its inverse generate a sequence of points $x_{n+1}=f(x_n)$. Then $\{x_n\}_{n\in\mathbf{Z}}$ is called the *orbit* through x_0. A sequence of points $\{y_n\}_{n\in\mathbf{Z}}$ is called a δ*-pseudo-orbit* of f if $|y_{n+1}-f(y_n)|\leqq\delta$ for all n, where $\delta>0$ is a constant. The Shadowing Lemma says that if $S\subset\mathbf{R}^k$ is a *hyperbolic* set for f then for every $\varepsilon>0$ there exists $\delta>0$ such that every δ-pseudo-orbit $\{y_n\}_{n\in\mathbf{Z}}$ in S is ε*-shadowed* by an orbit $\{x_n\}_{n\in\mathbf{Z}}$ of f, that is, $|x_n-y_n|\leqq\varepsilon$ for all n. This lemma was first stated and proved in Anosov [1] and Bowen [3] under slightly different conditions. Several different proofs were given later in Conley [4], Robinson [15], Guckenheimer, Moser, and Newhouse [6], Ekeland [5], Lanford [10], Shub [16], and Palmer [14].

A δ-pseudo-orbit can be thought of as an orbit generated numerically by a computer. If this orbit is in or near a hyperbolic set for f, the Shadowing Lemma implies that an orbit for f can be found near such a "noisy" numerical orbit for an arbitrarily long time. In fact, Hammel, Yorke, and Grebogi [7] showed how we may apply the ideas of the Shadowing Lemma to prove that "noisy" numerical orbits are actually near real orbits for a finite but fixed time even in the nonhyperbolic case. In [12], Palmer showed that the complicated behavior of the orbits of a diffeomorphism near a transversal homoclinic point can be explained by the sole use of the Shadowing Lemma. This has been generalized by Blazquez [2] to infinite-dimensional systems generated by parabolic evolution equations.

When considered abstractly, the problem of finding a shadowing orbit can be approached by Newton's method for finding zeros of functions. To see this, let X be the Banach space of all bounded $\mathbf{R}^k$-valued sequences $\mathbf{x}=\{x_n\}_{n\in\mathbf{Z}}$ with the usual sup norm and define $\mathcal{F}:X\to X$ by $(\mathcal{F}(\mathbf{x}))_n=x_n-f(x_{n-1})$, where $(\mathcal{F}(\mathbf{x}))_n$ denotes the nth element of the sequence $\mathcal{F}(\mathbf{x})\in X$. Thus $\mathbf{x}=\{x_n\}$ is an orbit of f if and only if $\mathcal{F}(\mathbf{x})=0$ and $\mathbf{y}=\{y_n\}$ is a δ-pseudo-orbit if and only if $\|\mathcal{F}(\mathbf{y})\|\leqq\delta$. The Shadowing Lemma says that if f is hyperbolic and there exists a good approximate (δ sufficiently small) solution $\mathbf{y}$ of the equation $\mathcal{F}=0$, then there exists a solution $\mathbf{x}$ near $\mathbf{y}$.

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For an analogue in the continuous time case, consider the abstract ordinary differential equation in a Banach space X:

$$\dot{x} + Ax = f(x, t), \tag{1.1}$$

where A is linear and f nonlinear. A typical example of (1.1) is the nonlinear heat equation. Suppose the real line $\mathbf{R}$ is partitioned as $\mathbf{R} = \bigcup_{n\in\mathbf{Z}} [\tau_{n-1}, \tau_n]$ and each $x_n(t): [\tau_{n-1}, \tau_n] \to X$ is an approximate solution, that is,

$$\dot{x}_n(t) + Ax_n(t) = f(x_n(t), t) + h_n(t),$$

$$x_n(\tau_n) - x_{n+1}(\tau_n) = g_n,$$

where $h_n(t)$ and g_n are the error terms. The problem is to find an analogue of the hyperbolicity condition to guarantee that there exists a solution of (1.1) which, for each n in $\mathbf{Z}$, is close to $x_n(t)$ in the interval $[\tau_{n-1}, \tau_n]$.

In this paper, we will show that a Shadowing Lemma may be derived using Newton's method and that the lemma is applicable to the just-mentioned situation (see § 6). Because of the applications, we will work with C^1 maps in Banach spaces that are not necessarily diffeomorphisms. In fact, we consider a sequence $\{f_n\}_{n\in\mathbf{Z}}$ of mappings rather than a single mapping f. We set up the problem abstractly in a Banach space of sequences and apply a variant of Newton's method. The key tool is Lemma 3.2 in which we show that a certain linear operator is invertible (in the finite-dimensional case, this can also be proved by the perturbation theorem for exponential dichotomies in Palmer [13]). Lin has proved a similar lemma in [11], where the application is to a problem in ordinary differential equations. Here Lemma 3.2 is proved by an iteration method, which means, we believe, that it could be implemented on the computer. Newhouse [6] also used an iteration process but it is rather more involved in that at each step it uses the intersection of the stable and unstable manifolds.

Finally we should mention that as this paper was being written Walther sent us the preprint [18] where he proves the Shadowing Lemma for noninvertible maps. Stoffer [17] has also proved such a theorem. Both of these authors use the methods of Kirchgraber [9], which are quite different from ours.

2. Definition and statement of the Shadowing Lemma. What we are going to prove is a "nonautonomous" Shadowing Lemma for a sequence $f_n: X_n \to X_{n+1} (n \in \mathbf{Z})$ of C^1 maps. Here X_n is a Banach space with norm $|\cdot|_{X_n}$ (or simply $|\cdot|$ if no confusion should arise). Assume $S_n \subset X_n$, $n \in \mathbf{Z}$, is *invariant* under f_n in the sense that $f_n(S_n) \subset S_{n+1}$. Also we assume that $f_n(x)$, $Df_n(x)$ are bounded and continuous in a closed Δ-neighborhood O_n of S_n uniformly in $x \in O_n$ and $n \in \mathbf{Z}$.

We want to define what is meant by saying $\{S_n\}_{n\in\mathbf{Z}}$ is *hyperbolic.* First there is a *splitting* into closed subspaces

$$X_n = E_n^s(x) \oplus E_n^u(x) \tag{2.1}$$

for x in S_n. We require this splitting to be *invariant* in the sense that

$$Df_n(x)E_n^s(x) \subset E_{n+1}^s(f_n(x)),\ Df_n(x)E_n^u(x) \subset E_{n+1}^u(f_n(x))$$

for all x in S_n, and also *continuous*; that is, if $\mathbf{P}_n(x)$ is the projection with range $E_n^s(x)$ and nullspace $E_n^u(x)$, $\mathbf{P}_n(x)$ is continuous in the operator norm, uniformly with respect to $x \in S_n$ and $n \in \mathbf{Z}$. In terms of $\mathbf{P}_n(x)$, the invariance of the splitting is equivalent to

$$Df_n(x)\mathbf{P}_n(x) = \mathbf{P}_{n+1}(f_n(x))Df_n(x) \tag{2.2}$$

for all x in S_n. We also assume that $Df_n(x): E_n^u(x) \to E_{n+1}^u(f_n(x))$ is an isomorphism with a (bounded) inverse $(Df_n(x))^{-1}: E_{n+1}^u(f_n(x)) \to E_n^u(x)$.

