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Smooth Invariant Foliations in Infinite Dimensional Spaces

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1. INTRODUCTION

One of the most useful properties of dynamical systems is the existence of invariant manifolds and their invariant foliations near an equilibrium or a periodic orbit. These manifolds and foliations serve as a convenient setting to describe the qualitative behavior of the local flows, and in many cases they are useful tools for technical estimates which facilitate the study of the local bifurcation diagram (see [6]). Many other important concepts in dynamical systems are closely related to the invariant manifolds and foliations. In finite dimensional space, the relations among invariant manifolds, invariant foliations, λ-lemma, linearization, and homoclinic bifurcation have been studied in [11]. It is well known that if each leaf is used as a coordinate, the original system is completely decoupled and the linearization follows easily (for example, see [27, 22]).

As a motivation, let us consider a linear system in $\mathbb{R}^{m+n}$

$$\begin{aligned} \dot{u} &= Au, \qquad u \in \mathbb{R}^m \\ \dot{v} &= Bv, \qquad v \in \mathbb{R}^n \end{aligned}$$

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with $\operatorname{Re} \sigma(A) > \gamma > \operatorname{Re} \sigma(B)$, where A and B are matrices, $\sigma(A)$ and $\sigma(B)$ are spectra of A and B with Re denoting the real parts, and $\gamma \in \mathbb{R}$ is a constant. For a given $u_0 \in \mathbb{R}^m$, after $t > 0$ the n-dimensional submanifold

$$M_0 = \{(u, v) \mid u = u_0, v \in \mathbb{R}^n\}$$

is carried by the flow to a new submanifold

$$M_t = \{(u, v) \mid u = e^{At}u_0, v \in \mathbb{R}^n\}.$$

Moreover, if (u_1, v_1) and $(u_2, v_2) \in M_0$, then

$$\|(e^{At}u_1, e^{Bt}v_1) - (e^{At}u_2, e^{Bt}v_2)\| = O(e^{\gamma t}), \qquad \text{as } t \to +\infty,$$

while points not in M_0 depart more rapidly than $Ce^{\gamma t}$, as $t \to +\infty$. Thus, we are able to group points in $\mathbb{R}^{m+n}$ as equivalent classes according to their asymptotic behavior as $t \to +\infty$, and each asymptotic class is a submanifold $u = \text{constant}$. We expect that these observations will persist after adding small nonlinear terms.

Let X be a Banach space, $\gamma \in \mathbb{R}$, and $T(\cdot, \cdot)$: $X \times \mathbb{R}^+ \to X$ be a nonlinear semigroup. We say that W^s_γ is a γ-stable fiber if $|T(x_1, t) - T(x_2, t)| = O(e^{\gamma t})$ as $t \to +\infty$ for any $x_1, x_2 \in W^s_\gamma$. We use $W^s_\gamma(x)$ to denote a γ-stable fiber passing through $x \in X$.

Let $Y \subset X$ be such that the backward flow $T(\cdot, \cdot)$; $Y \times \mathbb{R}^+ \to Y$ is uniquely defined. We say that $W^u_\gamma(y)$ is a γ-unstable fiber passing through $y \in Y$ if $|T(y_1, t) - T(y_2, t)| = O(e^{\gamma t})$ as $t \to -\infty$ for any $y_1, y_2 \in W^u_\gamma$.

If W^s_γ is an invariant manifold, we say W^s_γ is a γ-stable manifold. Similarly, we have γ-unstable manifolds. It follows from the definition that γ-stable (resp., γ-unstable) fibers are invariant under the forward (resp., backward) flow T; i.e.,

$$T(W^s_\gamma(x), t) \subset W^s_\gamma(T(x, t)), \qquad \text{for } t \in \mathbb{R}^+,$$

$$T(W^u_\gamma(y), t) \subset W^u_\gamma(T(y, t)), \qquad \text{for } t \in \mathbb{R}^-.$$

The purpose of this paper is to show that for some dynamical systems generated by partial differential equations, $W^s_\gamma(x)$ and $W^u_\gamma(y)$ are manifolds and to study the smoothness of those manifolds. We are also interested in the smooth dependence of $W^s_\gamma(x)$ (or $W^u_\gamma(y)$) on x (or y).

