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A SHADOWING LEMMA FOR MAPS IN INFINITE DIMENSIONS

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1. INTRODUCTION

One of the most important properties of hyperbolic sets is the *pseudo-orbit property*. This is embodied in the *shadowing lemma*. Many proofs of this have been given. Here we mention those of Bowen (1975), Conley (1975), Robinson (1977), Newhouse (1980), Ekeland (1983), Lanford (1983), Shub (1987), and Palmer (1988). In Meyer and Sell (1987), a simple analytical proof was given; however, it seems there is a gap on page 129. The argument there is not valid unless $DF(x) - I$ maps S_x and U_x into complementary subspaces. Our original motivation was to remedy this. Then we discovered that our methods could easily be modified to yield a version of the shadowing lemma for a not necessarily invertible map on a Banach space. Full details of the proofs are to appear in Chow, Lin, and Palmer (1989), and an application to semilinear parabolic equations is also given there. In this paper, we present the proof for diffeomorphisms in finite dimensional space.

2. DEFINITIONS AND STATEMENT OF THE SHADOWING LEMMA

Let $f: X \to X$ be a C^1 mapping of a Banach space X into itself. Let $S \subset X$ be an *invariant* set, that is, $f(S) \subset S$, such that $f(x)$ and $Df(x)$ are bounded and uniformly continuous in a closed -neighborhood O of S. We want to define what is meant by saying that S is *hyperbolic*. First, there is a *splitting*,

$$X = E_x^s \oplus E_x^u \tag{1}$$

for $x \in S$. We require this splitting to be *invariant*, that is,

$$Df(x)(E_x^s) \subset E_{f(x)}^s, Df(x)(E_x^u) \subset E_{f(x)}^u$$

for all x in S and also *continuous*, that is, if $P(x)$ is the projection with range E_x^s and null space E_x^u, (x) is uniformly continuous. In terms of $P(x)$, the invariance means that

$$Df(x)P(x) = P(f(x))Df(x) \tag{2}$$

for all x in S. We also assume that $Df(x)\colon E_x^u \to E_{f(x)}^u$ is an isomorphism with a bounded inverse $(Df(x))^{-1}\colon E_{f(x)}^u \to E_x^u$.

Second, we require that there exist constants $K \geq 1$, $\alpha > 0$ such that, for $n \geq 0$,

$$|Df^n(x)P(x)| \leq Ke^{-\alpha n}$$

$$|(Df^n(f^{-n}(x)))^{-1}(I - P(x))| \leq Ke^{-\alpha n} \tag{3}$$

If δ is a positive number, a sequence y_n, $n \in \mathbb{Z}$, in X is said to be a *δ-oseydi-orbit* for f if

$$|f(y_n) - y_{n+1}| < \delta$$

for all integers n.

THE SHADOWING LEMMA. Let X, f, S be defined as above and satisfy the conditions:

(i) $f(X) \subset S$.

(ii) $f(x)$ and $Df(x)$ are bounded and uniformly continuous in a closed σ-neighborhood 0 of S.

(iii) S is hyperbolic.

Then, given $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ such that if y_n is a δ-pseudo-orbit for f, there is a unique x in X such that

$$|f^n(x) - y_n| \leq \varepsilon$$

for all integers n.

REMARK In Chow, Lin, and Palmer (1989), a more general result for sequences of mappings is proved. Here, for the sake of simplicity, we have stated the result for a single mapping.

Also, here we prove the shadowing lemma stated above only for the case in which $X = \mathbb{R}^k$ and $f\colon \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism. Then $f(S) = S$ and the second inequality in equation (3) can be rewritten as

$$|Df^{-n}(x)(I - P(x))| \le Ke^{-\alpha n}$$

The proof given below uses facts about linear difference equations and a variant of Newton's method for finding the zeros of a function.

3. FACTS ABOUT LINEAR DIFFERENCE EQUATIONS

For each integer n, let C_n be an invertible $k \times k$ matrix. The *linear difference equation*

$$u_{n+1} = C_n u_n (u_n \in \mathbf{R}^k, \ n \in \mathbb{Z}) \tag{4}$$

is said to have an *exponential dichotomy* if there is a projection-valued function P_n and constants $K \ge 1$, $\alpha > 0$ such that

$$\Phi(n,m)P_m = P_n\Phi(n,m) \text{ for all } n, m, \tag{5}$$

$$|\Phi(n,m)P_m| \le Ke^{-\alpha(n-m)} \text{ for } n \ge m, \tag{6}$$

$$|\Phi(n,m)(I - P_m)| \le Ke^{-\alpha(m-n)} \text{ for } m \ge n \tag{7}$$

Here $\Phi(n,m)$ is the *transition matrix* for system (4). That is,

$$\Phi(n,m) = \begin{cases} C_{n-1}C_{n-2}\cdots C_m & n > m \\ I & n = m \\ \Phi(m,n)^{-1} & n < m \end{cases}$$

We denote by $\ell^\infty(\mathbb{Z})$ the Banach space of bounded $\mathbf{R}^k$-valued sequences $\{u_n\}_{n\in\mathbb{Z}}$ with norm

$$\|u\| = \sup_{n\in\mathbb{Z}} |u_n|$$

and make the following simple observation.