Second, we require that there exist constants $K \geqq 1$, $0 \leqq \lambda < 1$ such that for any finite sequence $x_m, x_{m+1} = f_m(x_m), x_{m+2} = f_{m+1}(x_{m+1}), \cdots, x_n = f_{n-1}(x_{n-1})$ with $x_m \in S_m$ and any integers $n \geqq m$,

$$(2.3)\qquad \begin{aligned} &|Df_n(x_n)Df_{n-1}(x_{n-1}) \cdots Df_m(x_m)\mathbf{P}_m(x_m)| \leqq K\lambda^{n-m+1}, \\ &|Df_m(x_m)^{-1}Df_{m+1}(x_{m+1})^{-1} \cdots Df_n(x_n)^{-1}(I - \mathbf{P}_{n+1}(x_{n+1}))| \leqq K\lambda^{n-m+1}. \end{aligned}$$

Also we assume that $|\mathbf{P}_n(x)| \leqq K$, $|I - \mathbf{P}_n(x)| \leqq K$ for $x \in S_n$, $n \in \mathbf{Z}$.

An *orbit* for $\{f_n\}_{n \in \mathbf{Z}}$ is a sequence $\{x_n\}_{n \in \mathbf{Z}}$ with $x_n \in X_n$ and $x_{n+1} = f_n(x_n)$ for all $n \in \mathbf{Z}$. If $\delta > 0$ is a constant, a sequence $\{y_n\}_{n \in \mathbf{Z}}$ with $y_n \in X_n$ is said to be a *δ-pseudo-orbit* for $\{f_n\}$ if

$$|f_n(y_n) - y_{n+1}| \leqq \delta$$

for all integers n. A sequence $\{x_n\}_{n \in \mathbf{Z}}$ with $x_n \in X_n$ is said to *ε-shadow* $\{y_n\}_{n \in \mathbf{Z}}$, $y_n \in X_n$, if

$$|x_n - y_n| \leqq \varepsilon$$

for all $n \in \mathbf{Z}$.

THE SHADOWING LEMMA. *Let $\{X_n\}$, $\{f_n\}$, $\{S_n\}$, $n \in \mathbf{Z}$, be defined as above and satisfy all the properties listed above, that is,*

(i) *S_n is invariant under f_n;*

(ii) *There is a closed Δ-neighborhood O_n of S_n such that $f_n(x)$ and $Df_n(x)$ are bounded and continuous uniformly with respect to x in O_n and n in $\mathbf{Z}$;*

(iii) *$\{S_n\}_{n \in \mathbf{Z}}$ is hyperbolic.*

Then there exists $\varepsilon_0 > 0$ with the property that if $0 < \varepsilon \leqq \varepsilon_0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $\{y_n\}$, $y_n \in S_n$, is a δ-pseudo-orbit for $\{f_n\}$ then there is a unique orbit $\{x_n\}$ which ε-shadows $\{y_n\}$.

To prove the Shadowing Lemma, we will use some facts about linear difference equations and a variant of Newton's method for solving nonlinear equations.

3. Facts about linear difference equations. For each integer n let $A_n : X_n \to X_{n+1}$ be a bounded linear mapping. Denote by $\Phi(n, m)(n \geqq m)$ the *transition matrix* for the *linear difference equation*

$$(3.1)\qquad x_n = A_{n-1}x_{n-1}, \qquad x_n \in X_n, \quad n \in \mathbf{Z},$$

that is,

$$\Phi(n, m) = \begin{cases} A_{n-1}A_{n-2} \cdots A_m, & n > m, \\ I, & n = m. \end{cases}$$

Equation (3.1) is said to have an *exponential dichotomy* if there is a projection valued function $P_n : X_n \to X_n$ and constants $K \geqq 1$, $0 \leqq \lambda < 1$ such that

$$(3.2)\qquad \Phi(n, m)P_m = P_n\Phi(n, m) \quad \text{for } n \geqq m,$$

$$(3.3)\qquad |\Phi(n, m)P_m| \leqq K\lambda^{n-m} \quad \text{for } n \geqq m.$$

Moreover, it is required that $\Phi(n, m) : \mathcal{N}(P_m) \to \mathcal{N}(P_n)$ ($\mathcal{N}$ denotes nullspace) be an isomorphism. Then for $n \geqq m$ we define $\Phi(m, n) : \mathcal{N}(P_n) \to \mathcal{N}(P_m)$ as the inverse of $\Phi(n, m) : \mathcal{N}(P_m) \to \mathcal{N}(P_n)$ and require that

$$(3.4)\qquad |\Phi(m, n)(I - P_m)| \leqq K\lambda^{n-m} \quad \text{for } n \geqq m.$$

It is clear from the definition of hyperbolicity that the following lemma holds.

LEMMA 3.1. *Let $\{S_n\}_{n \in \mathbf{Z}}$ be a hyperbolic set for a sequence $f_n : X_n \to X_{n+1}$ of C^1 mappings. Then if $\{x_n\}$ is an orbit of $\{f_n\}$ with $x_n \in S_n$ for all n, the linear difference equation*

$$u_n = Df(x_{n-1})u_{n-1}$$

has an exponential dichotomy with projections $P_n = \mathbf{P}(x_n)$ *and constants* K, λ, *the projections* $\mathbf{P}(x)$ *and the constants* K, λ *being those defining the hyperbolicity of* $\{S_n\}$.

We denote by ΠX_n the Banach space of bounded sequences $\mathbf{x} = \{x_n\}_{n\in\mathbf{Z}}$, $x_n \in X_n$, with norm

$$\|\mathbf{x}\| = \|\{x_n\}\| = \sup_{n\in\mathbf{Z}} |x_n|_{X_n}.$$

If $\sup_{n\in\mathbf{Z}} |A_n| < \infty$, we can associate with the linear difference equation (3.1) the linear operator $L: \Pi X_n \to \Pi X_n$ defined by

$$(L\mathbf{x})_n = x_n - A_{n-1}x_{n-1}.$$

It turns out that if (3.1) has an exponential dichotomy then L is invertible. Now in the proof of the Shadowing Lemma we are confronted with a linear difference equation for which the existence of an exponential dichotomy is not obvious. For this reason we need the following lemma.

