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Transition Layers for Singularly Perturbed Delay Differential Equations with Monotone Nonlinearities

Shui-Nee Chow,[1] Xiao-Biao Lin,[2] and John Mallet-Paret[3]

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Transition layers arising from square-wave-like periodic solutions of a singularly perturbed delay differential equation are studied. Such transition layers correspond to heteroclinic orbits connecting a pair of equilibria of an associated system of transition layer equations. Assuming a monotonicity condition in the nonlinearity, we prove these transition layer equations possess a unique heteroclinic orbit, and that this orbit is monotone. The proof involves a global continuation for heteroclinic orbits.

KEY WORDS: Differential delay equations; singular perturbations; transition layers.

1. INTRODUCTION AND PRELIMINARIES

Heteroclinic solutions of delay differential equations are of great interest, both in applications and in theory. A particular example arises in the study of slowly oscillating solutions of the scalar delay equation

$$\dot{z}(t) = -\lambda z(t) + \lambda f(z(t-1))$$

which was studied by Chow and Mallet-Paret (1983) and by Mallet-Paret and Nussbaum (1986a, b). Among other hypotheses, it was assumed that $f(0)=0$, $f'(0)<-1$, $zf(z)<0$ for all $z\neq 0$, and $f(a)=-b$ and $f(-b)=a$ for some $a>0, b>0$. It was shown by Mallet-Paret and Nussbaum (1986a, b) that, if also the set $\{-b, a\}$ attracts iterates $f^k(z)$ of each nonzero z, then, as $\lambda \to +\infty$, the slowly oscillating periodic solutions of this

[1] Center for Dynamical Systems and Nonlinear Studies, School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30333.

[2] Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695.

[3] Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912.

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differential equation approach a step function taking the value a on intervals $(2n, 2n+1)$, and $-b$ on $(2n+1, 2n+2)$, for $n \in \mathbb{Z}$. Near integer values of t, such a solution $z(t)$ possesses a transition layer of width $O(\lambda^{-1})$. Upon rescaling the time and passing to the limit $\lambda \to +\infty$, one obtains a solution $(x(t), y(t))$ of the transition layer system

$$\begin{aligned} \dot{x}(t) &= x(t) - f(y(t-r)) \\ \dot{y}(t) &= y(t) - f(x(t-r)) \end{aligned} \tag{1.1$_r$}$$

for some value of the (new) delay parameter $r > 0$. In fact, the solution of $(1.1)_r$ so obtained is a heteroclinic solution joining the equilibria $(x^-, y^-) = (-b, a)$ and $(x^+, y^+) = (a, -b)$. This paper shows that, under an additional hypothesis of monotonicity,

$$f'(z) \leqslant 0 \qquad \text{for all} \quad z \in \mathbb{R}$$

there exists a unique r such that $(1.1)_r$ possesses a heteroclinic solution connecting (x^-, y^-) to (x^+, y^+), and that also this solution is unique.

A related problem, the singularly perturbed integral equation

$$z(t) = \frac{\lambda}{2} \int_{t-1/\lambda}^{t+1/\lambda} f(z(s-1))\, ds$$

was studied by Chow *et al.* (1985). The nonlinearity f was assumed to be odd, monotone decreasing, and convex (for $z \geqslant 0$), with $f(\pm a) = \mp a$ for some $a > 0$. Existence and uniqueness of a transition layer corresponding to a special class of symmetric slowly oscillating periodic solutions were proved. However, the special properties of f as well as the restricted class of solutions played a prominent role in the proof. In the present paper, we use only the monotonicity of f to establish uniqueness for $(1.1)_r$ among all heteroclinic solutions. In addition, we develop some general techniques for global continuation of heteroclinic solutions based on exponential dichotomies and the method of Lyapunov–Schmidt.

To be specific, a homotopy method is developed to tackle our problem. To describe this method abstractly, let $F: \mathbb{R}^n \times I \to \mathbb{R}^n$ be a C^1 function, where $I = [0, 1]$, and consider a family of delay differential equations

$$\dot{X}(t) = X(t) + F(X(t-r), \alpha), \qquad \alpha \in I.$$

(The results of this section actually hold for much more general classes of delay equations; however, we state our results for the equation above as its special form will play a role later.) Two parameters, $\alpha \in I$ and the delay r, are present. If $r>0$, then we may rescale the time t and obtain the equivalent system

$$\dot{X}(t)=rX(t)+rF(X(t-1),\alpha). \tag{$(1.2)_{r,\alpha}$}$$

We assume for each $\alpha\in I$ that $(1.2)_{r,\alpha}$ possesses equilibria $X^-=X^-(\alpha)$ and $X^+=X^+(\alpha)$ varying smoothly with α. Our object is to determine, for each $\alpha\in I$, all values of $r>0$ for which there exists a heteroclinic solution $X(t)$ of $(1.2)_{r,\alpha}$ connecting $X^-(\alpha)$ to $X^+(\alpha)$, that is, for which $\lim_{t\to\pm\infty}X(t)=X^{\pm}(\alpha)$. In fact, we seek conditions under which there is a unique such $r=r(\alpha)$ and a unique (up to time translation) such solution $X(t)=X(t,\alpha)$ for each α.

To obtain such a result, we introduce the following four hypotheses. By a heteroclinic solution of $(1.2)_{r,\alpha}$, we mean a heteroclinic solution of this system connecting $X^-(\alpha)$ to $X^+(\alpha)$, as above. By a unique such solution, we mean uniqueness up to time translation.

H1. There is a unique $r=r(0)>0$ such that $(1.2)_{r(0),0}$ has a heteroclinic solution $X(t)=X(t,0)$; furthermore, this heteroclinic solution is unique.

H2. Suppose for some $\alpha^*\in I$ and $r^*>0$ that $(1.2)_{r^*,\alpha^*}$ has a heteroclinic solution $X(t,\alpha^*)$. Then, given $\varepsilon>0$, there exists $\delta>0$ such that, for any $\alpha\in(\alpha^*-\delta,\alpha^*+\delta)\cap I$, there exists $r=r(\alpha)\in(r^*-\varepsilon,r^*+\varepsilon)$ with the property that $(1.2)_{r(\alpha),\alpha}$ has a heteroclinic solution $X(t)=X(t,\alpha)$ satisfying $|X(t,\alpha)-X(t,\alpha^*)|<\varepsilon$ for all $t\in\mathbb{R}$. Furthermore, if ε is sufficiently small, then $r(\alpha)$ is unique in the interval $(r^*-\varepsilon,r^*+\varepsilon)$, and $X(t,\alpha)$ is unique among those heteroclinic solutions satisfying this inequality.

H3. There exist constants $0<r_m<r_M$ such that, if $(1.2)_{r,\alpha}$ has a heteroclinic solution for some $\alpha\in I$ and $r>0$, then $r_m\leqslant r\leqslant r_M$.

H4. Let $\alpha^j\in I$ and $r^j>0$ be convergent sequences, say $\alpha^j\to\alpha^*$ and r^j-r^* with $r^*>0$, and let $X^j(t)$ be a hetroclinic solution of $(1.2)_{r^j,\alpha^j}$. Then, for some subsequence of $j\to+\infty$ and for some values $\theta^j\in\mathbb{R}$, the limit $X^j(t+\theta^j)\to X^*(t)$ of the time translates exists uniformly for $t\in\mathbb{R}$. In particular, $X^*(t)$ is a heteroclinic solution of $(1.2)_{r^*,\alpha^*}$.

Theorem 1.1. *Under the hypotheses* H1–H4. *there exists a unique* $r(1)>0$ *such that Eq.* $(1.2)_{r(1),1}$ *has a heteroclinic solution. Furthermore, this solution is unique.*

Proof. The proof is standard. For $\alpha\in I$, let $n(\alpha)$ denote the total number of heteroclinic solutions of $(1.2)_{r,\alpha}$ for all values of $r>0$. The fact that, for fixed α, heteroclinic solutions are locally isolated, in the sense of H2, and form a sequentially compact set, in the sense of H4, together with the bounds on r in H3, implies that $n(\alpha)$ is finite. Next, note that $\lim_{\alpha \to \alpha^*} \inf n(\alpha) \geq n(\alpha^*)$ by H2, and that $\lim_{\alpha \to \alpha^*} \sup n(\alpha) \leq n(\alpha^*)$ by H3 and H4. Thus, $n(\alpha)$ is a continuous function of α. Since $n(0) = 1$ by H1, we have $n(1) = 1$. ▌

A local perturbation technique [see Hale and Lin (1986)] related to the Mel'nikov method will be employed to verify H2 for the system $(1.1)_r$ of interest. To describe this method for our general system $(1.2)_{r,\alpha}$, consider the C^1 map $\mathscr{F}: C^1(\mathbb{R}, \mathbb{R}^n) \times (0, \infty) \times I \to C^0(\mathbb{R}, \mathbb{R}^n)$ defined by

$$[\mathscr{F}(X, r, \alpha)](t) = \dot{X}(t) - rX(t) - rF(X(t-1), \alpha).$$

Here, $C^0 = C^0(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of continuous bounded functions with the supremum norm $\|X(\cdot)\|_{C^0} = \sup |X(t)|$ while $C^1 = C^1(\mathbb{R}, \mathbb{R}^n)$ is the corresponding space of C^1 functions with the norm $\|X(\cdot)\|_{C^1} = \|X(\cdot)\|_{C^0} + \|\dot{X}(\cdot)\|_{C^0}$. Of course, if $X(t)$ is a heteroclinic solution of $(1.2)_{r,\alpha}$, then $\mathscr{F}(X, r, \alpha) = 0$. Consider in particular the solution $X(t, \alpha^*)$ of $(1.2)_{r^*,\alpha^*}$ in H2, and let $\mathscr{L}(\alpha^*)$ denote the Fréchet derivative

$$\mathscr{L}(\alpha^*) = \frac{\partial \mathscr{F}(X(\cdot, \alpha^*), r^*, \alpha^*)}{\partial X}$$

of $\mathscr{F}$ with respect to its first argument evaluated at this solution. That is, $\mathscr{L}(\alpha^*)$: $C^1 \to C^0$ is the linear operator

$$[\mathscr{L}(\alpha^*)\, \Xi](t) = \dot{\Xi}(t) - r^* \Xi(t) - r^* A(t, \alpha^*)\, \Xi(t-1)$$

where

$$A(t, \alpha^*) = \frac{\partial F(X(t-1, \alpha^*), \alpha^*)}{\partial X}$$

is the Jacobian matrix of F evaluated along the heteroclinic solution. The variational equation along this solution is simply $\mathscr{L}(\alpha^*)\, \Xi = 0$, and relevant to this are the two limiting equations

$$\dot{\Xi}(t) = r^* \Xi(t) + r^* A(\pm\infty, \alpha^*)\, \Xi(t-1) \tag{1.3$_\pm$}$$

that is, the variational equations about the equilibria $X^{\pm}(\alpha^*)$. We also note here the formal adjoint

$$\dot{\Psi}(t) = -r^* \Psi(t) - r^* \Psi(t+1)\, A(t+1, \alpha^*) \tag{1.4}$$

of the variational equation along $X(t, \alpha^*)$. Here $\Psi(t)$ is a row vector, and (1.4) is solved in the decreasing direction of t. See Hale (1977).

The results described by Hale and Lin (1986) describe the Fredholm structure of the operator $\mathscr{L}(\alpha^*)$, and in particular characterize its kernel and range. Assume that both limiting Eqs. $(1.3)_\pm$ have hyperbolic spectra, that is, $\operatorname{Re}\mu \neq 0$ for all characteristic exponents μ. Let $N^\pm$ denote the number of such μ with $\operatorname{Re}\mu > 0$, that is, $N^\pm$ is the dimension of the unstable manifold of $X^\pm(\alpha^*)$. Then, $\mathscr{L}(\alpha^*)$ is a Fredholm operator, and its index is

$$\dim \mathscr{N}(\alpha^*) - \operatorname{codim} \mathscr{R}(\alpha^*) = N^- - N^+ \tag{1.5}$$

where $\mathscr{N}(\alpha^*)$ and $\mathscr{R}(\alpha^*)$ are the kernel and range, respectively, of $\mathscr{L}(\alpha^*)$. It is furthermore the case that

$$\mathscr{R}(\alpha^*) = \left\{ h \in C^0 \;\middle|\; \int_{-\infty}^{\infty} \Psi(t)\, h(t)\, dt = 0 \quad \text{for all solutions } \Psi(t) \text{ of (1.4) bounded on } \mathbb{R} \right\} \tag{1.6}$$

all bounded solutions of (1.4) in fact decaying exponentially as $t \to \pm\infty$. Note in particular that, if $N^- = N^+$ and $\dim \mathscr{N}(\alpha^*) = 1$, then (1.5) and (1.6) imply that the adjoint equation (1.4) has a unique (up to scalar multiple) nontrivial solution $\Psi^*(t)$ bounded on $\mathbb{R}$.

With the above remarks, we are able to give a sufficient condition for H2 to hold.