The smoothness of the invariant foliations suffers from two restrictions. First, if the nonlinear term in the equation is C^k, each fiber can only be a C^r submanifold with $r \leqslant k$. Second, there is a gap condition which requires that a gap between the real part of the spectrum of the linear equation has to be large compared with the module of the nonlinear term. The examples given in [9] show that if the gap conditions fail, then the invariant manifolds lose smoothness. It is well known that these gap conditions are always satisfied in the study of center, center-stable, and center-unstable manifolds (see [3, 5–7, 20, 19, 21, 24, 30]). Thus we are able to obtain the same smoothness, i.e., $r = k$. See, for example, [6, 7, 30, 31]. The theorems we give are closely related to the theory of inertial manifolds and generalize some recent results. For the theory of inertial manifolds, see [9, 10, 13, 15–17, 22, 25, 26]; also see Hale [18], Temám [29], and their references.

More delicate is the smooth dependence of $W^s_\gamma(x)$ (or $W^u_\gamma(y)$) with respect to x (or y). Here we do not have C^k dependence on x (or y) even if the vector field is C^k, otherwise we would have obtained a C^k linearization theorem which is in general not true (see [28]). A general condition is given in this paper which is similar to the gap condition we mentioned before. Accordingly, we can prove under very general assumptions that $W^s_\gamma(x)$ is Hölder continuous in x. And in some special cases, such as $\operatorname{Re} \sigma(A) \leqslant 0$, then $W^s_\gamma(x)$ is C^{k-1} with respect to x. In general we do not have a C^{k-1} foliation of the whole space but we do have a C^{k-1} foliation on the center-stable manifold W^{cs} and the center-unstable manifold W^{cu}. For application of this fact see [6, 12].

There have been some geometric proofs of the invariant foliations in finite dimensional spaces which are based on the concept of graph transforms, see [2, 14, 21], for example. Ours is an analytic proof which is based on the variation of constants formula (i.e., Liapunov–Perron formula) and generalized exponential dichotomies for semiflows in infinite dimensional spaces (see [20]). After an integral equation is written, the smallness of the nonlinear term usually guarantees the existence of a fixed point of the derived mapping by contraction mapping theorem. The smoothness of the fixed point with respect to the parameters is then studied from the integral equation. This allows a unified treatment of the whole problems.

We introduce the main notations and definitions in Section 2. Section 3 contains some basic theorems and lemmas which are used throughout the paper. A study of the abstract parabolic evolution equations is given in Section 4.

2. Notations

Let X and Y be Banach spaces and $U \subset X$ be an open subset. We define the following Banach spaces:

(1) For any integer $k > 0$, define the Banach space

$$C^k(U, Y) = \{f \mid f\colon U \to Y \text{ is } k\text{-times differentiable and } D^i f \text{ is bounded and continuous for } 0 \leqslant i \leqslant k\}$$

with the norm $\|f\|_k = \sum_{i=0}^{k} \sup_{x \in U} |D^i f(x)|_Y$, where D is the differential operator.

(2) Let $0 < \alpha \leqslant 1$; we define the Banach space

$$C^{k,\alpha}(U, Y) = \left\{ f \in C^k(U, Y) \mid |D^k f|_\alpha = \sup_{x_1 \neq x_2} \frac{|D^k f(x_1) - D^k f(x_2)|_Y}{|x_1 - x_2|_X^\alpha} < \infty \right\}$$

with the norm $|f|_{k,\alpha} = |f|_k + |D^k f|_\alpha$. For simplicity, we write C^α for $C^{0,\alpha}$.

(3) Let $\gamma \in \mathbb{R}$ and $\tau \in \mathbb{R}$ be fixed. We define the Banach space

$$E_\tau^-(\gamma, X) = \{ f\colon (-\infty, \tau] \to X \text{ is continuous and } \sup_{t \in (-\infty, \tau]} |e^{\gamma t} f(t)|_X < \infty \}$$

with the norm $|f|_{E_\tau^-(\gamma, X)} = \sup_{t \in (-\infty, \tau]} |e^{\gamma t} f(t)|_X$.

(4) Similarly, we define the Banach space

$$E_\tau^+(\gamma, X) = \{ f\colon [\tau, \infty) \to X \text{ is continuous and } \sup_{t \in [\tau, \infty)} |e^{\gamma t} f(t)|_X < \infty \}$$

with the norm $|f|_{E_\tau^+(\gamma, X)} = \sup_{t \in [\tau, \infty)} |e^{\gamma t f(t)|_X}$.