LEMMA For $n \in \mathbb{Z}$, let C_n be an invertible $k \times k$ matrix such that $\sup_{n\in\mathbb{Z}}|C_n| < \infty$. Then, if equation (4) has an exponential dichotomy with constants K, α the operator L: $\ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ defined by

$$(Lu)_n = u_{n+1} - C_n u_n$$

is invertible and 9

$$\|L^{-1}\| \le K(1 - e^{-\alpha})^{-1}(1 + e^{-\alpha}) \tag{8}$$

Proof: First we show that L is one to one. If $Lu = 0$, then u_n is a bounded solution of equation (4); that is, for all n

$$u_n = \Phi(n,0)u_0 = \Phi(n,0)P_0u_0 + \Phi(n,0)(I - P_0)u_0$$

Then, for $n \geq 0$,

$$\begin{aligned}
|(I - P_0)u_0| &= |\Phi(0,n)\Phi(n,0)(I - P_0)u_0| \\
&= |\Phi(0,n)\Phi(n,0)(I - P_0)(I - P_0)u_0| \\
&= |\Phi(0,n)(I - P_n)\Phi(n,0)(I - P_0)u_0| \quad \text{by (5)} \\
&\leq |\Phi(0,n)(I - P_n)||\Phi(n,0)(I - P_0)u_0| \\
&\leq |\Phi(0,n)(I - P_n)|\{|u_n| + |\Phi(n,0)P_0u_0|\} \\
&\leq Ke^{-\alpha n}\{\|u\| + Ke^{-\alpha n}|u_0|\} \quad \text{by (6), (7)} \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence, $(I - P_0)u_0 = 0$. Similarly, we can show $P_0u_0 = 0$. Thus, $u_0 = 0$ and $u = 0$; that is, L is one to one.

To show that L is onto, let $h \in \ell^\infty(\mathbb{Z})$. Then, we set for $n \in \mathbb{Z}$,

$$u_n = \sum_{m=-\infty}^{n} \Phi(n,m)P_mh_m - \sum_{m=n+1}^{\infty} \Phi(n,m)(I - P_m)h_m$$

Since, for $m \leq n$,

$$|\Phi(n,m)P_mh_m| \leq Ke^{-\alpha(n-m)}\|h\|$$

and for $m > n$,

$$|\Phi(n,m)(I - P_m)h_m| \leq Ke^{-\alpha(m-n)}\|h\|$$

u_n is well defined and

$$\begin{aligned}
|u_n| &\leq K\|h\|\left\{\sum_{m=-\infty}^{n} e^{-\alpha(n-m)} + \sum_{m=n+1}^{\infty} e^{-\alpha(m-n)}\right\} \\
&= K\|h\| \sum_{m=-\infty}^{\infty} e^{-\alpha|m|} \\
&= K(1 - e^{-\alpha})^{-1}(1 + e^{-\alpha})\|h\| \qquad (9)
\end{aligned}$$

Thus, $u = \{u_n\}_{n\in\mathbb{Z}}$ is in $\ell^\infty(\mathbb{Z})$, and it is easy to check that, for all n,

$$u_{n+1} = C_nu_n + h_n$$

Hence, u is the unique sequence in $\ell^\infty(\mathbb{Z})$ such that $Lu = h$ and inequality (8) follows from (9).

Now we state a *perturbation theorem* for exponential dichotomies in linear difference equations. This is an exact analog to that for linear differential equations (cf. Theorem 3 in Palmer, 1987), and so the proof is omitted.

PERTURBATION THEOREM Suppose that, for each i in an index set I and for each integer n, $C_n^{(i)}$ is an invertible $k \times k$ matrix satisfying

$$|C_n^{(i)}|, |C_n^{(i)-1}| \le M$$

and such that the linear difference equation

$$u_{n+1} = C_n^{(i)} u_n$$

has an exponential dichotomy, with the constants K, α and the rank of the projection both independent of i.