H2′. Suppose for some $\alpha^* \in I$, $r^* > 0$ that $(1.2)_{r^*,\alpha^*}$ has a heteroclinic solution $X(t, \alpha^*)$. Then both equilibria $X^\pm(\alpha^*)$ are hyperbolic and their unstable manifolds have the same dimension $N^- = N^+$. Furthermore, $\dim \mathscr{N}(\alpha^*) = 1$, that is, the varitional equation $\mathscr{L}(\alpha^*)\,\Xi = 0$ has $\Xi(t) = \dot{X}(t, \alpha^*)$ as its unique bounded solution up to scalar multiple. Finally,

$$\int_{-\infty}^{\infty} \Psi^*(t)\, \dot{X}(t, \alpha^*)\, dt \neq 0 \tag{1.7}$$

where $\Psi^*(t)$ is the unique (up to scalar multiple) nontrivial bounded solution of the adjoint equation.

Proposition 1.1. Hypothesis H2′ implies Hypothesis H2.

Proof. As in Hale and Lin (1986), this is an application of the implicit function theorem and the method of Lyapunov–Schmidt. For completeness, we outline the arguments.

We wish to determine all solutions of $\mathscr{F}(X, r, \alpha)=0$ for (X, r, α) near (X^*, r^*, α^*), where we write $X^*(t)=X(t, \alpha^*)$. First fix a linear functional $\Lambda: C^0 \to \mathbb{R}$ such that $\Lambda(\dot{X}^*) \neq 0$. Observe that $\Lambda(\dot{X}^*(\cdot+\theta))$ is continuous in θ and hence nonzero for all $\theta \in [-\theta_0, \theta_0]$, for some $\theta_0 > 0$. Then note that, if (X, r, α) is a solution near (X^*, r^*, α^*), then X is near X^* in the space C^1. Therefore, in a small enough neighborhood,

$$\frac{d}{d\theta} \Lambda(X(\cdot+\theta)-X^*)=\Lambda(\dot{X}(\cdot+\theta)) \neq 0$$

for all $\theta \in [-\theta_0, \theta_0]$ and so there exists a unique $\theta \in [-\theta_0, \theta_0]$ such that

$$\Lambda(X(\cdot+\theta)-X^*)=0.$$

Thus, without loss, we may consider (X, r, α) with

$$X=X^*+\Xi, \qquad \Xi \in \mathscr{N}_\Lambda$$

where $\mathscr{N}_\Lambda \subseteq C^1$ is the kernel of Λ restricted to C^1.

The Fréchet derivative $\mathscr{L}(\alpha^*)=\partial \mathscr{F}(X^*, r^*, \alpha)/\partial X$ is an isomorphism from $\mathscr{N}_\Lambda$ onto $\mathscr{R}(\alpha^*)$, since $\dot{X} \notin \mathscr{N}_\Lambda$ is a basis for its one-dimensional kernel in C^0. Further,

$$\frac{\partial \mathscr{F}(X^*, r^*, \alpha^*)}{\partial r}=\dot{X}^*$$

belongs to a one-dimensional complement of $\mathscr{R}(\alpha^*)$ in C^0, by (1.7). Therefore, by the implicit function theorem, there exists a unique $(X(\cdot, \alpha), r(\alpha))$ near (X^*, r^*), depending smoothly on α near α^*, such that $\mathscr{F}(X(\cdot, \alpha), r(\alpha), \alpha)=0$. With this, H2 holds, and our proposition is proved. ∎

2. THE TRANSITION LAYER SYSTEM AND VERIFICATION OF H2′

In this section, we begin our study of the system

$$\begin{aligned} \dot{x}(t) &= rx(t)-rf(y(t-1)) \\ \dot{y}(t) &= ry(t)-rf(x(t-1)) \end{aligned} \qquad (2.1)_r$$

of two delay differential equations. These, equations, with $r>0$, are equivalent to the transition layer system $(1.1)_r$ by a time rescaling. We

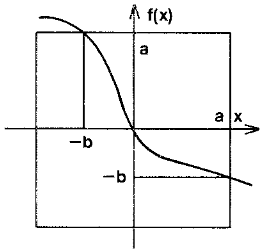

Fig. 1

introduce the following conditions on f: Fig. 1 depicts the graph of such a function f.

B1. $f: \mathbb{R} \to \mathbb{R}$ is C^1.

B2. There exist $a > 0$, $b > 0$ such that $f(a) = -b$ and $f(-b) = a$; also, $f(0) = 0$.

B3. $f'(x) \leqslant 0$ for all $x \in \mathbb{R}$.

B4. $f'(0) < -1$ and $0 \leqslant f'(a) f'(-b) < 1$.

B5. $|f(f(x))| > |x|$ for $x \in (-b, 0) \cup (0, a)$, and $|f(f(x))| < |x|$ for $x \in (-\infty, -b) \cup (a, +\infty)$. [Note that $f(f(x))$ and x have the same sign for all x.]

The system $(2.1)_r$ has three equilibria: $(x, y) = (-b, a)$, $(0, 0)$, and $(a, -b)$. Writing $X = (x, y)$, we shall denote two of these equilibria by $X^- = (-b, a)$ and $X^+ = (a, -b)$. We are interested in the existence of heteroclinic solutions $X(t)$ with $X(-\infty) = X^-$ and $X(+\infty) = X^+$. Under the above hypotheses, it turns out there is a unique such solution; moreover, the components $x(t)$ and $y(t)$ are monotone for all $t \in \mathbb{R}$. This is the content of the following theorem, which is the main result of our paper.

Theorem 2.1. *Assume that f satisfies* B1–B5. *Then, there exists a unique $r > 0$ such that* $(1.1)_r$, *or equivalently* $(2.1)_r$ *possesses a heteroclinic solution $X(t) = (x(t), y(t))$ joining $X(-\infty) = X^-$ to $X(+\infty) = X^+$. Furthermore, this is the only such solution, up to time translation. Finally, this solution has the additional property of monotonicity*:

$$\dot{x}(t) \geqslant 0 \qquad \textit{and} \qquad \dot{y}(t) \leqslant 0 \qquad \textit{for all} \quad t \in \mathbb{R}$$

with strict inequality for $\dot{x}(t)$ as long as $x(t) < a$, and for $\dot{y}(t)$ as long as $y(t) > -b$.

If $r \leqslant 0$, then $(1.1)_r$ *does not possess any heteroclinic solution joining X^- to X^+.*

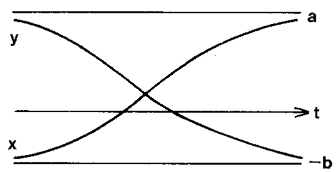

Fig. 2

Figure 2 depicts the graphs of the solutions $x(t)$ and $y(t)$ of Theorem 2.1. To prove Theorem 2.1, we will use the abstract Theorem 1.1 with Proposition 1.1. This involves verifying Hypotheses H1, H2′, H3, and H4 for a parametrized system

$$\begin{aligned} \dot{x}(t) &= rx(t) - rf(y(t-1), \alpha \\ \dot{y}(t) &= ry(t) - rf(x(t-1), \alpha \end{aligned} \qquad (2.2)_{r,\alpha}$$

with equilibria

$$X^-(\alpha) = (-b(\alpha), a(\alpha)) \qquad \text{and} \qquad X^+(\alpha) = (a(\alpha), -b(\alpha)).$$

When $\alpha = 1$, the above system yields $(2.1)_r$. The precise form of the homotopy, and in particular the trivial system at $\alpha = 0$, will be given later, but we do note here that the function $f(\cdot, \alpha)$ will be constructed so as to satisfy B1–B5 for each $\alpha \in I$.

For simplicity, we shall often suppress the parameter α and work directly with $(2.1)_r$. As standing hypotheses on $(2.1)_r$, we assume henceforth, without comment, that B1–B5 hold. With these, we shall verify H2′, H3, and H4. An analysis of the trivial system at $\alpha = 0$, where the nonlinearity will take a special form, will prove H1.

We begin our analysis with a discussion of a general linear system of the form

$$\begin{aligned} \dot{\xi}(t) &= r\xi(t) - rp(t)\,\eta(t-1) \\ \dot{\eta}(t) &= r\eta(t) - rq(t)\,\xi(t-1) \end{aligned} \qquad (2.3)$$

Such a system can arise from the linearization of $(2.1)_r$. In particular, the linearization of $(2.1)_r$ about one of the equilibria $X^{\pm}$ yields an autonomous system

$$\begin{aligned} \dot{\xi}(t) &= r\xi(t) - rp^*\eta(t-1) \\ \dot{\eta}(t) &= r\eta(t) - rq^*\xi(t-1) \end{aligned} \qquad (2.4)$$

where

$$p^* \leqslant 0, \qquad q^* \leqslant 0, \qquad \text{and} \qquad 0 \leqslant p^* q^* < 1. \tag{2.5}$$

If μ is an eigenvalue of this system, with eigenfunction

$$(\xi(t), \eta(t)) = (c_1 e^{\mu t}, c_2 e^{\mu t}) \tag{2.6}$$

then

$$\begin{pmatrix} \mu - r & rp^* e^{-\mu} \\ rq^* e^{-\mu} & \mu - r \end{pmatrix} \begin{pmatrix} c^1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and μ satisfies the characteristic equation

$$\begin{aligned} &\mu = r \pm rke^{-\mu}, && \text{where} \\ &k = (p^* q^*)^{1/2}, && 0 \leqslant k < 1. \end{aligned} \tag*{(2.7)$_\pm$}$$

A simple homotopy argument, with k as the homotopy parameter, shows that, if $r > 0$, then each of the two equations $(2.7)_\pm$ has exactly one root $\mu^\pm > 0$ in the right-half plane, and none on the imaginary axis. In particular, with $r > 0$, the equilibria $X^\pm$ of $(2.1)_r$ are hyperbolic with unstable manifolds of dimension

$$N^\pm = \dim W^u(X^\pm) = 2.$$

With a little extra analysis, we easily obtain the following result.

Proposition 2.1. If $r > 0$ and p^* and q^* satisfy (2.5), then the linear system (2.4) is hyperbolic, with exactly two unstable eigenvalues

$$\mu^+ \geqslant \mu^- > 0.$$

If $p^* q^* \neq 0$, then $\mu^+ > \mu^-$, and the coefficients $c_j^\pm$ in the eigenfunctions (2.6) are all nonzero and satisfy

$$c_1^+ c_2^+ > 0 \qquad \text{and} \qquad c_1^- c_2^- < 0.$$

If $p^* = q^* = 0$, then $\mu^+ = \mu^- = r$ is a double eigenvalue, with two independent eigenfunctions, so any choice of $(c_1, c_2) \neq (0, 0)$ can be made.

If $p^* = 0$, but $q^* \neq 0$, then $\mu^+ = \mu^- = r$ is a double eigenvalue, but with only one independent eigenfunction. The general solution on the generalized eigenspace has the form

$$(\xi(t), \eta(t)) = (c_1 e^{rt}, (c_2 - c_1 r e^{-r} q^* t)\, e^{rt})$$

with arbitrary $(c_1, c_2) \neq (0, 0)$. A similar result holds if $p^* \neq 0$ and $q^* = 0$.

In any case, if $\Xi(t) = (\xi(t), \eta(t))$ is a nontrivial solution of (2.4) which is bounded as $t \to -\infty$, then the limit

$$\lim_{t \to -\infty} \frac{\Xi_t}{\|\Xi_t\|} = (\xi^*, \eta^*) \in C([-1, 0], \mathbb{R}^2)$$

exists and satisfies either

$$\xi^*(\theta) \neq 0 \quad \text{for all} \quad \theta \in [-1.0], \qquad \text{or else}$$

$$\eta^*(\theta) \neq 0 \quad \text{for all} \quad \theta \in [-1, 0].$$

If $r \leqslant 0$, then a similar analysis of the linearization of Eq. $(1.1)_r$ shows that, after reversing time $t \to -t$ in this advanced equation, both equilibria $X^\pm$ are asymptotically stable. In particular, this proves the final statement in Theorem 2.1.

Proposition 2.2. If $r \leqslant 0$, then $(1.1)_r$ does not possess any heteroclinic solution joining X^- to X^+.

In light of the above result, henceforth we shall assume without comment that $r > 0$.

Now consider the nonautonomous system (2.3), which we regard as a perturbation of (2.4) as $t \to -\infty$.

Proposition 2.3. Assume $p(t)$ and $q(t)$ are continuous for sufficiently negative t, that p^* and q^* satisfy (2.5), and that

$$\lim_{t \to -\infty} p(t) = p^*, \qquad \lim_{t \to -\infty} q(t) = q^*. \tag{2.8}$$

Then, for all sufficiently negative t, say $t \leqslant -T$, there exists a two-dimensional subspace $P(t) \subseteq C$ such that any solution $\Xi(t) = (\xi(t), \eta(t))$ of (2.3) which is bounded as $t \to -\infty$ satisfies

$$\Xi_t \in P(t) \tag{2.9}$$

for all $t \leqslant -T$. Conversely, if an initial condition satisfies (2.9) for some $t = t_0 \leqslant -T$, then there exists a backward extension that is bounded as $t \to -\infty$. The subspace $P(t)$ varies continuously in t and approaches, as $t \to -\infty$, the canonical unstable subspace $P(-\infty)$ of the autonomous system (2.4).