LEMMA 3.2. *Assume* $\sup_{n\in\mathbf{Z}} |A_n| < \infty$. *For each* $n \in \mathbf{Z}$ *let* Q_n *be a projection such that* $|Q_n| \leqq K$, $|I - Q_n| \leqq K$ *and* $|Q_{n+1}A_n(I-Q_n)| \leqq \delta$, $|(I - Q_{n+1})A_nQ_n| \leqq \delta$. *Suppose also that for all* $n \in \mathbf{Z}$ $|A_nQ_n| \leqq \lambda$ *and that* $(I - Q_{n+1})A_n : \mathcal{N}(Q_n) \to \mathcal{N}(Q_{n+1})$ *has an inverse* B_n *with* $|B_n(I - Q_{n+1})| \leqq \lambda$. *Then if* $8K\lambda \leqq 1$, $8\delta \leqq 1$ *the operator* $L: \Pi X_n \to \Pi X_n$ *defined by* $(L\mathbf{x})_n = x_n - A_{n-1}x_{n-1}$ *is invertible with* $\|L^{-1}\| \leqq 2K + 1$.

Proof. First we show L is onto. To do this we define the linear mapping $S: \Pi X_n \to \Pi X_n$ by $(S\mathbf{h})_n = Q_nh_n - B_n(I - Q_{n+1})h_{n+1}$. Then S is bounded with $\|S\| \leqq K + \lambda$ and for all n

$$\begin{aligned}
|(LS\mathbf{h})_n - h_n| &= |Q_nh_n - B_n(I-Q_{n+1})h_{n+1} - A_{n-1}\{Q_{n-1}h_{n-1} - B_{n-1}(I-Q_n)h_n\} - h_n| \\
&= |-B_n(I-Q_{n+1})h_{n+1} - A_{n-1}Q_{n-1}h_{n-1} + Q_nA_{n-1}B_{n-1}(I-Q_n)h_n| \\
&\qquad\qquad \text{since } (I-Q_n)A_{n-1}B_{n-1}(I-Q_n) = I - Q_n \\
&= |-B_n(I-Q_{n+1})h_{n+1} - A_{n-1}Q_{n-1}h_{n-1} + Q_nA_{n-1}(I-Q_{n-1})B_{n-1}(I-Q_n)h_n| \\
&\leqq |B_n(I-Q_{n+1})|\,|h_{n+1}| + |A_{n-1}Q_{n-1}|\,|h_{n-1}| \\
&\quad + |Q_nA_{n-1}(I-Q_{n-1})|\,|B_{n-1}(I-Q_n)|\,|h_n| \\
&\leqq \lambda(2+\delta)\|\mathbf{h}\| \\
&\leqq \tfrac{1}{2}\|\mathbf{h}\|.
\end{aligned}$$

Hence $\|LS - I\| \leqq \frac{1}{2}$ and so LS has an inverse T with $\|T\| \leqq (1 - \|LS - I\|)^{-1} \leqq 2$. Then $ST = L_R^{-1}$ is a right inverse of L with

$$\|L_R^{-1}\| \leqq \|S\|\,\|T\| \leqq 2(K+\lambda) \leqq 2K + 1.$$

All that remains is to show that L is one-to-one. First note that for all $x \in X_n$,

$$\begin{aligned}
|(I-Q_n)x| &= |B_n(I-Q_{n+1})A_n(I-Q_n)x| \\
&\leqq \lambda|(I-Q_{n+1})A_n(I-Q_n)x|
\end{aligned}$$

so that

$$|(I-Q_{n+1})A_n(I-Q_n)x| \geqq \lambda^{-1}|(I-Q_n)x|. \tag{3.5}$$

Suppose L is not one-to-one. Then there exists a nonzero bounded sequence $\{x_n\}$ in ΠX_n such that $x_n = A_{n-1}x_{n-1}$ for all n. Suppose that for some n

$$|(I-Q_n)x_n| > |Q_n x_n|. \tag{3.6}$$

Then, using (3.5),

$$\begin{aligned}
|(I-Q_{n+1})x_{n+1}| - |Q_{n+1}x_{n+1}| &= |(I-Q_{n+1})A_n x_n| - |Q_{n+1}A_n x_n| \\
&\geqq |(I-Q_{n+1})A_n(I-Q_n)x_n| - |(I-Q_{n+1})A_n Q_n x_n| \\
&\quad - |Q_{n+1}A_n Q_n x_n| - |Q_{n+1}A_n(I-Q_n)x_n| \\
&\geqq \lambda^{-1}|(I-Q_n)x_n| - \delta|x_n| - K\lambda|x_n| - \delta|x_n| \\
&\geqq (\lambda^{-1} - 4\delta - 2K\lambda)|(I-Q_n)x_n| \quad \text{since } |x_n| \leqq 2|(I-Q_n)x_n| \\
&\geqq 7|(I-Q_n)x_n|.
\end{aligned}$$

This implies that $|(I-Q_{n+1})x_{n+1}| > |Q_{n+1}x_{n+1}|$ and that $|(I-Q_{n+1})x_{n+1}| \geqq 7|(I-Q_n)x_n|$. So if (3.6) holds for some $n=m$, it holds for all $n \geqq m$ and

$$\begin{aligned}
|x_n| \geqq K^{-1}|(I-Q_n)x_n| &\geqq K^{-1}7^{n-m}|(I-Q_m)x_m| \\
&\geqq \tfrac{1}{2}K^{-1}7^{n-m}|x_m|.
\end{aligned}$$

Thus $|x_n| \to \infty$ as $n \to \infty$, contradicting the boundedness of $\{x_n\}$.

Hence it must be that $|Q_n x_n| \geqq |(I-Q_n)x_n|$ *for all* n and then

$$\begin{aligned}
|Q_{n+1}x_{n+1}| &= |Q_{n+1}A_n x_n| \\
&\leqq |Q_{n+1}A_n Q_n x_n| + |Q_{n+1}A_n(I-Q_n)x_n| \\
&\leqq (K\lambda + \delta)|x_n| \\
&\leqq 2(K\lambda + \delta)|Q_n x_n| \\
&\leqq \tfrac{1}{2}|Q_n x_n|.
\end{aligned}$$

Now there exists some m such that $Q_m x_m \neq 0$. Then for all $n \leqq m$, $|Q_n x_n| \geqq 2^{-(n-m)}|Q_m x_m| \to \infty$ as $n \to -\infty$. Again this contradicts the boundedness of $\{x_n\}$. So L must be one-to-one.

4. Newton's method for solving nonlinear equations. In this section we prove the following variant of Newton's method for solving nonlinear equations.