(5) We use $L^k(X, Y)$, $k > 0$, to denote the Banach space of all k-linear maps from X to Y with the norm $|\cdot|_{L^k(X,Y)}$.

(6) Let $n > k > 0$ be integers and Λ be an index set. Let M^n be an n-dimensional manifold and M_λ^k, $\lambda \in \Lambda$, be k-dimensional submanifolds of M^n. We say that M^n has a C^r foliation indexed by $\lambda \in \Lambda$ if $M^n = \bigcup_{\lambda \in \Lambda} M_\lambda^k$ and M_λ^k are mutually disjoint. Each M_λ^k is called a leaf through $\lambda \in \Lambda$ and is an injectively immersed connected submanifold. Moreover, M^n is covered by C^r chart $\phi\colon D^k \times D^{n-k} \to M^n$ with $\phi(D^k \times y) \subset M_\lambda^k$, where $\phi(0, y) \in M_\lambda^k$ and D^s is the unit s-dimensional disk. Let $\pi(t, x)$, $t > 0$ and $x \in M^n$, be a semiflow on M^n. The foliation $M^n = \bigcup_{\lambda \in \Lambda} M_\lambda^k$ is said to be invariant under π if $\pi(t, M_\lambda^k)$ is contained in a leaf for every $t \geqslant 0$.

3. Main Results

Let X, Y, and Z be Banach spaces. Assume that $X \subset Y \subset Z$, X is continuously embedded in Y and Y is continuously embedded in Z. Let $T(t, s)$ be an evolution operator on Z, which means that $T(t, s) \in L(Z, Z)$ $(t \geqslant s)$ is defined on interval $J \subset \mathbb{R}$; ordinarily $J = \mathbb{R}$ or $[\tau, \infty)$ or $(-\infty, \tau]$ and satisfies: (a) $T(t, t) = I = identity$; (b) $T(t, s)\, T(s, r) = T(t, r)$ if $t \geqslant s \geqslant r$; (c) $T(t, s)$ is strongly continuous in (t, s).

We say that $T(t, s)$ has a pseudo-dichotomy on the triplet (X, Y, Z), or on Z for short, if there exist continuous projection $P(t)$, $t \in J$, and constants α, $\beta > 0$, $\alpha < \beta$, $0 \leqslant \rho < 1$, and $M_i > 0$, $i = 1, 2, 3, 4$, such that

(i) $T(t, s)\, P(s) = P(t)\, T(t, s)$, $t \geqslant s$, $T(t, s)\, Y \subset X$, $t > s$, and $R(P(t)) \subset X$, where $R(P(t))$ denotes the range of the operator $P(t)$,

(ii) the restriction $T(t, s)|_{R(P(s))}$, $t \geqslant s$ is an isomorphism from $R(P(s))$ onto $R(P(t))$, and we define $T(s, t)$ as the inverse map from $R(P(t))$ to $R(P(s))$, and

(iii) the following equalities hold:

$$|T(t, s)\, P(s)\, x|_X \leqslant M_1 e^{-\alpha(t-s)}\, |x|_X, \qquad \text{for } t \leqslant s \tag{3.1}$$

$$|T(t, s)\, P(s)\, y|_X \leqslant M_2 e^{-\alpha(t-s)} |y|_Y, \qquad \text{for } t \leqslant s \tag{3.2}$$

$$|T(t, s)(I - P(s))\, x|_X \leqslant M_3 e^{-\beta(t-s)}\, |x|_X, \qquad \text{for } t \geqslant s \tag{3.3}$$

$$|T(t, s)(I - P(s))\, y|_X \leqslant M_4 (t-s)^{-\rho}\, e^{-\beta(t-s)}\, |y|_Y, \qquad \text{for } t > s. \tag{3.4}$$

Remark. Condition (3.4) is a smooth property of the evolution operator $T(t, s)$. Condition (ii) is not very restrictive since in many cases the unstable space is finite dimensional.

Let $\tau \in \mathbb{R}$ and $J = (-\infty, \tau]$. Define an operator L as

$$(Lf)(t) = \int_\tau^t T(t, s)\, P(s)\, f(s)\, ds + \int_{-\infty}^t T(t, s)\, Q(s)\, f(s)\, ds, \tag{3.5}$$

where $f \in E_\tau^-(\gamma, Y)$, $\alpha < \gamma < \beta$, and $Q(s) = I - P(s)$.