If, in addition to (2.8), we assume

$$\int_{-\infty} |t| \; |p(t) - p^*| \, dt < \infty \qquad \text{and} \qquad \int_{-\infty} |t| \; |q(t) - q^*| \, dt < \infty \tag{2.10}$$

then, for each nontrivial solution $\Xi(t) = (\xi(t), \eta(t))$ of (2.3) that is bounded as $t \to -\infty$, there exists a nontrivial solution $\tilde{\Xi}(t) = (\tilde{\xi}(t), \tilde{\eta}(t))$ of (2.4) that is bounded as $t \to -\infty$, and satisfies

$$\Xi_t = \tilde{\Xi}_t + o(\|\tilde{\Xi}_t\|) \qquad \text{as} \quad t \to -\infty. \tag{2.11}$$

In particular, there exists T such that either

$$\begin{aligned} &\xi(t) \neq 0 \qquad \text{for all} \quad t \leqslant -T, \qquad \text{or else} \\ &\eta(t) \neq 0 \qquad \text{for all} \quad t \leqslant -T. \end{aligned}$$

The proof of Proposition 2.3 is based in part on the theory of exponential dichotomies; we shall be somewhat sketchy in our exposition, as many of the ideas can be found elsewhere. In particular, see Palmer (1984) (ordinary differential equations) and Lin (1986) (delay equations) where the relation between exponential dichotomies and homoclinic orbits is explored. Another tool that we need for our proof is the following finite dimensional result. It is a special case of a general result given in Theorem 10.13.2 of Hartman (1964).

Proposition 2.4. Consider the N-dimensional linear ordinary differential equation

$$\dot{u} = (A + B(t))\, u \tag{2.12}$$

where the matrix A is hyperbolic and the matrix $B(t)$ is continuous, satisfying

$$\lim_{t \to -\infty} B(t) = 0 \qquad \text{and}$$

$$\int_{-\infty} |t|^{h-1}\, |B(t)|\, dt < \infty.$$

Here $h \geqslant 1$ is the size of the largest Jordan block, in the canonical form of A, corresponding to eigenvalues with positive real part. Then, to every nontrivial solution $u(t)$ of (2.12) that is bounded as $t \to -\infty$, there exists a nontrivial solution $\tilde{u}(t)$ of $\dot{u} = Au$ bounded as $t \to -\infty$, satisfying

$$u(t) = \tilde{u}(t) + o(|\tilde{u}(t)|) \qquad \text{as} \quad t \to -\infty. \tag{2.13}$$

Proof of Proposition 2.3. Consider first the more general problem of a linear delay equation

$$\dot{\Xi}(t) = [L + R(t)]\, \Xi_t \qquad \text{for} \quad t \leqslant 0 \tag{2.14}$$

in the n-dimensional variable Ξ, where, for each t,

$$L,\ R(t)\colon C([-1,0],\mathbb{R}^n)\to\mathbb{R}^n$$

are continuous linear maps. Assume that $R(t)$ varies continuously (in the operator topology) in t, and

$$\lim_{t\to-\infty}\|R(t)\|=0.$$

Assume also that the limiting equation

$$\dot{\Xi}(t)=L\Xi_t$$

is hyperbolic and [following Hale (1977) here and in the rest of the proof] denote the canonical decomposition of the phase space $C=C([-1,0],\mathbb{R}^n)$ by

$$C=P\oplus Q,\qquad \dim P=N<\infty$$

$$\Xi_t=Y_t+Z_t.$$

Here, N is the unstable dimension. Relative to a fixed basis $\Phi=(\varphi^1,\varphi^2,\dots,\varphi^N)$ of P, we may write

$$Y_t=\Phi u(t),\qquad u(t)\in\mathbb{R}^N.$$

We wish to obtain, for each sufficiently negative t, say $t\leqslant -T$, a linear map

$$M(t)\colon P\to Q$$

varying continuously in t, and approaching zero (in the operator norm) as $t\to-\infty$. The map is characterized by the property that all solutions $\Xi(t)$ of (2.14) that are bounded as $t\to-\infty$ satisfy

$$Z_t=M(t)\,Y_t \tag{2.15}$$

for all $t\leqslant -T$; and, conversely, if (2.15) holds for some $t_0\leqslant -T$, then there exists a solution through $\Xi_{t_0}=Y_{t_0}+Z_{t_0}$ that is bounded as $t\to-\infty$.

The map $M(t)$ is obtained by a simple contraction mapping argument. This argument will also furnish an upper bound for $\|M(t)\|$, which we shall need. In a standard fashion, we write the variation of constants formulas

$$u(t)=e^{A(t-t_0)}u(t_0)+\int_{t_0}^{t}e^{A(t-s)}R(s)[\Phi u(s)+Z_s]\,ds \tag{2.16}$$

$$Z_t=\int_{-\infty}^{t}[T(t-s)\,X_0^Q]\,R(s)[\Phi u(s)+Z_s]\,ds \tag{2.17}$$

which hold for any solution of (2.14) that is bounded as $t \to -\infty$. Here, A is the N by N matrix representing the infinitesimal generator of the limiting equation relative to the basis Φ of P: all eigenvalues of A have positive real part. Fixing $t_0 \leqslant -T$ and $u(t_0) = u_0$, we consider the class of continuous functions

$$Z\colon (-\infty, t_0] \to Q$$

which are bounded as $t \to -\infty$. For such a function Z_t [which is not necessarily of the form $Z_t(\theta) = Z(t+\theta)$], we let $u(t) = u(t, t_0, x_0, Z)$ denote the unique solution of the first equation (2.16). The estimate

$$|u(t, t_0, u_0, Z)| \leqslant K(|u_0| + \rho(t_0)\,\|Z\|)$$

for $t \leqslant t_0$ is easily obtained, where

$$\rho(t_0) = \sup_{t \leqslant t_0} \|R(t)\|,$$

$$\|Z\| = \sup_{t \leqslant t_0} \|Z_t\|$$

and K is a positive constant depending only on the equation (2.14).

Upon inserting $u(t) = u(t, t_0, u_0, Z)$ into the right-hand side of the second equation (2.17), the integral defines a new bounded function $(\mathscr{T}Z)_t$ satisfying

$$\|\mathscr{T}Z\| \leqslant K\rho(t_0)(|u_0| + \|Z\|) \tag{2.18}$$

for a possibly different K. Furthermore, if $-T$ is sufficiently negative, so that

$$K\rho(t_0) \leqslant K\rho(-T) < 1$$

then $\mathscr{T}$ is a contraction mapping and yields a unique fixed point $Z_t = Z_t(t_0, u_0)$ depending linearly on u_0. The map $M(t_0)$ is defined then by

$$M(t_0)\,\Phi u_0 = Z_{t_0}(t_0, u_0)$$

and one has, from (2.18),

$$\|M(t_0)\,\Phi u_0\| \leqslant K\rho(t_0)(1 - K\rho(t_0))^{-1}\,|u_0|.$$

Hence, for a larger K,

$$\|M(t)\| \leqslant K\rho(t) \qquad \text{for} \quad t \leqslant -T.$$

Continuity of $M(t)$ in t can also be verified:

With $M(t)$ so obtained, we have a one-to-one correspondence between solutions of (2.14) bounded at $-\infty$, and solutions of the N-dimensional ordinary differential equation

$$\dot{u} = (A + B(t))\, u.$$

Here, the matrix

$$B(t) = \Omega(0)\, R(t)(\Phi + M(t)\, \Phi)$$

satisfies

$$|B(t)| \leqslant K\rho(t) \qquad \text{as} \quad t \to -\infty.$$

where Ω is the appropriate adjoint basis dual to Φ. Indeed, solutions of (2.14) bounded at $-\infty$ are precisely those satisfying (2.9) for some, or equivalently all, $t \leqslant -T$, where $P(t) \subseteq C = P \oplus Q$ is the N-dimensional graph of $M(t)$. Now, in the system of interest in the statement of Proposition 2.3, we have $N = 2$, as well as

$$\int_{-\infty} |t| \;\; |B(t)|\, dt < \infty$$

from the estimates (2.10). As $h \leqslant N = 2$, Proposition 2.4 applies to give (2.13) for some $u(t)$. Upon noting that $\tilde{\Xi}_t = \Phi\tilde{u}(t)$ satisfies (2.4) and that $\Xi_t = \Phi u(t) + M(t)\, \Phi u(t)$ is our given solution, we easily obtain (2.11).

The final statement of Proposition 2.3 follows easily from Proposition 2.1. ▌

In the case of interest here, both $p(t)$ and $q(t)$ are nonnegative, and that leads to further results for (2.3).

Proposition 2.5. Assume the hypotheses (2.8) and (2.10) of Proposition 2.3, and in addition that

$$p(t) \leqslant 0 \qquad \text{and} \qquad q(t) \leqslant 0 \tag{2.19}$$

for all sufficiently negative t. Let $\Xi(t) = (\xi(t), \eta(t))$ be a nontrivial solution of (2.3) that is bounded as $t \to -\infty$. Then there exists T such that either

$$\begin{array}{llllll} \xi(t) \neq 0 & \text{and} & \eta(t) \neq 0 & \text{for all} & t \leqslant -T, & \text{or} \\ \xi(t) = 0 & \text{and} & \eta(t) \neq 0 & \text{for all} & t \leqslant -T, & \text{or} \\ \xi(t) \neq 0 & \text{and} & \eta(t) = 0 & \text{for all} & t \leqslant -T. & \end{array} \tag{2.20}$$

Assume now the inequalities (2.19) hold for all $t \in \mathbb{R}$ (with $p(t)$ and $q(t)$ continuous there). If $\Xi(t)$ is any solution of (2.3) which satisfies, for some t_0, either

$$\xi(t) \geqslant 0 \qquad \text{and} \qquad \eta(t) \geqslant 0 \qquad \text{for all} \quad t \in [t_0 - 1, t_0] \tag{2.21}$$

or else

$$\xi(t) \leqslant 0 \qquad \text{and} \qquad \eta(t) \leqslant 0 \qquad \text{for all} \quad t \in [t_0 - 1, t_0] \tag{2.22}$$

and in addition $\Xi(t_0) \neq 0$, then $\Xi(t)$ is unbounded as $t \to +\infty$. Furthermore, the inequalities (2.21) or (2.22) hold for all $t \geqslant t_0 - 1$.

Finally, if $\Xi(t)$ is a nontrivial solution of (2.3) bounded for $t \in \mathbb{R}$, then in fact

$$\xi(t)\, \eta(t) < 0 \qquad \text{for all} \quad t \leqslant -T. \tag{2.23}$$

Proof. Without loss, $\xi(t) > 0$ for all $t \leqslant -T$, by Proposition 2.3. Now, for all sufficiently negative t, one has

$$\frac{d}{dt}(\eta(t)\, e^{-rt}) = -rq(t)\, \xi(t-1)\, e^{-rt} \geqslant 0$$

that is, $\eta(t)\, e^{-rt}$ is nondecreasing. The result (2.20) follows immediately, with possibly a larger value of T.

To prove the second part of the proposition, assume that $p(t) \leqslant 0$ and $q(t) \leqslant 0$ for all $t \in \mathbb{R}$, and that $\xi(t) \geqslant 0$ and $\eta(t) \geqslant 0$ for all $t \in [t_0 - 1, t_0]$, with the strict inequality $\xi(t_0) > 0$. Then, arguing as above, one shows inductively that $\xi(t)\, e^{-rt}$ and $\eta(t)\, e^{-rt}$ are nondecreasing and hence nonnegative on $[t_0 + n - 1, t_0 + n]$ for each $n \geqslant 1$. Therefore,

$$\dot{\xi}(t) \geqslant r\xi(t) > 0 \qquad \text{for} \quad t \geqslant t_0$$

and hence $\xi(t) \to +\infty$ as $t \to +\infty$.

The inequality (2.23) for solutions bounded on $\mathbb{R}$ follows from the results above. ▌

An immediate application of the above result is to heteroclinic solutions of $(2.1)_r$. Letting $\xi(t) = \dot{x}(t)$ and $\eta(t) = \dot{y}(t)$ where $(x(t), y(t))$ is a heteroclinic solution of $(2.1)_r$ connecting X^- to X^+, one notes the exponential approach to the equilibrium X^-, and concludes from Proposition 2.5 applied to the variational equation of $(2.1)_r$ that

$$\dot{x}(t)\, \dot{y}(t) < 0 \qquad \text{for all} \quad t \leqslant -T$$

for some T. In fact, a stronger result holds: that both $x(t)$ and $y(t)$ are monotone for all $t \in \mathbb{R}$, with strict monotonicity holding for much (possibly all) of this interval. The proof of this result, stated below, is long and technical, so is deferred to Section 5.

Proposition 2.6. Let $X(t) = (x(t), y(t))$ be a heteroclinic solution connecting $X^- = (-b, a)$ to $X^+ = (a, -b)$. Then $\dot{x}(t) \geqslant 0$ and $\dot{y}(t) \leqslant 0$ for all $t \in \mathbb{R}$. Furthermore, $\dot{x}(t) > 0$ as long as $x(t) < a$, and $\dot{y}(t) < 0$ as long as $y(t) > -b$.

We shall freely use Proposition 2.6, as its proof does not rely on the theory developed below.

Let us now fix a heteroclinic solution $X(t) = (x(t), y(t))$ of our system $(2.1)_r$. We then have the following result.

