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BIFURCATION OF A HOMOCLINIC ORBIT WITH A SADDLE-NODE EQUILIBRIUM

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1. Introduction. Bifurcation of periodic orbits from an orbit Γ_0 homoclinic to a hyperbolic equilibrium point was extensively studied by Silnikov in [12, 13, 14]. In this paper, we study the codimension two unfolding of an orbit Γ_0 homoclinic to a saddle-node equilibrium in $\mathbb{R}^d$. We allow the eigenvalues of the equilibrium to have both positive and negative real parts, except for a simple eigenvalue $\lambda_0 = 0$. Several new methods are employed. The bifurcation diagram is obtained by the bifurcation function which is derived by the method of Liapunov-Schmidt decomposition. The idea of exponential trichotomy (see Hale and Lin [7]) is employed to study the linearized equation around Γ_0. We show that the linear operator induced by the linearization around Γ_0 is Fredholm, which is essential for applying the Liapunov-Schmidt method. We use the method of cross sections and Poincare mappings to study the bifurcation of the periodic orbits from Γ_0. This requires the extension of the domain of the Poincare mapping to a whole neighborhood of Γ_0 since flows starting on a cross section near Γ_0 do not always return. Although extended mapping is not very nice as a point mapping, submanifolds of certain type, called the u-slices, behave nicely under the extended mapping. Roughly speaking, they are stretched in the unstable directions and compressed in the center and stable directions. Thus, a u-slice under the Poincare mapping can be found to be fixed and hence bifurcation of periodic orbits is proved.

Lukyanov [10] has considered the situation that we discuss here in the special case where the equilibrium point has a simple zero eigenvalue and all others have negative real parts. As pointed out by Schecter [11], the bifurcation diagram of Lukyanov is incorrect. Using the Melnikov function, Schecter [11] gave a correct bifurcation diagram for differential equations in $\mathbb{R}^2$. Vegter [17], using the theory of singularities of maps, has studied a related problem in $\mathbb{R}^2$.

Our main results are presented in §2. Basic facts concerning exponential trichotomies and Fredholm properties of the linearized system around Γ_0 are given in §3. In §4, bifurcations near Γ_0 are classified geometrically in terms of center

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manifold, center stable and unstable manifold and fibrations of these manifolds. This is based on many previous works on the center (center stable and unstable) manifold. See for example van Gils and Vanderbauwhede [6] and Chow and Lu [4]. In §5, bifurcation functions for the existence of homoclinic and heteroclinic orbits are given which clarify the geometrical conditions in §4. The novelty here is the generalization of classical methods to the case of a nonhyperbolic equilibrium. The most difficult part of the paper is in §6 in which bifurcation of periodic orbits is studied. Although the basic idea of reducing the problem to the study of certain maps comes from the work of Silnikov [11, 12, 13], several extensions are needed. Lemma 6.1 is essential for the bifurcation analysis and presents a local analysis using Silnikov's method of reparametrization for a nonhyperbolic equilibrium.

We received Deng's preprint [18] shortly before submitting this paper. In his work, Deng obtained the bifurcation diagram for homoclinic solution interacting with saddle-node, transcritical or pitchfork equilibrium. However, as noted in [18], his method is different from ours.

2. Main results. Consider the following ordinary differential equation in $\mathbb{R}^d$:

$$\dot{x}(t) = f(x, \mu), \quad x \in \mathbb{R}^d \tag{2.1}$$

where $f : \mathbb{R}^d \times M \to \mathbb{R}^d$, $\mu = (\mu_1, \mu_2) \in M$ and M is a neighborhood of $\mu = 0$ in $\mathbb{R}^2$. Let D be the differentiation operator and $\sigma(A)$ be the spectrum of a matrix A. Let $(a, \mu) \in \mathbb{R}^d \times M$ be an equilibrium of (2.1); i.e., $f(a, \mu) = 0$. Let $W^s(a; \mu)$, $W^u(a, \mu)$, $W^{ss}(a; \mu)$ and $W^{uu}(a, \mu)$ be the stable, unstable, strong stable and strong unstable manifolds of $x = a$ in $\mathbb{R}^d$ at $\mu \in M$ for equation (2.1). Let $W^s_{loc}(a; \mu)$ be the local stable manifold of the equilibrium $x = a$ at parameter μ for equation (2.1). Other local invariant manifolds will be denoted in a similar way.

Assume the $f(0, 0) = 0$. We rewrite equation (2.1) as an autonomous equation in $\mathbb{R}^d \times M$:

$$\begin{cases} \dot{x} = f(x, \mu) \\ \dot{\mu} = 0. \end{cases} \tag{2.2}$$

Local center, local center stable and center unstable manifolds at $(x, \mu) = (0, 0)$ of equation (2.2) are well-defined (see §7 Appendix) and denoted by W^c_{loc}, W^{cs}_{loc}, and W^{cu}_{loc} respectively. We note that these local invariant manifolds W^c_{loc}, W^{cs}_{loc}, and $W^{cu}_{loc} \subseteq \mathbb{R}^d \times M$ are not uniquely defined. If $|\mu|$ is sufficiently small (we allow $\mu = 0$), then define

$$W^c_{loc}(\mu) = \{x \in \mathbb{R}^d \mid (x, \mu) \in W^c_{loc}\}.$$

Since $\dot{\mu} = 0$, $W^c_{loc}(\mu)$ is a local invariant manifold near $x = 0$ for equation (2.1). Similarly, we define $W^{cs}_{loc}(\mu)$ and $W^{cu}_{loc}(\mu)$. Let $T(t; \mu)$ be the flow generated by equation (2.1). Given any local center manifold $W^c_{loc}(\mu)$, we define

$$W^c(\mu) = \bigcup_{t \in \mathbb{R}} T(t, \mu) W^c_{loc}(\mu).$$

Similarly, we define $W^{cs}(\mu)$ and $W^{cu}(\mu)$. For the existence and smoothness of these invariant manifolds, we refer the readers to Carr [1], Chow and Hale [2], Chow and Lu [3, 4], Chow, Lin and Lu [5], Hirsch, Pugh and Shub [9] and van Gils and Vanderbauwhede [16].

Consider the following hypotheses:

(i) $f \in C^k$, $k \geq 4$. $D_\mu f(x,\mu)$ is uniformly bounded in $\mathbb{R}^n \times M$.

(ii) $f(0,0) = 0$ and $\sigma(D_x f(0,0)) = \{\lambda_{-n}, \cdots, \lambda_{-1}, \lambda_0, \lambda_1, \cdots, \lambda_m\}$, with $\operatorname{Re}\lambda_{-n} \leq \cdots \leq \operatorname{Re}\lambda_{-1} < \lambda_0 = 0 < \operatorname{Re}\lambda_1 \leq \cdots \leq \operatorname{Re}\lambda_m$, $n + m + 1 = d$,

(iii) $\lambda_0 = 0$ is simple with right and left eigenvectors e_r and e_ℓ,

(iv) for $\mu = 0$, (2.1) has a homoclinic orbit $\Gamma_0 : x = q(t)$ such that $q(t) \in W^c_{loc}(0)$ as $t \to -\infty$ and $q(t) \in W^{ss}_{loc}(0;0)$ as $t \to +\infty$,

(v) $\dot q(t)/|\dot q(t)| \to e_r$ as $t \to -\infty$ and $e_\ell \cdot e_r > 0$,

(vi) $e_\ell \cdot D_{xx} f(0,0)(e_r, e_r) > 0$,

(vii) $e_\ell \cdot D_{\mu 1} f(0,0) > 0, \quad e_\ell \cdot D_{\mu 2} f(0,0) = 0$,

(viii) $W^{cs}(0)$ and $W^{cu}(0)$ intersect transversely along Γ_0.

By Theorem A in the Appendix (§7), even though local center stable or unstable manifolds are not unique, assumption (viii) is meaningful and is independent of the choice of local center stable (or unstable) manifolds.

By (ii), $\dim W^c_{loc}(\mu) = 1$ for all small μ. By (vi) and (vii), we have a generic saddle-node bifurcation of $x = 0$ at $\mu = (0,0)$. In fact, by choosing a suitable coordinate system for μ (see Lemma 4.2 in §4) we may assume that for $\mu_1 < 0$, there exist two equilibria on $W^c_{loc}(\mu)$; one is a source $S(\mu)$ in $W^c_{loc}(\mu)$, another is a node $N(\mu)$ in $W^c_{loc}(\mu)$. $S(\mu)$ and $N(\mu)$ are hyperbolic equilibria in $\mathbb{R}^d$. For $\mu_1 = 0$, $S(\mu)$ and $N(\mu)$ collapse and form a saddle-node $SN(\mu)$ on $W^c_{loc}(\mu)$. For $\mu_1 > 0$, there are no equilibria on $W^c_{loc}(\mu)$. The solutions on $W^c_{loc}(0;\mu)$ are denoted by $q_1(t,\mu)$, $q_2(t,\mu)$ and $q_3(t,\mu)$ (see in Figure 1.). Note that if $\mu_1 > 0$, then there exists a unique solution $q_1(t,\mu)$ in $W^c_{loc}(\mu)$.

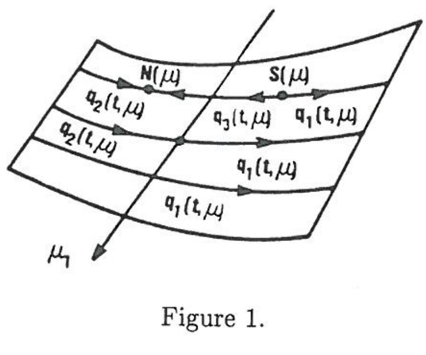

Figure 1.

For a homoclinic solution $q(t,\mu)$ of equation (2.1), we say that the orbit $\Gamma(\mu)$ of $q(t,\mu)$ is tangent to $W^c(\mu)$ (or $W^s(a;\mu)$, or $W^u(a;\mu)$) at $t = \pm\infty$ if the tangent space of $\Gamma(\mu)$ approaches the tangent space of $W^c(\mu)$ (or $W^s(a;\mu)$, or $W^u(a;\mu)$) as $t \to \pm\infty$. Our main result is the following:

Theorem 2.1. *There exists a C^1 function $G(\mu)$ (see (5.7) in §5) such that if $D_{\mu 2}G(0) > 0$ (explicit formula for $D_{\mu 2}G(0)$ can be found in (5.10)) then there exists C^1 change of variables in the parameter space M such that bifurcation of phase flows of (2.1) in a neighborhood of $x = 0$ is completely determined by the*

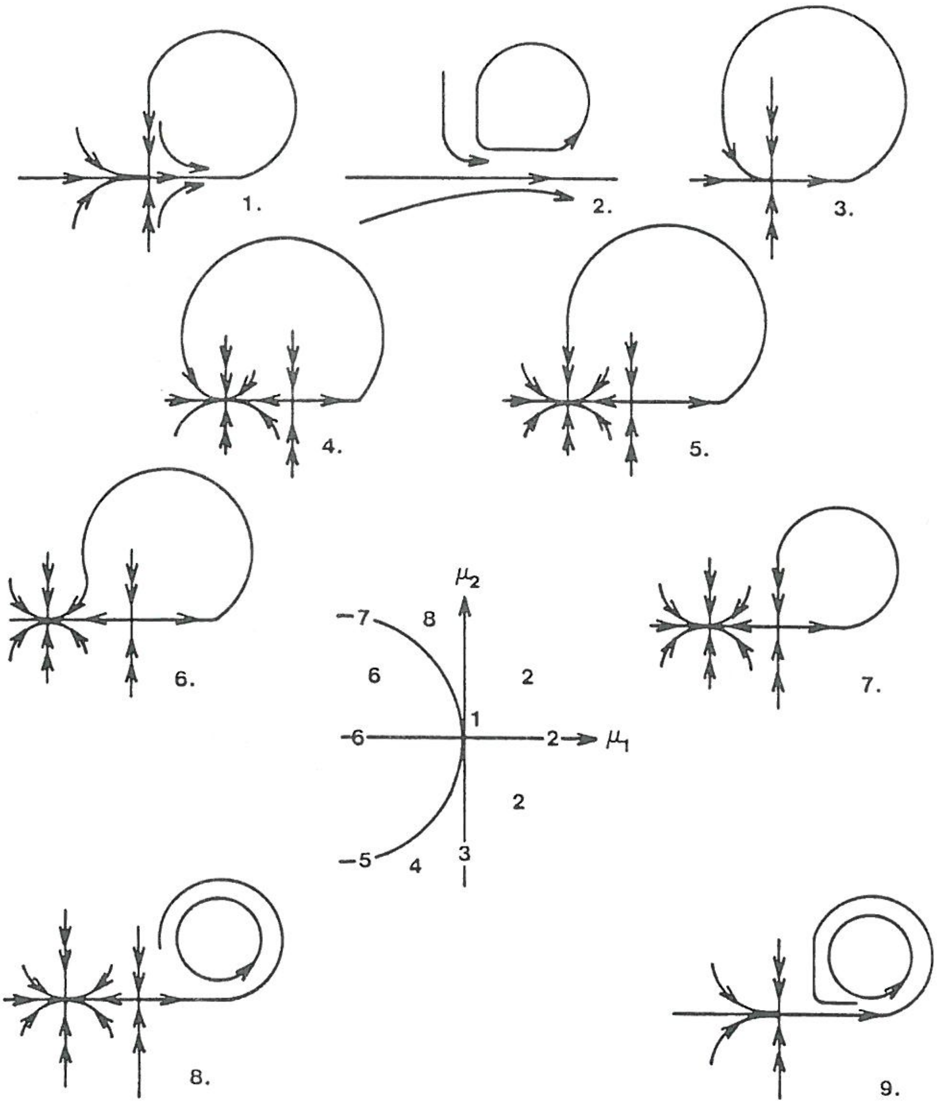

Figure 2. Unfolding of saddle-node homoclinic.

signs of μ_1, μ_2 and a C^1 curve $\mu_1 = \xi(\mu_2)$ which is quadratically tangent ot μ_2 axis and is contained in the half plane $\{\mu \mid \mu_1 \le 0\}$. The bifurcation diagram is depicted in (μ_1, μ_2)–space (Figure 2) and in phase space (Figures 2.1–2.9). There are 9 cases:

Case 1. *Figure 2.1: $\mu_1 = \mu_2 = 0$. There exists a saddle-node equilibrium and a homoclinic orbit Γ_0 such that $\Gamma_0 \in W^c_{loc}(0)$ as $t \to -\infty$ and $\in W^{ss}_{loc}(0)$ as $t \to +\infty$.*

Case 2. *Figure 2.2: $\mu_1 > 0$. There are no equilibria and there exists a unique periodic orbit Π_μ near Γ_0.*

Case 3. *Figure 2.3: $\mu_1 = 0$, $\mu_2 < 0$. There exists a saddle-node equilibrium $SN(\mu)$ and a homoclinic orbit $\Gamma(\mu)$ near Γ_0 such that $\Gamma(\mu)$ is tangent to $W^c_{loc}(\mu)$ at $t = -\infty$ and $\Gamma(\mu) \in W^{cs}_{loc}(\mu)$ as $t \to +\infty$*

Case 4. *Figure 2.4: $\xi(\mu_2) < \mu_1 < 0$, $\mu_2 < 0$. There are* (a) *two equilibria, a saddle $S(\mu)$, a node $N(\mu)$ in $W^c_{loc}(\mu)$;* (b) *a (local) heteroclinic orbit $\Gamma_{loc}(\mu) \subseteq W^c_{loc}(\mu)$ which is the orbit of $q_3(t,\mu)$ in Fig. 1 that goes from $S(\mu)$ to $N(\mu)$;* (c) *a (global) heteroclinic orbit $\Gamma(\mu)$ close to Γ_0 that goes from $S(\mu)$ to $N(\mu)$ and $\Gamma(\mu)$ is tangent to the orbit of $q_1(t,\mu)$ in $W^c_{loc}(\mu)$ at $t = -\infty$ and $\Gamma(\mu)$ is tangent to the orbit of $q_2(t,\mu)$ in $W^c_{loc}(\mu)$ at $t = +\infty$.*

Case 5. *Figure 2.5: $\mu_1 = \xi(\mu_2) < 0$, $\mu_2 < 0$. This case is topologically equivalent to Case 4 and $\Gamma(\mu) \in W^{ss}_{loc}(N(\mu);\mu)$ as $t \to +\infty$.*

Case 6. *Figure 2.6: $\mu_1 < \xi(\mu_2) < 0$. This case is topologically equivalent to Case 4 and $\Gamma(\mu)$ is tangent to the orbit of $q_3(t,\mu)$ in $W^c_{loc}(\mu)$ at $t = +\infty$.*

Case 7. *Figure 2.7: $\mu_1 = \xi(\mu_2) < 0$, $\mu_2 > 0$. This case is the same as Case 4 except that the global heteroclinic orbit becomes a homoclinic orbit $\Gamma(\mu)$ which is tangent to $W^c_{loc}(\mu)$ at $t = -\infty$ and is in $W^{ss}_{loc}(S(\mu);\mu)$ as $t \to +\infty$.*

Case 8. *Figure 2.8: $\xi(\mu_2) < \mu_1 < 0$, $\mu_2 > 0$. There exist $S(\mu)$, $N(\mu)$ and $\Gamma_{loc}(\mu)$ as in Case 4; however, there is no homoclinic or heteroclinic orbit $\Gamma(\mu)$ close to Γ_0. There is a periodic orbit $\Pi(\mu)$ near Γ_0.*

Case 9. *Figure 2.9: $\mu_1 = 0$, $\mu_2 > 0$. There exists a saddle-node equilibrium $SN(\mu)$ on $W^c_{loc}(\mu)$. There is no heteroclinic or homoclinic orbit, but there is a periodic orbit $\Pi(\mu)$ near Γ_0.*

In all the cases, one and only one periodic, homoclinic or heteroclinic orbit can bifurcate from Γ_0. The periodic orbit $\Pi(\mu)$, if it exists, is hyperbolic with $\dim W^u(\Pi(\mu)) = \dim W^{cu}(0)$, where $W^u(\Pi(\mu))$ is the unstable manifold of the peridic orbit $\Pi(\mu)$.