PROPOSITION 4.1. *Let X be a Banach space, $U \subset X$ an open subset and $\mathcal{F}: U \to X$ a C^1 mapping. Let y be a point in U such that $D\mathcal{F}(y)^{-1}$ exists and let $\varepsilon_0 > 0$ be chosen so that*

$$\|D\mathcal{F}(x) - D\mathcal{F}(y)\| \leqq (2\|D\mathcal{F}(y)^{-1}\|)^{-1} \tag{4.1}$$

for $\|x-y\| \leqq \varepsilon_0$. Then if $0 < \varepsilon \leqq \varepsilon_0$ and

$$\|\mathcal{F}(y)\| \leqq \varepsilon(2\|D\mathcal{F}(y)^{-1}\|)^{-1}, \tag{4.2}$$

the equation

$$\mathcal{F}(x) = 0 \tag{4.3}$$

has a unique solution x such that $\|x-y\| \leqq \varepsilon$.

Proof. We write

$$\mathcal{F}(x)=\mathcal{F}(y)+D\mathcal{F}(y)(x-y)+\eta(x).$$

When $\|x_1-y\|$, $\|x_2-y\|\leqq\varepsilon_0$,

$$\begin{aligned}\|\eta(x_1)-\eta(x_2)\|&=\|\mathcal{F}(x_1)-\mathcal{F}(x_2)-D\mathcal{F}(y)(x_1-x_2)\|\\&\leqq\left\|\int_0^1 D\mathcal{F}(x_2+\theta(x_1-x_2))-D\mathcal{F}(y)\,d\theta\right\|\cdot\|x_1-x_2\|\\&\leqq(2\|D\mathcal{F}(y)^{-1}\|)^{-1}\|x_1-x_2\|,\end{aligned}\tag{4.4}$$

using (4.1).

We can rewrite (4.3) as

$$x=y-D\mathcal{F}(y)^{-1}\{\mathcal{F}(y)+\eta(x)\}:=T(x).$$

For $0<\varepsilon\leqq\varepsilon_0$, we define $B_\varepsilon=\{x\in X\colon \|x-y\|\leqq\varepsilon\}$ and show that T is a contraction on B_ε. The proposition will then follow immediately from the contraction mapping principle.

Note first if $x\in B_\varepsilon$ then

$$\begin{aligned}\|T(x)-y\|&=\|D\mathcal{F}(y)^{-1}\{\mathcal{F}(y)+\eta(x)\}\|\\&\leqq\|D\mathcal{F}(y)^{-1}\|\{\varepsilon(2\|D\mathcal{F}(y)^{-1}\|)^{-1}+(2\|D\mathcal{F}(y)^{-1}\|)^{-1}\|x-y\|\}\\&=\varepsilon/2+\|x-y\|/2\\&\leqq\varepsilon/2+\varepsilon/2=\varepsilon,\end{aligned}$$

where we have used (4.2) and (4.4) with $x_1=x$, $x_2=y$. Hence T maps B_ε into itself. Moreover if $x_1, x_2\in B_\varepsilon$ then, using (4.4),

$$\begin{aligned}\|T(x_1)-T(x_2)\|&=\|D\mathcal{F}(y)^{-1}\{\eta(x_1)-\eta(x_2)\}\|\\&\leqq\|D\mathcal{F}(y)^{-1}\|\cdot(2\|D\mathcal{F}(y)^{-1}\|)^{-1}\|x_1-x_2\|\\&=1/2\|x_1-x_2\|.\end{aligned}$$

Thus T is indeed a contraction on B_ε and the proof is completed.

5. Proof of the Shadowing Lemma.

We need three lemmas for the proof.

LEMMA 5.1. *Let X be a Banach space and let $P, Q: X\to X$ be projections such that $|P|$, $|Q|\leqq K$. Then if $|P-Q|<1/2K$, the operator $J=PQ+(I-P)(I-Q)$ is invertible with $|J^{-1}|\leqq(1-2K|P-Q|)^{-1}$. Moreover, $J(\mathcal{R}(Q))=\mathcal{R}(P)$, $J(\mathcal{N}(Q))=\mathcal{N}(P)$.*

Proof.

$$\begin{aligned}|J-I|&=|J-P^2-(I-P)^2|\\&=|P(Q-P)+(I-P)(P-Q)|\leqq 2K|P-Q|<1.\end{aligned}$$

So J is invertible with $|J^{-1}|\leqq(1-|J-I|)^{-1}\leqq(1-2K|P-Q|)^{-1}$. Clearly $J(\mathcal{R}(Q))\subset\mathcal{R}(P)$, $J(\mathcal{N}(Q))\subset\mathcal{N}(P)$ and equality follows from the invertibility of J. So the proof of the lemma is complete.

Now by assumption $|Df_n(x)|$ is bounded in a closed Δ-neighborhood O_n of S_n, uniformly in $x\in O_n$ and $n\in\mathbf{Z}$. Let this bound be M. Then

$$|f_n(x)-f_n(y)|\leqq M|x-y|$$

for $x\in S_n$, $y\in X_n$, and $|x-y|\leqq\Delta$. This fact is used in the following two lemmas, which make precise a statement of Guckenheimer, Moser, and Newhouse [6] that in the Shadowing Lemma it is enough to shadow a δ-pseudo-orbit for the sequence of mappings $\{f_{nk+k-1}\circ\cdots\circ f_{nk+1}\circ f_{nk}\}_{n\in\mathbf{Z}}$.

LEMMA 5.2. *If $\{y_n\}_{n\in\mathbf{Z}}$ is a δ-pseudo-orbit for $\{f_n\}$ with $y_n \in S_n$ for all n, then $\{y_{nk}\}_{n\in\mathbf{Z}}$ is a $\delta(1+M+\cdots+M^{k-1})$-pseudo-orbit for $\{f_{nk+k-1}\circ\cdots\circ f_{nk+1}\circ f_{nk}\}_{n\in\mathbf{Z}}$.*

Proof. We prove by induction that

$$|y_{nk+i}-(f_{nk+i-1}\circ\cdots\circ f_{nk+1}\circ f_{nk})(y_{nk})|\leqq \delta(1+M+\cdots+M^{i-1})$$

for $1\leqq i\leqq k$. Since $\{y_n\}$ is a δ-pseudo-orbit, it certainly holds for $i=1$. Assuming it for $i\geqq 1$, we prove it for $i+1$ as follows:

$$\begin{aligned}
&|y_{nk+i+1}-(f_{nk+i}\circ\cdots\circ f_{nk+1}\circ f_{nk})(y_{nk})|\\
&\quad\leqq |y_{nk+i+1}-f_{nk+i}(y_{nk+i})|+|f_{nk+i}(y_{nk+i})-f_{nk+i}((f_{nk+i-1}\circ\cdots\circ f_{nk})(y_{nk}))|\\
&\quad\leqq \delta+M|y_{nk+i}-(f_{nk+i-1}\circ\cdots\circ f_{nk})(y_{nk})|\\
&\quad\leqq \delta+M\delta(1+M+\cdots+M^{i-1})=\delta(1+M+\cdots+M^{i}).
\end{aligned}$$