LEMMA 3.1. *If $T(t, s)$ has a pseudo-dichotomy on Z, then the operator L defined by (3.5) is a bounded linear operator from $E_\tau^-(\gamma, Y)$ to $E_\tau^-(\gamma, X)$ and the norm of L satisfies the estimate*

$$\|L\| \leqslant K(\beta - \gamma, \gamma - \alpha, \rho),$$

where K: $(0, \infty) \times (0, \infty) \times [0, 1) \to \mathbb{R}^+$ is defined by

$$K(a, b, c) = M_2 b^{-1} + M_4 \frac{2-c}{1-c} a^{c-1}. \tag{3.6}$$

Proof. By using (3.2) and (3.4), we have that

$$|Lf|_{E_\tau^-(\gamma,X)} \leqslant \sup_{t\in(-\infty,\tau]} \left\{ \int_t^\tau e^{\gamma t}\, |T(t,s)\, P(s)\, f(s)|_X\, ds + \int_{-\infty}^t e^{\gamma t}\, |T(t,s)\, Q(s)\, f(s)|_X\, ds \right\}$$

$$\leqslant M_2(\gamma-\alpha)^{-1} + M_4 \frac{2-\rho}{1-\rho} (\beta-\gamma)^{\rho-1}.$$

This completes this proof. ∎

LEMMA 3.2. *Suppose that $\alpha<\gamma<\beta$, $u\colon(-\infty,\tau]\to[0,\infty)$ is continuous, $\sup_{t\leqslant\tau} e^{\gamma t}u(t)<\infty$ and satisfies for $t\leqslant\tau$*

$$u(t) \leqslant C_1 e^{-\alpha(t-\tau)} + C_2 \int_\tau^t e^{-\alpha(t-s)} u(s)\, ds + C_3 \int_{-\infty}^t (t-s)^{-\rho} e^{-\beta(t-s)} u(s)\, ds, \tag{3.7}$$

where C_1, C_2, and C_3 are positive constants satisfying

$$C_2(\gamma-\alpha)^{-1} + C_3 \frac{2-\rho}{1-\rho} (\beta-\gamma)^{\rho-1} < 1. \tag{3.8}$$

Then

$$u(t) \leqslant \left(1 - C_2(\gamma_1-\alpha)^{-1} - C_3 \frac{2-\rho}{1-\rho} (\beta-\gamma_1)^{\rho-1}\right)^{-1} C_1 e^{-\gamma_1(t-\tau)} \tag{3.9}$$

for any γ_1, $\alpha<\gamma_1<\beta$, satisfying

$$C_2(\gamma_1-\alpha)^{-1} + C_3 \frac{2-\rho}{1-\rho} (\beta-\gamma_1)^{\rho-1} < 1.$$

Proof. Without losing generality, we assume $\tau=0$. We prove that if $v(t)$, $t\leqslant 0$, satisfies

$$v(t) = C_1 e^{-\alpha t} + C_2 \int_0^t e^{-\alpha(t-s)} v(s)\, ds + C_3 \int_{-\infty}^t (t-s)^{-\rho} e^{-\beta(t-s)} v(s)\, ds \tag{3.10}$$

and $\sup_{t \leqslant 0} |e^{\gamma t} v(t)| < \infty$, then

$$0 \leqslant v(t) \leqslant \left(1 - C_2(\gamma_1 - \alpha)^{-1} - C_3 \frac{2-\rho}{1-\rho} (\beta - \gamma_1)^{\rho - 1}\right)^{-1} C_1 e^{-\gamma_1 t}$$

and $u(t) \leqslant v(t)$ for $t \leqslant 0$.

By using the contraction mapping theorem and (3.8), we have that (3.10) has a unique solution $v(t)$ satisfying $\sup_{t \leqslant 0} |e^{\gamma t} v(t)| < \infty$. If $v(t) < 0$ for some $t = t_0$, then $\inf_{t \leqslant 0} \{e^{\gamma t} v(t)\} < 0$. Hence

$$e^{\gamma t} v(t) \geqslant C_1 e^{(\gamma - \alpha) t} + \left(C_2(\gamma - \alpha)^{-1} + C_3 \frac{2-\rho}{1-\rho} (\beta - \gamma)^{\rho - 1}\right) \inf_{t \leqslant 0} e^{\gamma t} v(t).$$

Thus $e^{\gamma t} v(t) \geqslant 0$. This is a contradiction and proves that $v(t) \geqslant 0$.