Remark 2.2. For simplicity, our main result is stated in a two dimensional parameter space. Let $\tilde{f}$ be a C^k vector field in a neighborhood of f. We can show that there exist C^1 nonlinear functionals $\mu_1^*(\tilde{f})$ and $\mu_2^*(\tilde{f})$ such that $D\mu_1^*(f)$ and $D\mu_2^*(f)$ are linearly independent and the bifurcation diagram is completely determined by μ_1^* and μ_2^*.

3. Exponential trichotomies and Fredholm operators. Consider a linear nonautonomous system

$$\dot{x}(t) - A(t)x(t) = h(t) \tag{3.1}$$

and its associated homogeneous system

$$\dot{x}(t) - A(t)x(t) = 0. \tag{3.2}$$

Assume that $A(t)$ is a continuous $n \times n$ matrix function of $t \in \mathbb{R}$. Let $X(t,s)$ be a fundamental matrix of equation (3.2). Assume that $X(t,t) = I$, the identity matrix, for all $t \in \mathbb{R}$. Note that $X(t,\tau)X(\tau,s) = X(t,s)$, for all $t, \tau, s \in \mathbb{R}$. We say that $X(t,s)$ (or equation 3.2)) has an exponential trichotomy on an interval $J \subseteq \mathbb{R}$, is there exist constants $\alpha < \nu - \epsilon < \nu + \epsilon < \beta$ and $K > 0$ and continuous projections $P_u(t)$, $P_s(t)$ and $P_c(t)$ satisfying $I = P_c(t) + P_u(t) + P_s(t)$, for all $t \in J$ and

$$X(t,s)P_i(s) = P_i(t)X(t,s), \qquad \text{for } i = c, u, s \tag{3.3}$$

and $t, s \in J$ such that for all $t, s \in J$

$$\begin{aligned} |X(t,s)P_s(s)| &\le Ke^{\alpha(t-s)}, \quad t \ge s \\ |X(s,t)P_u(t)| &\le Ke^{-\beta(t-s)}, \quad t \ge s \\ |X(t,s)P_c(s)| &\le Ke^{(\nu+\epsilon)(t-s)}, \quad t \ge s \\ |X(s,t)P_c(t)| &\le Ke^{(-\nu+\epsilon)(t-s)}, \quad t \ge s. \end{aligned} \tag{3.4}$$

The adjoint system of (3.2) is

$$\dot{y}(t) + A^*(t)y(t) = 0 \tag{3.5}$$

which has a fundamental matrix $Y(t,s) = [X(t,s)^*]^{-1} = X(s,t)^*$. If $X(t,s)$ has an exponential trichotomy on J, then $Y(t,s)$ also has an exponential trichotomy on J with projections being $P_u^*(t)$, $P_s^*(t)$ and $P_c^*(t)$, $t \in J$. Properties similar to (3.3) and (3.4) may be obtained by simply taking adjoints in (3.3) and (3.4).

Lemma 3.1. *Let $a \in \mathbb{R}^n$ be an equilibrium of a nonlinear equation*

$$\dot{x}(t) = f(x(t)), \tag{3.6}$$

and $q(t)$ be a solution of (3.6). *Suppose that $q(t) \to a$ as $t \to +\infty$ (or $t \to -\infty$); then the linearized equation*

$$\dot{x}(t) - D_x f(q(t))x(t) = 0 \tag{3.7}$$

has an exponential trichotomy on $[0, +\infty)$ (or $(-\infty, 0]$). Moreover, let $\eta > 0$ be so small that if $\lambda \in \sigma\{D_x f(a)\}$ and $-\eta \le \operatorname{Re}\lambda \le \eta$, then $\operatorname{Re}\lambda = 0$. Then one can choose $\nu = 0$, $\alpha < -\eta < -\epsilon < +\epsilon < \eta < \beta$ in (3.4), *provided that the constant $K > 0$ is sufficiently large. Furthermore, $\dim P_i(t) = \dim W^i(a)$, where $i = u, c, s$.*

Proof: See Hale and Lin [7] (Lemma 4.3). ∎

Let γ_1 and γ_2 be two real constants. Let $C^0(\gamma_1, \gamma_2)$ denote the set of continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ such that $|x(t)|e^{-\gamma_1 t}$, $t \le 0$ and $|x(t)|e^{-\gamma_2 t}$, $t \ge 0$ are finite. The set of such functions form a Banach space $C^0(\gamma_1, \gamma_2)$ with the norm

$$|x|_{C^0(\gamma_1,\gamma_2)} = \sup_{t \ge 0}\{|x(t)|e^{-\gamma_2 t},\ |x(-t)|e^{\gamma_1 t}\}.$$

Let $C^k(\gamma_1, \gamma_2)$ be the Banach space of all the C^k functions $x(t)$ such that $D^i x(\cdot) \in C^0(\gamma_1, \gamma_2)$, $i = 0, 1, \cdots k$, with the norm

$$|x|_{C^k(\gamma_1,\gamma_2)} = \sum_{i=0}^{k} |D^i x|_{C^0(\gamma_1,\gamma_2)}.$$

Suppose a and $b \in \mathbb{R}^n$ are two equilibria for equation (3.6) (possibly $a = b$), and $q(t)$ is a solution of (3.6) with $q(t) \to a$ as $t \to -\infty$ and $q(t) \to b$ as $t \to +\infty$. Let $\gamma > 0$ be so small that if $\lambda \in \sigma\{D_x f(a)\} \cup \sigma\{D_x f(b)\}$ and $-\gamma \le \operatorname{Re}\lambda \le \gamma$, then $\operatorname{Re}\lambda = 0$. Then we have the following lemma.

Lemma 3.2. *Let $\mathcal{F} : C^1(\pm\gamma, \pm\gamma) \to C^0(\pm\gamma, \pm\gamma)$ be a linear operator defined by $\mathcal{F}(y) = h$, where $y \in C^1(\pm\gamma, \pm\gamma)$ and $h \in C^0(\pm\gamma, \pm\gamma)$ are related by the following equation:*

$$\dot{y}(t) - D_x f(q(t))y(t) = h(t), \tag{3.8}$$

and "$\pm\gamma$" means we may choose either $+\gamma$ or $-\gamma$. Then $\mathcal{F}$ is a Fredholm operator, and index of $\mathcal{F}$ is determined by the following relations:

(1) *If $\mathcal{F} : C^1(\gamma, \gamma) \to C^0(\gamma, \gamma)$, then $\operatorname{Ind}\mathcal{F} = \dim W^u(a) - \dim W^u(b)$;*
(2) *If $\mathcal{F} : C^1(\gamma, -\gamma) \to C^0(\gamma, -\gamma)$, then $\operatorname{Ind}\mathcal{F} = \dim W^u(a) - \dim W^{cu}(b)$;*
(3) *If $\mathcal{F} : C^1(-\gamma, \gamma) \to C^0(-\gamma, \gamma)$, then $\operatorname{Ind}\mathcal{F} = \dim W^{cu}(a) - \dim W^u(b)$;*
(4) *If $\mathcal{F} : C^1(-\gamma, -\gamma) \to C^0(-\gamma, -\gamma)$, then $\operatorname{Ind}\mathcal{F} = \dim W^{cu}(a) - \dim W^{cu}(b)$.*

The range of $\mathcal{F}$ is determined by:

$$\mathcal{R}(\mathcal{F}) = \Big\{ h : \int_{-\infty}^{+\infty} \psi(t)^* h(t)dt = 0, \text{ for all solutions } \psi(t)$$
$$\text{of the adjoint equation of (3.8) and } \psi \in C^0(\pm\gamma, \pm\gamma) \Big\},$$

where the sign of γ for $\psi \in C^0(\pm\gamma, \pm\gamma)$ is chosen to be the opposite of the sign of γ in the range of $\mathcal{F} \subseteq C^0(\pm\gamma, \pm\gamma)$.

Proof: See Hale and Lin [7] (Lemma 4.5). ∎

Remark. If $h \in \mathcal{R}(\mathcal{F}) \subseteq C^0(\gamma, -\gamma)$ and $\psi \in C^0(-\gamma, \gamma)$ is a solution of the adjoint equation of (3.8), then $\psi \in C^0(-\gamma + \delta, \gamma - \delta)$ for some $\delta > 0$ and

$$\int_{-\infty}^{+\infty} \psi(t)^* h(t)dt$$

converges. The same remark holds for all other choices of $\pm\gamma$.

4. Center manifolds, fiberations and heteroclinic orbits.

Consider equation (2.2):

$$\begin{cases} \dot{x} = f(x, \mu) \\ \dot{\mu} = 0. \end{cases} \tag{4.1}$$

From the conditions on the eigenvalues of $D_x f(0, 0)$, there is a linear change of coordinates such that $x = (y, w)$, $y \in \mathbb{R}$, $w \in \mathbb{R}^{m+n}$, and (4.1) becomes

$$\begin{cases} \dot{y} = h(y, w, \mu) \\ \dot{w} = Aw + g(y, w, \mu) \\ \dot{\mu} = 0 \end{cases} \tag{4.2}$$

where h is real-valued, $w = (u, v)$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, $A = \operatorname{diag}(A_1, A_2)$, A_1 is an $m \times m$ matrix and A_2 is an $n \times n$ matrix, $\operatorname{Re}\sigma(A_1) > 0$, $\operatorname{Re}\sigma(A_2) < 0$, $g = (g_1, g_2)$, $h(0, 0, 0) = 0$, $g(0, 0, 0) = 0$, $D_{(y,w)}h(0, 0, 0) = 0$, $D_{(y,w)}g(0, 0, 0) = 0$.

By assumption (ii), $\sigma(A_1) = \{\lambda_1, \cdots, \lambda_m\}$ and $\sigma(A_2) = \{\lambda_{-n}, \cdots, \lambda_{-1}\}$. Hypothesis (vi) implies that $\partial^2 h(0, 0, 0)/\partial y^2 > 0$, and (vii) implies that $\partial h(0, 0, 0)/\partial \mu_1 > 0$, and $\partial h(0, 0, 0)/\partial \mu_2 = 0$.

We are interested in the flow of (4.2) near $\mu = 0$ and $x = 0$. By a C^k change of coordinates, we may assume in a small neighborhood of the origin $x = 0$ that

$$\begin{aligned} W^{cs}_{loc}(\mu) &= \{x \mid x = (y,w),\ u = 0\}, \\ W^{cu}_{loc}(\mu) &= \{x \mid x = (y,w),\ v = 0\}, \quad \text{and} \\ W^{c}_{loc}(\mu) &= \{x \mid x = (y,w),\ u = 0,\ v = 0\}. \end{aligned}$$

We now perform another change of variable to facilitate further investigation. It is known that $W^c_{loc}(\mu) \subset W^{cs}_{loc}(\mu)$, and $W^{cs}_{loc}(\mu)$ is invariantly fibered by C^k submanifolds $\mathcal{W}^s_{(y,0)}(\mu)$ which passes through $x = (y,0) \in W^c_{loc}(\mu)$. We shall use $\mathcal{W}^s_y(\mu)$ to denote the fibers. Any two points x_1 and $x_2 \in \mathcal{W}^s_y(\mu)$ satisfy the property that

$$|T(t;\mu)x_1 - T(t;\mu)x_2| \le Ce^{\gamma t} \quad \text{as } t \to +\infty,$$

where $T(t;\mu)$ is the flow generated by equation (2.1), $\sup \operatorname{Re} \sigma(A_2) = \lambda_{-1} < \gamma < 0$ and γ can be chosen to be as close, but not equal, to $\sup \operatorname{Re} \sigma(A_2)$ as we want by allowing the constant C to be large. Note that points in $W^c_{loc}(\mu)$ are not contracted as fast. We may write in a small neighborhood of $x = 0$:

$$\mathcal{W}^s_{y_0}(\mu) = \{(y,0,v) \mid y = y_0 + \phi(y_0,v,\mu)\}.$$

It is known that $\phi(y_0,v,\mu)$ is C^{k-1} in the variables (y_0,v,μ). Since $\phi(y_0,0,\mu) = 0$ and the nonlinear terms g and h in equation (4.2) are small together with all the derivatives up to order $k-1$, by using the fibers as coordinates on $W^{cs}_{loc}(\mu)$ we can make the fibers flat. Consequently, $h(y,0,v,\mu) = h(y,0,0,\mu)$. Similarly, $W^{cu}(\mu)$ is invariantly fibered by $\mathcal{W}^u_y(\mu)$. Using fibers as coordinates, we have $h(y,u,0,\mu) = h(y,0,0,\mu)$. Thus, we have the following.

Lemma 4.1. *There exists a small neighborhood of $x = 0$ and a C^{k-1} change of coordinates such that in the new coordinate system and in that neighborhood we have for equation* (4.2)

$$\begin{aligned} W^{cs}_{loc}(\mu) &= \{x \mid x = (y,w),\ u = 0\}, \\ W^{cu}_{loc}(\mu) &= \{x \mid x = (y,w),\ v = 0\}, \quad \textit{and} \\ W^{c}_{loc}(\mu) &= \{x \mid x = (y,w),\ u = 0,\ v = 0\}. \end{aligned}$$

Moreover, $W^{cs}_{loc}(\mu)$ is invariantly fibered by

$$\mathcal{W}^s_{y_0}(\mu) = \{(y,w) \mid y = y_0,\ u = 0\},$$

$W^{cu}(\mu)$ is invariantly fibered by

$$\mathcal{W}^u_{y_0}(\mu) = \{(y,w) \mid y = y_0,\ v = 0\},$$

and the nonlinear terms in equation (4.2) *satisfy: $h(y,u,0,\mu) = h(y,0,v,\mu) = h(y,0,0,\mu)$, $g_1(y,0,v,\mu) = 0$ and $g_2(y,u,0,\mu) = 0$.*

Proof: Although the results presented here are not new, see Hirsch, Pugh and Shub [9] for mappings, and Takens (1971), Fenichel (1974) for flows. We give a proof for the C^{k-1} foliation on $W^{cs}_{loc}(\mu)$ by $\mathcal{W}^s_{y_0}(\mu)$ in Theorem B of Appendix B of this paper for completeness. The idea of the proof is from Chow, Lin and Lu [5] on a more general case. ∎

We now look for equilibria of (4.2) in a small neighborhood of $x = 0$. This is equivalent to solving

$$h(y, 0, 0, \mu) = 0.$$

Here we have an elementary saddle-node bifurcation in $\mathbb{R}^1$ and the following is well known.

Lemma 4.2. *There is a C^{k-1} change of coordinates in the parameter space such that in the new coordinate system, (4.2) has a saddle-node equilibrium $SN(\mu)$ for $\mu_1 = 0$; two equilibria $S(\mu)$ and $N(\mu)$ for $\mu_1 < 0$, $N(\mu)$ is an attractor and $S(\mu)$ is a repeller on $W^c_{loc}(\mu)$; no equilibria for $\mu_1 > 0$.*

Proof: See Chow and Hale [2]. ∎

We note that $S(\mu)$ (or $N(\mu)$) is not defined for $\mu_1 > 0$ and is not smooth at $\mu_1 = 0$. We now introduce another coordinate system to obtain smooth dependence on parameters. Let $\mu_1 = -\delta_1^2$ and $\mu_2 = \delta_2$ in the half space $\mu_1 \leq 0$. Then there is a C^{k-3} function $r : M \to \mathbb{R}^{1+m+n}$ such that $\partial r_1(0,0)/\partial\delta_1 > 0$ and $r_2(\delta_1, \delta_2) = 0$, where $r = (r_1, r_2)$, $r_1 \in \mathbb{R}$ and $r_2 \in \mathbb{R}^{m+n}$. Moreover,

$$N(\mu) = N(\delta_1, \delta_2) = \begin{cases} r(\delta_1, \delta_2) & \text{if } \delta_1 < 0 \\ r(-\delta_1, \delta_2) & \text{if } \delta_1 > 0, \end{cases}$$

$$S(\mu) = S(\delta_1, \delta_2) = \begin{cases} r(\delta_1, \delta_2) & \text{if } \delta_1 > 0 \\ r(-\delta_1, \delta_2) & \text{if } \delta_1 < 0, \end{cases}$$

$$SN(\mu) = SN(\delta_1, \delta_2) = r(0, \delta_2).$$

The specific fibers $\mathcal{W}^s_y(\mu)$, $\mu_1 \leq 0$, passing through $(y, 0) = N(\mu)$, $S(\mu)$ or $SN(\mu)$ will be very useful for determining the bifurcation diagram of (4.2). In the (δ_1, δ_2) coordinates, these fibers are $\mathcal{W}^s_{r(\delta_1,\delta_2)}(\mu)$ and $\mathcal{W}^s_{r(-\delta_1,\delta_2)}(\mu)$.