LEMMA 5.3. *Let $\{y_n\}$ be a δ-pseudo-orbit for $\{f_n\}$ with $y_n\in S_n$, and let $\{x_n\}$ be an orbit of $\{f_n\}$ such that $\{x_{nk}\}$ ε-shadows $\{y_{nk}\}$, $k\geqq 1$ being fixed. Set $\varepsilon_1=\max\{\varepsilon,\delta\}$. Then if $\varepsilon_1(1+M+\cdots+M^k)\leqq\Delta$, $\{x_n\}$ $\varepsilon_1(1+M+\cdots+M^k)$-shadows $\{y_n\}$.*

Proof. Note first that

$$\begin{aligned}
|y_{nk+1}-f_{nk}(x_{nk})|&\leqq |y_{nk+1}-f_{nk}(y_{nk})|+|f_{nk}(y_{nk})-f_{nk}(x_{nk})|\\
&\leqq \delta+M\varepsilon\leqq \varepsilon_1(1+M).
\end{aligned}$$

Then we show by induction, as in the proof of Lemma 5.2, that

$$|y_{nk+i}-(f_{nk+i-1}\circ\cdots\circ f_{nk})(x_{nk})|\leqq \varepsilon_1(1+M+\cdots+M^{i})$$

for $1\leqq i\leqq k$.

Proof of the Shadowing Lemma. Let k be a positive integer such that $16K^3\lambda^k\leqq 1$. We first prove the Shadowing Lemma for the sequence of maps $F_n=f_{nk+k-1}\circ\cdots\circ f_{nk}$ and hyperbolic sets $\{S_{nk}\}$. If we define $\omega(\eta)=\sup\{|DF_n(y)-DF_n(x)|: x\in S_{nk}, y\in X_{nk}, |y-x|\leqq\eta, n\in\mathbf{Z}\}$, it follows from the uniform continuity and boundedness of Df_n that $\omega(\eta)\to 0$ as $\eta\to 0$. Then we choose $\varepsilon_0>0$ so that $\varepsilon_0<\Delta$ and $\omega(\varepsilon_0)\leqq 1/(4K+2)$. Also if we define $\bar\omega(\eta)=\sup\{\|\mathbf{P}_n(y)-\mathbf{P}_n(x)\|: x\in S_n, y\in X_n, |y-x|\leqq\eta, n\in\mathbf{Z}\}$ it follows from the uniform continuity of $\mathbf{P}_n(x)$ that $\bar\omega(\eta)\to 0$ as $\eta\to 0$. Then given $0<\bar\varepsilon\leqq\varepsilon_0$, we let $\delta_1>0$ be such that $\delta_1\leqq\Delta$, $(4K+2)\delta_1\leqq\bar\varepsilon$, $8M^kK\omega(\delta_1)\leqq 1$, $4K\omega(\delta_1)\leqq 1$. (Note: M is defined before Lemma 5.2.)

Now suppose $\{\bar y_n\}_{n\in\mathbf{Z}}$ is a δ_1-pseudo-orbit for F_n with $\bar y_n\in S_{nk}$ for all n. We show the existence of a unique orbit of F_n that $\bar\varepsilon$-shadows $\{\bar y_n\}$. First we apply Lemma 3.2 to $A_n=DF_n(\bar y_n)$, $Q_n=\mathbf{P}_{nk}(\bar y_n)$. For all n, $|A_n|\leqq M^k$, $|Q_n|\leqq K$, $|I-Q_n|\leqq K$. Also by the hyperbolicity, $|A_nQ_n|\leqq K\lambda^k$ for all n and $A_n(I-Q_n)$: $\mathcal{N}(Q_n)\to\mathcal{N}(\mathbf{P}_{(n+1)k}(F_n(\bar y_n)))$ is invertible with inverse having norm bounded by $K\lambda^k$. Using the invariance property of $\mathbf{P}_n$,

$$\begin{aligned}
|Q_{n+1}A_n(I-Q_n)|&=|[\mathbf{P}_{(n+1)k}(\bar y_{n+1})-\mathbf{P}_{(n+1)k}(F_n(\bar y_n))]A_n(I-Q_n)|\\
&\leqq \omega(\delta_1)\cdot M^kK\leqq 1/8
\end{aligned}$$

and, similarly, $|(I-Q_{n+1})A_nQ_n|\leqq 1/8$. Also since $2K\omega(\delta_1)\leqq 1/2$ it follows from Lemma 5.1 that

$$J_n=Q_{n+1}\mathbf{P}_{(n+1)k}(F_n(\bar y_n))+(I-Q_{n+1})(I-\mathbf{P}_{(n+1)k}(F_n(\bar y_n)))$$

is invertible with $|J_n^{-1}|\leqq(1-2K\omega(\delta_1))^{-1}\leqq 2$ and that $J_n(\mathcal{N}(\mathbf{P}_{(n+1)k}(F_n(\bar y_n)))=\mathcal{N}(Q_{n+1})$. Hence

$$(I-Q_{n+1})A_n(I-Q_n)=J_nA_n(I-Q_n):\mathcal{N}(Q_n)\to\mathcal{N}(Q_{n+1})$$

is invertible with inverse B_n satisfying $|B_n| \leqq 2K\lambda^k$ so that $|B_n(I-Q_n)| \leqq 2K^2\lambda^k$. Thus the conditions of Lemma 3.2 are satisfied with X_{nk} instead of X_n and $2K^2\lambda^k$ instead of λ. So if we define L: $\prod_{n=-\infty}^{\infty} X_{nk} \to \prod_{n=-\infty}^{\infty} X_{nk}$ by

$$(L\mathbf{u})_n = u_n - A_{n-1}u_{n-1} = u_n - DF_{n-1}(\bar{y}_{n-1})u_{n-1},$$

L is invertible with $\|L^{-1}\| \leqq 2K+1$.