For γ_1, (3.10) has a unique solution $w(t)$ which satisfies

$$w(t) \leqslant \left(1 - C_2(\gamma_1 - \alpha)^{-1} - C_3 \frac{2-\rho}{1-\rho} (\beta - \gamma_1)^{\rho - 1}\right)^{-1} C_1 e^{-\gamma_1 t}.$$

By the uniqueness, we have that $v(t) = w(t)$.

Next observe that

$$\begin{aligned} u(t) - v(t) \leqslant\; & C_2 \int_0^t e^{-\alpha(t-s)}(u(s) - v(s))\, ds \\ & + C_3 \int_{-\infty}^t (t-s)^{-\rho} e^{-\beta(t-s)}(u(s) - v(s))\, ds. \end{aligned} \tag{3.11}$$

If $u(t) - v(t) > 0$ for some $t = t_0$, then

$$\sup_{t \leqslant 0} \{e^{\gamma t}(u(t) - v(t))\} > 0.$$

From (3.11), we have that

$$\begin{aligned} e^{\gamma t}(u(t) - v(t)) \leqslant & \left(C_2(\gamma_1 - \alpha)^{-1} + C_3 \frac{2-\rho}{1-\rho} (\beta - \gamma_1)^{\rho - 1}\right) \\ & \times \sup_{t \leqslant 0} \{e^{\gamma t}(u(t) - v(t))\}. \end{aligned}$$

By (3.8), $e^{\gamma t}(u(t) - v(t)) \leqslant 0$. This contradiction proves that $u(t) - v(t) > 0$ for some $t = t_0$ is impossible. This completes this proof. ∎

Let Λ be a Banach space. Let $\tau \in \mathbb{R}$ and J be either the interval $(-\infty, \tau]$ or the interval $[\tau, \infty)$. Consider the nonlinear map

$$F: J \times X \times \Lambda \to Y.$$

We assume that the nonlinear operator F satisfies

HYPOTHESIS A. *F is a continuous mapping from $J \times X \times \Lambda$ to Y* with bounded *continuous Frechét derivatives $D_u^{k_1} D_\lambda^{k_2} F(t, u, \lambda)$ with respect to u and λ, $k_1 + k_2 \leqslant k$, where k is a given positive integer.*

For the above nonlinear mapping F with $J = (-\infty, \tau]$, we consider the nonlinear integral equation

$$u(t) = T(t, \tau)\, P(\tau)\, \xi + \int_\tau^t T(t, s)\, P(s)\, F(s, u(s), \lambda)\, ds$$

$$+ \int_{-\infty}^t T(t, s)\, Q(s)\, F(s, u(s), \lambda)\, ds, \tag{3.12}$$

where $\xi \in X$, $u \in E_\tau^-(\gamma, X)$, and $\lambda \in \Lambda$. We assume that $T(t, s)$ is an evolution operator with a pseudo-dichotomy on Z. It is not hard to see from the definition of pseudo-dichotomy and Hypothesis A that the right-hand side of (3.12) is well defined.

Our first theorem is

THEOREM 3.3. *Assume that the evolution operator $T(t, s)$ has a pseudo-dichotomy on Z and F satisfies Hypothesis* A *with $J = (-\infty, \tau]$. If there is $\gamma > 0$ such that*

$$\alpha < \gamma \leqslant k\gamma < \beta, \tag{3.13}$$

$$K(\beta - k\gamma, \gamma - \alpha, \rho)\, \mathrm{Lip}_u F < 1, \tag{3.14}$$

where $\mathrm{Lip}_u F$ *is the Lipschitz constant of F with respect to u, then for every $(\xi, \lambda) \in X \times \Lambda$ the integral equation* (3.12) *has a unique solution $u(\cdot\,; \xi, \lambda) \in E_\tau^-(\gamma, X)$. Moreover*

$$u: X \times \Lambda \to E_\tau^-(k\gamma, X)$$

is a C^k mapping.