Then there exists a positive integer N and a positive number Δ, both depending only on M, K, α, with the following property: Let B_n be a sequence of invertible $k \times k$ matrices such that, for each integer m, there exists $i_m \in I$, for which

$$|B_n - C_n^{(i_m)}| < \Delta \quad \text{for } m \le n \le m + N$$

Then, the equation

$$u_{n+1} = B_n u_n$$

has an exponential dichotomy with constants $2K^4$, $\alpha/2$.

4. A VARIANT OF NEWTON'S METHOD

In this section, we prove the following variant of Newton's method.

PROPOSITION Let E be a Banach space and $F: E \to E$ a C^1 map. Let y be a point in E such that $DF(y)^{-1}$ exists and let $\varepsilon_0 > 0$ be chosen so that

$$\|DF(x) - DF(y)\| \le 1/2\|DF(y)^{-1}\| \tag{10}$$

for $\|x - y\| \le \varepsilon_0$. Then, if $0 < \varepsilon \le \varepsilon_0$ and

$$\|F(y)\| \le \varepsilon/2\|DF(y)^{-1}\| \tag{11}$$

the equation

$$F(x) = 0 \tag{12}$$

has a unique solution x such that $\|x - y\| \le \varepsilon$.

Proof: We write

$$F(x) = F(y) + DF(y)(x - y) + \eta(x)$$

When $\|x_1 - y\|, \|x_2 - y\| \leq \varepsilon_0$,

$$\begin{aligned}\|\eta(x_1) - \eta(x_2)\| &= \|F(x_1) - F(x_2) - DF(y)(x_1 - x_2)\| \\ &= \left\|\int_0^1 \{DF(x_2 + \theta(x_1 - x_2)) - DF(y)\}d\theta(x_1 - x_2)\right\| \\ &\leq \int_0^1 \|DF(x_2 + \ (x_1 - x_2)) - DF(y)\|d\theta\|x_1 - x_2\| \\ &\leq (1/2\|DF(y)^{-1}\|)\|x_1 - x_2\| \qquad (13)\end{aligned}$$

using equation (10).

We can rewrite equation (12) as

$$x = y - DF(y)^{-1}\{F(y) + \eta(x)\} := T(x)$$

For $0 < \varepsilon \leq \varepsilon_0$, we define $B_\varepsilon = \{x \in E : \|x - y\| \leq \varepsilon\}$ and show that T is a contradiction of B_ε. The proposition will then follow immediately from the contraction mapping principle.

Note, first, that if $x \in B_\varepsilon$, then

$$\begin{aligned}\|T(x) - y\| &= \|DF(y)^{-1}\{F(y) + \eta(x)\}\| \\ &\leq \|DF(y)^{-1}\|\{\varepsilon/2\|DF(y)^{-1}\| + \|x - y\|/2\ DF(y)^{-1}\|\} \\ &= \varepsilon/2 + \|x - y\|/2 \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

where we have used equations (11) and (13) with $x_1 = x$, $x_2 = y$. Hence, T maps B_ε into itself. Moreover, if $x_1, x_2 \in B_\varepsilon$, then, using equation (13),

$$\begin{aligned}\|T(x_1) - T(x_2)\| &= \|DF(y)^{-1}\{\eta(x_1) - \ (x_2)\}\| \\ &\leq \|DF(y)^{-1}\| \cdot (1/2\|DF(y)^{-1}\|)\|x_1 - x_2\| \\ &= 1/2\|x_1 - x_2\|\end{aligned}$$

Thus, T is indeed a contraction on B_ε, and the proof is finished.

5. PROOF OF THE SHADOWING LEMMA

We write

$$M = \sup\{|Df(x)| : x \in O\}$$

and for $\delta \geq 0$,

$$\omega(\delta) = \sup\{|Df(y) - Df(x)| : x \in S,\ y \in \mathbb{R}^k,\ |y - x| \leq \delta\}$$

By our assumptions, $M < \infty$ and $\omega(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Next, we define ε_0 to be the largest positive number satisfying

$$\omega(\varepsilon_0) \leq 1/2M_1$$

where

$$M_1 = 2K^4(1 - e^{-\alpha/2})^{-1}(1 + e^{\alpha/2})$$

Let N, Δ be the numbers in the perturbation theorem corresponding to M, K, α. Then, for $0 < \varepsilon \leq \varepsilon_0$, choose $\delta = \delta(\varepsilon)$ to be the largest $\delta > 0$ satisfying

$$\delta \leq \frac{\varepsilon}{2} M_1, \quad \delta(1 + M + \cdots + M^{N-1}) \leq \sigma, \quad \omega(\delta(1 + M + \cdots + M^N)) \leq \Delta$$