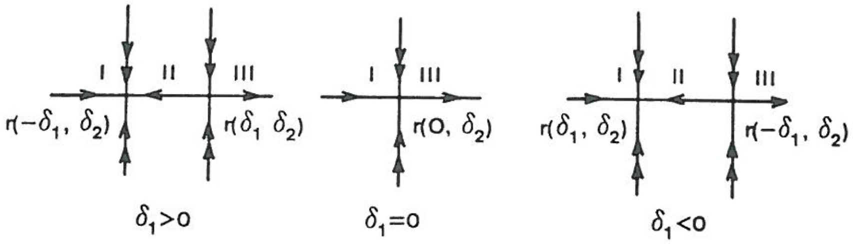

Figure 3.

Figure 3 shows the flows on $W^c_{loc}(\mu)$ for $\mu_1 \leq 0$. The horizontal direction corresponds to y in the center manifold. Note that $S(\delta_1, \delta_2)$ is always on the right of $N(\delta_1, \delta_2)$ in $W^c_{loc}(\mu)$. The local center stable manifold $W^{cs}_{loc}(\mu)$ is divided into three regions, I, II and III: region I consists of all the points that are to the left of $\mathcal{W}^s_{r(\delta_1,\delta_2)}(\delta_1,\delta_2)$ and $\mathcal{W}^s_{r(-\delta_1,\delta_2)}(\delta_1,\delta_2)$; region III consists of all the points that are to the right of $\mathcal{W}^s_{r(\delta_1,\delta_2)}(\delta_1,\delta_2)$ and $\mathcal{W}^s_{r(-\delta_1,\delta_2)}(\delta_1,\delta_2)$; region II consists of all the points that are in between $\mathcal{W}^s_{r(\delta_1,\delta_2)}(\delta_1,\delta_2)$ and $\mathcal{W}^s_{r(-\delta_1,\delta_2)}(\delta_1,\delta_2)$. For the case $\delta_1 = 0$, $N(\delta_1,\delta_2)$ and $S(\delta_1,\delta_2)$ collapse into $SN(0,\delta_2)$ and II disappears while I and III persist.

Consider the homoclinic orbit Γ_0. By assumption (iv), in a neighborhood of 0, Γ_0 has two connected parts, Γ_0^- and Γ_0^+. $\Gamma_0^- \subseteq W^c_{loc}(0)$ as $t \to -\infty$ and $\Gamma_0^+ \subseteq W^{ss}_{loc}(0;0)$ as $t \to +\infty$. Let Σ be a transverse cross section to Γ_0^+. Assume that $\mathcal{U} \subset \Sigma$ is a small $(d-1)$ dimensional disc centered at $\Gamma_0^+ \cap \Sigma$. We have that $\Gamma_0^+ \cap \mathcal{U}$ is a unique isolated point if $\mathcal{U}$ is sufficiently small. Under assumption (viii) in §2, $W^{cu}(0)$ and $W^{cs}(0)$ intersect transversely along Γ_0. Thus, for $|\mu|$ sufficiently small, $\mathcal{U}$ intersects $W^{cs}(\mu)$ transversely and the intersection is an n-dimensional submanifold. Also, for each fiber $\mathcal{W}^s_x(\mu)$, $\mathcal{U} \cap \mathcal{W}^s_x(\mu)$ is an $(n-1)$-dimensional submanifold. We have a similar situation for $\mathcal{U} \cap W^{cu}(\mu)$. Assumption (viii) and the smooth dependence of $W^{cu}_{loc}(\mu)$ and $W^{cs}_{loc}(\mu)$ on μ imply that for $|\mu|$ sufficiently small $\{\mathcal{U} \cap W^{cu}(\mu)\}$ and $\{\mathcal{U} \cap W^{cs}(\mu)\}$ intersect transversely and the intersection is a unique isolated point

$$\{E(\mu)\} = \{\mathcal{U} \cap W^{cu}(\mu)\} \cap \{\mathcal{U} \cap W^{cs}(\mu)\}$$

which is C^k in μ. Let the first component of $E(\mu)$ be $E(\mu)_1$ which is also C^k in μ. Let r_1 be the first component of $r(\delta_1,\delta_2)$ as before. This gives the following.

Lemma 4.3. *If $\mu_1 \le 0$ and $-\mu_1 = \delta_1^2$, $\delta_1 \ge 0$, then*

$E(\mu) \in$ region I if $E(\mu)_1 < r_1(-\delta_1,\delta_2)$

$E(\mu) \in$ region II if $r_1(-\delta_1,\delta_2) < E(\mu)_1 < r_1(\delta_1,\delta_2)$

$E(\mu) \in$ region III if $E(\mu)_1 > r_1(\delta_1,\delta_2)$

$E(\mu) \in W^{SS}_{loc}(N(\mu);\mu)$ if $E(\mu)_1 = r_1(-\delta_1,\delta_2)$

$E(\mu) \in W^{SS}_{loc}(S(\mu);\mu)$ if $E(\mu)_1 = r_1(\delta_1,\delta_2)$.

Similar results hold for $\delta_1 \le 0$.

Corollary 4.4. *For equation (4.2), in a neighborhood of Γ_0,*

(a) *there exists a homoclinic orbit $\Gamma(\mu) \subseteq W^{SS}_{loc}(S(\mu);\mu)$ as $t \to +\infty$ and $\Gamma(\mu) \subseteq W^u_{loc}(S(\mu);\mu)$ as $t \to -\infty$ if and only if $\mu_1 < 0$ and $E(\mu)_1 = r_1(\delta_1,\delta_2)$, $\delta_1 > 0$;*

(b) *there exists a heteroclinic orbit $\Gamma(\mu) \subseteq W^{SS}_{loc}(N(\mu);\mu)$ as $t \to +\infty$ and $\Gamma(\mu) \subseteq W^u_{loc}(S(\mu);\mu)$ as $t \to -\infty$ if and only if $\mu_1 < 0$, $E(\mu)_1 = r_1(-\delta_1,\delta_2)$, $\delta_1 > 0$;*

(c) *there exists a heteroclinic orbit $\Gamma(\mu) \subseteq W^S_{loc}(N(\mu);\mu)$ which is tangent to the orbit of $q_2(t,\mu)$ in region I (see Figs. 1 and 3) at $t = +\infty$ and $\Gamma(\mu) \subseteq W^u_{loc}(S(\mu);\mu)$ as $t \to -\infty$ if and only if $\mu_1 < 0$, $E(\mu)_1 < r_1(-\delta_1,\delta_2)$, $\delta_1 > 0$;*

(d) *there exists a heteroclinic orbit $\Gamma(\mu) \subseteq W^S_{loc}(N(\mu);\mu)$ which is tangent to the orbit of $q_3(t,\mu)$ in region II (see Figs. 1 and 3) at $t = +\infty$ and $\Gamma(\mu) \subseteq W^u_{loc}(S(\mu);\mu)$ as $t \to -\infty$ if and only if $\mu_1 < 0$, $r_1(-\delta_1,\delta_2) < E(\mu)_1 < r_1(\delta_1,\delta_2)$, $\delta_1 > 0$;*

(e) *there exists a homoclinic orbit to $SN(\mu)$, which is tangent to the orbit of $q_2(t,\mu)$ in region I (see Figs. 1 and 3) at $t = +\infty$ and is tangent to the orbit of $q_1(t,\mu)$ in region III (see Figs. 1 and 3) at $t = -\infty$ if and only if $\delta_1 = 0$ and $E(\mu)_1 < r_1(0,\delta_2)$.*

Proof: We only need to see that $\Gamma(\mu) \subseteq W^u_{loc}(S(\mu);\mu)$ as $t \to -\infty$. This can be shown by continuous dependence. ∎

Corollary 4.4 allows us to relate the bifurcation diagram with the sign of the C^{k-3} functions of μ. In §5, we shall show that the conditions in Lemma 4.3 and Corollary 4.4 can be described by an integral along Γ_0, known as the bifurcation function.

5. Liapunov-Schmidt reduction and bifurcation functions. In this section we consider only $\mu_1 \leq 0$. Let $\delta_1^2 = -\mu_1$, $\delta_2 = \mu_2$. Thus $(\mu_1, \mu_2) = (-\delta_1^2, \delta_2)$. Let $E(\mu)$ be the point defined in §4. We will find conditions on μ under which $E(\mu) \in \mathcal{W}^s_{r(\delta_1,\delta_2)}(\mu)$, $\mathcal{W}^s_{r(-\delta_1,\delta_2)}(\mu)$, region I, region II, or region III. First, we will find μ for which $E(\mu) \in \mathcal{W}^s_{r(\delta_1,\delta_2)}(\mu)$. Assume that $\gamma > 0$ is a constant, $\gamma < \min\{\operatorname{Re}\lambda_1, -\operatorname{Re}\lambda_{-1}\}$ (see assumption (ii) in §2). It is clear that in this case, there exists a heteroclinic (or homoclinic) solution $x(t)$ of (2.1), $|x(t) - r(\delta_1, \delta_2)| \leq Ce^{-\gamma t}$ as $t \to +\infty$ (C is some fixed constant depending on γ), and the orbit of $x(t)$, denoted by $\Gamma(\mu)$ is tangent to that of $q_1(t, \mu)$ as $t \to -\infty$, where $\mu = (-\delta_1^2, \delta_2)$.

Let $\theta_1(t)$ be a smooth real-valued function satisfying $\theta_1(t) = 0$ for $t \geq -1$ and $\theta_1(t) = 1$ for $t \leq -2$. Let $\theta_2(t) = \theta_1(-t)$. The solution $x(t)$ can be written as

$$x(t) = q(t) + \theta_1(t)\big(q_1(t,\mu) - q(t)\big) + \theta_2(t)r(\delta) + z(t) \tag{5.1}$$

where $\mu = (-\delta_1^2, \delta_2)$ and $r(\delta) = r(\delta_1, \delta_2)$. Note that $q_1(t,\mu) = q_1(t, (-\delta_1^2, \delta_2))$ is smooth in δ for t in any compact subset of $\mathbb{R}$. We shall assume that $q_1(t, 0) = q(t)$. Observe that the center manifold is not unique so this can always be achieved by choosing a center manifold to contain $q(t)$. After a phase shift in $x(t)$, we may assume that $|z(t)| \leq Ce^{-\gamma|t|}$ as $t \to \pm\infty$, for some constant $C \geq 1$. This suggests that we search for a solution $x(t)$ in the form (5.1) with $z \in C^1(\gamma, -\gamma)$. Substituting (5.1) into (2.1), the equation for $z(t)$ becomes

$$\dot{z}(t) - D_x f(q(t), 0)z(t) = N(z, \delta)(t) \tag{5.2}$$

with

$$\begin{aligned} N(z,\delta)(t) &= f(q(t) + \theta_1(t)(q_1(t,\mu) - q(t)) + \theta_2(t)r(\delta) + z(t), \delta) \\ &\quad - f(q(t), 0) - \frac{\partial}{\partial t}(\theta_1(t)(q_1(t,\mu) - q(t)) + \theta_2(t)r(\delta)) - D_x f(q(t),0)z(t). \end{aligned}$$

It is not hard to see that for small $z \in C^0(\gamma, -\gamma)$, $N(z, \delta) \in C^0(\gamma, -\gamma)$ and

$$\big|N(z,\delta)\big|_{C^0(\gamma,-\gamma)} = O\Big(\big|z\big|^2_{C^0(\gamma,-\gamma)} + |\delta|\Big).$$

Moreover, $N : C^0(\gamma, -\gamma) \to C^0(\gamma, -\gamma)$ is C^k and is continuous with respect to $\delta \in M$. By Lemma 3.2, (5.2) can be written as

$$\mathcal{F}z = N(z, \delta) \tag{5.3}$$

where $\mathcal{F} : C^1(\gamma, -\gamma) \to C^0(\gamma, -\gamma)$ is Fredholm with Fredholm index $\operatorname{Ind}\mathcal{F} = -1$. From assumption (viii), $x = \dot{q}(t)$ is the only solution of (3.6) which approaches zero as $t \to \pm\infty$. However, $\dot{q}(t) \to 0$ is slower than $e^{\gamma t}$ as $t \to -\infty$. Therefore, the null space of $\mathcal{F}$ is $\{0\}$ and $\operatorname{codim}\mathcal{R}(\mathcal{F}) = 1$. In other words, there exists, up to a scalar factor, a unique nontrivial solution $\psi(t)$ for the adjoint equation

$$\dot{x}(t) + [D_x f(q(t), 0)]^* x(t) = 0 \tag{5.4}$$

with $\psi \in C^1(-\gamma, \gamma)$. We actually know more about $\psi(t)$.

Lemma 5.1. (i) $|\psi(t) - e_\ell| \to 0$ *as* $t \to +\infty$, *where* e_ℓ *is the left eigenvector of the eigenvalue* λ_0 *(assumption (iii))*

(ii) $|\psi(t)| \le Ce^{\gamma t}$ *as* $t \to -\infty$ *where* C *is a constant.*

Proof: Let $A(t) = D_x f(q(t), 0)$ and $A(\infty) = D_x f(0,0)$. We note that $A(t) \to A(\infty)$ as $t \to \pm\infty$. However, $A(t) \to A(\infty)$ exponentially as $t \to \infty$ while $A(t) \to A(\infty)$ at a slower rate which is not exponential as $t \to -\infty$. Let $\mathcal{F}$ be the linear operator in Lemma 3.2. Since we will consider $\mathcal{F}$ as operators on different spaces, we will use subscripts to indicate the differences. Consider $\mathcal{F}_1 : C^1(-\gamma, -\gamma) \to C^0(-\gamma,-\gamma)$, we have $\operatorname{Ind}\mathcal{F}_1 = 0$. Since $\dim \mathcal{N}(\mathcal{F}_1) = 1$ ($\mathcal{N}(\mathcal{F}_1)$ is the null space of $\mathcal{F}_1$); i.e., $\mathcal{N}(\mathcal{F}_1)$ is spanned by $\{\dot{q}(t)\}$. We have $\operatorname{codim} \mathcal{R}(\mathcal{F}_1) = 1$. Therefore, there exists a unique solution (up to scalar factor) $\psi(t)$ of (5.4) such that $\psi \in C^0(\gamma, \gamma)$. This proves (ii) of Lemma 5.1. It remains to show (i). Next consider $\mathcal{F}_2 : C^1(\gamma,\gamma) \to C^0(\gamma,\gamma)$. $\operatorname{Ind}\mathcal{F}_2 = 0$. Since $\dim \mathcal{N}(\mathcal{F}_2) = 0$, we have $\operatorname{codim} \mathcal{R}(\mathcal{F}_2) = 0$. Thus, $\psi(t) \in C^0(\gamma,\gamma) \backslash C^0(-\gamma,-\gamma)$; i.e., $|\psi(t)| \le Ce^{\gamma t}$ as $t \to +\infty$ (slowly growing) but $\psi(t)$ does not approach 0 like $e^{-\gamma t}$ as $t \to +\infty$.

Write (5.4) as $\dot{x}(t) = B(t)x(t)$, $B(t) = -A^*(t)$. Since $|B(t) - B(\infty)| \to 0$ exponentially, we have

$$\int^{\infty} |t|^h |B(t)|\, dt < \infty$$

for any integer h. From a general result given in Theorem 10.13.2 of Hartman [8], to every nontrivial solution $x(t)$ of (5.4), there exists a nontrivial solution $y(t)$ of $\dot{y}(t) = B(\infty)y(t)$ satisfying

$$|x(t) - y(t)| = o(|y(t)|)$$

as $t \to +\infty$. Let $\hat{y}(t)$ be the solution of $\dot{y}(t) = B(\infty)y(t)$ with

$$|\psi(t) - \hat{y}(t)| = o(|\hat{y}(t)|).$$

$\hat{y}(t)$ is slowly growing or bounded, but not decaying like $e^{-\gamma t}$ as $t \to +\infty$. From the properties of linear autonomous equations, it is clear that $\hat{y}(t) \to Ce_\ell$ as $t \to +\infty$, for some $c \neq 0$, and we may set $c = 1$. ∎

Let Q be a projection onto $\mathcal{R}(\mathcal{F})$. Equation (5.3) is equivalent to the following:

$$\begin{cases} \mathcal{F}z - QN(z,\delta) = 0 \\ (I - Q)N(z,\delta) = 0. \end{cases} \tag{5.6}$$

Using the uniform contraction mapping theorem, we obtain a continuous function $\hat{z}(\delta) \in C^0(\gamma, -\gamma)$ such that $\hat{z}(0) = 0$ and $\hat{z}(\delta)$ solves the first equation in (5.6) for small z and δ. Substituting into the second equation in (5.6), we obtain in a neighborhood of $\delta = 0$ a continuous function $G(\delta)$ and the equation:

$$G(\delta) = \int_{-\infty}^{\infty} \psi(t)^* N\big(\hat{z}(\delta), \delta\big)(t)\, dt = 0 \tag{5.7}$$

is equivalent to equation (5.6). We shall call $G(\delta)$ the Melnikov function and $G(\delta) = 0$ the bifurcation equation for the existence of a heteroclinic (or homoclinic) solution of (2.1).