Lemma 2.1. *There exists a solution $\Xi(t) = (\xi(t), \eta(t))$ of the variational equation*

$$\begin{aligned} \dot{\xi}(t) &= r\xi(t) - rf'(y(t-1))\,\eta(t-1) \\ \dot{\eta}(t) &= r\eta(t) - rf'(x(t-1))\,\xi(t-1) \end{aligned} \tag{2.24}$$

which is bounded as $t \to -\infty$ and unbounded as $t \to +\infty$. In addition, this solution may be chosen so that

$$\xi(t) \geqslant 0 \quad \text{and} \quad \eta(t) \geqslant 0 \quad \text{for all } t \in \mathbb{R}. \tag{2.25}$$

Proof. With Proposition 2.1, and $p^* = f'(a)$ and $q^* = f'(-b)$, a careful examination of the solutions of the constant coefficient system (2.4) that are bounded as $t \to -\infty$ reveals that there exists one $\Xi^*(t) = (\xi^*(t), \eta^*(t))$ satisfying

$$\xi^*(\theta) > 0 \quad \text{and} \quad \eta^*(\theta) > 0 \quad \text{for all } \theta \in [-1, 0].$$

Letting $P(t)$ be the subspace as in Proposition 2.3, we have $\Xi_0^* \in P(-\infty)$ (where Ξ_0^* of course is the restriction of $\Xi^*(\theta)$ to $[-1, 0]$). Thus, for any sufficiently negative t_0, there exists a solution $\Xi(t) = (\xi(t), \eta(t))$ of the variational equation (2.24), which is bounded as $t \to -\infty$, with $\Xi_{t_0} \in P(t_0)$ near enough to Ξ_0^* that $\xi(t_0 + \theta) > 0$ and $\eta(t_0 + \theta) > 0$ for all $\theta \in [-1, 0]$. From Proposition 2.5, it follows that

$$\xi(t) \geqslant 0 \quad \text{and} \quad \eta(t) \geqslant 0 \quad \text{for all } t \geqslant t_0 - 1.$$

Now consider a sequence $\Xi^n(t)$ of such solutions, with quantities $t_n \to -\infty$ in place of t_0. Of course, $\Xi_t^n \in P(t)$ for all $t \leqslant -T$, and without loss $\|\Xi_0^n\| = 1$. By fixing two independent solutions, say Ξ_t^1 and Ξ_t^2, forming a basis in each $P(t)$, we have

$$\Xi_t^n = k_1^n \Xi_t^1 + k_2^n \Xi_t^2 \qquad \text{for} \quad t \geqslant t_n$$

for bounded sequences k_1^n and k_2^n. Upon taking convergent subsequences, we obtain $\Xi^n(t) \to \Xi(t)$ uniformly on compact sets. By construction, this limiting solution $\Xi(t) = (\xi(t), \eta(t))$ satisfies (2.25). Also, $\Xi_t \in P(t)$ for $t \leqslant -T$, and hence it is bounded as $t \to -\infty$. As $\|\Xi_0\| = 1$, it is nontrivial, and hence unbounded as $t \to +\infty$ by Proposition 2.5. ∎

Let $\Xi^u(t) = (\xi^u(t), \eta^u(t))$ be any solution as in Lemma 2.1; we keep this fixed from now on. Also, let $\Xi^b(t) = (\xi^b(t), \eta^b(t))$ be the solution

$$\xi^b(t) = \dot{x}(t), \qquad \eta^b(t) = \dot{y}(t) \tag{2.26}$$

of (2.24). As these are the only two independent solutions of (2.24) bounded as $t \to -\infty$ (by Proposition 2.3), it follows that (2.24) has only one independent solution that is bounded on $\mathbb{R}$, namely, $\Xi^b(t)$. With this observation, and the more elementary results on eigenvalues, in Proposition 2.1, all of Hypothesis H2′ is verified except for the integral condition (1.7).

To prove (1.7), we turn our attention to the adjoint equation

$$\begin{aligned} \dot{\psi}(t) &= -r\psi(t) + rf'(x(t))\,\zeta(t+1) \\ \dot{\zeta}(t) &= -r\zeta(t) + rf'(y(t))\,\psi(t+1). \end{aligned} \tag{2.27}$$

As noted earlier, this equation has exactly one independent solution that is bounded on $\mathbb{R}$, because the variational equation (2.24) also does. Let us fix a choice of such a (nontrivial) solution, which we denote by $\Psi^b(t) = (\psi^b(t), \zeta^b(t))$.

Upon reversing time $t \to -t$, the system (2.27) assumes the form (2.3) and satisfies the conditions of Proposition 2.5. Indeed, this result implies that $\psi^b(t)\,\zeta^b(t) < 0$ for large t; without loss, we assume

$$\psi^b(t) > 0 \qquad \text{and} \qquad \zeta^b(t) < 0 \qquad \text{for all} \quad t \geqslant T$$

for some T. We claim in fact $\psi^b(t) \geqslant 0$ and $\zeta^b(t) \leqslant 0$ for all $t \in \mathbb{R}$, with strict inequality for enough values of t that the integral condition in H2′ holds. To show this, we begin by defining quantities $t_1, t_2 \in (-\infty, +\infty]$ and $t_3, t_4 \in [-\infty, T)$ by

$$\begin{aligned} t_1 &= \sup\{t \mid x(t) < a\} \\ t_2 &= \sup\{t \mid y(t) > -b\} \\ t_3 &= \inf\{t \mid \psi^b(s) > 0 \text{ for all } s > t\} \\ t_4 &= \inf\{t \mid \zeta^b(s) < 0 \text{ for all } s > t\}. \end{aligned}$$

Clearly $\xi^b(t) > 0$ if and only if $t < t_1$ and $\eta^b(t) < 0$ if and only if $t < t_2$.

The full force of the following result will not be needed. We do present it, however, as it gives an interesting necessary and sufficient condition for a heteroclinic solution to reach the equilibrium X^+ in finite time.

Proposition 2.7. The quantities t_1 and t_2 are either both finite or both infinite. They are finite if and only if f is constant on either $[-b, -b+\varepsilon]$ or on $[a-\varepsilon, a]$ for some ε, and necessarily either $f'(-b)=0$ or $f'(a)=0$ in this case. Further, $|t_1-t_2|\leqslant 1$ if t_1 and t_2 are finite. Finally, if $f'(a)\neq 0$ and $f'(-b)=0$, and if t_1 and t_2 are finite, then $t_1=t_2-1$, with the analogous result holding if $f'(a)=0$ and $f'(-b)\neq 0$.

Proof. Suppose that $t_1<\infty$. Then, from $(2.1)_r$, one has $\dot{y}(t)=ry(t)+rb$ for $t\geqslant t_1+1$, and hence $y(t)=-b$ for all such t since $y(t)$ is bounded; thus, $t_2\leqslant t_1+1$. Similarly, $t_1\leqslant t_2+1$, showing $|t_1-t_2|\leqslant 1$.

Now assume that $t_1, t_2<\infty$; without loss, $t_1\leqslant t_2$. Then, for $t_1\leqslant t\leqslant t_1+1$, one has from $(2.1)_r$ that $f(y(t-1))=a$, but that $y(t-1)>-b$. This proves that f is constant on $[-b, -b+\varepsilon]$ for some ε.

If, on the other hand, one assumes that f is constant on $[-b, -b+\varepsilon]$, then $\dot{x}(t)=rx(t)-ra$ for all t large enough that $y(t-1)\leqslant -b+\varepsilon$. For such t necessarily $x(t)=a$; hence, $t_1, t_2<\infty$.

Finally, suppose that $f'(a)\neq 0, f'(-b)=0$, and $t_1, t_2<\infty$. Then, $t\geqslant t_2$ implies [from $(2.1)_r$] that $f(x(t-1))=-b$, and hence [as $f'(a)\neq 0$] that $x(t-1)=a$. Thus, $t_1\leqslant t_2-1$. As $|t_1-t_2|\leqslant 1$, we conclude that $t_1=t_2-1$. ∎

Lemma 2.2. $t_3<t_1$ *and* $t_4<t_2$.

Proof. We prove only that $t_3<t_1$, so suppose that $t_1\leqslant t_3<\infty$. Two cases arise: first suppose that $f'(a)=0$. Then, $t\geqslant t_1$ implies, from the adjoint system (2.27), that $\dot{\psi}^b(t)=-r\psi^b(t)$. As $\psi^b(t)\neq 0$ for large t, we have $\psi^b(t)\neq 0$ for all $t\geqslant t_1$, thereby proving $t_3<t_1$.

Now suppose that $f'(a)\neq 0$. By Proposition 2.7, $f'(-b)=0$. If $t\geqslant t_1-1$, then $f(y(t))=a$ [from $(2.1)_r$]; hence, $f'(y(t))=0$. [Note: we are not claiming $y(t)=-b$ here.] Therefore, $\dot{\zeta}^b(t)=-r\zeta^b(t)$ and so $\zeta^b(t)<0$ for $t\geqslant t_1-1$, as $\zeta^b(t)<0$ for all large t. The first equation in (2.27) now implies

$$\dot{\psi}^b(t)\geqslant -r\psi^b(t) \qquad \text{for} \qquad t\geqslant t_1-2.$$

As $\psi^b(t_3)=0$, it follows that $\psi^b(t)\leqslant 0$ for $t_1-2\leqslant t\leqslant t_3$, and hence

$$\psi^b(t)\leqslant 0 \qquad \text{and} \qquad \zeta^b(t)<0 \qquad \text{for} \quad t_1-1\leqslant t\leqslant t_1 \tag{2.28}$$

as $t_1 \leqslant t_3$. From the inequalities (2.28) on the unit interval $[t_1 - 1, t_1]$, one may reverse time in (2.28) and use Proposition 2.5 to conclude that $\psi^b(t)$ is unbounded as $t \to -\infty$. This contradiction completes the proof. ∎

***Lemma* 2.3.** $\psi^b(t) \geqslant 0$ *and* $\zeta^b(t) \leqslant 0$ *for all* $t \in \mathbb{R}$.

Proof. Set

$$t_5 = \inf\{t \mid \psi^b(s) \geqslant 0 \text{ and } \zeta^b(s) \leqslant 0 \text{ for all } s \geqslant t\}$$

and assume $t_5 > -\infty$. We seek a contradiction. Clearly, either

$$\psi^b(t) < 0 \qquad \text{for some} \quad t < t_5 \quad \text{arbitrarily near} \quad t_5 \tag{2.29}$$

or else

$$\zeta^b(t) > 0 \qquad \text{for some} \quad t < t_5 \quad \text{arbitrarily near} \quad t_5 \tag{2.30}$$

or else both (2.29) and (2.30) hold. Without loss, assume that (2.29) holds. Assume also that (2.30) does not hold, that is, assume

$$\zeta^b(t) \leqslant 0 \qquad \text{for all} \quad t \geqslant t_5 - \varepsilon \tag{2.31}$$

for some ε. The case in which both (2.29) and (2.30) hold will be dealt with later.

Observe that $t_5 \leqslant t_3 < t_1$ and so $\xi^b(t) > 0$ in $[t_5 - \varepsilon, t_5]$. Also recall the inequalities $\eta^b(t) \leqslant 0$, from Proposition 2.6, and (2.25) and (2.26) for $\xi^u(t)$ and $\eta^u(t)$. From these, it easily follows that, when $t_5 - \varepsilon \leqslant t \leqslant t_5$, there exist nonnegative quantities $k_1 = k_1(t)$ and $k_2 = k_2(t)$ such that

$$\begin{aligned} &k_1 \xi^b(t) + k_2 \xi^u(t) = 1 \qquad \text{and} \\ &k_1 \eta^b(t) + k_2 \eta^u(t) = 0. \end{aligned} \tag{2.32}$$

Furthermore, k_1 and k_2 can be chosen so that $k_1, k_2 \in L^\infty(t_5 - \varepsilon, t_5)$ [we do not claim that k_1 and k_2 depend continuously on t; of course, they do if the determinent of (2.32) is nonzero].

Let

$$\begin{aligned} &\xi(t, s) = k_1(t)\, \xi^b(s) + k_2(t)\, \xi^u(s) \qquad \text{and} \\ &\eta(t, s) = k_1(t)\, \eta^b(s) + k_2(t)\, \eta^u(s). \end{aligned}$$

For $t_5 - \varepsilon \leqslant t \leqslant t_5$ and all $s \in \mathbb{R}$, we have $\xi(t, s) \geqslant 0$, from the nonnegativity of $\xi^b(s)$, $\xi^u(s)$, k_1, and k_2. We claim also that

$$\eta(t, s) \leqslant 0 \qquad \text{for} \quad s \leqslant t \qquad \text{and} \qquad t_5 - \varepsilon \leqslant t \leqslant t_5. \tag{2.33}$$

To prove (2.33), fix t and note that $\partial\eta(t, s)/\partial s \leqslant r\eta(t, s)$ from the differential equation (2.24), and the nonnegativity of $\xi(t, s)$; then (2.33) holds because $\eta(t, t) = 0$.