Let U be the open set in $\prod_{n=-\infty}^{\infty} X_{nk}$ consisting of those $\{x_n\}$, $x_n \in X_{nk}$ satisfying $\sup_n |x_n - \bar{y}_n| < \Delta$. Then we define $\mathcal{F}: U \to \prod_{n=-\infty}^{\infty} X_{nk}$ by

$$(\mathcal{F}(\mathbf{x}))_n = x_n - F_{n-1}(x_{n-1}).$$

$\mathcal{F}$ is C^1 with $(D\mathcal{F}(\mathbf{x})\mathbf{h})_n = h_n - DF_{n-1}(x_{n-1})h_{n-1}$. $L = D\mathcal{F}(\bar{\mathbf{y}})(\bar{\mathbf{y}} = \{\bar{y}_n\})$ is invertible with $\|L^{-1}\| \leqq 2K+1$. Condition (4.1) is satisfied by choice of ε_0 and since $\|\mathcal{F}(\bar{y})\| \leqq \delta_1 \leqq \bar{\varepsilon}/(4K+2)$, condition (4.2) is also satisfied with $\bar{\varepsilon}$ instead of ε. Then it follows from Proposition 4.1 that there exists a unique $\bar{\mathbf{x}} = \{\bar{x}_n\}$ in $\prod_{n=-\infty}^{\infty} X_{nk}$ such that $\mathcal{F}(\mathbf{x}) = 0$ and $\|\bar{\mathbf{x}} - \bar{\mathbf{y}}\| \leqq \bar{\varepsilon}$. That is, $\{\bar{x}_n\}$ is the unique orbit of $\{F_n\}$ such that $|\bar{x}_n - \bar{y}_n| \leqq \bar{\varepsilon}$ for all n.

Now let $0 < \varepsilon \leqq \varepsilon_0$ and let δ_1 correspond to $\bar{\varepsilon} = (1+M+\cdots+M^k)^{-1}\varepsilon$. Set $\delta = (1+M+\cdots+M^k)^{-1}\delta_1$ and let $\{y_n\}$ be a δ-pseudo-orbit for $\{f_n\}$ with $y_n \in S_n$ for all n. Then $\{\bar{y}_n\}_{n\in\mathbf{Z}}$, with $\bar{y}_n = y_{nk}$, is a δ_1-pseudo-orbit for $\{F_n\}$ by virtue of Lemma 5.2. So there exists a unique orbit $\{\bar{x}_n\}$ of $\{F_n\}$ which $\bar{\varepsilon}$-shadows $\{\bar{y}_n\}$. Then we define $x_{nk} = \bar{x}_n$, $n \in \mathbf{Z}$, and each x_{nk+i}, $1 \leqq i \leqq k-1$, $n \in \mathbf{Z}$, in the obvious way so that $\{x_n\}_{n\in\mathbf{Z}}$ is an orbit for $\{f_n\}$. It follows from Lemma 5.3 and the fact that $\max\{\bar{\varepsilon}, \delta\} = \bar{\varepsilon}$ that $\{x_n\}\varepsilon$-shadows $\{y_n\}$. $\{x_n\}$ must be unique because $\{x_{nk}\}$ ε-shadows $\{y_{nk}\}$ and there is a unique such orbit because $\varepsilon \leqq \varepsilon_0$.

6. Application to parabolic evolution equations. Consider the following parabolic evolution equation

$$\dot{x} + Ax = f(x, t) \tag{6.1}$$

in a Banach space X with norm $|\cdot|$. Suppose A is a sectorial operator in X (see Henry [8] for general reference in this section) with $\operatorname{Re}\sigma(A) > 0$. We can define the fractional powers $A^\alpha: \mathcal{D}(A^\alpha) \to X$, $0 \leqq \alpha \leqq 1$, and then $X^\alpha = \mathcal{D}(A^\alpha)$, the domain of A^α, becomes a Banach space with the graph norm $|x|_\alpha = |A^\alpha x|$.

We also assume that $f \in C^1(X^\alpha \times \mathbf{R}, X)$ and that f and $D_x f: X^\alpha \times \mathbf{R} \to \mathcal{L}(X^\alpha, X)$ are Lipschitzian in x and locally Hölder continuous in t. Under these conditions the initial value problem

$$\dot{x} + Ax = f(x, t), \qquad x(t_0) = x_0$$

has for all $(x_0, t_0) \in X^\alpha \times \mathbf{R}$ a unique solution

$$x(t; x_0, t_0) \in C^0([t_0, T), X^\alpha) \cap C^1((t_0, T), X) \cap C^0((t_0, T), \mathcal{D}(A)),$$

where $[t_0, T)$ is the maximal interval of existence. We denote the solution map of (6.1) by $\mathbf{T}(t, t_0)(x_0) = x(t; x_0, t_0)$.

Let $S \subset X^\alpha \times \mathbf{R}$ be a *forward invariant set* for (6.1), that is, if $(x_0, t_0) \in S$ then $\mathbf{T}(t, t_0)(x_0)$ is defined for all $t \geqq t_0$ and $(\mathbf{T}(t, t_0)(x_0), t) \in S$. This means that $\mathbf{T}(t, t_0)S_{t_0} \subset S_t$ for all $t \geqq t_0$, where $S_t = \{x \in X^\alpha : (x, t) \in S\}$ is the t-section of S. We say S is *hyperbolic* if:

(i) For $x \in S_t$, $t \in \mathbf{R}$, there is a splitting

$$X^\alpha = E_t^s(x) \oplus E_t^u(x) \tag{6.2}$$

which is *invariant*, that is,

$$\begin{aligned} D_x\mathbf{T}(t, t_0)(x)E_{t_0}^s(x) &\subset E_t^s(\mathbf{T}(t, t_0)(x)), \\ D_x\mathbf{T}(t, t_0)(x)E_{t_0}^u(x) &\subset E_t^u(\mathbf{T}(t, t_0)(x)) \end{aligned} \tag{6.3}$$

for all $x \in S_{t_0}$ and $t, t_0 \in \mathbf{R}$ with $t \geqq t_0$, and also *continuous*, that is, if $\mathbf{P}_t(x)$ is the projection with range $E_t^s(x)$ and nullspace $E_t^u(x)$, $\mathbf{P}_t(x)$ is continuous uniformly with respect to $x \in S_t$, $t \in \mathbf{R}$. We also assume that $D_x\mathbf{T}(t, t_0)(x): E_{t_0}^u(x) \to E_t^u(\mathbf{T}(t, t_0)(x))$ is an isomorphism with (bounded) inverse

$$(D_x\mathbf{T}(t, t_0)(x))^{-1}: E_t^u(\mathbf{T}(t, t_0)(x)) \to E_{t_0}^u(x).$$