We now indicate how to show that $G(\delta)$ is in fact C^1 in δ. First, it is not hard to see that there is a small $\epsilon > 0$ such that $\hat{z}(\delta) \in C^0(\gamma+\epsilon, -\gamma-\epsilon)$ and $\hat{z}(\delta)$ is continuous in δ in the new norm. We then show $N : C^0(\gamma+\epsilon, -\gamma-\epsilon) \times M \to C^0(\gamma, -\gamma)$ is C^1 with respect to (z, δ) jointly. The difficulty here is that $q_1(t, \mu)$ is not smooth in δ_1 in the uniform norm for $t \in \mathbb{R}^-$. However, observe that for $t \leq -2$,

$$\begin{aligned} f(q(t) &+ \theta_1(q_1(t,\mu) - q(t)) + \theta_2(t)r(\delta) + z(t), \delta) \\ &- \frac{\partial}{\partial t}(\theta_1(t)(q_1(t,\mu) - q(t)) - f(q(t), 0) \\ &= f(q_1(t,\mu) + z(t), \delta) - \frac{\partial}{\partial t} q_1(t,\mu) = \int_0^1 f_x(q_1(t,\mu) + \nu z(t), \delta) z(t)\, d\nu. \end{aligned}$$

Observe that $|\frac{\partial}{\partial \delta} q_1(t,\mu)| e^{\epsilon t} \to 0$ as $t \to -\infty$ for any $\epsilon > 0$. Based on this, it is not hard to show that $N : C^0(\gamma+\epsilon, -\gamma-\epsilon) \times M \to C^0(\gamma, -\gamma)$ is C^1.

We are now able to show $\hat{z}(\delta) \in C^0(\gamma, -\gamma)$ and $N(\hat{z}(\delta), \delta) \in C^0(\gamma, -\gamma)$ are both C^1 in δ. The idea here is similar to the use of scaled Banach spaces in proving the smoothness of center manifolds (see [18]). Hence,

$$\begin{aligned} DG(\delta)\big|_{\delta=0} &= \int_{-\infty}^{\infty} \psi(t)^* \{\frac{\partial N(0,0)}{\partial \delta}(t)\} dt \\ &= \int_{-\infty}^{\infty} \psi(t)^* \Big\{ D_x f(q(t), 0) \Big[\theta_1(t) \frac{\partial q_1(t,\delta)}{\partial \delta} + \theta_2(t) \frac{\partial r(0)}{\partial \delta}\Big] \\ &\quad + D_\mu f(q(t), 0) \cdot \frac{\partial \mu}{\partial \delta} - \frac{\partial}{\partial t}\Big[\theta_1(t)\frac{\partial q_1(t,0)}{\partial \delta} + \theta_2(t)\frac{\partial r(0)}{\partial \delta}\Big]\Big\} dt. \end{aligned}$$

Since $\psi(t)$ is the solution of the adjoint equation, integrating by parts, we obtain for any $t_1 > 0$:

$$\int_{-t_1}^{t_1} \psi(t)^* \Big[D_x f(q(t), 0) \frac{\partial \tilde{q}(t,\mu)}{\partial \delta} - \frac{\partial}{\partial t}\big[\frac{\partial \tilde{q}(t,0)}{\partial \delta}\big]\Big] dt = -\Big[\psi(t)^* \frac{\partial \tilde{q}(t,0)}{\partial \delta}\Big]_{-t_1}^{t_1}, \quad (5.8)$$

where $\tilde{q}(t,\mu) = \tilde{q}(t, \mu(\delta)) = \theta_1(t) q_1(t,\mu) + \theta_2(t) r(\delta)$. Note that $\tilde{q}(t,\mu) = q_1(t,\mu)$ for $t < -2$. Hence, $\partial \tilde{q}(t, \mu(0))/\partial \delta$ satisfies the following equation for $t \leq -\tau$, where $\tau > 0$ is sufficiently large:

$$\dot{w} = D_x f(q_1(t, \mu(0)), 0) w.$$

Since $q_1(t, \mu(0)) = (y(t), 0, 0)$, it follows from Gronwall's inequality that

$$\frac{\partial \tilde{q}(t, \mu(0))}{\partial \delta} = O\big(e^{-L(\epsilon)t}\big), \quad \text{as } t \to -\infty.$$

This implies

$$\lim_{t \to -\infty} \psi(t)^* [\partial \tilde{q}(t, \mu(0))/\partial \delta] = 0. \quad (5.9a)$$

Next, consider $t > 2$. We have that $\tilde{q}(t,\mu) = r(\delta)$ and $\partial \tilde{q}(t, \mu(0))/\partial \delta \to \partial r(0)/\partial \delta$ as $t \to +\infty$. By Lemma 5.1,

$$\lim_{t \to \infty} \psi(t)^* [\partial \tilde{q}(t, \mu(0))/\partial \delta] = \psi(+\infty)^* \frac{\partial r(0)}{\partial \delta}. \quad (5.9b)$$

It follows from (5.8) and (5.9) that by letting $t_1 \to +\infty$ we have

$$\int_{-\infty}^{\infty} \psi(t)^* \left[D_x f(q(t),0) \frac{\partial \tilde{q}(t,0)}{\partial \delta} - \frac{\partial}{\partial t} \frac{\partial \tilde{q}(t,0)}{\partial \delta} \right] dt = -\psi(+\infty)^* \frac{\partial r(0)}{\partial \delta}.$$

Therefore,

$$DG(0) = -\psi(+\infty)^* \frac{\partial r(0)}{\partial \delta} + \int_{-\infty}^{\infty} \psi(t)^* D_\mu f(q(t),0) \frac{\partial \mu}{\partial \delta} dt$$

and the second integral converges. Since $\frac{\partial \mu}{\partial \delta_1}\big|_{\delta=0} = 0$, we have

$$\frac{\partial G(0)}{\partial \delta_1} = -\psi(+\infty)^* \frac{\partial r(0)}{\partial \delta_1} \neq 0$$

here we use the facts that $\frac{\partial r(0)}{\partial \delta_1} = Ce_r$, $C > 0$, $\psi(+\infty) = e_\ell$ and $e_\ell \cdot e_r > 0$. Hence,

$$\frac{\partial G(0)}{\partial \delta_1} < 0.$$

We assume the following generic hypothesis on G:

$$\frac{\partial G(0)}{\partial \delta_2} = \int_{-\infty}^{\infty} \psi^*(t) D_{\mu 2} f(q(t),0)\, dt > 0. \tag{5.10}$$

We have shown that the existence of a heteroclinic (homoclinic) trajectory connecting $S(\delta)$ to $r(\delta)$ and tangent to W^c as $t \to -\infty$, tangent to $\mathcal{W}^s_{r(\delta)}$ as $t \to +\infty$ is equivalent to $G(\delta) = 0$, which defines a codimension one hypersurface; i.e., curve, in (δ_1, δ_2)-space. The existence of the homoclinic orbit with a saddle-node equilibrium is governed by the system of equations

$$\begin{cases} \delta_1 = 0 \\ G(\delta_1, \delta_2) = 0 \end{cases}$$

which defines two curves intersecting transversely at $\delta = 0$ in (δ_1, δ_2)-space.

In §4 we have shown that the fibers $\mathcal{W}^s_y$ of $W^{cs}(\mu)$ are well-ordered and we can speak of the right or left side of a fiber. We have shown that $E(\mu) \in \mathcal{W}^s_{r(\delta)}(\delta)$ is equivalent to $G(\delta) = 0$. Thus, we know that $E(\mu)$ is to the right of $\mathcal{W}^s_{r(\delta)}(\delta)$ if one of the conditions $G(\delta) > 0$ or $G(\delta) < 0$ is satisfied. Similarly, $E(\mu)$ is to the left of $\mathcal{W}^s_{r(\delta)}(\delta)$ if $G(\delta) < 0$ or $G(\delta) > 0$ is satisfied. Let us fix $\delta_2 = 0$ and increase δ_1 slightly from $\delta_1 = 0$. Thus, $r(\delta)$ moves to the right, like $O(\delta_1)$ and the perturbation of $W^{cu}(\mu)$ is like $O(\delta_1^2)$. Therefore, we expect to have $E(\mu)$ moved to the left of $\mathcal{W}^s_{r(\delta)}(\delta)$ for $\delta = (\delta_1, 0)$ $0 < \delta_1 < \epsilon$. Observe that $\partial G(0)/\partial \delta_1 < 0$. We conclude that $E(\mu)$ is on the left of $\mathcal{W}^s_{r(\delta)}(\delta)$ if and only if $G(\delta) < 0$, and $E(\mu)$ is on the right of $\mathcal{W}^s_{r(\delta)}(\delta)$ if and only if $G(\delta) > 0$. These facts can be proved rigorously, but we shall not render it here.

We know that for every $\mu_1 < 0$, there are two equilibria $r(\delta_1, \delta_2)$ and $r(-\delta_1, \delta_2)$ with $\delta_1^2 = -\mu_1$. So it is obvious that the condition for $E(\mu) \in \mathcal{W}^s_{r(-\delta_1,\delta_2)}(\delta)$ is that

$G(-\delta_1, \delta_2) = 0$. The graph defined by this equation is a submanifold of codimension one in (δ_1, δ_2)-space and is a reflection of the graph of $G(\delta_1, \delta_2) = 0$ with respect to $\delta_1 = 0$. The neighborhood of $(\delta_1, \delta_2) = 0$ is divided by the two submanifold $G(\delta_1, \delta_2) = 0$ and $G(-\delta_1, \delta_2) = 0$ into four parts:

$$\begin{cases} G(\delta_1, \delta_2) > 0 \\ G(-\delta_1, \delta_2) > 0 \end{cases} \quad \text{if and only if } E(\mu) \text{ belongs to Region III;}$$

$$\begin{cases} G(\delta_1, \delta_2) < 0 \\ G(-\delta_1, \delta_2) > 0 \end{cases} \quad \text{or}$$

$$\begin{cases} G(\delta_1, \delta_2) > 0 \\ G(-\delta_1, \delta_2) < 0 \end{cases} \quad \text{if and only if } E(\mu) \text{ belongs to Region II;}$$

$$\begin{cases} G(\delta_1, \delta_2) < 0 \\ G(-\delta_1, \delta_2) < 0 \end{cases} \quad \text{if and only if } E(\mu) \text{ belongs to Region I.}$$

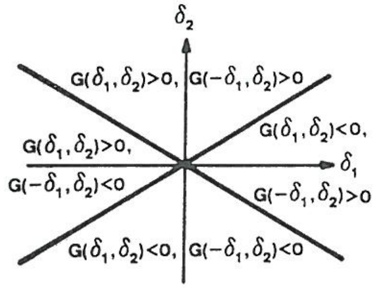

Figure 4.

For a description of Regions I, II, and III see Figure 3 in §4.

The following is the main result of this section.

Lemma 5.2. *Assume that $D_{\mu_2}G(0) > 0$. Then there exists a C^1 curve $\mu_1 = \xi(\mu_2)$ with $\xi(0) = 0$, $\xi(\cdot) \leq 0$ and ξ is quadratically tangential to μ_2-axis at $\mu = 0$. In a neighborhood of 0, if $\mu_1 < \xi(\mu_2)$, $E(\mu) \in$ Region II, if $0 \geq \mu_1 > \xi(\mu_2)$ and $\mu_2 < 0$, then $E(\mu) \in$ Region I; if $0 \geq \mu_1 > \xi(\mu_2)$, and $\mu_2 > 0$, then $E(\mu) \in$ Region III. If $\mu_1 = \xi(\mu_2)$, and $\mu_2 > 0$, then $E(\mu) \in W^{SS}(S(\mu), \mu)$. If $\mu_1 = \xi^*(\mu_2)$ and $\mu_2 < 0$, then $E(\mu) \in W^{SS}(N(\mu), \mu)$.*

Proof: Consider the equation $G(\delta_1, \delta_2) = 0$. Since $G(0,0) = 0$ and $\frac{\partial G(0)}{\partial \delta_1} < 0$, we can solve δ_1 as a function of δ_2; i.e., $\delta_1 = \Lambda(\delta_2)$. Moreover, since $\frac{\partial G(0)}{\partial \delta_2} > 0$, $D\Lambda(0) \neq 0$. Note that $\mu_1 = -\delta_1^2$ and and $\mu_2 = \delta_2$, the desired curve is $\mu_1 = -(\Lambda(\mu_2))^2$. The rest of the proof follows from Figure 4.

6. Periodic orbits. In this section we show that if $\mu_1 > 0$ or if $\mu_1 \leq 0$ and $E(\mu) \in$ Region III, then there is a unique periodic orbit Π_μ bifurcating from Γ_0. Moreover, Π_μ is hyperbolic with $\dim W^u(\Pi_\mu) = \dim W^{cu}(0)$.

We will use the same notation as in §4. We assume that $\epsilon > 0$ is sufficiently small and the assumptions on the smallness of ϵ will be described later. We assume that for the study of the local flow near $x = 0$, a truncation has been made to equation (4.1) so that the vector field f has compact support in the ball $\{x : |x| \le 2\epsilon\}$ and $|Df| \le L(\epsilon)$ with $L(\epsilon) \to 0$ as $\epsilon \to 0$. Moreover, assume that a C^{k-1} change of variable has been made so that Lemma 4.1 is valid. Let $t_0 > 0$ be a sufficiently large constant. Any constant independent of ϵ and t_0 will be denoted by $C_1, C_2, \cdots$. We use $C_1(\epsilon), C_2(\epsilon), \cdots$, to denote constants that depend on ϵ and approach 0 as $\epsilon \to 0$. Let α, α_1 and β be constants satisfying $0 < \beta < -\alpha < \min\{\mathrm{Re}\,\lambda_1, -\mathrm{Re}\,\lambda_{-1}\}$, and $\alpha < \alpha_1 < -3\beta/2 < -\beta < 0$.

Consider equation (4.2). As in Silnikov [12], we consider a two-point boundary value problem in studying the flow near the equilibrium $x = 0$. The following lemma is crucial in this section and is a genralization of Silnikov's reparametrization theorem to a nonhyperbolic equlibrium.

Lemma 6.1. *There exists $\bar\epsilon > 0$ and $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, then for each $v_0 \in \mathbb{R}^n$, $u_1 \in \mathbb{R}^m$, $y_1 > 0$ and $t_0 > 0$, with $|v_0| \le \bar\epsilon$ and $|u_1| \le \bar\epsilon$ there exists a unique solution $(y(t), u(t), v(t))$ for (4.2), $0 \le t \le t_0$ with $y(t_0) = y_1$, $u(t_0) = u_1$ and $v(0) = v_0$. Denote the solution by*

$$\begin{aligned} y(t) &= y^*(t, t_0, y_1, u_1, v_0, \mu) + y_c(t, t_0, y_1, \mu) \\ u(t) &= u^*(t, t_0, y_1, u_1, v_0, \mu), \\ v(t) &= v^*(t, t_0, y_1, u_1, v_0, \mu) \end{aligned}$$

where $y_c(t, t_0, y_1, \mu)$ is the solution on $W^c(\mu)$, with $y_c(t_0, t_0, y_1, \mu) = y_1$. We have the following estimates:

$$\begin{aligned} |y^*(t)| &\le C_1(\bar\epsilon) e^{\alpha_1 t_0} e^{\beta(t_0 - t)} \\ |u^*(t)| &\le C_1(\bar\epsilon) e^{\alpha_1 (t_0 - t)} \\ |v^*(t)| &\le C_1(\bar\epsilon) e^{\alpha_1 t}, \end{aligned}$$

$$\begin{aligned} &|\frac{\partial y^*}{\partial t}| + |\frac{\partial y^*}{\partial t_0}| + |\frac{\partial y^*}{\partial u_1}| + |\frac{\partial y^*}{\partial v_0}| \le C_1 e^{\beta(t_0 - t)} e^{\alpha_1 t_0} \qquad (6.1) \\ &|\frac{\partial u^*}{\partial t}| + |\frac{\partial u^*}{\partial t_0}| + |\frac{\partial u^*}{\partial u_1}| + |\frac{\partial u^*}{\partial v_0}| \le C_1 e^{\alpha_1 (t_0 - t)} \\ &|\frac{\partial v^*}{\partial t}| + |\frac{\partial v^*}{\partial t_0}| + |\frac{\partial v^*}{\partial u_1}| + |\frac{\partial v^*}{\partial v_0}| \le C_1 e^{\alpha_1 t} \\ &\qquad \frac{\partial}{\partial t_0}(y^* + y_c)(t) \le -C_2^{-\frac{\beta}{2}(t_0 - t)}, \quad \text{if } y_1 > |r(\delta)|. \end{aligned}$$

Proof: Consider the following integral equation on $0 \le t \le t_0$:

$$y(t) = \int_{t_0}^{t} \left[h(y_c + y, u, v, \mu)(s) - h(y_c, 0, 0, \mu)(s)\right] ds, \qquad (6.2)$$

$$u(t) = e^{A(t - t_0)} u_1 + \int_{t_0}^{t} e^{A(t - s)} g_1(y_c + y, u, v, \mu)(s)\, ds, \qquad (6.3)$$

$$v(t) = e^{Bt}v_0 + \int_0^t e^{B(t-s)} g_2(y_c + y, u, v, \mu)(s)\, ds. \tag{6.4}$$

Observe that

$$\left|h(y_c, u, v, \mu) - h(y_c, 0, 0, \mu)\right| \le C_3|u|\,|v|, \quad |h_y| \le L(\epsilon),$$

$$|g_{1u}| \le L(\epsilon), \quad |g_{2v}| \le L(\epsilon),$$

$$\left|e^{A(t-s)}\right| \le C_4 e^{\alpha(t_0-t)}, \quad |e^{Bt}| \le C_4 e^{\alpha t}.$$

Let $L(\epsilon) \le \beta/2$, $C_3 \cdot C_1(\bar\epsilon)/\beta \le 1/2$, $L(\epsilon)C_4 < |\alpha - \alpha_1|/2$ and $2C_4\bar\epsilon = C_1(\bar\epsilon)$.