Now consider the two expressions

$$\Gamma^v(t) = \psi^b(t)\,\xi^v(t) + \zeta^b(t)\,\eta^v(t)$$
$$- r\int_{t-1}^{t} \psi^b(s+1) f'(y(s))\,\eta^v(s) + \zeta^b(s+1) f'(x(s))\,\xi^v(s)\,ds$$

where $v = b$ or u. The quantity $\Gamma^v(t)$ represents the duality product between the solutions

$$\Psi^b(t) = (\psi^b(t), \zeta^b(t)) \qquad \text{and} \qquad \Xi^v(t) = (\xi^v(t), \eta^v(t))$$

of (2.24) and (2.27). In particular, $\Gamma^v(t)$ is constant, and letting $t \to +\infty$ shows that $\Gamma^v(t) = \Gamma^v + \infty) = 0$ for all $t \in \mathbb{R}$. Therefore, upon taking the linear combination

$$k^1(t)\,\Gamma^b(t) + k_2(t)\,\Gamma^u(t) \tag{2.34}$$

we obtain

$$\psi^b(t) = r\int_{t-1}^{t} \psi^b(s+1)\,f'(y(s))\,\eta(t, s) + \zeta^b(s+1)\,f'(x(s))\,\xi(t, s)\,ds$$
$$\geqslant r\int_{t-1}^{t} \psi^b(s+1)\,f'(y(s))\,\eta(t, s)\,ds \tag{2.35}$$

where (2.31) has been used. The inequality (2.35) can be rewritten as

$$\psi^b(t) = \int_t^{t+1} \psi^b(s)\,A(t, s)\,ds + b(t) \tag{2.36}$$

where the kernel $A(t, s)$ and the forcing function $b(t)$ are both nonnegative in the appropriate range. Regarding (2.36) as an initial value problem for $\psi^b(t)$, to be solved backward in time, we see also that the initial datum, $\psi^b(t_5 + \theta)$, $0 \leqslant \theta \leqslant 1$, is nonnegative. An elementary result based on the existence theorem for (2.36) implies that the solution $\psi^b(t)$ is nonnegative for $t_5 - \varepsilon \leqslant t \leqslant t_5$. However, this contradicts (2.29).

To complete the proof of Lemma 2.3, we consider the case in which both (2.29) and (2.30) hold. In this case, both $t_5 < t_1$ and $t_5 < t_2$ hold, and so we have both $\xi^b(t) > 0$ and $\eta^b(t) < 0$ in some interval $[t_5 - \varepsilon, t]$. Therefore, in addition to the functions k_1 and k_2 obtained above, we may also obtain functions $h_1, h_2 \in L^\infty(t_5 - \varepsilon, t_5)$ satisfying

$$h_1(t)\,\xi^b(t) + h_2(t)\,\xi^u(t) = 0$$
$$h_1(t)\,\eta^b(t) + h_2(t)\,\eta^u(t) = 1$$

with $h_1(t) \leqslant 0$ and $h_2(t) \geqslant 0$. Again taking the linear combination (2.34), we obtain (2.35); however, instead of deleting the term involving $\zeta^b(s+1)$, we retain it and write

$$\psi^b(t) = \int_t^{t+1} \psi^b(s)\, A(t,s) + \zeta^b(s)\, B(t,s)\, ds \tag{2.37}$$

where $A(t,s) \geqslant 0$ and $B(t,s) \leqslant 0$. Similarly, replacing k_1 and k_2 with h_1 and h_2 gives

$$\zeta^b(t) = \int_t^{t+1} \psi^b(s)\, C(t,s) + \zeta^b(s)\, D(t,s)\, ds \tag{2.38}$$

where $C(t,s) \leqslant 0$ and $D(t,s) \geqslant 0$. Treating (2.37) and (2.38) as a problem with initial data $\psi^b(t_5+\theta) \geqslant 0$ and $\zeta^b(t_5+\theta) \leqslant 0$ for $0 \leqslant \theta \leqslant 1$ yields, as before, $\psi^b(t) \geqslant 0$ and $\zeta^b(t) \leqslant 0$ on $[t_5-\varepsilon, t_5]$. Again, this contradicts our assumptions (2.29) and (2.30). ∎

The following result completes this section.

Lemma 2.4. *Hypothesis* H2′ *holds for the system* $(2.1)_r$.

Proof. This result follows directly from Proposition 2.6, and Lemmas 2.2 and 2.3. ∎

3. VERIFICATION OF H3 AND H4

We first verify H4. We begin with the following technical lemma, which was presented in a slightly more general form as Lemma 4.2 in Mallet-Paret and Nussbaum (1986a).

Lemma 3.1. *There do not simultaneously exist nontrivial solutions* $(x^1(t), y^1(t))$ *and* $(x^2(t), y^2(t))$, $t \in \mathbb{R}$, *of* $(2.1)_r$ *such that*

$$x^1(t) \leqslant 0, \qquad y^1(t) \geqslant 0$$
$$\dot{x}^1(t) \geqslant 0 \qquad \text{and}$$
$$(x^1(+\infty), y^1(+\infty)) = (0,0)$$

and such that

$$x^2(t) \geqslant 0, \qquad y^2(t) \leqslant 0$$
$$\dot{x}^2(t) \geqslant 0 \qquad \text{and}$$
$$(x^2(-\infty), y^2(-\infty)) = (0,0).$$

We remark that the result in Mallet-Paret and Nussbaum (1986a) concerned solutions defined on a half-interval $J = (-\infty, -T]$ or $[T, +\infty)$. However, any solution of $(2.1)_r$ on $\mathbb{R}$, which vanishes identically on J, in fact vanishes identically on $\mathbb{R}$. Thus, the result reported by Mallet-Paret and Nussbaum implies directly the lemma above.

Lemma 3.2. *Hypothesis* H4 *holds for the parametrized system* $(2.2)_{r,\alpha}$.

Proof. Let $\alpha^j \to \alpha^*$ and $r^j \to r^* > 0$ be as in H4, and $X^j(t) = (x^j(t), y^j(t))$ be a heteroclinic solution of $(2.2)_{r^j,\alpha^j}$ connecting $X^-(\alpha^j) = (-b(\alpha^j), a(\alpha^j))$ to $X^+(\alpha^j) = (a(\alpha^j), -b(\alpha^j))$. These solutions are uniformly bounded (by monoticity, as in Proposition 2.6) and hence any sequence $X^j(t+\theta^j)$ of time translates has a subsequence converging uniformly on compact sets. [What is not clear is that the limit solution connects $X^-(\alpha^*)$ to $X^+(\alpha^*)$, nor that the convergence is uniform on $\mathbb{R}$.]

Fix $\varepsilon > 0$. Let $t^{1,j} < t^{2,j}$ satisfy

$$x^j(t^{1,j}) = -b(\alpha^j) + \varepsilon$$
$$x^j(t^{2,j}) = a(\alpha^j) - \varepsilon$$

and let $t^{3,j} < t^{4,j}$ satisfy

$$y^j(t^{3,j}) = a(\alpha^j) - \varepsilon$$
$$y^j(t^{4,j}) = -b(\alpha^j) + \varepsilon.$$

To prove our result, it is sufficient, in view of the monotonicity of the solutions, to show that

$$\max_k \{t^{k,j}\} - \min_k \{t^{k,j}\} \qquad \text{is bounded as } j \to +\infty. \tag{3.1}$$

We first show that $t^{2,j} - t^{1,j}$ is bounded. Assume that it is not, and consider the limits

$$\begin{aligned} X^j(t + t^{1,j}) &\to X^{1,*}(t) = (x^{1,*}(t), y^{1,*}(t)) \\ X^j(t + t^{2,j}) &\to X^{2,*}(t) = (x^{2,*}(t), y^{2,*}(t)) \end{aligned} \tag{3.2}$$

obtained from a subsequence with $t^{2,j} - t^{1,j} \to +\infty$. The monotonicity of the limit functions $x^{k,*}(t)$ and $y^{k,*}(t)$ implies that $X^{1,*}(t)$ and $X^{2,*}(t)$ are both heteroclinic solutions connecting some equilibria. The properties of $t^{1,j}$ and $t^{2,j}$ imply that

$$x^{1,*}(0) = -b(\alpha^*) + \varepsilon \qquad \text{and}$$
$$-b(\alpha^*) + \varepsilon \leqslant x^{1,*}(+\infty) \leqslant a(\alpha^*) - \varepsilon.$$

Necessarily then, $X^{1,*}(t)$ connects $X^-(\alpha^*)$ to $(0, 0)$; in fact, for all $t \in \mathbb{R}$,

$$\begin{aligned} x^{1,*}(t) \leqslant 0, \qquad & y^{1,*}(t) \geqslant 0 \\ \dot{x}^{1,*}(t) \geqslant 0 \qquad & \text{and} \\ (x^{1,*}(+\infty), y^{1,*}(+\infty)) = (0, 0). & \end{aligned} \tag{3.3}$$

In a similar fashion, one shows that $X^{2,*}(t)$ connects $(0, 0)$ to $X^+(\alpha^*)$, and that, for all $t \in \mathbb{R}$,

$$\begin{aligned} x^{2,*}(t) \geqslant 0,\ & y^{2,*}(t) \leqslant 0 \\ \dot{x}^{2,*}(t) \geqslant 0 \qquad & \text{and} \\ (x^{2,*}(-\infty), y^{2,*}(-\infty)) = (0, 0). & \end{aligned} \tag{3.4}$$

However, the properties (3.3) and (3.4), and the fact that both solutions $X^{1,*}(t)$ and $X^{2,*}(t)$ are nontrivial, immediately contradict Lemma 3.1. This completes the proof that $t^{2,j} - t^{1,j}$ is bounded.

A similar proof shows that $t^{4,j} - t^{3,j}$ is bounded.

To complete the proof of the lemma, assume that (3.1) fails: then, $|t^{3,j} - t^{1,j}|$ is unbounded; taking the case $t^{3,j} - t^{1,j} > 0$, and passing to a subsequence, we have in fact $t^{3,j} - t^{1,j} \to +\infty$. Now consider the limit $X^{1,*}(t)$ given by (3.2) (again with a subsequence). One has

$$\begin{aligned} & x^{1,*}(-\infty) \leqslant -b(\alpha^*) + \varepsilon \\ & x^{1,*}(+\infty) \geqslant a(\alpha^*) - \varepsilon \\ & \dot{x}^{1,*}(t) \geqslant 0 \qquad \text{for all} \quad t \in \mathbb{R} \end{aligned} \tag{3.5}$$

since $t^{2,j} - t^{1,j}$ is bounded, and

$$\begin{aligned} & y^{1,*}(t) \geqslant -b(\alpha^*) + \varepsilon \qquad \text{and} \\ & \dot{y}^{1,*}(t) \leqslant 0 \qquad \text{for all} \quad t \in \mathbb{R} \end{aligned} \tag{3.6}$$

because $t^{3,j} - t^{1,j} \to +\infty$. However, (3.5) and (3.6) are together impossible as $X^{1,*}$ $(\pm\infty)$ are both equilibria. This contradiction completes the proof. ▮

We now obtain the bounds $0 < r_m \leqslant r \leqslant r_M$ that establish H3. For simplicity, we suppress α and work with the system $(2.1)_r$, making the easy observation that the estimates obtained are uniform in $\alpha \in I$.

Lemma 3.3. *Hypothesis* H3 *holds for the system* $(2.1)_r$.

Proof. Assume that r_m does not exist; then there is a sequence $r^j \to 0$ and a sequence $X^j(t) = (x^j(t), y^j(t))$ of heteroclinic solutions of $(2.1)_{r^j}$ connecting X^- to X^+. Assume, without loss, that $x^j(0) = c = a/2$. Set $\tilde{X}^j(t) = X^j(t/r^j)$; then, $\tilde{X}^j(t) = (\tilde{x}^j(t), \tilde{y}^j(t))$ is a heteroclinic solution of $(1.1)_{r^j}$ connecting X^- to X^+. There exists a subsequence $\tilde{X}^j(t) \to \tilde{X}^*(t) = (\tilde{x}^*(t), \tilde{y}^*(t))$ converging uniformly on compact subsets of $\mathbb{R}$, where $\tilde{X}^*(t)$ satisfies the system of ordinary differential equations

$$\dot{x} = x - f(y)$$
$$\dot{y} = y - f(x).$$

Examining the system at $t = 0$ and using the monotonicity of $\tilde{x}^*(t)$ and $\tilde{y}^*(t)$, we have

$$0 \leqslant \dot{\tilde{x}}^*(0) = \tilde{x}^*(0) - f(\tilde{y}^*(0))$$
$$0 \geqslant \dot{\tilde{y}}^*(0) = \tilde{y}^*(0) - f(\tilde{x}^*(0)).$$

But then, using the monotonicity of f, we have, with the above inequalities,

$$f(f(c)) = f(f(\tilde{x}^*(0))) \leqslant f(\tilde{y}^*(0)) \leqslant \tilde{x}^*(0) = c$$

contradicting B5.