Remark. Condition (3.13) describes the spectral gap. The examples in [9] imply that if (3.13) fails, then the solutions of (3.12) will lose smoothness. We shall apply this theorem to get invariant manifolds for evolutionary equations in the next section.

Proof. Let $\mathscr{F}(u, \xi, \lambda)$ be the right-hand side of (3.12); i.e.,

$$\mathscr{F}(u, \xi, \lambda) = T(t, \tau) P(\tau) \xi + \int_\tau^t T(t, s) P(s) F(s, u(s), \lambda) \, ds$$
$$+ \int_{-\infty}^t T(t, s) Q(s) F(s, u(s), \lambda) \, ds.$$

From the definition of pseudo-dichotomy and Hypothesis A on F, with $J = (-\infty, \tau]$, we have that $\mathscr{F}$ is well defined from $E_\tau^-(\gamma, X) \times X \times \Lambda$ to $E_\tau^-(\gamma, X)$ and Lipschitz continuous in (ξ, λ). For each $u, \bar{u} \in E_\tau^-(k\gamma, X)$, by using Lemma 3.1, we have that

$$|\mathscr{F}(u, \xi_1, \lambda) - \mathscr{F}(\bar{u}, \xi_2, \lambda)|_{E_\tau^-(\gamma, X)}$$
$$\leqslant K(\beta - \gamma, \gamma - \alpha, \rho) \operatorname{Lip}_u F \, |u - \bar{u}|_{E_\tau^-(\gamma, X)}.$$

Since $K(\beta - \gamma, \gamma - \alpha, \rho) \operatorname{Lip}_u F < 1$ from the assumption of this theorem, we have that $\mathscr{F}$ is a uniform contraction with respect to the parameters ξ and λ. Using the uniform contraction principle, we have that for each $(\xi, \lambda) \in X \times \Lambda$, $\mathscr{F}(\cdot, \xi, \lambda)$ has a unique fixed point $u(\cdot; \xi, \lambda) \in E_\tau^-(\gamma, X)$ and $u(\cdot; \xi, \lambda)$ is Lipschitz continuous in (ξ, λ). Moreover, we have that

$$|u(\cdot; \xi_1, \lambda) - u(\cdot; \xi_2, \lambda)|_{E_\tau^-(\gamma, X)}$$
$$\leqslant \frac{M_1 e^{\gamma\tau}}{1 - K(\beta - \gamma, \gamma - \alpha, \rho) \operatorname{Lip}_u F} |\xi_1 - \xi_2|_X. \tag{3.15}$$

In other words, $u(t; \xi, \lambda)$ is a solution of (3.12) which satisfies (3.15). Next we want to show that u is C^k from $X \times \Lambda$ to $E_\tau^-(k\gamma, X)$. We prove this by induction on k. The method we shall use to show the smoothness of the solution u is different from those used in [7–9].

First let us consider $k = 1$. Since $K(\beta - \gamma, \gamma - \alpha, \rho) \operatorname{Lip}_u F < 1$, there is a small $\delta > 0$ such that $\alpha < \gamma - 2\delta$ and

$$K(\beta - \gamma_1, \gamma_1 - \alpha, \rho) \operatorname{Lip}_u F < 1, \qquad \text{for } \gamma - 2\delta \leqslant \gamma_1 \leqslant \gamma. \tag{3.16}$$

Using Lemma 3.2, we have that $u(\cdot; \xi, \lambda) \in E_\tau^-(\gamma_1, X)$ for $\gamma - 2\delta \leqslant \gamma_1 \leqslant \gamma$. Let

$$I=\Big\{\int_\tau^t T(t,s)\,P(s)[F(s,u(s;\xi_1,\lambda),\lambda)-F(s,u(s;\xi_2,\lambda),\lambda)$$
$$-D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]\,ds$$
$$+\int_{-\infty}^t T(t,s)\,Q(s)[F(s,u(s;\xi_1,\lambda),\lambda)-F(s,u(s;\xi_2,\lambda),\lambda)$$
$$-D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]\,ds\Big\}.$$