The right-hand side of the integral equation (6.2)–(6.4) defines a mapping

$$\mathcal{H} : (y, u, v) \to (\bar y, \bar u, \bar v).$$

If $|y(t)| \le C_1(\bar\epsilon)e^{\alpha_1 t_0}e^{\beta(t_0-t)}$, $|u(t)| \le C_1(\bar\epsilon)e^{\alpha_1(t_0-t)}$, and $|v(t)| \le C_1(\bar\epsilon)e^{\alpha_1 t}$, then we have

$$\begin{aligned}
|\bar y(t)| &\le \int_t^{t_0} \left(\frac{\beta}{2}|y| + C_3|u|\,|v|\right) ds \\
&\le \int_t^{t_0} \left\{\frac{\beta}{2}C_1(\bar\epsilon)e^{\alpha_1 t_0}e^{\beta(t_0-s)} + C_3[C_1(\bar\epsilon)]^2 e^{\alpha_1 t_0}\right\} ds \\
&\le \frac{C_1(\bar\epsilon)}{2}e^{\alpha_1 t_0}e^{\beta(t_0-t)} + \frac{C_3[C_1(\bar\epsilon)]^2}{\beta}e^{\alpha_1 t_0}e^{\beta(t_0-t)} \\
&\le C_1(\bar\epsilon)e^{\alpha_1 t_0}e^{\beta(t_0-t)},
\end{aligned}$$

$$\begin{aligned}
|\bar u(t)| &\le C_4|u_1|e^{\alpha(t_0-t)} + \int_t^{t_0} C_4 e^{\alpha(s-t)}L(\epsilon)|u(s)|ds \\
&\le C_4\bar\epsilon e^{\alpha(t_0-t)} + \frac{L(\epsilon)C_1(\bar\epsilon)C_4}{|\alpha-\alpha_1|}e^{\alpha_1(t_0-t)} \le C_1(\bar\epsilon)e^{\alpha_1(t_0-t)}
\end{aligned}$$

$$\begin{aligned}
|\bar v(t)| &\le C_4|v_0|e^{\alpha t} + \int_0^t C_4 e^{\alpha(t-s)}L(\epsilon)|v(s)|ds \\
&\le C_4\bar\epsilon e^{\alpha t} + \frac{L(\epsilon)C_1(\bar\epsilon)C_4}{|\alpha-\alpha_1|}e^{\alpha_1 t} \le C_1(\bar\epsilon)e^{\alpha_1 t}.
\end{aligned}$$

Let X be the Banach spaces of continuous functions (y, u, v) in $0 \le t \le t_0$ with norm

$$\|(y,u,v)\|_X = \max\left\{\begin{array}{l}
\sup\limits_{0\le t\le t_0} \left|y(t)e^{-\alpha_1 t_0}e^{-\beta(t_0-t)}\right|, \\
\sup\limits_{0\le t\le t_0} \left|u(t)e^{-\alpha_1(t_0-t)}\right|, \\
\sup\limits_{0\le t\le t_0} \left|v(t)e^{-\alpha_1 t}\right|
\end{array}\right\}.$$

Thus, $\mathcal{H}: X \to X$ maps a $C_1(\bar{\epsilon})$-ball in X into itself provided that $|u_1| \le \bar{\epsilon}$, $|v_0| \le \bar{\epsilon}$ and $\epsilon > 0$ is small. Furthermore, $\mathcal{H}$ is a uniform contraction provided that $\bar{\epsilon} > 0$ and $\epsilon > 0$ are small. Therefore, equation (6.2)–(6.4) admits a unique solution (y^*, u^*, v^*) in the $C_1(\bar{\epsilon})$-ball in X.

The estimates for $|\frac{\partial y^*}{\partial t}|$, $|\frac{\partial u^*}{\partial t}|$, and $|\frac{\partial v^*}{\partial t}|$ come directly from (4.2). Except for the estimates involving partials with respect to t_0 in (6.1), all the other estimates can be proved similarly. Thus, we will only prove the estimates for $|\frac{\partial y^*}{\partial t_0}|$, $|\frac{\partial u^*}{\partial t_0}|$, and $|\frac{\partial v^*}{\partial t_0}|$. We start with the following integral equation

$$\frac{\partial y}{\partial t_0}(t) = \int_{t_0}^{t} \left(h_y \frac{\partial y}{\partial t_0} + h_u \frac{\partial u}{\partial t_0} + h_v \frac{\partial v}{\partial t_0} \right)(s)ds,$$

$$\begin{aligned} \frac{\partial u}{\partial t_0}(t) &= e^{A(t-t_0)}\left(-Au_1 - g_1(y,u,v,\mu)(t_0)\right) \\ &\quad + \int_{t_0}^{t} e^{A(t-s)} \left(g_{1y} \frac{\partial y}{\partial t_0} + g_{1u} \frac{\partial u}{\partial t_0} + g_{1v} \frac{\partial v}{\partial t_0} \right)(s)ds, \end{aligned}$$

$$\frac{\partial v}{\partial t_0} = \int_{0}^{t} e^{B(t-s)} \left(g_{2y} \frac{\partial y}{\partial t_0} + g_{2u} \frac{\partial u}{\partial t_0} + g_{2v} \frac{\partial v}{\partial t_0} \right)(s)ds.$$

Since $|h_u| \le C_5|v|$, $|h_v| \le C_5|u|$, $|g_{1y}| + |g_{1v}| \le C_5|u|$ and $|g_{2y}| + |g_{2u}| \le C_5|v|$, we have

$$|\frac{\partial y}{\partial t_0}(t)| \le \left\{ \int_{t}^{t_0} \left(L(\epsilon)e^{\alpha_1 t_0} e^{\beta(t_0-s)} + C_5 C_1(\bar{\epsilon}) e^{\alpha_1 t_0} \right) ds \right\} \cdot \|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X$$

$$\begin{aligned} |\frac{\partial u}{\partial t_0}(t)| &\le C_5 e^{\alpha(t_0-t)} + \left\{ \int_{t}^{t_0} \left(C_5 C_1(\bar{\epsilon}) e^{\alpha(s-t)} e^{\alpha_1(t_0-s)} e^{\alpha_1 t_0} e^{\beta(t_0-s)} \right.\right. \\ &+ \left.\left. C_5 L(\epsilon) e^{\alpha(s-t)} e^{\alpha_1(t_0-s)} + C_5 C_1(\bar{\epsilon}) e^{\alpha(s-t)} e^{\alpha_1(t_0-s)} e^{\alpha_1 s} \right) ds \right\} \cdot \|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X \end{aligned}$$

$$\begin{aligned} |\frac{\partial v}{\partial t_0}(t)| &\le \left\{ \int_{0}^{t} \left(C_5 C_1(\bar{\epsilon}) e^{\alpha(t-s)} e^{\alpha_1 s} e^{\alpha_1 t_0} e^{\beta(t_0-s)} + C_5 C_1(\bar{\epsilon}) e^{\alpha(t-s)} e^{\alpha_1 s} e^{\alpha_1(t_0-s)} \right.\right. \\ &\quad + \left.\left. C_5 L(\epsilon) e^{\alpha(t-s)} e^{\alpha_1 s} \right) ds \right\} \cdot \|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X. \end{aligned}$$

If

$$\frac{L(\epsilon)}{\beta} + \frac{C_5 C_1(\bar{\epsilon})}{\beta} \le \frac{1}{2} \quad \text{and} \quad \frac{2C_5 C_1(\bar{\epsilon}) + C_5 L(\epsilon)}{|\alpha - \alpha_1|} \le \frac{1}{2},$$

then

$$|\frac{\partial y}{\partial t_0}(t)| \le \frac{1}{2} \|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X e^{\alpha_1 t_0} e^{\beta(t_0-t)},$$

$$|\frac{\partial u}{\partial t_0}(t)| \le C_5 e^{\alpha_1(t_0-t)} + \frac{1}{2} \|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X e^{\alpha_1(t_0-t)},$$

$$|\frac{\partial v}{\partial t_0}(t)| \le \frac{1}{2}\|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X e^{\alpha_1 t}.$$

Thus, $\|(\frac{\partial y}{\partial t_0}, \frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0})\|_X \le C_5$. This gives the desired estimates for $|\frac{\partial y^*}{\partial t_0}|$, $|\frac{\partial u^*}{\partial t_0}|$, and $|\frac{\partial v^*}{\partial t_0}|$.

We now prove the last estimate in (6.1). Observe that $\partial y_c/\partial t_0$ satisfies the following initial value problem:

$$\begin{cases} \frac{d}{dt}\frac{\partial y_c}{\partial t_0}(t) = h_y \cdot \frac{\partial y_c}{\partial t_0}(t) \\ \frac{\partial y_c}{\partial t_0}(t_0) = -h(y,0,0,\mu), \end{cases}$$

where $-h(y,0,0,\mu) = e_1 > 0$ is a constant. Since $|h_y| \le \beta/2$, we have

$$\frac{\partial y_c}{\partial t_0}(t) \le -e_1 e^{-\frac{\beta}{2}(t_0-t)} < 0, \quad 0 \le t \le t_0.$$

Thus,

$$\frac{(\partial y^* + y_c)}{\partial t_0}(t) \le -e_1 e^{-\frac{\beta}{2}(t_0-t)} + C_5 e^{\alpha_1 t_0} e^{\beta(t_0-t)}.$$

Let t_0 be so large that

$$C_5 e^{(\alpha_1 + \frac{3\beta}{2})t_0} \le \frac{1}{2}e_1.$$

We have that

$$\frac{\partial}{\partial t_0}(y^* + y_c)(t) \le -\frac{e_1}{2}e^{-\frac{\beta}{2}(t_0-t)}. \qquad \blacksquare$$

We note that although Lemma 6.1 is proved for the truncated system, it is valid for the original system, if y_1, $\bar{\epsilon}$ and $|\mu|$ are small, and $-\epsilon < y_0 < y_1$ in the case $\mu_1 > 0$.

Let $y^*(0, t_0, y_1, u_1, v_0, \mu) + y_c(0, t_0, y, \mu) = y_0$. Estimate (6.1) allows us to invert t_0 as a function of $(y_0, y_1, u_1, v_0, \mu)$. In the case of $\mu_1 \le 0$, such inversion is possible only if $y_0 \le y_1$ and y_0 is in the range of $y^*(0)+y_c(0)$. If $\mu_1 \le 0$ and $y_0 < y_1$ and y_0 is not in the range of $y^*(0)+y_c(0)$, we set $t_0 = +\infty$. By a shift in the y direction, continuous with respect to μ, we assume that: (i) for $\mu_1 \le 0$, $S(\mu) = \{(y,u,v) = (0,0,0)\}$; (ii) for $\mu_1 = 0$, $SN(\mu) = \{(y,u,v) = (0,0,0)\}$; (iii) for $\mu_1 > 0$, the solution of $\frac{\partial h}{\partial y}(y,0,0,\mu) = 0$ is $y = 0$. We have the following:

Lemma 6.2. *Assume that $\epsilon > 0$, $\bar{\epsilon} > 0$ are sufficiently small and the assumptions of Lemma 6.1 are true. For fixed $y_1 > 0$, u_1, v_0 and μ, $y_0 = y^*(0, t_0, y_1, u_1, v_0, \mu) + y_c(0, t_0, y_1, \mu)$ is strictly decreasing as $t_0 \to +\infty$. Moreover, for $\mu_1 \le 0$ we have $y_0 \to 0$ as $t_0 \to +\infty$. On the other hand, for $\mu_1 > 0$, there exists $\bar{t}_0$ such that $\lim_{t_0 \to \bar{t}_0} y_0 = -\epsilon$. Moreover, if $\mu_1 > 0$ and y_0, μ are allowed to change, with $|y_0| \le \epsilon_3$ and $|\mu| \le \epsilon_3$, then $\bar{t}_0 = \bar{t}_0(y_0, \mu) \to +\infty$ as $\epsilon_3 \to 0$.*

Proof: The monotonicity is obvious from (6.1). From the property of the flow on the center manifold, it is clear that if $\mu_1 \le 0$ then $\lim_{t_0 \to \infty} y_c(0, t_0, y_1, \mu) = 0$. Furthermore, $\lim_{t_0 \to +\infty} y^*(0, t_0, y_1, u_1, v_0, \mu) = 0$ from Lemma 6.1. Thus, $\lim_{t_0 \to \infty} y_0 = 0$, provided $\mu_1 \le 0$.

To show the last assertion, suppose it is false. Then there exist sequences $\{t_0^j\}$, $\{y_0^j\}$, $\{\epsilon_3^j\}$, $j = 1, 2, \cdots$, and $t_1 > 0$ such that $|y_0^j| \le \epsilon_3^j$, $t_0^j \le t_1$ and $\epsilon_3^j \to 0$ as $j \to \infty$. Let $\mu_1 = 0$. Let $\bar{\epsilon}$ be so small that $C_1(\bar{\epsilon}) \le y_1/2$. It can be shown that if $|y^*(0, t_0, \cdots)| \le C_1(\bar{\epsilon})e^{(\alpha_1+\beta)t_1}$, then $y_c(0, t_0, y_1, \mu_1) \ge y_1 e^{-\beta t_1/2} > 0$. We have that $y_0 = y^*(0) + y_c(0) \ge \frac{y_1}{2} e^{-\beta t_1/2}$. Thus, y_0 is bounded below by a positive constant for $\mu_1 = 0$. By the uniform continuity of y_0 with respect to t_0 and μ, y_0 is bounded below by a positive constant for $0 < \mu_1 < \epsilon_4$ where $\epsilon_4 > 0$ is a small constant. This contradicts $|y_0^j| \le \epsilon_3^j \to 0$. ∎

As in §4, for $\mu = 0$, the homoclinic orbit Γ_0 has two connected components Γ_0^+ and Γ_0^- in a small neighborhood of $x = 0$. Consider a cross section $\Sigma_1 = \{x \,|\, x = (y, u, v),\ y = d_1\}$ for some $0 < d_1 < \epsilon$. Thus, $\Gamma_- \cap \Sigma_1 = M_1 = (d_1, 0, 0)$. Also consider a cross section $\Sigma_0 = \{x \,|\, x = (y, u, v),\ v = (v^1, \cdots, v^n),\ v^n = d_0\}$ for some $0 < d_0 < \epsilon$. Hence, $\Sigma_0 \cap \Gamma_+ = M_0 = (0, 0, \mathfrak{v})$, where $\mathfrak{v} = (0, \cdots, d_0)$. In a δ_i-neighborhood of $M_i \in \Sigma_i$ the flow of (4.2) is transverse to Σ_i, where $i = 0, 1$, and δ_0 and δ_1 can be chosen independently of μ if μ and ϵ are sufficiently small. By following the trajectories of (4.2), a diffeomorphism $T_1(\mu) : \Sigma_1 \to \Sigma_0$ can be defined with the domain being a δ_1-neighborhood of M_1 in Σ_1. Also, by following the local flow a diffeomorphism $T_0(\mu) : \Sigma_0 \to \Sigma_1$ can be defined in the domain σ_0 which is in a δ_0-neighborhood of M_0 in Σ_0. In the above definitions for T_0 and T_1 we assume that δ_0 and δ_1 are sufficiently small. Observe that not all the points in the δ_0-neighborhood will return to Σ_1 when $\mu_1 \le 0$ even if δ_0 is very small.