We now give a direct construction of the upper bound r_M. Fix positive constants ε, γ_1, and γ_2 such that

$$|f(x) - a| \leqslant \gamma_1 \, |x + b| \qquad \text{if} \quad |x + b| \leqslant \varepsilon$$
$$|f(x) + b| \leqslant \gamma_2 \, |x - a| \qquad \text{if} \quad |x - a| \leqslant \varepsilon$$

and such that [recall $0 \leqslant f'(a) f'(b) < 1$]

$$\gamma_1 \gamma_2 < 1.$$

Let $X(t) = (x(t), y(t))$ be a heteroclinic solution of $(2.1)_r$ connecting X^- to X^+ and set $u(t) = x(t) + b$ and $v(t) = y(t) - a$. Note the bounds and monotonicity conditions

$$0 \leqslant u(t) \leqslant a + b, \qquad \dot{u}(t) \geqslant 0$$
$$-(a + b) \leqslant v(t) \leqslant 0, \qquad \dot{v}(t) \leqslant 0$$

which hold for all $t \in \mathbb{R}$. Since $(u(t), v(t)) \to (0, 0)$ as $t \to -\infty$, we may assume by means of a time translation that

$$|u(t)|, \ |v(t)| \leqslant \varepsilon \qquad \text{for all} \quad t \leqslant 0, \qquad \text{and}$$
$$\max\{|u(0)|, |v(0)|\} = \varepsilon.$$

Now restrict $t \in [0, 1]$ and use the above properties to obtain the following estimates. First,

$$\begin{aligned}\dot{u}(t) &= ru(t) - r[f(a + v(t-1)) + b] \\ &\geqslant ru(t) + r\gamma_2 v(t-1) \\ &\geqslant ru(t) + r\gamma_2 v(t)\end{aligned}$$

and similarly

$$\dot{v}(t) \leqslant rv(t) + r\gamma_1 u(t).$$

Therefore,

$$\begin{aligned}(1+\gamma_1)\,\dot{u}(t) - (1+\gamma_2)\,\dot{v}(t) &\geqslant r(1-\gamma_1\gamma_2)(u(t) - v(t)) \\ &\geqslant r(1-\gamma_1\gamma_2)(u(0) - v(0)) \\ &\geqslant \varepsilon r(1-\gamma_1\gamma_2)\end{aligned}$$

so integrating from $t = 0$ to 1 gives

$$(1+\gamma_1)\,u(1) - (1+\gamma_2)\,v(1) \geqslant \varepsilon r(1-\gamma_1\gamma_2).$$

From the bounds on $u(t)$ and $v(t)$, we obtain

$$(2+\gamma_1+\gamma_2)(a+b) \geqslant \varepsilon r(1-\gamma_1\gamma_2)$$

which gives the desired bound r_M for r. ∎

4. CONSTRUCTION OF THE HOMOTOPY AND VERIFICATION OF H1

We first construct a nonlinearity $f(x, 0)$ for which H1 holds. Then, we construct a homotopy $f(x, \alpha)$ between $f(x, 0)$ and our given function $f(x, 1)$.

The trivial function, at $\alpha = 0$, will be odd:

$$f(-x, 0) = -f(x, 0) \qquad \text{for all} \quad x \in \mathbb{R}.$$

We begin by showing that all heteroclinic solutions for such a nonlinearity enjoy a corresponding symmetry property.

Proposition 4.1. Let f satisfy B1–B5. Assume also that f is odd:

$$f(-x) = -f(x) \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then any heteroclinic solution $X(t) = (x(t), y(t))$ of $(2.1)_r$ connecting $X^- = (-a, a)$ to $X^+ = (a, -a)$ necessarily satisfies

$$y(t) = -x(t) \qquad \text{for all} \quad x \in \mathbb{R}.$$

In particular, $x(t)$ is a heteroclinic solution of the equation

$$\dot{x}(t) = rx(t) + rf(x(t-1))$$

connecting the equilibrium $x = -a$ to $x = a$.

Proof. Let $u(t) = x(t) + y(t)$ where $(x(t), y(t))$ is a heteroclinic solution of $(2.1)_r$ connecting X^- to X^+. We first show that there does not exist $t_0 \in \mathbb{R}$ such that

$$u(t) > 0 \qquad \text{for all} \quad t \in [t_0, t_0 + 1]. \tag{4.1}$$

Suppose in fact that (4.1) holds for some t_0. Then, for $t \in [t_0 + 1, t_0 + 2]$, we have

$$\begin{aligned} \dot{u}(t) &= ru(t) - r[f(x(t-1)) + f(y(t-1))] \\ &\geqslant ru(t) - r[f(x(t-1)) + f(-x(t-1))] \\ &= ru(t) \end{aligned} \tag{4.2}$$

and so both $u(t)$ and $\dot{u}(t)$ are positive in $[t_0 + 1, t_0 + 2]$. By induction, we see that $u(t) \to +\infty$ as $t \to +\infty$, which is a contradiction. A similar proof shows that there does not exist $t_0 \in \mathbb{R}$ such that $u(t) < 0$ in $[t_0, t_0 + 1]$.

Now fix positive quantities ε and γ such that

$$|f'(x)| \leqslant \gamma < 1 \qquad \text{if} \quad |x + a| \leqslant \varepsilon. \tag{4.3}$$

Also assume (by a time translation) that

$$|x(t) + a| \leqslant \varepsilon \qquad \text{and} \qquad |y(t) - a| \leqslant \varepsilon \qquad \text{for all} \quad t \leqslant -1. \tag{4.4}$$

It is enough to show that $u(t) = 0$ for all $t \leqslant 0$, for then integrating the first equation in (4.2) shows that $u(t) = 0$ for all $t \in \mathbb{R}$. Assume then that $u(t)$ is not identically zero on $(-\infty, 0]$. From the oscillatory behavior of $u(t)$ obtained in the beginning of our proof, there exists $t_1 < 0$ such that $\dot{u}(t_1) = 0$. Without loss, $u(t_1) > 0$. From (4.2),

$$\begin{aligned} u(t_1) &= f(x(t_1 - 1)) + f(y(t_1 - 1)) \\ &= f(x(t_1 - 1)) - f(-y(t_1 - 1)) \\ &= f'(\sigma)\, u(t_1 - 1) \end{aligned}$$

for some σ between $x(t_1-1)$ and $-y(t_1-1)$. From (4.3) and (4.4), it follows that

$$u(t_1-1)=\frac{u(t_1)}{f'(\sigma)}\leqslant -\frac{u(t_1)}{\gamma}<0.$$

Let $\tau<t_1-1<\tau'<t_1$ be such that $u(t)<0$ on (τ, τ'), with $u(\tau)=u(\tau')=0$. Let $t_2\in(\tau, \tau')$ be the point in that interval where $u(t)$ attains its minimum:

$$u(t_2)=\min_{[\tau, \tau']} u(t)\leqslant u(t_1-1).$$

We have $\dot{u}(t_2)=0$ and $|u(t_2)|\geqslant|u(t_1)|/\gamma$. By induction, we can find a sequence $t_1>t_2>\cdots$ such that

$$|u(t_n)|\geqslant\frac{|u(t_1)|}{\gamma^{n-1}}\to +\infty$$

as $n\to +\infty$. This contradicts the boundedness of $u(t)$. ▮

We now construct the trivial function $f(x, 0)$, for which H1 will hold. Let $g\colon \mathbb{R}\to\mathbb{R}$ satisfy B1–B5 and assume also that g is odd:

$$g(-x)=-g(x) \qquad \text{for all} \quad x\in\mathbb{R}.$$

Assume in addition that

$$g(x)=-a \qquad \text{for all} \quad x\in(a, +\infty).$$

Keeping g fixed for the remainder of this section, we let

$$g^\beta(x)=g(x/\beta)$$

for some $\beta\in(0, 1)$ to be chosen later, and study the equation

$$\dot{x}(t)=rx(t)+rg^\beta(x(t-1)). \tag{4.5$_r$}$$

We shall show that, if β is small enough, then H1 holds for a homotopy constructed with $f(x, 0)=g^\beta(x)$.

Lemma 4.1. *If $x(t)$ is a heteroclinic solution of* (4.5)$_r$ *connecting* $-a$ *to* a, *then*

$$r>(1-2\beta)/3. \tag{4.6}$$

Proof. Let $x(t_0) = a/2$. Then, from the monotonicity condition $\dot{x}(t) \geqslant 0$, and from the differential equation, we have $\dot{x}(t) \leqslant r(a/2 + a) = 3ra/2$ for all $t \leqslant t_0$, and hence $x(t_0 - 1) \geqslant (1 - 3r)a/2$. Therefore,

$$g^{\beta}((1-3r)a/2) \geqslant g^{\beta}(x(t_0 - 1)) = \frac{1}{r}\dot{x}(t_0) - x(t_0) \geqslant -\frac{a}{2}$$

which implies that $(1 - 3r)a/2 < \beta a$, which is equivalent to (4.6). ∎

Lemma 4.2. *There is a constant $0 < \beta_0 < 1$, independent of r, such that, if $0 < \beta \leqslant \beta_0$ and if $x(t)$ is a heteroclinic solution connecting $-a$ to a, then $t_2 - t_1 \leqslant 1$ where $x(t_1) = -\beta a$ and $x(t_2) = \beta a$.*

Proof. If $t \leqslant t_1 + 1$, then the differential equation $(4.5)_r$ becomes $\dot{x}(t) = rx(t) + ra$, so, without loss,

$$x(t) = -a + e^{rt} \tag{4.7}$$

for this range of t. Now $x(t_1) = -a + e^{rt_1} = -\beta a$, and hence

$$e^{rt_1} = (1 - \beta)\, a. \tag{4.8}$$

Also,

$$\begin{aligned} x(t_1 + 1) &= -a + e^{r(t_1 + 1)} \\ &= -a + (1 - \beta)\, ae^{r} \\ &\geqslant -a + (1 - \beta)\, a \exp[(1 - 2\beta)/3] \end{aligned} \tag{4.9}$$

using Lemma 4.1. There exists β_0 such that, if $0 < \beta \leqslant \beta_0$, then the right-hand side of (4.9) is greater than or equal to βa. This is equivalent to $t_2 - t_1 \leqslant 1$. ∎

Lemma 4.3. *If $0 < \beta \leqslant \beta_0$, there exists a unique r such that $(4.5)_r$ has a heteroclinic solution connecting $-a$ to a. Furthermore, this heteroclinic solution is unique. Thus,* H1 *holds for Eq.* $(4.5)_r$.

Proof. With t_1 and t_2 as in Lemma 4.2, and (4.7) holding for $t \leqslant t_1 + 1$, we have $x(t_2) = -a + e^{rt_2} = \beta a$; hence

$$e^{rt_2} = (1 + \beta)\, a. \tag{4.10}$$

Also, $\dot{x}(t) = rx(t) - ra$ for $t \geqslant t_2 + 1$, and hence

$$x(t_2 + 1) = a. \tag{4.11}$$

Integrating the equation $(e^{-rt}x(t))^{\cdot} = re^{-rt}g^{\beta}(x(t-1))$ from t_1+1 to t_2+1, and using (4.8)–(4.11), we have

$$\begin{aligned} &e^{-r(t_2+1)}a - e^{-r(t_1+1)}[-1+(1-\beta)\,e^r]\,a \\ &= \frac{2e^{-r}}{1-\beta^2} - 1 = re^{-r}\int_{t_1}^{t_2} e^{-rs}g^{\beta}(x(s))\,ds \\ &= \beta e^{-r}\int_{-a}^{a} \frac{g(x)}{(a+\beta x)^2}\,dx \end{aligned}$$

and so

$$e^r = \frac{2}{1-\beta^2} - \beta\int_{-a}^{a} \frac{g(x)}{(a+\beta x)^2}\,dx.$$

This uniquely determines r as a function of β. It is also easy to see that, for this value of r, Eq. $(4.5)_r$ has a unique heteroclinic solution. ∎

To complete this section, we construct the homotopy $f(x, \alpha)$ between the trivial function $f(x, 0) = g^{\beta}(x)$ and our given function $f(x, 1)$. We construct $f(x, \alpha)$ in two parts, beginning with the range $\frac{1}{2} \leqslant \alpha \leqslant 1$.

With the period two points $-b$ and a as in B2 for the given function $f(x, 1)$, let $k(x, \alpha)$ be a C^1 function $k\colon \mathbb{R} \times [\frac{1}{2}, 1] \to \mathbb{R}$ satisfying the following properties for all x and α:

$$\begin{aligned} k(0, \alpha) = 0 \qquad &\text{and} \qquad k(x, 1) = x \\ &\frac{\partial k(x, \alpha)}{\partial x} > 0 \\ &\lim_{x \to \pm\infty} k(x, \alpha) = \pm\infty \\ k(-b, \tfrac{1}{2}) = -a \qquad &\text{and} \qquad k(a, \tfrac{1}{2}) = a \qquad \text{and} \\ &\frac{\partial k(x, \frac{1}{2})}{\partial \alpha} = 0. \end{aligned} \tag{4.12}$$

Thus, as a function of x, $k(x, \alpha)$ is a diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, for each fixed α. Let $h(x, \alpha)$ denote the inverse diffeomorphism, that is, $k(h(x, \alpha), \alpha) = x$ for all x and α, and set

$$f(x, \alpha) = k(f(h(x, \alpha), 1), \alpha) \qquad \text{for} \quad x \in \mathbb{R} \qquad \text{and} \qquad \tfrac{1}{2} \leqslant \alpha \leqslant 1.$$

It is elementary to verify that B1–B5 hold for $f(x, \alpha)$, for each α, but with $b(\alpha) = -k(-b, \alpha)$ replacing b in B2. Note $b(\frac{1}{2}) = a$.