(ii) There exist constants $K \geqq 1$, $\beta > 0$ such that for $x \in S_{t_0}$ and $t, t_0 \in \mathbf{R}$ with $t \geqq t_0$,

$$\begin{aligned} &|D_x\mathbf{T}(t, t_0)(x)\mathbf{P}_{t_0}(x)|_{\mathscr{L}(X^\alpha, X^\alpha)} \leqq K\, e^{-\beta(t-t_0)}, \\ &|(D_x\mathbf{T}(t, t_0)(x))^{-1}(I - \mathbf{P}_t(\mathbf{T}(t, t_0)(x)))|_{\mathscr{L}(X^\alpha, X^\alpha)} \leqq K\, e^{-\beta(t-t_0)}. \end{aligned} \tag{6.4}$$

Now we want to define pseudosolutions of (6.1). Let $\bigcup_{n \in \mathbf{Z}} [\tau_{n-1}, \tau_n] = \mathbf{R}$ be a partition of $\mathbf{R}$ with $\inf\{\tau_n - \tau_{n-1}: n \in \mathbf{Z}\} = \tau > 0$. Then, if δ is positive, we say the sequence $\{x_n(t)\}$, $t \in [\tau_{n-1}, \tau_n]$, $n \in \mathbf{Z}$ is a *δ-pseudosolution* of (6.1) if for all n

$$x_n(\cdot) \in C^0([\tau_{n-1}, \tau_n], X^\alpha) \cap C^1((\tau_{n-1}, \tau_n), X) \cap C^0((\tau_{n-1}, \tau_n), \mathscr{D}(A))$$

and

$$\sup\{|h_n(t)|: \tau_{n-1} \leqq t \leqq \tau_n\} \leqq \delta, \qquad |g_n|_\alpha \leqq \delta, \tag{6.5}$$

where $h_n \in C^0([\tau_{n-1}, \tau_n], X)$, defined by

$$h_n(t) = \dot{x}_n(t) + Ax_n(t) - f(x_n(t), t) \tag{6.6}$$

is the *residual error* and

$$g_n = x_n(\tau_n) - x_{n+1}(\tau_n) \tag{6.7}$$

is the *jump* at τ_n.

If ε is positive, a solution $x(t)$ of (6.1) is said to *ε-shadow* the δ-pseudosolution $\{x_n(t)\}$ if $x(t)$ is defined for all t and $|x(t) - x_n(t)| \leqq \varepsilon$ for $\tau_{n-1} \leqq t \leqq \tau_n$, $n \in \mathbf{Z}$.

THEOREM 6.1. *Let A, X, X^α, $f(t, x)$ be as above and suppose $S \subset X^\alpha \times \mathbf{R}$ is a forward invariant hyperbolic set for* (6.1) *such that $f(x, t)$ and $D_x f(x, t)$ are bounded and Lipschitz continuous in a Δ-neighborhood O of S in $X^\alpha \times \mathbf{R}$.*

Let $\{x_n(t)\}$, $\tau_{n-1} \leqq t \leqq \tau_n$, $n \in \mathbf{Z}$, be a δ-pseudosolution of (6.1) *such that for $\tau_{n-1} \leqq t \leqq \tau_n$ and $n \in \mathbf{Z}$, $x_n(t)$ is in a δ-neighborhood of S_t in the X^α norm.*

Then there exist $\varepsilon_0 > 0$ and a positive function $\delta(\varepsilon)$, both depending only on $A, f, \tau = \inf_n (\tau_n - \tau_{n-1})$, such that if $0 < \varepsilon \leqq \varepsilon_0$ and $\delta \leqq \delta(\varepsilon)$, there is a unique solution of (6.1) *that ε-shadows $\{x_n(t)\}$.*

For the proof of Theorem 6.1, we need a lemma.

LEMMA 6.2. *Let the hypotheses of Theorem* 6.1 *hold and let M be the bound for $\mathscr{D}_x f(t, x)$ in O. Let $x(t)$ be a solution of* (6.1) *in S and let $y(t)$ be a solution of the initial value problem*

$$\dot{y} + Ay = f(y, t) + h(t), \qquad y(t_0) = \xi \tag{6.8}$$

for $t \in [t_0, t_0 + 2\tau]$, where $|\xi - x(t_0)|_\alpha \leqq \delta$ and $h \in C^0([t_0, t_0 + 2\tau], X)$ with $\sup|h(t)| \leqq \delta$. If $(y(t), t) \in O$ for $t \in [t_0, t_0 + 2\tau]$, then there exists a constant $C \geqq 1$ depending only on A and M such that $|y(t) - x(t)|_\alpha \leqq C\delta$ on $[t_0, t_0 + 2\tau]$.

Proof. Our assumptions on A imply the existence of positive constants C_1, C_2 and a such that for $t \geqq 0$

$$|e^{-At}|_{\mathscr{L}(X^\alpha, X^\alpha)} \leqq C_1\, e^{at}, \qquad |e^{-At}|_{\mathscr{L}(X, X^\alpha)} \leqq C_2 t^{-\alpha}\, e^{at}.$$

Now $z(t)=y(t)-x(t)$ satisfies the integral equation

$$z(t)=e^{-A(t-t_0)}z(t_0)+\int_{t_0}^{t} e^{-A(t-s)}\{f(x(s)+z(s), s)-f(x(s), s)+h(s)\}\,ds.$$

Then for $t_0 \leqq t \leqq t_0+2\tau$,

$$|z(t)|_\alpha \leqq C_1\, e^{a(t-t_0)}\delta+\int_{t_0}^{t} MC_2(t-s)^{-\alpha}\, e^{a(t-s)}|z(s)|_\alpha\, ds+\int_{t_0}^{t} C_2(t-s)^{-\alpha}\, e^{a(t-s)}\delta\, ds.$$

It follows from an inequality in Henry [8, Lemma 7.1.1, p. 188] that $|z(t)|_\alpha < C\delta$ for $t_0 \leqq t \leqq t_0+2\tau$. The proof is completed.

Proof of Theorem 6.1. The hypotheses on A and f, Lemma 6.2, and Henry [8] imply that there exists a closed Δ_1-neighborhood O_1 of S in $X^\alpha \times \mathbf{R}$ such that for $(x, t_0) \in O_1$, $\mathbf{T}(t, t_0)(x)$ is defined for $t_0 \leqq t \leqq t_0+2\tau$ and both $\mathbf{T}(t, t_0)(x)$ and $D_x\mathbf{T}(t, t_0)(x)$ are bounded and continuous, uniformly with respect to $(x, t_0) \in O_1$ and $t \in [t_0, t_0+2\tau]$. (These functions have ranges in X^α and $\mathscr{L}(X^\alpha, X^\alpha)$, respectively, and the continuity is with respect to these norms.)