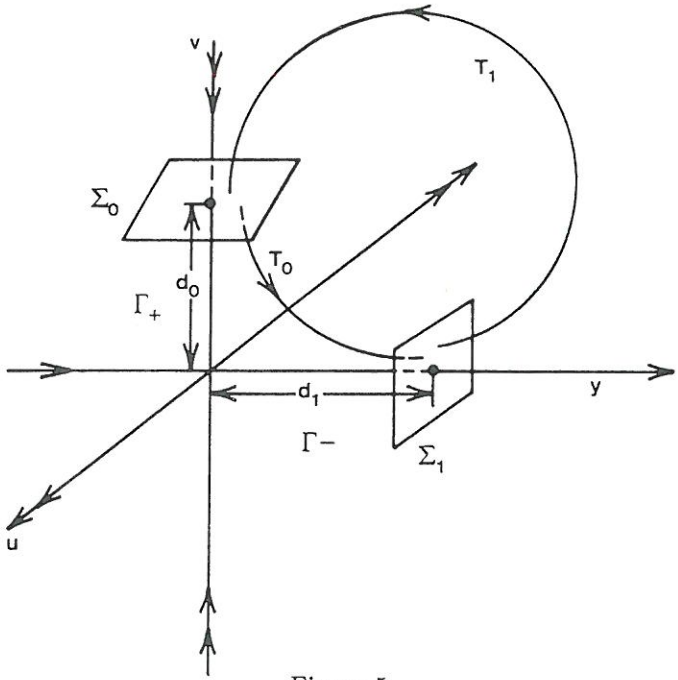

Figure 5.

The mapping $T_0(\mu) : \sigma_0 \subset \Sigma_0 \to \Sigma_1$ can easily be described by the parametrized version (see [10]). Set $v_0 = (v_0^1, \cdots, v_0^n) = (\bar{v}_0, v_0^n)$, where $\bar{v}_0 = (v_0^1, \cdots, v_0^{n-1})$. If $(y_0, u_0, v_0) \in \sigma_0$, then $v_0^n = d_0$ and $|\bar{v}_0| < \delta_0$. Let $0 < d_1 < \epsilon$, $|u_1| < \delta_1 \le \bar{\epsilon}$, $\delta_0 \le \bar{\epsilon}$,

$|d_0| \le \bar{\epsilon}$, where ϵ and $\bar{\epsilon}$ are so small that Lemmas 6.1 and 6.2 are valid. By Lemma 6.2, if $\mu_1 \le 0$ and $\delta_0 > y_0 > 0$ (or $\mu_1 > 0$, $|y_0| \le \delta_0$), then we can solve for t_0 as a function of y_0 from $y_0 = y^*(0, t_0, d_1, u_1, v_0, \mu) + y_c(0, t_0, y, \mu)$, and obtain the function $t_0^*(y_0) = t_0^*(y_0, d_1, u_1, v_0, \mu)$ which is C^1. Substituting into u^* and v^*, we define:

$$\kappa(y_0, \bar{v}_0, u_1, \mu) = u^*\big(0, t_0^*(y_0), u_1, (\bar{v}_0, d_0), \mu\big),$$
$$\eta(y_0, \bar{v}_0, u_1, \mu) = v^*\big(t_0^*(y_0), t_0^*(y_0), u_1, (\bar{v}_0, d_0), \mu\big).$$

Let $T_0(\mu) : (y_0, u_0, (\bar{v}_0, d_0)) \to (d_1, u_1, v_1)$. It is clear that the domain of T_0 is

$$\sigma_0 = \bigcup \Big\{ (y_0, u_0, (\bar{v}_0, d_0)) \,|\, u_0 = \kappa(y_0, \bar{v}_0, u_1, \mu), \text{ where } |\bar{v}_0| < \delta_0,\ |u_1| < \delta_1 \text{ and } 0 < y_0 < \delta_0 \text{ (or } |y_0| < \delta_0) \text{ for } \mu_1 \le 0 \text{ (or } \mu_1 > 0) \Big\}, \tag{6.5}$$

and the range of $T_0(\mu)$ is

$$T_0(\mu)\sigma_0 = \bigcup \Big\{ (d_1, u_1, v_1) \,|\, v_1 = \eta(y_0, \bar{v}_0, u_1, \mu), \text{ where } |\bar{v}_0| < \delta_0,\ |u_1| < \delta_1 \text{ and } 0 < y_0 < \delta_0 \text{ (or } |y_0| < \delta_0) \text{ for } \mu_1 \le 0 \text{ (or } \mu_1 > 0) \Big\}.$$

Corollary 6.3. *We have*

$$|\kappa| + |\eta| \le 2C_1(\bar{\epsilon})e^{\alpha_1 t_0},$$
$$\Big|\frac{\partial \kappa}{\partial \bar{v}_0}\Big| + \Big|\frac{\partial \kappa}{\partial u_1}\Big| + \Big|\frac{\partial \eta}{\partial \bar{v}_0}\Big| + \Big|\frac{\partial \eta}{\partial u_1}\Big| \le Ce^{\alpha_1 t_0},$$
$$\Big|\frac{\partial \kappa}{\partial y_0}\Big| + \Big|\frac{\partial \kappa}{\partial y_0}\Big| \le Ce^{(\alpha_1 + \frac{\beta}{2})t_0}.$$

The proof is obvious from the definition of κ and η, and Lemma 6.1, and is therefore omitted.

From Corollary 6.3, it is clear that we can extend the domain of κ and η to $y_0 \le 0$ and $\mu_1 \le 0$ by letting $\kappa = 0$ and $\eta = 0$ for $y_0 \le 0$ and $\mu_1 \le 0$. The extended functions, still denoted by κ and η, are C^1 with respect to y_0, $\bar{v}_0$ and u_1. Accordingly, the domain of the mapping T_0 has been extended; see (6.5). The extended map T_0 is not very nice if $\mu_1 \le 0$. It is not injective and it is multivalued. The set $\{(y_0, u_0, (\bar{v}_0, d_0)) \,|\, -\delta_0 \le y_0 \le 0,\ u_0 = 0,\ |\bar{v}_0| \le \delta_0\}$ is mapped onto the set $\{(d_1, u_1, v_1) \,|\, v_1 = 0,\ |u_1| \le \delta_1\}$. However, we shall introduce a set of C^1 submanifolds of dimension m, called u-slices, whose images under T_0 are very nice.

For $i = 0, 1$, a C^1 submanifold $\tilde{\phi}^i$ of dimension m will be called a u-slice of size $(\epsilon_i^1 \times \epsilon_i^2, L_i)$ in a neighborhood of M_i if it is the graph of a C^1 function $\tilde{\phi}^i(u) = (\tilde{\phi}_1^i(u), \tilde{\phi}_2^i(u))$ and ϵ_i^1, $\epsilon_i^2 > 0$, $L_i \ge 0$ with

$$\tilde{\phi}^i = \big\{(y, u, v) \,|\, y = \tilde{\phi}_1^i(u),\ v = \tilde{\phi}_2^i(u),\ |u| \le \epsilon_i^1,\ |\tilde{\phi}^i(0) - M_i| \le \epsilon_i^2,\ |D\tilde{\phi}^i(u)| \le L_i\big\}.$$

The set of all the u-slices of size $(\epsilon_i^1 \times \epsilon_i^2, L_i)$ in a neighborhood of M_i shall be denoted by $\mathcal{J}_i(\epsilon_i^1 \times \epsilon_i^2, L_i)$, $i = 0, 1$.

Lemma 6.4. *Given $L_0 > 0$, there exist sufficiently small positive constants $\bar{\epsilon}$, δ_0, δ_1 ϵ_0^1 and ϵ_0^2 such that if $|\mu| \leq \bar{\epsilon}$ and $\tilde{\phi}^0 \in \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L) \subset \{\delta_0$ neighborhood of M_0 on $\Sigma_0\}$, then $T_0(\mu)\tilde{\phi}^0$ contains a u-slice in $\mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$. Moreover, $L_1 \to 0$ as δ_0, δ_1, $\bar{\epsilon}$, ϵ_0^1 and $\epsilon_0^2 \to 0$.*

Proof: Consider

$$u_0 = \kappa(\tilde{\phi}_1(u_0), u_1, \tilde{\phi}_2(u_0), \mu), \quad v_1 = \eta(\tilde{\phi}_1(u_0), u_1, \tilde{\phi}_2(u_0), \mu).$$

For each u_1, $|u_1| < \epsilon_1^1$, u_0 may be solved from the first equation: $u_0 = u_0^{\#}(u_1, \mu)$. $\frac{\partial u_0}{\partial u_1} \to 0$ if $\partial\kappa/\partial u_1 \to 0$. Substituting into the second equation, we have $v_1 = v_1^{\#}(u_1, \mu)$. Clearly, $\partial v_1/\partial u_1 \to 0$ as $\frac{\partial \eta}{\partial \mu}$ and $\partial\kappa/\partial u_1 \to 0$. We note that for a u-slice $\tilde{\phi}^0$ on Σ_0, the n-th coordinate of $\tilde{\phi}_2^0(u_0)$ is d_0 and the y-coordinate of $T_0(\mu)\tilde{\phi}^0$ is d_1 for all $|u_0| \leq \epsilon_0^1$.

As can be seen from Lemma 6.4, $T_0(\mu)$ induced a mapping from a subset of the graph of $\tilde{\phi}^0(u)$ onto the graph of a C^1 function $\tilde{\phi}^1(u)$. This in turn induces a mapping from a subset of $|u| \leq \epsilon_0^1$ onto $|u| \leq \epsilon_1^1$. We say that $T_0(\mu) : \tilde{\phi}^0 \to T_0(\mu)\tilde{\phi}^0$ is an expansion if the induced mapping is an expansion from a subset of $|u| \leq \epsilon_0^1$ onto $|u| \leq \epsilon_1^1$. ∎

We define the distance between two u-slices $\tilde{\phi}^{(1)}$ and $\tilde{\phi}^{(2)}$ as

$$d(\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}) = |\tilde{\phi}_1^{(1)} - \tilde{\phi}_1^{(2)}|_{C^0} + |\tilde{\phi}_2^{(1)} - \tilde{\phi}_2^{(2)}|_{C^0}.$$

Lemma 6.5. (i) *$T_0(\mu) : \tilde{\phi}^0 \to T_0(\mu)\tilde{\phi}^0$ is an expansion if $\tilde{\phi}^0 \in \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0)$, $|\mu| \leq \bar{\epsilon}$ and ϵ_0^1, ϵ_0^2, $\bar{\epsilon}$ are small. The rate of expansion approaches ∞ as $\bar{\epsilon}$, ϵ_0^1 and ϵ_0^2 approach zero.*

(ii) *There exists a continuous function $\lambda = \lambda(\bar{\epsilon}, \epsilon_0^1, \epsilon_0^2) \geq 0$ such that $T_0(\mu)$: $\tilde{\phi}^0 \to T_0(\mu)\tilde{\phi}^0$ is a λ-contraction with the distances of two u-slices defined as above. Moreover, $\lambda \to 0$ as $\bar{\epsilon}$, ϵ_0^1 and $\epsilon_0^2 \to 0$.*

Proof: The proof of (i) can be seen from the proof of Lemma 6.4 that $u_0 = u_0^{\#}(u_1, \mu)$ and which is a contraction from $|u_1| \leq \epsilon_1^1$ to u_0, and $\frac{\partial u_0}{\partial u_1} \to 0$ as $\bar{\epsilon}$, ϵ_0^1 and $\epsilon_0^2 \to 0$. For a proof of (ii), let $\tilde{\phi}_i^{(t)} = \tilde{\phi}_i^{(1)} + (t-1)(\tilde{\phi}_i^{(2)} - \tilde{\phi}_i^{(1)})$, $1 \leq t \leq 2$, $i = 1, 2$. For each fixed $|u_1| < \epsilon_1^1$, we have

$$u_0^t = \kappa(\tilde{\phi}_1^{(t)}(u_0^t), u_1, \tilde{\phi}_2^{(t)}(u_0^t), \mu), \quad v_1^t = \eta(\tilde{\phi}_1^{(t)}(u_0^t), u_1, \tilde{\phi}_2^{(t)}(u_0^t), \mu).$$

$$\begin{aligned} \frac{\partial u_0^t}{\partial t} &= \frac{\partial \kappa}{\partial y_0}\left[D\tilde{\phi}_1^{(t)} \cdot \frac{\partial u_0^t}{\partial t} + (\tilde{\phi}_1^{(2)} - \tilde{\phi}_1^{(1)})(u_0^t)\right] \\ &\quad + \frac{\partial \kappa}{\partial v_0}\left[D\tilde{\phi}_2^{(t)} \cdot \frac{\partial u_0^t}{\partial t} + (\tilde{\phi}_2^{(2)} - \tilde{\phi}_2^{(1)})(u_0^t)\right]. \end{aligned} \tag{6.7}$$

$$\begin{aligned} \frac{\partial v_1^t}{\partial t} &= \frac{\partial \eta}{\partial y_0}\left[D\tilde{\phi}_1^{(t)} \cdot \frac{\partial u_0^t}{\partial t} + (\tilde{\phi}_1^{(2)} - \tilde{\phi}_1^{(1)})(u_0^t)\right] \\ &\quad + \frac{\partial \eta}{\partial v_0}\left[D\tilde{\phi}_2^{(t)} \cdot \frac{\partial u_0^t}{\partial t} + (\tilde{\phi}_2^{(2)} - \tilde{\phi}_2^{(1)})(u_0^t)\right]. \end{aligned} \tag{6.8}$$

Observe that $|\frac{\partial\kappa}{\partial y_0}| + |\frac{\partial\kappa}{\partial v_0}| + |\frac{\partial\eta}{\partial y_0}| + |\frac{\partial\eta}{\partial v_0}| \le \hat{C}$, with $\hat{C} \to 0$ as ϵ_0^1, ϵ_0^2 and $\bar{\epsilon} \to 0$. From (6.7), we have

$$|\frac{\partial u_0^t}{\partial t}|(1 - 2\hat{C}L) \le 2\hat{C}d(\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}).$$

From (6.8), we have

$$|\frac{\partial v_1^t}{\partial t}| \le 2\hat{C}L|\frac{\partial u_0^t}{\partial t}| + 2\hat{C}d(\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}).$$

Therefore

$$|\frac{\partial v_1^t}{\partial t}| \le \left(\frac{(2\hat{C})^2 L}{1 - 2\hat{C}L} + 2\hat{C}\right) d(\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}).$$

Define

$$\lambda = \frac{(2\hat{C})^2 L}{1 - 2\hat{C}L} + 2\hat{C}.$$

Hence, $\lambda \to 0$ as $\hat{C} \to 0$. Finally,

$$d\big(T_0\tilde{\phi}^{(1)}, T_0\tilde{\phi}^{(2)}\big) \le \max_{|u_1| \le \epsilon} \int_1^2 |\frac{\partial v_1^t}{\partial t}|\, dt \le \lambda d\big(\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}\big). \quad ∎$$

We have defined $T_1(\mu) : \Sigma_1 \to \Sigma_0$ from a δ_1-neighborhood of M_1 to a δ_0-neighborhood of M_0, for $|\mu| \le \bar{\epsilon}$. $T_1(\mu)$ is a diffeomorphism which is continuous with respect to the parameter μ. We now show that $T_1(\mu)$ induces a mapping from $\mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$ into $\mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0)$ for certain constants ϵ_i^1, ϵ_i^2, L_i and $|\mu| \le \bar{\epsilon}$. First assume that $\mu = 0$. It is clear that $\Sigma_1 \cap W_{loc}^{cu}(0) = \{(y, u, v) \mid y = d_1,\ v = 0\}$ is a u-slice of Lipschitz constant 0. Since $W^{cu}(0)$ transversely intersects with $W^{cs}(0)$ along Γ_0^+, and Σ_0 transversely intersects with Γ_0^+ at M_0, we have $(\Sigma_0 \cap W^{cu}(0))$ transversely intersects with $(\Sigma_0 \cap W^{cs}(0))$ at M_0. Since $\Sigma_0 \cap W^{cs}(0) = \{(y, u, v) \mid u = 0,\ v^n = d_0\}$, we conclude that locally near M_0, $\Sigma_0 \cap W^{cu}(0)$ is a u-slice. In terms of the map $T_1(\mu)$, locally we have

$$\Sigma_0 \cap W^{cu}(0) = T_1(0)\big(\Sigma_1 \cap W^{cu}(0)\big).$$

Lemma 6.6. *Assume that $\Sigma_0 \cap W^{cu}(0)$ contains a u-slice of size $(\epsilon_0^1 \times 0, L_0)$. There exists ϵ_0^2, ϵ_1^1, ϵ_1^2, L_1, and $\bar{\epsilon}$ such that for each $\tilde{\phi}^1 \in \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$ and $|\mu| \le \bar{\epsilon}$, $T_1(\mu)\tilde{\phi}^1$ contains a u-slice in $\mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0 + 1)$.*

Proof: We have seen that for $\mu = 0$ and $\tilde{\phi}^1 \in \Sigma_1 \cap W^{cu}(0)$ the assertion is valid. The rest of the proof follows easily from the continuity of $T_1(\mu)$.