To construct the homotopy in the range $0 \leqslant \alpha \leqslant \frac{1}{2}$, we assume β is such that $|f(x, \frac{1}{2})| \leqslant |g^{\beta}(x)|$ for $|x| \leqslant a$. This is certainly the case if β is small enough. Note also that $|f(x, \frac{1}{2})| \geqslant |g^{\beta}(x)|$ for $|x| \geqslant a$. Let $\rho(\alpha)$ be a C^1 function ρ: $[0, \frac{1}{2}] \to [0, 1]$ satisfying

$$\begin{aligned} &\rho(0) = 0 \quad \text{and} \quad \rho(\tfrac{1}{2}) = 1 \\ &\rho'(\alpha) \geqslant 0 \quad \text{and} \\ &\rho'(\tfrac{1}{2}) = 0 \end{aligned} \tag{4.13}$$

and set

$$f(x, \alpha) = \rho(\alpha) f(x, \tfrac{1}{2}) + [1 - \rho(\alpha)] f(x, 0)$$

where $f(x, 0) = g^{\beta}(x)$. Again, it is easily verified that B1–B5 hold throughout $0 \leqslant \alpha \leqslant \frac{1}{2}$. In particular, to verify B5, one uses the monotonicity of $f(x, \alpha)$ in α for fixed x in the ranges $(-\infty, -a]$, $[-a, 0]$, $[0, a]$, and $[a, +\infty)$. Finally, note that $f(x, \alpha)$ is smooth in α at $\alpha = \frac{1}{2}$ by (4.12) and (4.13).

With the exception of the proof of Proposition 2.6 (given in the next section), this completes the proof of our main result, Theorem 2.1.

5. PROOF OF PROPOSITION 2.6

This section is devoted to the proof of Proposition 2.6. It is easier to work with the system

$$\begin{aligned} \dot{x}(t) &= x(t) - f(y(t-r)) \\ \dot{y}(t) &= y(t) - f(x(t-r)). \end{aligned} \tag{5.1$_r$}$$

Throughout this section, we assume that $X(t) = (x(t), y(t))$ is a heteroclinic solution of $(5.1)_r$ connecting X^- to X^+.

Lemma 5.1. $\overline{\text{range }(x)} = \overline{\text{range }(y)} = [-b, a]$.

Proof. Let $I = \overline{\text{range }(x)}$ and $J = \overline{\text{range }(y)}$. Clearly I and J are nonempty compact connected sets satisfying $[-b, a] \subseteq I, J$. Denote $I = [-b^*, a^*]$ where $a^* \geqslant a$ and $b^* \geqslant b$, and choose a sequence $t^j \in \mathbb{R}$ such that

$$x(t^j) \to a^* = \sup x(t).$$

As $x(t)$ is equicontinuous on $\mathbb{R}$ [since $\dot{x}(t)$ is uniformly bounded, by the differential equation], we have $\dot{x}(t^j) \to 0$; hence, $f(y(t^j - r)) \to a^*$. As

$y(t^j-r)\in J$, we conclude $a^*\in f(J)$. Similarly $-b^*\in f(J)$ and so $I\subseteq f(J)$ since $f(J)$ is connected. Likewise, we have $J\subseteq f(I)$, and so

$$[-b, a]\subseteq I\subseteq f(f(I)). \tag{5.2}$$

The inclusions (5.2) and the properties of f easily yield $I=[-b, a]$ as claimed. In a similar fashion, $J=[-b, a]$. ▮

***Lemma* 5.2.** *There exists $T\in\mathbb{R}$ such that*

$$\dot{x}(t)>0 \quad \text{and} \quad \dot{y}(t)<0 \quad \text{if} \quad t\leqslant -T.$$

Proof. This follows immediately from Proposition 2.5 and Lemma 5.1. ▮

Fix a constant $c>0$, and set

$$u(t)=x(t+c)-x(t) \qquad \text{and}$$
$$v(t)=y(t+c)-y(t).$$

Observe that $(u(t), v(t))$ satisfies the system

$$\dot{u}(t)=u(t)+g(t, v(t-r))$$
$$\dot{v}(t)=v(t)+h(t, u(t-r))$$

where

$$g(t, v)= -f(v+y(t-r))+f(y(t-r))$$
$$h(t, u)= -f(u+x(t-r))+f(x(t-r)).$$

Note also that the functions g and h satisfy

$$vg(t, v)\geqslant 0, \qquad g(t, 0)=0, \qquad \text{and}$$
$$uh(t, u)\geqslant 0, \qquad h(t, 0)=0$$

for any (t, v), $(t, u)\in\mathbb{R}^2$, and that

$$\begin{aligned} &u(t)>0 \quad \text{and} \quad v(t)<0 \quad \text{if} \quad t\leqslant -T-c \\ &\lim_{t\to\pm\infty}(u(t), v(t))=(0, 0). \end{aligned} \tag{5.3}$$

In what follows, think of $c>0$ as fixed, but arbitrary. A precise value of c will be chosen later.

Lemma 5.3. (a) *Suppose for some $t^* \in \mathbb{R}$ and $\varepsilon > 0$ that*

$$u(t^*) = 0 \qquad \textit{and} \qquad u(t) > 0 \qquad \textit{if} \quad t \in (t^*, t^* + \varepsilon].$$

Then there exists a sequence $\mu^j > 0$, $\mu^j \to 0$ such that

$$v(t^* + \mu^j - r) > 0.$$

(b) *Suppose instead that*

$$u(t^*) = 0 \qquad \textit{and} \qquad u(t) > 0 \qquad \textit{if} \quad t \in [t^* - \varepsilon, t^*).$$

Then there exist $\mu^j > 0$, $\mu^j \to 0$ such that

$$v(t^* - \mu^j - r) < 0 \text{ (note the sign reversal here).}$$

(c) *Each of the above two results holds if the inequalities for u and v (but not $\mu^j > 0$) are reversed.*

(d) *Also, the roles of u and v may be exchanged in the above results.*

Figure 3 illustrates two of the situations described in Lemma 5.3. In fact, the pictures are a bit deceiving, as $v(t)$ need not be positive/negative on an interval to the right/left of $t^* - r$, but only for some sequence $\mu^j \neq t^* - r$.

Proof. (a) Suppose that the conclusion is false; then,

$$v(t - r) \leqslant 0 \qquad \text{for all} \quad t \in [t^*, t^* + \mu]$$

for some $\mu > 0$. For this range of t, we have

$$\dot{u}(t) = u(t) + g(t, v(t - r)) \leqslant u(t)$$

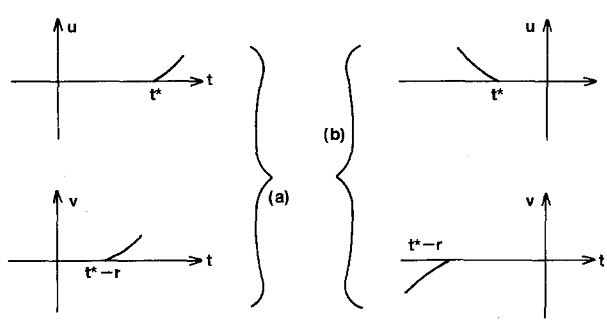

Fig. 3

and since $u(t^*)=0$, we conclude from this differential inequality that $u(t)\leqslant 0$ on $[t^*, t^*+\mu]$. This is a contradiction.

The proofs of (b), (c), (d) are similar. ∎

Remark. We need not assume $u(t)>0$ on $(t^*, t^*+\varepsilon]$ in Lemma 5.3. It is enough to assume that $u(t^*+\varepsilon^j)>0$ for some $\varepsilon^j>0$, $\varepsilon^j\to 0$, and $u(t^*)=0$, to make the same conclusion in (a); and similarly for (b), (c), (d).

Lemma 5.4. *Define the zero sets*

$$Z_u=\{t\in\mathbb{R}\mid u(t)=0\}$$
$$Z_v=\{t\in\mathbb{R}\mid v(t)=0\}$$

of the functions u and v. Then, any bounded interval contains only finitely many connected components of Z_u. The same conclusion holds for Z_v.

Remark. The above result holds for any fixed $c>0$, as the functions u and v, and hence the sets Z_u and Z_v, depend implicitly on this parameter. We do not claim any uniformity in the number of components as c varies.

Proof. Define sets A_u and B_u by

$$A_u=\{t^*\in\mathbb{R}\mid \text{there exist } t^j\to t^* \text{ monotonically, with } u(t^{2k})>0 \text{ and } u(t^{2k+1})<0\}$$
$$B_u=\{t^*\in\mathbb{R}\mid \text{there exist } t^j\to t^* \text{ monotonically, with } u(t^{2k})=0 \text{ and } u(t^{2k+1})\neq 0\}$$

and sets A_v and B_v in an analogous fashion. We allow the sequence t^j to be either monotonically increasing or decreasing. Observe that

$$A_u\subseteq B_u\subseteq Z_u \qquad \text{and}$$
$$A_v\subseteq B_v\subseteq Z_v.$$

One easily sees that it is sufficient to prove

$$B_u=B_v=\varnothing$$

in order to prove the proposition. And in order to prove that B_u and B_v are empty, it is enough to prove the implications

$$t^*\in B_u \text{ implies } t^*-r\in A_v \qquad \text{and}$$
$$t^*\in B_v \text{ implies } t^*-r\in A_u.$$

The reason for this is that, if $t^* \in B_u$, then the above two implications would yield $t^* - 2nr \in A_u$ for each $n \geqslant 0$, contradicting (5.3).

We shall only prove the first implication. Assume, therefore, that $t^* \in B_u$. We shall show that $t^* - r \in A_v$. To be definite, assume t^j (as in the definition of B_u) decreases to t^* and consider the points t^{2k+1}. Each t^{2k+1} lies in a maximal interval (σ^k, τ^k) on which $u(t) \neq 0$, and which approaches t^* as $k \to \infty$. That is,

$$\begin{aligned} u(t) &\neq 0 \qquad \text{if} \quad t \in (\sigma^k, \tau^k) \\ u(\sigma^k) &= u(\tau^k) = 0 \\ t^* &\leqslant \sigma^k < \tau^k \qquad \text{and} \\ \tau^k &\to t^*. \end{aligned}$$

Lemma 5.3 applied at the points σ^k and τ^k (*not* at t^*) shows that there exist $\tilde{\sigma}^k$ and $\tilde{\tau}^k$ satisfying

$$\begin{aligned} \sigma^k < \tilde{\sigma}^k < \tilde{\tau}^k < \tau^k \qquad \text{and} \\ v(\tilde{\sigma}^k - r)\, v(\tilde{\tau}^k - r) < 0. \end{aligned}$$

Thus, $v(t)$ changes sign infinitely often to the right of, and arbitrarily close to, $t^* - r$; hence, $t^* - r \in A_v$ as claimed. This completes the proof of the lemma. ∎

If I and J are nonempty closed intervals (or points), we write $I < J$ to mean $s < t$ for each $s \in I$ and $t \in J$.

Lemma 5.4, together with the fact that $u(t) > 0$ for all $t \leqslant -T - c$, imply that the set Z_u has the form

$$Z_u = \bigcup_{j=1}^{p} I^j, \qquad 0 \leqslant p \leqslant \infty, \qquad \text{or}$$

$$Z_u = \left(\bigcup_{j=1}^{p} I^j \right) \cup I^*, \qquad 0 \leqslant p < \infty$$

where

$$\begin{aligned} I^j &= [\alpha^j, \beta^j] && \text{is a nonempty compact interval or point} \\ I^* &= [\alpha^*, \infty] && \text{for some} \quad \alpha^* \\ I^j &< I^{j+1} && \text{and} \quad I^j < I^* \quad \text{whenever these intervals exist,} \quad \text{and} \\ \alpha^j, \beta^j &\to \infty && \text{if} \quad p = \infty. \end{aligned}$$

Similarly Z_v has the form

$$Z_v = \bigcup_{j=1}^{q} J^j, \qquad 0 \leqslant q \leqslant \infty, \qquad \text{or}$$

$$Z_v = \left(\bigcup_{j=1}^{q} J^j\right) \cup J^*, \qquad 0 \leqslant q < \infty$$

where

$$J^j = [\gamma^j, \delta^j]$$
$$J^* = [\gamma^*, \infty)$$
$$J^j < J^{j+1} \qquad \text{and} \qquad J^j < J^* \qquad \text{and}$$
$$\gamma^j, \delta^j \to \infty \qquad \text{if} \quad q = \infty.$$

Let

$$B = \{\beta^j\}_{j=1}^{p} \qquad \text{and} \qquad D = \{\delta^j\}_{j=1}^{q}$$

denote the sets of right-hand endpoints of I^j and J^j, respectively; thus, B and D are discrete subsets of $\mathbb{R}$ that are bounded below. Motivated by the integer-valued Lyapunov function defined in Mallet-Paret (1988), we define functions

$$N_u, N_v \colon \mathbb{R} \to \{0, 1, 2, \ldots\}$$
$$N_u(t) = \operatorname{card}((-\infty, t] \cap B) \qquad \text{and}$$
$$N_v(t) = \operatorname{card}((-\infty, t] \cap D)$$

and also define

$$V_u, V_v \colon \mathbb{R} \to \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots\} \qquad \text{by}$$

$$V_u(t) = \begin{cases} N_u(b) - N_v(b-r), & \text{where} \quad b = \inf\{\beta \in B \mid t \leqslant \beta\} \\ & \text{if this exists, or} \\ \operatorname{card} B - \operatorname{card} D & \text{if} \quad \beta < t \qquad \text{for all} \quad \beta \in B \end{cases}$$

and

$$V_v(t) = \begin{cases} N_v(d) - N_u(d-r), & \text{where} \quad d = \inf\{\delta \in D \mid t \leqslant \delta\} \\ & \text{if this exists, or} \\ \operatorname{card} D - \operatorname{card} B & \text{if} \quad \delta < t \qquad \text{for all} \quad \delta \in D. \end{cases}$$

It is obvious that N_u and N_v are nondecreasing functions of t. What is not so obvious is that V_u and V_v are nonincreasing functions of t.