Now, suppose that $0 < \varepsilon \leq \varepsilon_0$ and y_n is a δ-pseudo-orbit for f. We define $F\colon \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by

$$[F(x)]_n = x_{n+1} - f(x_n)$$

Clearly, F is C^1 with derivative given by

$$[DF(x)h]_n = h_{n+1} - Df(x_n)h_n$$

Then, if y is the sequence $\{y_n\}$ and $x = \{x_n\}$ is another sequence with $\|x - y\| \leq \varepsilon_0$,

$$\begin{aligned}\|DF(x) - DF(y)\| &\leq \sup_{n\in\mathbb{Z}} |Df(x_n) - Df(y_n)| \\ &\leq \omega(\varepsilon_0) \\ &\leq 1/2M_1\end{aligned}$$

Below we shall prove that $DF(y)^{-1}$ exists with $\|DF(y)^{-1}\| \leq M_1$. Also, by the definition of pseudo-orbit,

$$\|F(y)\| = \sup_{n\in\mathbb{Z}} |y_{n+1} - f(y_n)| \leq \delta \leq \varepsilon/2M_1$$

Then by the proposition, the equation $F(x) = 0$ has a unique solution x satisfying $\|x - y\| \leq \varepsilon$. This is exactly what we have to prove.

So all that remains to show is that $DF(y)^{-1}$ exists and that $\|DF(y)^{-1}\| \leq M_1$.

It follows from the chain rule that, for $x \in S$, the linear difference equation

$$u_{n+1} = Df(f^n(x))u_n \tag{14}$$

has the transition matrix $\Phi(n,m) = Df^{n-m}(f^m(x))$. Then, if we define $P_n = P(f^n(x))$, we see, using equation (2), that for all n, m,

$$\Phi(n,m)P_m = Df^{n-m}(f^m(x))P(f^m(x)) = P(f^n(x))Df^{n-m}(f^m(x)) = P_n\Phi(n,m)$$

and, using equation (3), that

$$|\Phi(n,m)P_m| = |Df^{n-m}(f^m(x))P(f^m(x))| \le Ke^{-\alpha(n-m)} \quad (n \ge m)$$

$$|\Phi(n,m)(I - P_m)| = |Df^{n-m}(f^m(x))(I - P(f^m(x)))| \le Ke^{-\alpha(m-n)} \quad (m \ge n)$$

That is, equation (14) has an exponential dichotomy with the constants K, α and the rank of the projection both independent of x in S. This is the family of equations that we want to use in the perturbation theorem—the index set, in this case, is S.

We want to use the perturbation theorem to show that the linear difference equation

$$u_{n+1} = Df(y_n)u_n \tag{15}$$

has an exponential dichotomy.

First, we claim that, for all integers m,

$$|y_n - f^{n-m}(y_m)| \le \delta(1 + M + \cdots + M^{n-m-1}) \tag{16}$$

for $m + 1 \le n \le m + N$ (N chosen as at the beginning of this section). We prove this by induction on n. Since y_n is a δ-pseudo-orbit, it certainly holds for $n = m + 1$. Assuming it for some n in $m + 1 \le n \le m + N - 1$, we prove it for $n + 1$ as follows:

$$\begin{aligned} |y_{n+1} - f^{n+1-m}(y_m)| &\le |y_{n+1} - f(y_n)| + |f(y_n) - f(f^{n-m}(y_m))| \\ &\le \delta + M|y_n - f^{n-m}(y_m)| \\ &\le \delta + M\delta(1 + M + \cdots + M^{n-m-1}) \quad \text{by the induction hypothesis} \\ &= \delta(1 + M + M^2 + \cdots + M^{n-m}) \end{aligned}$$

This completes the induction and so equation (16) holds for $m + 1 \le n \le m + N$.

So, if $m \le n \le m + N$ (N, Δ chosen as at beginning of this section),

$$\begin{aligned} |Df(y_n) - Df(f^n(f^{-m}(y_m))| &\le \omega(|y_n - f^n(f^{-m}(y_m))|) \\ &= \omega(|y_n - f^{n-m}(y_m)|) \\ &\le \omega(\delta(1 + M + \cdots + M^{n-m})) \\ &\le \omega(\delta(1 + M + \cdots + M^N)) \\ &\le \Delta \end{aligned}$$

Then, the perturbation theorem implies that the difference equation (15) has an exponential dichotomy with constants $2K^4$, $\alpha/2$. But then it follows from the lemma that the operator $DF(y)$: $\ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$, which is defined by $[DF(y)h]_n = h_{n+1} - Df(y_n)h_n$, is invertible and that

$$\|DF(y)^{-1}\| \le 2K^4(1 - e^{-\alpha/2})^{-1}(1 + e^{-\alpha/2}) = M_1$$

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