Lemma 6.7. *Let L_0 be given as in Lemma 6.6. There exist positive constants ϵ_0^1, ϵ_0^2, ϵ_1^1, ϵ_1^2, L_1 and $\bar{\epsilon}$ such that for $|\mu| \le \bar{\epsilon}$, $T_1(\mu) : \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1) \to \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0 + 1)$ is well defined in the sense that for each $\tilde{\phi}^1 \in \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$, $T_1(\mu)\tilde{\phi}^1$ contains a u-slice in $\mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0 + 1)$. Moreover, $T_0(\mu) : \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0 + 1) \to \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$*

is well defined in the sense that for each $\phi \in \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0+1)$, $T_0(\mu)\tilde{\phi}^0$ *contains a u-slice in* $\mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$. *Thus, we have a composed map*

$$T_0(\mu)T_1(\mu) : \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1) \to \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$$

for $|\mu| \le \bar{\epsilon}$. *Moreover, there exists a unique fixed element* $\tilde{\phi}_*^1 \in \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$ *for the map* $T_0(\mu)T_1(\mu)$.

Proof: Notice in the statement of Lemma 6.6, we can choose $L_1 > 0$ to be a fixed small constant, $\epsilon_1^1 = \epsilon_1^2 = \tau \le \tau_0$ where $\tau_0 > 0$ is a small constant, $\epsilon_0^1 = C_1\tau$, $\epsilon_0^2 = C_2\tau$ and $\bar{\epsilon} = C_3\tau$. We now prove that if τ is sufficiently small, and choose $\bar{\epsilon}$ smaller than $C_3\tau$ if necessary, then for each $\tilde{\phi}^0 \in \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0+1)$, $T_0(\mu)\tilde{\phi}^0$ contains a u-slice in $\mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$. The key to the proof is that as $\tau \to 0$, the time t_0 required for the flow to go from $\Sigma_0 \to \Sigma_1$ approaches ∞ (see Lemma 6.2).

(i) As $\tau \to 0$, the Lipschitz constant for $T_0(\mu)\tilde{\phi}^0$ approaches 0 from Lemma 6.4, and is therefore less than L_1 if τ is small.

(ii) From Lemma 6.5, the rate of expansion of the map $\tilde{\phi}^0 \to T_0(\mu)\tilde{\phi}^0$ is bounded below by $C(\tau)$ which approaches ∞ as $\tau \to 0$. We shall choose τ to be sufficiently small such that $C(\tau)C_1 > 1$.

(iii) It remains to show that for the proper choices of τ and $\bar{\epsilon}$, $M_2 = T_0(\mu)\tilde{\phi}^0 \cap \{u = 0\}$ is in the τ-neighborhood of M_1. It is clear that the point $M_3 = \tilde{\phi}^0 \cap \{u = 0\}$ is in $W^{cs}_{loc}(\mu)$, therefore we are able to obtain an estimate for t_0 easily. Consider $\frac{dy}{dt} = h(y,0,v,\mu) = h(y,0,0,\mu) < a\mu_1 + by^2$, where a, b are positive constants for the case $\mu_1 > 0$. We set $\bar{\epsilon} = (\epsilon_0^2)^2 \le C_3\tau$ for sufficiently small $\tau > 0$, then

$$t_0 \ge \int_{\epsilon_0^2}^{d_1} \frac{dy}{h} \ge \int_{\epsilon_0^2}^{d_1} \frac{dy}{a\mu + by^2} \ge \int_{\epsilon_0^2}^{d_1} \frac{dy}{a\bar{\epsilon} + by^2} = \frac{1}{ab\epsilon_0^2}\Big[\arctan(y)\Big]_{\sqrt{b}/\sqrt{a}}^{d_1\sqrt{b}/\epsilon_0^2\sqrt{a}}$$
$$\ge \frac{1}{2ab\epsilon_0^2}\Big(\frac{\pi}{2} - \arctan\frac{\sqrt{b}}{\sqrt{a}}\Big) = C_4/\epsilon_0^2,$$

provided that $\tau > 0$ is small and $\arctan(d\sqrt{b}/\epsilon_0^2\sqrt{a}) \approx \frac{\pi}{2}$. Since the flow on $W^{cs}_{loc}(\mu)$ is an exponential contraction towards $W^c_{loc}(\mu)$, we have

$$|M_2| \le d_0ce^{\alpha_1 t_0} \le d_0ce^{c_4\alpha_1/\epsilon_0^2} = d_0ce^{c_4\alpha_1/(c_2\tau)},$$

which is less than τ if $\tau > 0$ is sufficiently small. The case $\mu_1 \le 0$ can be treated similarly using the fact $h(y,0,0,\mu) < ay(y + \sqrt{\mu_1})$ where $a > 0$ is a constant, which also yields that $|M_2| \le \tau$ if $\tau > 0$ is sufficiently small.

The existence of a unique fixed element $\tilde{\phi}_*^1 \in \mathcal{J}_1(\epsilon_1^1 \times \epsilon_2^2, L_1)$ follows from the contraction of $T_0(\mu)T_1(\mu)$, provided that $\tau > 0$ is sufficiently small. From Lemma 6.5, (ii), the rate of contraction $T_0(\mu) : \mathcal{J}_0(\epsilon_0^1 \times \epsilon_0^2, L_0) \to \mathcal{J}_1(\epsilon_1^1 \times \epsilon_1^2, L_1)$ approaches 0 as $\tau \to 0$. ∎

We know that for $|\mu| \le \bar{\epsilon}$, $T_0(\mu)T_1(\mu) : \tilde{\phi}_*^1 \to \tilde{\phi}_*^1$ is an expansion. When $\mu = 0$, clearly $T_0(\mu)T_1(\mu)\tilde{\phi}_*^1 \supset \tilde{\phi}_*^1$. Therefore, $T_0(\mu)T_1(\mu)\tilde{\phi}_*^1 \supset \tilde{\phi}_*^1$ for $|\mu| \le \bar{\epsilon}$. Therefore, we obtain a unique fixed point $\tilde{x}_* \in \tilde{\phi}_*^1$ such that $T_0(\mu)T_1(\mu)\tilde{x}_* = \tilde{x}_*$. We almost have a periodic orbit for the whole system, except that $T_0(\mu)$ has been extended artificially. The following is the basic result for this section.

Theorem 6.8. *If $|\mu| \le \bar{\epsilon}$ is small, there exists a unique periodic orbit Π_μ intersecting Σ_1 at $x^* = \Pi_\mu \cap \Sigma_1$, x^* is in the δ_1-neighborhood of M_1, if and only if $\mu_1 > 0$, or $\mu_1 \le 0$ and $E(\mu) \in$ Region III. Moreover, the periodic orbit Π_μ is hyperbolic with $\dim W^u(\Pi_\mu) = \dim W^{cu}(0)$ and $\dim W^s(\Pi_\mu) = \dim W^{cs}(0)$.*

Proof: For simplicity we write T_i for $T_i(\mu)$, $i = 0, 1$. Assume that Π_μ is a periodic orbit and $\Pi_\mu \cap \Sigma_1 = x^*$ in the δ_1-neighborhood of M_1. Construct a u-slice ϕ^* passing through x^*, of size $(\epsilon_1^1 \times \epsilon_1^2, L_1)$. By Lemma 6.5, $(T_0T_1)^n\phi^* \to \tilde{\phi}^*$ with $(T_0T_1)\tilde{\phi}^* = \tilde{\phi}^*$. Since $\tilde{\phi}^*$ is unique and $x^* \in \tilde{\phi}_*$, we have $x^* = \tilde{x}_*$, thus the uniqueness of Π_μ has been proved.

Suppose $\mu_1 \le 0$ and $E(\mu) \notin$ Region III. Then $\Sigma_1 \cap W^{cu}_{loc}(\mu)$ contains a u-slice which is a fixed element of T_0T_1. Thus, $\tilde{x}_* \in \tilde{\phi}^1_* \subset \Sigma_1 \cap W^{cu}_{loc}(\mu)$ and there is no periodic orbit Π_μ.

Suppose $\mu_1 \le 0$ and $E(\mu) \in$ Region III. Then the y coordinate of $T_1\tilde{x}_*$, $y(T_1\tilde{x}_*) > 0$. If not, $T_0(T_1\tilde{x}_*)$ is defined in the extended sense of T_0; i.e., the v coordinate of $T_0(T_1\tilde{x}_*)$ is zero and $T_0T_1\tilde{x}_* \in W^{cu}_{loc} \cap \Sigma_1$. Therefore, $T_1(T_0T_1\tilde{x}_*) = E(\mu)$, and $0 < y(E(\mu)) = y(T_1(T_0T_1\tilde{x}_*)) = y(T_1\tilde{x}_*)$ which is a contradiction. Since $y(T_1\tilde{x}_*) > 0$, the trajectory passing through $\tilde{x}_*$ is a real periodic trajectory for the original system.

We now prove the hyperbolicity of Π_μ, if it exists. Let $\tilde{\phi}^1_*$ be as in Lemma 6.7. Since T_0T_1 is an expansion of $\tilde{\phi}^1_*$, $\tilde{\phi}^1_* \subset W^u(\Pi_\mu)$ and $\dim W^u(\Pi_\mu) \ge \dim \tilde{\phi}^1_* + 1 = \dim W^{cu}(0)$. Similarly, consider $(T_0T_1)^{-1}$ and the $(T_0T_1)^{-1} : \tilde{\psi} \to (T_0T_1)^{-1}\tilde{\psi}$, where $\tilde{\psi}$ is a v-slice on Σ_1. We can prove that $(T_0T_1)^{-1}$ is a contraction on v-slice and there is a fixed element $\tilde{\psi}_*$. Moreover, $(T_0T_1)^{-1}$ is an expansion on $\tilde{\psi}_*$. Thus, $\tilde{\psi}_* \subset W^s(\Pi_\mu)$ and $\dim W^s(\Pi_\mu) \ge \dim \tilde{\psi}_* + 1 = \dim W^{cs}(0)$. Therefore, we must have $\dim W^u(\Pi_\mu) = \dim W^{cu}(0)$ and $\dim W^s(\Pi_\mu) = \dim W^{cs}(0)$.

Remark 6.9. The equilibria, homoclinic, heteroclinic and periodic solutions found in this paper are the only solutions that remain in a neighborhood of Γ_0. The proof of this follows from the proof of the uniqueness of the periodic orbit.

7. Appendix A.

Consider the following equation:

$$\dot{w}(t) = f(w(t)), \qquad w \in \mathbb{R}^d \tag{A1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is C^∞, $f(0) = 0$, and 0 nonhyperbolic.

Let $w = (u, v)$, $f = (g, h)$ and rewrite (A1) as

$$\begin{aligned} \dot{u}(t) &= A_1u(t) + g(u(t), v(t)) \\ \dot{v}(t) &= A_2v(t) + h(u(t), v(t)) \end{aligned} \tag{A2}$$

where $u, g \in \mathbb{R}^n$, $v, h \in \mathbb{R}^m$, $n + m = d$, $A_1 : \mathbb{R}^n \to \mathbb{R}^n$, $A_2 : \mathbb{R}^m \to \mathbb{R}^m$ are linear maps with $\operatorname{Re}\sigma(A_1) \ge 0$ and $\operatorname{Re}\sigma(A_2) < 0$. Assume that $g(0,0) = 0$, $h(0,0) = 0$, $Dg(0,0) = 0$ and $Dh(0,0) = 0$. Let $L(\epsilon)$ be the Lipschitz constant of (g, h) for $|(u,v)| \le \epsilon$, where $\epsilon > 0$. Note that $L(\epsilon) \to O(|\epsilon|)$ as $\epsilon \to 0$. By the assumptions on the matrices A_1 and A_2, there exist constants $\beta > \alpha > 0$ such that

$$\left|e^{-A_1t}\right| \le ce^{\alpha t} \quad \text{and} \quad \left|e^{A_2t}\right| \le ce^{-\beta t} \quad \text{for } t \ge 0.$$

Assume ϵ is sufficiently small so that $\beta - L(\epsilon) - cL(\epsilon) > \alpha$.

We define the local center stable and local center unstable manifolds as follows. Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be C^∞ with compact support. Assume that $\phi(w) = 1$ for all $|w| \leq \epsilon$ and $\phi(w) = 0$ for all $|w| \geq 2\epsilon$. Let $\tilde{f}(w) = \phi(w)f(w)$. Thus $\tilde{f}$ is globally small together with its first order derivatives. Consider the equation:

$$\dot{w} = \tilde{f}(w) \tag{A3}$$

Let $\alpha < \gamma < \beta$. Define

$$\tilde{W}^{cu} = \left\{ \begin{array}{l} x \,|\, \text{there is a solution } w(t) \text{ of (A3) such that} \\ \quad w(0) = x \text{ and } |w(t)| \leq ce^{-\gamma t} \text{ for } t \leq 0 \end{array} \right\}$$

$$\tilde{W}^{cs} = \left\{ \begin{array}{l} x \,|\, \text{there is a solution } w(t) \text{ of (A3) such that} \\ \quad w(0) = x \text{ and } |w(t)| \leq ce^{\gamma t} \text{ for } t \geq 0 \end{array} \right\}$$

where the constant $c \geq 1$ depends on the solution $w(t)$. It is well known that $\tilde{W}^{cu}$ and $\tilde{W}^{cs}$ are smooth invariant manifolds and are called global center unstable and global center stable manifolds of (A3).

Observe that the flows of (A1) and (A3) coincide in $B_\epsilon = \{w : |w| \leq \epsilon\}$. Define the local center stable and center unstable manifolds, W^{cs}_{loc} and W^{cu}_{loc}, of (A1) as

$$W^{cu}_{loc} = \tilde{W}^{cu} \cap B_\epsilon, \qquad W^{cs}_{loc} = \tilde{W}^{cs} \cap B_\epsilon.$$

These definitions clearly depend on the choice of the truncation function $\phi(w)$. Hence, local center stable (or unstable) manifolds are not unique. However, the following results are true.

Theorem A. *Let $\epsilon > 0$ be fixed and $\Gamma \subseteq B_\epsilon$ be a backward invariant set; i.e,*

$$\Gamma = \left\{ \begin{array}{l} x \,|\, \text{there is a solution } w(t) \text{ of (1) such that} \\ \quad w(0) = x,\ w(t) \in \Gamma \text{ and } |w(t)| \leq \epsilon \text{ for all } t \leq 0 \end{array} \right\}.$$

Then, for any local center unstable manifold W^{cu}_{loc}, we have $\Gamma \subset W^{cu}_{loc}$. Furthermore, if $\overline{W}^{cu}_{loc}$ is another local center unstable manifold, then they have the common tangent spaces at any $x \in \Gamma$; i.e., $T_x W^{cu}_{loc} = T_x W^{cu}_{loc}$ for any $x \in \Gamma$.

Proof: The first assertion $\Gamma \subset W^{cu}_{loc}$ comes directly from the definitions. Assume that $\overline{W}^{cu}_{loc}$ is flat; i.e.,

$$\overline{W}^{cu}_{loc} = \{w = (u, v) \,|\, v = 0,\ |w| \leq \epsilon\}.$$

Let $x \in \Gamma \subset \overline{W}^{cu}_{loc} \cap W^{cu}_{loc}$ be fixed. For any large $t_1 > 0$, there exists a solution $w(t)$ of (A1) such that $w(t_1) = x$ and $w(t) \in \Gamma$ for all $t \in [0, t_1]$. Since $\overline{W}^{cu}_{loc}$ is flat, the second component $v(t)$ of $w(t) = (u(t), v(t))$ equals 0 for all $t \in [0, t_1]$. Moreover, $h(u, 0) = 0$ provided $|u|,\ |v| \leq \epsilon$.

Let $(u_1, v_1) \in T_x W^{cu}_{loc} = T_{w(t_1)} W^{cu}_{loc}$. Since $\overline{W}^{cu}_{loc}$ is flat, the second assertion is proved if we show that $v_1 = 0$.

Let $(u_0, v_0) \in T_{w(0)} W^{cu}_{loc}$. Let $\pi(t, w)$ be the flow generated by (A1) with $\pi(0, w) = w$. We assume that

$$D_w \pi(t_1, x) \cdot (u_0, v_0) = (u_1, v_1).$$

The mapping $D_w \pi(t_1, x) : (u_0, v_0) \to (u_1, v_1)$ is determined by the following linear variational equation

$$\begin{aligned} \dot{u} &= A_1 u + D_u g(u(t), 0) u + D_v h(u(t), 0) v, \\ \dot{v} &= A_2 v + D_u h(u(t), 0) u + D_v h(u(t), 0) v. \end{aligned} \tag{A4}$$

Since $h(u, 0) = 0$ locally, we have

$$\dot{v} = A_2 v + D_v h(u(t), 0) v.$$

Thus,

$$|v(t)| \le c e^{-(\beta - L(\epsilon))t} |v(0)|$$

where $L(\epsilon)$ is the Lipschitz number for $f(w)$. Substituting into (A4), we have

$$\begin{aligned} |u(t)| &\le c e^{-\alpha(t - t_1)} |u(t_1)| + \int_t^{t_1} c e^{-\alpha(t-s)} L(|u(s)| + |v(s)|)\, ds \\ &\le c e^{-\alpha(t-t_1)} |u(t_1)| + \int_t^{t_1} c L e^{\alpha(s-t)} |u(s)|\, ds + \frac{c^2 L}{\beta - L} e^{-(\beta - L)t} |v(0)|. \end{aligned}$$

It follows that

$$|e^{\alpha t} u(t)| \le C_1 e^{cL(t_1 - t)} e^{\alpha t_1} |u(t_1)| + L C_2 e^{-(\beta - L - \alpha)t} |v(0)|,$$

$$|u(0)| \le C_1 e^{(\alpha + cL)t_1} |u(t_1)| + L C_2 |v(0)|,$$

where C_1 and C_2 are constants independent of ϵ and t_1. Note that $u(0) = u_0$, $v(0) = v_0$, $u(t_1) = u_1$, $v(t_1) = v_1$, and (u_0, v_0) is tangent to W^{cu}_{loc} and $|v_0| \le C_3 |u_0|$. Thus

$$|u(0)| \le C_1 (1 - L C_2 C_3)^{-1} e^{(\alpha + cL)t_1} |u(t_1)|,$$

for some constant C_3 independent of ϵ and t_1. If ϵ is small, then $L C_2 C_3 < 1$.