***Lemma* 5.5.** *If $t_1 < t_2$, then*

$$V_u(t_1) \geqslant V_u(t_2) \qquad \textit{and} \qquad V_v(t_1) \geqslant V_v(t_2).$$

Proof. We consider only V_u. By examining the definition of V_u, one sees that it is enough to prove that

$$V_u(\beta^j) \geqslant V_u(\beta^{j+1}) \qquad \text{for each} \quad j < p.$$

One has from the definitions

$$\begin{aligned} V_u(\beta^j) - V_u(\beta^{j+1}) &= N_u(\beta^j) - N_u(\beta^{j+1}) - N_v(\beta^j - r) + N_v(\beta^{j+1} - r) \\ &= -1 - N_v(\beta^j - r) + N_v(\beta^{j+1} - r) \\ &= -1 + \operatorname{card}((\beta^j - r, \beta^{j+1} - r] \cap D). \end{aligned}$$

Hence it is enough to show that

$$(\beta^j - r, \beta^{j+1} - r] \cap D \neq \varnothing.$$

This follows easily from Lemma 5.3 by examining the signs of $u(t)$ to the right of β^j and the left of $\alpha^{j+1} \leqslant \beta^{j+1}$. One concludes that $v(t)$ changes sign at least once in the interval $(\beta^j - r, \alpha^{j+1} - r) \subseteq (\beta^j - r, \beta^{j+1} - r]$; hence, this interval contains a point of D. ▮

***Lemma* 5.6.** *For all $t \in \mathbb{R}$, one has $V_u(t) \leqslant 1$ and $V_v(t) \leqslant 1$.*

Proof. If $B = \phi$, then $V_u(t) \leqslant 0$ for all t, from the definition. If $B \neq \phi$, then $V_u(t) \leqslant V_u(\beta^1)$ for $t \geqslant \beta^1$, and $V_u(t) = V_u(\beta^1)$ for $t \leqslant \beta^1$. Thus,

$$V_u(t) \leqslant V_u(\beta^1) \leqslant N_u(\beta^1) = 1$$

holds for all t. The proof for V_v is similar. ▮

Because Z_u consists of discrete sets of intervals, it is possible to divide the compact intervals I^j into two classes: those on which $u(t)$ changes sign, that is,

$$u(\alpha^j - \varepsilon)\, u(\beta^j + \varepsilon) < 0 \qquad \text{for small} \quad \varepsilon > 0$$

and those on which it does not,

$$u(\alpha^j - \varepsilon)\, u(\beta^j + \varepsilon) > 0 \qquad \text{for small} \quad \varepsilon > 0.$$

Fig. 4

Similarly, the intervals J^j in Z_v fall into these classes. See Fig. 4, where such intervals are depicted.

The next lemma says that V_u (or V_v), if positive, must strictly decrease if t passes an interval that is not a sign change.

***Lemma* 5.7.** *Let $I^j = [\alpha^j, \beta^j]$ be such that $u(t)$ has the same sign on either side, near I^j; that is,*

$$u(\alpha^j - \varepsilon)\, u(\beta^j + \varepsilon) > 0 \qquad \textit{for small} \quad \varepsilon > 0.$$

Then,

$$V_u(\beta^j) \leqslant 0.$$

A similar result holds for the function V_v for intervals $J^j = [\gamma^j, \delta^j]$ on which $v(t)$ does not change sign.

Proof. First suppose that $j \geqslant 2$. In view of Lemma 5.6, it is enough to prove the strict inequality

$$V_u(\beta^j) < V_u(\beta^{j-1}).$$

As in Lemma 5.5, we have

$$V_u(\beta^{j-1}) - V_u(\beta^j) = -1 + \operatorname{card}((\beta^{j-1} - r, \beta^j - r] \cap D).$$

We must therefore show that $(\beta^{j-1} - r, \beta^j - r]$ contains at least two points of D. There is one point (at least) of D in $(\beta^{j-1} - r, \alpha^j - r) \subseteq (\beta^{j-1} - r, \beta^j - r]$, as in Lemma 5.5; we claim that $[\alpha^j - r, \beta^j - r]$ contains another, necessarily different, point of D. This claim follows from Lemma 5.3: since $u(t)$ has the same sign on either side of $[\alpha^j, \beta^j]$, $v(t)$ has opposite signs on either side of $[\alpha^j - r, \beta^j - r]$. Thus, $J^k \subseteq [\alpha^j - r, \beta^j - r]$ for some k; hence, $[\alpha^j - r, \beta^j - r] \cap D \neq \varnothing$ as desired.

Now suppose that $j = 1$. We have

$$\begin{aligned} V_u(\beta^1) &= N_u(\beta^1) - N_v(\beta^1 - r) \\ &= 1 - N_v(\beta^1 - r) \\ &= 1 - \operatorname{card}((-\infty, \beta^1 - r] \cap D) \end{aligned}$$

and so must prove that $(-\infty, \beta^1 - r] \cap D \neq \varnothing$. This holds as in the case above, since $[\alpha^1 - r, \beta^1 - r] \cap D \neq \varnothing$. Thus, $V_u(\beta^1) \leqslant 0$, completing the proof. ▮

Lemma 5.8. *If* $I^j = [\alpha^j, \beta^j]$ *is as in Lemma* 5.7, *then* $\operatorname{dist}(\beta^j, D) \geqslant r$. *The analogous result for* $J^j = [\gamma^j, \delta^j]$ *and* B *holds.*

Proof. Suppose that $|\beta^j - \delta^k| < r$ for some $\delta^k \in D$. Consider

$$\begin{aligned} V_u(\beta^j) + V_v(\delta^k) &= N_u(\beta^j) - N_u(\delta^k - r) + N_v(\delta^k) - N_v(\beta^j - r) \\ &= \operatorname{card}((\delta^k - r, \beta^j] \cap B) + \operatorname{card}((\beta^j - r, \delta^k] \cap D) \\ &\geqslant 1 + 1 = 2 \end{aligned}$$

where $\delta^k - r < \beta^j \in B$ and $\beta^j - r < \delta^k \in D$ are used. However, by Lemmas 5.6 and 5.7,

$$V_u(\beta^j) + V_v(\delta^k) \leqslant 0 + 1 = 1$$

is a contradiction. ▮

Lemma 5.9. *There does not exist an interval* $I^j = [\alpha^j, \beta^j]$ *as in Lemma* 5.7, *that is, one on which* $u(t)$ *does not change sign. There likewise does not exist* J^j *on which* $v(t)$ *does not change sign.*

Proof. Suppose that I^j exists as stated. For definiteness, suppose that

$$u(t) > 0 \qquad \text{if} \quad t \in (\beta^j, \beta^j + \varepsilon]$$

for some $\varepsilon > 0$. Lemma 5.3 implies that $v(t) > 0$ at some point $t = \beta^j + \mu - r$ close to, and to the right of, $\beta^j - r$. This and the fact that $(\beta^j - r, \beta^j + r) \cap D = \varnothing$ (which follows from Lemma 5.8) implies that

$$v(t) \geqslant 0 \qquad \text{if} \quad t \in [\beta^j - r, \beta^j + r].$$

Therefore, $\dot{u}(t) \geqslant u(t)$ for all $t \in [\beta^j, \beta^j + r]$, implying that

$$u(t) > 0 \qquad \text{if} \quad t \in (\beta^j, \beta^j + r].$$

since $u(t) > 0$ holds immediately to the right of β^j.

As in the proof of Proposition 2.5, one easily shows by integrating the differential equations for $u(t)$ and $v(t)$ on steps of length r, and using the inequalities $ug(t, u) \geqslant 0$ and $vh(t, v) \geqslant 0$, that $\dot{u}(t) \geqslant u(t) > 0$ for $t > \beta^j$. This contradicts the boundedness of $u(t)$. ▮

Proof of Proposition 2.6. We shall show first that $\dot{x}(t) \geqslant 0$ for all t. In fact, we prove strict monotonicity of $x(t)$ for a certain range of t. Fix $\varepsilon > 0$ sufficiently small, and let $t_1 < t_2$ be the numbers uniquely determined by the conditions

$$\begin{aligned} x(t_1) &= -b + \varepsilon \\ x(t_2) &= a - \varepsilon \qquad \text{and} \\ -b + \varepsilon < x(t) &< a - \varepsilon \qquad \text{if} \quad t \in (t_1, t_2). \end{aligned}$$

Define the set

$$C = \{c > 0 \mid x(t) = x(t + c) \text{ for some } t \text{ satisfying } t_1 \leqslant t < t + c \leqslant t_2\}$$

and observe that $C \subseteq (0, t_2 - t_1]$ is closed in the relative topology of $(0, t_2 - t_1]$. We show that $C = \varnothing$, thereby proving strict monotonicity of $x(t)$ in (t_1, t_2). Suppose that $C \neq \varnothing$ and let $c^* = \sup C$. As $c^* \in C$, we may choose $t = t^*$ as in the definition of C, that is,

$$x(t^*) = x(t^* + c^*)$$

with t^* and $t^* + c^*$ both between t_1 and t_2. In fact, the strict inequalities

$$t_1 < t^* < t^* + c^* < t_2$$

hold from the properties of t_1 and t_2. See Fig. 5.

Consider now the function $u(t) = x(t + c^*) - x(t)$ in the interval $I = (t_1, t_2 - c^*)$. Clearly, $t^* \in I$ and $u(t^*) = 0$. We claim that

$$u(t) \geqslant 0 \qquad \text{at each} \quad t \in I. \tag{5.4}$$

If this claim were false, then $u(\tilde{t}) < 0$ would hold at some $\tilde{t} \in I$, giving

$$x(\tilde{t} + c^*) < x(\tilde{t}) < x(t_2).$$

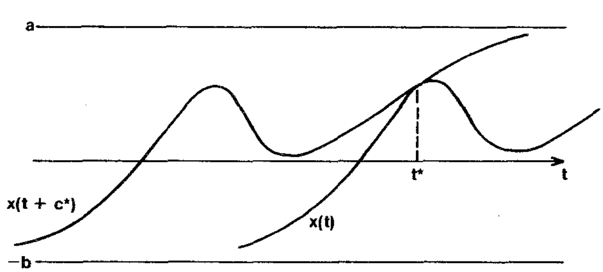

Fig. 5

Then there would exist $\tilde{c} \in (c^*, t_2 - \tilde{t})$ such that $x(\tilde{t}) = x(\tilde{t} + \tilde{c})$. However, $\tilde{c} \in C$ would hold, contradicting the definition of c^*. This establishes (5.4). From this we see, upon evaluating $u(t)$ at the endpoints of I, that

$$\begin{aligned} u(t) &\geqslant 0 \qquad \text{at each} \quad t \in I \\ u(t) &> 0 \qquad \text{at the endpoints of } I, \qquad \text{and} \\ u(t^*) &= 0 \qquad \text{at some } t^* \in I. \end{aligned}$$

However, the above three relations immediately contradict Lemma 5.9, as they imply the existence of an interval $I^j \subseteq I$ as in Lemma 5.7. With this contradiction, the monotonicity of $x(t)$ in (t_1, t_2) is proved. In fact, as ε is arbitrary and $\dot{x}(t) > 0$ for sufficiently negative t, we have shown that $\dot{x}(t) \geqslant 0$ for all t and $x(t)$ is strictly monotone as long as $x(t) < a$. The analogous result for $y(t)$, of course, also holds.

To complete the proof of the proposition, assume $\dot{x}(t) = 0$ for some t; let t_0 be the first such time. It is enough to show $\dot{x}(t) \leqslant 0$ for all $t \geqslant t_0$, for that would imply $\dot{x}(t) = 0$; hence, $x(t) = x(+\infty) = a$ for all $t \geqslant t_0$. Denoting $k(t) = -f(y(t-1))$, we note that $k(t)$ is nonincreasing in t; hence, $k(t) \leqslant k(t_0)$ for $t \geqslant t_0$. Also,

$$\dot{x}(t) = x(t) + k(t)$$

and so $x(t_0) = -k(t_0)$. Therefore,

$$x(t) = -k(t_0)\, e^{t - t_0} + \int_{t_0}^{t} e^{t-s} k(s)\, ds.$$

Hence,

$$\begin{aligned} \dot{x}(t) &= -k(t_0)\, e^{t - t_0} + k(t) + \int_{t_0}^{t} e^{t-s} k(s)\, ds \\ &= k(t) - k(t_0) + \int_{t_0}^{t} e^{t-s} [k(s) - k(t_0)]\, ds \\ &\leqslant 0 \qquad \text{for} \quad t \geqslant t_0. \end{aligned}$$

This completes the proof. ∎

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