Without loss of generality we may assume that $0 \leqq \tau \leqq \tau_n-\tau_{n-1} \leqq 2\tau$ for all n. We first consider the case where $h_n(t)=0$ and $x_n(t) \in S_t$ for all t and n. Then if we let X_n be X^α for all n and f_n be $\mathbf{T}(\tau_n, \tau_{n-1}): X^\alpha \to X^\alpha$ the domain of f_n contains a closed Δ_1-neighborhood of $S_{\tau_{n-1}}$ in which f_n and Df_n are both bounded and uniformly continuous, uniformly with respect to $n \in \mathbf{Z}$. From the hyperbolicity of S with respect to (6.1), we see that $\{S_{\tau_{n-1}}\}_{n\in\mathbf{Z}}$ is invariant ($f_n(S_{\tau_{n-1}}) \subset S_{\tau_n}$) and hyperbolic for $\{f_n\}_{n\in\mathbf{Z}}$ with projections $\mathbf{P}_{\tau_{n-1}}(x)$ and constants K, $e^{-\beta\tau}$. Hence conditions (i), (ii), (iii) of the Shadowing Lemma hold. Set $y_n=x_n(\tau_{n-1})$, $n \in \mathbf{Z}$. Then $y_n \in S_{\tau_{n-1}}$ for all n and

$$|f_n(y_n)-y_{n+1}|_\alpha=|x_n(\tau_n)-x_{n+1}(\tau_n)|_\alpha \leqq \delta.$$

So if $0<\varepsilon \leqq \varepsilon_1$ and $\delta \leqq \delta_1(\varepsilon)$ (ε_1 and $\delta_1(\varepsilon)$ correspond to ε_0 and $\delta(\varepsilon)$ in the Shadowing Lemma) there is a unique solution $x(t)$ of (6.1) such that $|x(\tau_{n-1})-x_n(\tau_{n-1})|_\alpha \leqq \varepsilon$ for all n.

Now we consider the general case. We suppose $0<\varepsilon \leqq \varepsilon_0=\frac{1}{2}\min\{\Delta, \varepsilon_1\}$ and $\delta \leqq \delta(\varepsilon)=\min\{(2C+1)^{-1}\delta_1(\varepsilon/2C), \varepsilon/2C\}$. Let $\{x_n(\cdot)\}$ be a δ-pseudosolution as in the statement of the theorem. Since for all n, $x_n(\tau_{n-1})$ is in a δ-neighborhood of $S_{\tau_{n-1}}$, we can choose y_n in $S_{\tau_{n-1}}$ so that $|x_n(\tau_{n-1})-y_n|_\alpha \leqq \delta$. Then let $\bar{x}_n(t)$ be the solution of (6.1) satisfying $\bar{x}_n(\tau_{n-1})=y_n$. By Lemma 6.2 with $t_0=\tau_{n-1}$, $x(t)=\bar{x}_n(t)$, $h(t)=h_n(t)$, $y(t)=x_n(t)$ we have

$$|\bar{x}_n(t)-x_n(t)|_\alpha \leqq C\delta$$

for $\tau_{n-1} \leqq t \leqq \tau_n$. This holds for $n \in \mathbf{Z}$. Moreover,

$$\begin{aligned} |\bar{x}_n(\tau_n)-\bar{x}_{n+1}(\tau_n)|_\alpha &\leqq |\bar{x}_n(\tau_n)-x_n(\tau_n)|_\alpha+|x_n(\tau_n)-x_{n+1}(\tau_n)|_\alpha+|x_{n+1}(\tau_n)-\bar{x}_{n+1}(\tau_n)|_\alpha \\ &\leqq (2C+1)\delta \leqq \delta_1(\varepsilon/2C). \end{aligned} \tag{6.9}$$

Hence $\{\bar{x}_n(t)\}$ is a $\delta_1(\varepsilon/2C)$-pseudosolution of (6.1) with $\bar{x}_n(t) \in S_t$ for all t and n, where $\varepsilon/2C<\varepsilon_1$. It follows from the first part of the proof that there is a unique solution $x(t)$ of (6.1) such that

$$|x(\tau_{n-1})-\bar{x}_n(\tau_{n-1})|_\alpha \leqq \varepsilon/2C \tag{6.10}$$

for all n. Then for all n

$$\begin{aligned} |x(\tau_{n-1})-x_n(\tau_{n-1})|_\alpha &\leqq |x(\tau_{n-1})-\bar{x}_n(\tau_{n-1})|_\alpha+|\bar{x}_n(\tau_{n-1})-x_n(\tau_{n-1})|_\alpha \\ &\leqq \varepsilon/2C+\delta. \end{aligned}$$

By Lemma 6.2 with $t_0=\tau_{n-1}$, $x(t)=\bar{x}_n(t)$, $y(t)=x(t)$ and $\varepsilon/2C+\delta$ instead of δ (note: $C(\varepsilon/2C+\delta)=\varepsilon/2+C\delta<\Delta$), we deduce for $\tau_{n-1}\leqq t\leqq\tau_n$, $n\in\mathbf{Z}$ that

$$|x(t)-x_n(t)|_\alpha\leqq\varepsilon/2+C\delta\leqq\varepsilon.$$

That is, $x(t)$ does ε-shadow the δ-pseudosolution $\{x_n(t)\}$.

Let $\tilde{x}(t)$ be another such solution. Then for all n

$$\begin{aligned}|\tilde{x}(\tau_{n-1})-\bar{x}_n(\tau_{n-1})|_\alpha&\leqq|\tilde{x}(\tau_{n-1})-x_n(\tau_{n-1})|_\alpha+|x_n(\tau_{n-1})-\bar{x}_n(\tau_{n-1})|_\alpha\\&\leqq\varepsilon+\delta\leqq3\varepsilon/2<\varepsilon_1,\end{aligned}$$

where for all n, using (6.9),

$$\begin{aligned}|f_n(\bar{x}_n(\tau_{n-1}))-\bar{x}_{n+1}(\tau_n)|_\alpha&=|\bar{x}_n(\tau_n)-\bar{x}_{n+1}(\tau_n)|_\alpha\\&\leqq\delta_1(\varepsilon/2C)\\&\leqq\delta_1(3\varepsilon/2).\end{aligned}$$

(Note: we assume without loss of generality that $\delta_1(\varepsilon)$ is nondecreasing in ε.) That is, the sequence $\{\tilde{x}(\tau_{n-1})\}$ is an orbit of $\{f_n\}$ that $3\varepsilon/2$-shadows the $\delta_1(3\varepsilon/2)$-pseudo-orbit $\{\bar{x}_n(\tau_{n-1})\}$, where $3\varepsilon/2<\varepsilon_1$. But by (6.10), $\{x(\tau_{n-1})\}$ is another such sequence and so it follows by uniqueness that $\tilde{x}(\tau_{n-1})=x(\tau_{n-1})$ for all n. Hence $x(t)$ is unique.

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