Next,

$$\begin{aligned} |v_1| = |v(t_1)| &\le C e^{-(\beta - L)t_1} |v(0)| \le C C_3 e^{-(\beta - L)t_1} |u(0)| \\ &\le C C_1 C_3 (1 - L C_2 C_3)^{-1} e^{-(\beta - L - LC - \alpha)t_1} |u(t_1)| \\ &= C_4 e^{-(\beta - L - LC - \alpha)t_1} |u_1|. \end{aligned}$$

Notice that C_4 does not depend on t_1. Letting $t_1 \to +\infty$, we have $v_1 = 0$. This completes the proof.

8. Appendix B.

Consider the following equation in $\mathbb{R}^{m+n}$:

$$\dot{x}(t) = A x(t) + f(x(t)), \tag{B1}$$

where $f : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is $C^{k,1}$ $(k \geq 0)$; i.e., f is k times continuously differentiable and its k-th derivatives are Lipschitz continuous. Let

$$|f|_{k,1} = \sup_{x\in\mathbb{R}^{m+n}} \left\{ |f|_0, \cdots, |f|_k, \sup_{x\neq y} \left[\frac{|D^k f(x) - D^k f(y)|}{|x-y|}\right] \right\}$$

where $|\cdot|_r$, $r = 0, \cdots, k$, denotes the usual C^r norm and D^k is the k-th differentiation operator. Let $L > 0$ be fixed and $|f|_{0,1} \leq L$.

Let $\operatorname{Re}\sigma(A) \leq 0$. $\sigma(A) = \sigma_- \cup \sigma_0$ with

$$\begin{cases} \operatorname{Re}\lambda = 0, & \lambda \in \sigma_0, \\ \operatorname{Re}\lambda < -\alpha, & \lambda \in \sigma_-, \end{cases}$$

$\alpha > 0$ is a constant.

Let P and $Q = I - P$ be the spectral projection matrices corresponding to σ_- and σ_0. Let $P\mathbb{R}^{m+n} = Y$ and $Q\mathbb{R}^{m+n} = Z$. Assume that after a change of coordinates, $Y = \mathbb{R}^m$ and $Z = \mathbb{R}^n$. Let α, β and K be positive constants satisfying $\beta < \alpha/(k+3)$ and

$$|e^{At}P| \leq Ke^{-\alpha t}, \quad t \geq 0,$$
$$|e^{At}Q| \leq Ke^{\beta|t|}, \quad t \in \mathbb{R}.$$

For each real constant γ, let

$$E(\gamma) = \{x(t) \mid x : \mathbb{R}^+ \to \mathbb{R}^{m+n}, \text{ continuous with } |x(t)| \leq Ce^{\gamma t}\}$$

where C is some positive constant. Let $|x|_{E(\gamma)}$ be the smallest of such constants. It is not hard to see that $E(\gamma)$ is a Banach space with norm $|\cdot|_{E(\gamma)}$. Let $\xi(t,x)$ denote the solution of (B1) with $\xi(0,x) = x \in \mathbb{R}^{m+n}$. The following is the main result in this section.

Theorem B. *If $\beta + KL \leq \delta = (\alpha - \beta)/(k+2)$ and $2KL < \delta$, then there exists a $C^k\phi : \mathbb{R}^{m+n} \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ with bounded C^k derivatives such that*

$$\phi(x, 0) = 0, \quad P\phi(x,y) = y$$

and

$$\xi(t,x) - \xi(t, x + \phi(x,y)) \in E(-\alpha + \delta) \quad \text{for all } y \in \overline{Y} = \mathbb{R}^m.$$

We note that the set $\mathcal{W}^s(x) = \{x + \phi(x,y) : y \in Y\}$ is called the stable fiber through $x \in \mathbb{R}^{m+n}$ and it satisfies the invariance property: $T(t)\mathcal{W}^s(x) = \mathcal{W}^s(T(t)x)$, where $T(t)$ denotes the nonlinear group generated by (B1). The condition $\beta + KL \leq (\alpha - \beta)/(k+2)$ can be satisfied if L is sufficiently small since $\beta < \alpha/(k+3)$.

We shall need the following lemmas in proving Theorem B.

Lemma B1. *If $|f|_{Lip} \le L$, then for any a, a_1 and $b \in \mathbb{R}^{m+n}$*

$$\big|[f(a+b)-f(a)]-[f(a_1+b)-f(a_1)]\big| \le |a-a_1|^\eta |b|^{1-\eta}$$

for any $0 \le \eta \le 1$.

Lemma B2. *If $|f|_{Lip} \le L$, y and $y_1 \in E(\epsilon)$, $z \in E(-\lambda)$, then*

$$\begin{aligned}\big|[f(y(t)+z(t))-f(y(t))]-[f(y_1(t)+z(t))-f(y_1(t))]\big|_{E(-\lambda+\eta(\epsilon+\lambda))}\\ \le L|y-y_1|^\eta_{E(\epsilon)}|z|^{1-\eta}_{E(-\lambda)},\end{aligned}$$

where ϵ, λ are real constants and $0 \le \eta \le 1$.

Lemma B3. *We have that $\xi(\cdot,x) \in E(\beta)$, $D^i_x\xi(\cdot,x) \in E(i\delta)$, $1 \le i \le k$, and*

$$\big|D^i_x\xi(\cdot,x)-D^i_x\xi(\cdot,z)\big|_{E((k+1)\delta)} \le C|x-z|$$

for some $C > 0$.

The proof of Lemma B1 shall be omitted. Lemma B2 follows from Lemma B1 easily. The proof of Lemma B3 uses the Gronwall inequality, and shall be omitted too.

Proof of Theorem B: Let $\xi(t,x)+\zeta(t)$ be a solution of (B1) with $\zeta(t) \in E(-\gamma)$, $\beta < \gamma < \alpha$. Then $\zeta(t)$ satisfies the integral equation

$$\begin{aligned}\zeta(t) = e^{At}P\zeta(0) + \int_0^t e^{A(t-s)}P[f(\xi(s,x)+\zeta(s))-f(\xi(s,x))]\,ds\\ + \int_\infty^t e^{A(t-s)}Q[f(\xi(s,x)+\zeta(s))-f(\xi(s,x))]\,ds.\end{aligned}$$

This indicates that we define a linear operator $\Phi : E(-\gamma) \to E(-\gamma)$, for any $\beta < \gamma < \alpha$ as follows

$$(\Phi x)(t) = \int_0^t e^{A(t-s)}P\zeta(s)\,ds + \int_\infty^t e^{A(t-s)}Q\zeta(s)\,ds.$$

It is not hard to see that

$$\|\Phi\|_{\mathcal{L}(E(-\gamma),E(-\gamma))} \le K\Big(\frac{1}{\alpha-\gamma}+\frac{1}{\gamma-\beta}\Big).$$

We are looking for a fixed point of the following equation

$$\zeta(t) = e^{At}y + \Phi[f(\xi(\cdot,x)+\zeta(\cdot))-f(\xi(\cdot,x))]. \tag{B2}$$

By the contraction mapping principle if $\beta < \gamma < \alpha$ and

$$LK\Big(\frac{1}{\alpha-\gamma}+\frac{1}{\gamma-\beta}\Big) < 1, \tag{B3}$$

(B2) has a unique fixed point $\zeta(t,x,y) \in E(-\gamma)$. (B3) is satisfied if $\beta+\delta \le \gamma \le \alpha-\delta$. Hence, $\zeta(t,x,y) \in E(-\alpha+\delta)$. Let $\phi(x,y) = \zeta(0,x,y)$. Thus, ϕ is the desired function.

We now consider the smoothness of $\phi(x,y)$. Formally, for each $1 \le i \le k$, $D_x^i\zeta(t,x,y)$ satisfies the following integral equation in Γ^i.

$$\Gamma^1(x) = \Phi\{Df(\xi(x)+\zeta(x))\Gamma^1(x)\} + \Phi\{(Df(\xi(x)+\zeta(x)) - Df(\xi(x)))\xi_1(x)\}, \quad (B4)_1$$

$$\begin{aligned} \Gamma^i(x) = {} & \Phi\{Df(\xi(x)+\zeta(x))\Gamma^i(x)\} + \Phi\{(Df(\xi(x)+\zeta(x)) - Df(\xi(x)))\xi_i(x)\} \\ & + \Phi\{F_i(\xi_1(x)+\zeta_1(x), \cdots \xi_{i-1}(x)+\zeta_{i-1}(x); \cdots D^i f(\xi(x)+\zeta(x)), \cdots) \\ & - F_i(\xi_1(x), \cdots \xi_{i-1}(x); \cdots D^i f(\xi(x)), \cdots)\}, \end{aligned} \quad (B4)_i$$

where for simplicity $\xi(x)$, $\zeta(x)$ and $\xi_i(x)$ denote $\xi(\cdot,x)$, $\zeta(\cdot,x,y)$ and $D_x^i\xi(\cdot,x,y)$ respectively. $F_i(\eta_1,\cdots,\eta_{i-1};A_\alpha,\cdots)$ is a sum of multilinear forms on $(\eta_1,\cdots,\eta_{i-1})$; each term has the form

$$A_\alpha\eta_1^{\alpha_1}\cdots\eta_{i-1}^{\alpha_{i-1}}$$

where α and α_j are multi-indices with

$$|\alpha| \le i, \quad \sum_{j=1}^{i-1}\alpha_j = \alpha$$

and $|\alpha_1| + 2|\alpha_2| + \cdots + (i-1)|\alpha_{i-1}| = i$.

Using the fact that $D_x\xi(x) \in E(\delta)$ (Lemma B3), the nonhomogeneous term in $(B4)_1$ is in $E(-\alpha+2\delta)$. By the contraction principle, $(B4)_1$ has a unique fixed point $\Gamma^1(\cdot,x,y) \in E(-\alpha+2\delta)$. It is straightforward to show by induction that the nonhomogeneous term in $(B4)_i$ is in $E(-\alpha+\delta+i\delta)$ and $(B4)_i$ admits a unique fixed point $\Gamma^i(\cdot,x,y) \in E(-\alpha+\delta+i\delta)$, if $1 \le i \le k$. To do this we also need the fact $D_x^i\xi(\cdot,x) \in E(i\delta)$ (Lemma B3). We leave the detail to the reader.

It remains to show that $\Gamma^i = D_x^i\zeta(x)$. We first show that if $f \in C^{1,1}$, then

$$\big|\zeta(x) - \zeta(z) - \Gamma^i(z)(x-z)\big|_{E(-\alpha+2\delta)} = o(|x-z|) \quad \text{as } x \to z. \quad \text{(B5)}$$

From (B2) and $(B4)_1$,

$$\begin{aligned} & \zeta(x) - \zeta(z) - \Gamma^1(z)(x-z) \\ & \quad = \Phi\{f(\xi(x)+\zeta(x)) - f(\xi(z)+\zeta(z)) - f(\xi(x)) + f(\xi(z)) \\ & \qquad - Df(\xi(z)+\zeta(z))[D_x\xi(z)+\Gamma^1(z)](x-z) + Df(\xi(z))D_x\xi(z)(x-z)\} \\ & \quad = \Phi\{Df(\xi(z)+\zeta(z)))(\zeta(x)-\zeta(z)-\Gamma^1(z)(x-z)\} \\ & \qquad + \Phi\{[f(\xi(x)+\zeta(x)) - f(\xi(x))] - [f(\xi(z)+\zeta(z)) - f(\xi(z))] \\ & \qquad - Df(\xi(z)+\zeta(z))(\zeta(x)-\zeta(z)) - Df(\xi(z)+\zeta(z))D_x\xi(z)(x-z) \\ & \qquad - Df(\xi(z))D_x\xi(z)(x-z)\} \\ & \quad = \Phi\{Df(\xi(z)+\zeta(z)))(\zeta(x)-\zeta(z)-\Gamma^1(z)(x-z)\} + I_1, \quad \text{say.} \end{aligned}$$

If we can prove that $I_1 = o(|x-z|)$ in $E(-\alpha+2\delta)$, then (B5) follows easily by the contraction mapping principle in the space $E(-\alpha+2\delta)$.

Consider

$$\begin{aligned}I_2 &= [f(\xi(x)+\zeta(z)) - f(\xi(z)+\zeta(z)(x))] - [f(\xi(z)+\zeta(z)) - f(\xi(z))] \\ &\quad - [Df(\xi(z)+\zeta(z)) - Df(\xi(z))](\xi(x)-\xi(z)) \\ &= \{[Df(\theta\xi(x)+(1-\theta)\xi(z))+\zeta(z)) - Df(\theta\xi(x)+(1-\theta)\xi(z))] \\ &\quad - [Df(\xi(z)+\zeta(z)) - Df(\xi(z))]\}(\xi(x)-\xi(z)) \\ &= I_3(\xi(x)-\xi(z)), \quad \text{say,}\end{aligned}$$

where θ is in fact a function of t for $t \in \mathbb{R}^+$. By Lemma B2, let $\epsilon = \delta$ and $\lambda = \alpha - \delta$,

$$|I_3| \le L|\xi(x)-\xi(z)|^{\eta}_{E(\delta)}|\zeta(z)|^{1-\eta}_{E(-\alpha+\delta)}e^{(-\alpha+\delta+\eta\alpha)t}.$$

Moreover, by Lemma B3,

$$|\xi(x)-\xi(z)|_{E(\delta)} \le C|x-z|.$$

Therefore, choosing $\eta\alpha = \delta$ we have

$$|I_2|_{E(-\alpha+2\delta)} \le C|x-z|^{1+\eta}|\zeta(z)|^{1-\eta}_{E(-\alpha+\delta)}, \qquad 0 \le \eta \le 1. \tag{B6}$$

Observe that from (B2), we can easily prove that

$$|\zeta(x)-\zeta(z)|_{E(-\alpha+\delta)} = O(|x-z|).$$

Thus

$$\begin{aligned}&|f(\xi(x)+\zeta(x)) - f(\xi(x)+\zeta(z)) - Df(\xi(x)+\zeta(z))(\zeta(x)-\zeta(z))|_{E(-\alpha+\delta)} \\ &\qquad \le L\,|\zeta(x)-\zeta(z)|^2_{E(-\alpha+\delta)} = O(|x-z|^2).\end{aligned} \tag{B7}$$

$$\begin{aligned}&|[Df(\xi(x)+\zeta(z)) - Df(\xi(z)+\zeta(z))](\zeta(x)-\zeta(z))|_{E(-\alpha+2\delta)} \\ &\qquad \le L\,|\xi(x)-\xi(z)|_{E(\delta)}\,|\zeta(x)-\zeta(z)|_{E(-\alpha+\delta)} = O(|x-z|^2).\end{aligned} \tag{B8}$$

$$\begin{aligned}&|Df(\xi(z)+\zeta(z)) - Df(\xi(z)))(\xi(x)-\xi(z)-D_x\xi(z)(x-z)|_{E(-\alpha+2\delta)} \\ &\qquad = O(|x-z|^2).\end{aligned} \tag{B9}$$

From (B6)–(B9), it is clear that I_1 is $o(|x-z|)$ in the space $E(-\alpha+2\delta)$, if η is such that $\eta\alpha = \delta$, (B5) has been proven.

It is now clear one can prove by induction that $D_x^{i-1}\zeta(x) - D_x^{i-1}\zeta(z) - \Gamma^i(x-z) = o(|x-z|)$ in the space $E(-\alpha+\delta+i\delta)$, $2 \le i \le k$, provided $f \in C^{k,1}$ with $|f|_{k,1} \le L$. The proof needs the fact that F_i is obtained from F_{i-1} by formally taking partial derivatives with respect to x.

We still have to show that $D_x^i\zeta(\cdot,x,y)$ is C^{k-i} with respect to y. However, this is a consequence of uniform contraction theorem in the space $E(-\alpha+\delta+i\delta)$. Finally, $\phi(x,y) = \zeta(0,x,y)$ is clearly C^k jointly in $\mathbb{R}^{m+n} \times Y$.

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