We claim that $|I|_{E_\tau^-(\gamma-\delta,X)}=o(|\xi_1-\xi_2|)$ as $\xi_1\to\xi_2$. Using this claim, we have that

$$\begin{aligned}
&u(\cdot;\xi_1,\lambda)-u(\cdot;\xi_2,\lambda)\\
&\quad-\int_\tau^t T(t,s)\,P(s)\,D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))\,ds\\
&\quad-\int_{-\infty}^t T(t,s)\,Q(s)\,D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))\,ds\\
&=T(t,\tau)\,P(\tau)(\xi_1-\xi_2)+I\\
&=\mathscr{J}(\xi_1-\xi_2)+o(|\xi_1-\xi_2|_X),\qquad \text{as } \xi_1\to\xi_2,
\end{aligned}\tag{3.17}$$

where $\mathscr{J}=T(t,\tau)\,P(\tau)$ is a bounded linear operator from X to $E_\tau^-(\gamma-\delta,X)$. Let

$$\mathscr{L}f=\int_\tau^t T(t,s)\,P(s)\,D_uF(s,u(s;\xi_2,\lambda),\lambda)\,f\,ds$$
$$+\int_{-\infty}^t T(t,s)\,Q(s)\,D_uF(s,u(s;\xi_2,\lambda),\lambda)\,f\,ds.$$

Using (3.16) and Lemma 3.1, we have that $\mathscr{L}$ is a continuous linear operator from $E_\tau^-(\gamma-\delta,X)$ to itself and $|\mathscr{L}|_{L(E_\tau^-(\gamma-\delta,X),E_\tau^-(\gamma-\delta,X))}<1$. Equation (3.17) implies that

$$u(\cdot;\xi_1,\lambda)-u(\cdot;\xi_2,\lambda)=\mathscr{L}^{-1}\mathscr{J}(\xi_1-\xi_2)+o(|\xi_1-\xi_2|_X),\quad \text{as}\quad \xi_1\to\xi_2.$$

This implies that $u(\cdot;\xi,\lambda)$ is differentiable in ξ and its derivative satisfies

$$\begin{aligned}
&D_\xi u(t;\xi,\lambda)\\
&\quad=T(t,\tau)\,P(\tau)+\int_\tau^t T(t,s)\,P(s)\,D_uF(s,u(s;\xi,\lambda),\lambda)\,D_\xi u(s;\xi,\lambda)\,ds\\
&\qquad+\int_{-\infty}^t T(t,s)\,Q(s)\,D_uF(s,u(s;\xi,\lambda),\lambda)\,D_\xi u(s;\xi,\lambda)\,ds.
\end{aligned}\tag{3.17$'$}$$

Now we prove that $|I|_{E_\tau^-(\gamma-\delta,X)}=o(|\xi_1-\xi_2|)$ as $\xi_1\to\xi_2$. Let

$$I_1=e^{(\gamma-\delta)t}\left\{\left|\int_N^t T(t,s)\,P(s)[F(s,u(s;\xi_1,\lambda),\lambda)-F(s,u(s;\xi_2,\lambda),\lambda)\right.\right.$$

$$\left.\left.-D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]\,ds\right|_X\right\}$$

for $t\leqslant N<\tau$ and $I_1=0$ for $t>N$;

$$I_2=e^{(\gamma-\delta)t}\left\{\left|\int_\tau^N T(t,s)\,P(s)[F(s,u(s;\xi_1,\lambda),\lambda)-F(s,u(s;\xi_2,\lambda),\lambda)\right.\right.$$

$$\left.\left.-D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]\,ds\right|_X\right\}$$

for $t\leqslant N\leqslant\tau$, but change N to t if $t>N$;

$$I_3=e^{(\gamma-\delta)t}\left\{\left|\int_N^t T(t,s)\,Q(s)[F(s,u(s;\xi_1,\lambda),\lambda)-F(s,u(s;\xi_2,\lambda),\lambda)\right.\right.$$

$$\left.\left.-D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]\,ds\right|_X\right\};$$

$$I_4=e^{(\gamma-\delta)t}\left\{\left|\int_{-\infty}^N T(t,s)\,Q(s)[F(s,u(s;\xi_1,\lambda),\lambda)-F(s,u(s;\xi_2,\lambda),\lambda)\right.\right.$$

$$\left.\left.-D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]\,ds\right|_X\right\};$$

where $-N$ is a large number to be chosen later.

It is sufficient to show that for any $\varepsilon>0$ there is a $\sigma>0$ such that if $|\xi_1-\xi_2|\leqslant\sigma$, then $|I|_{E_\tau^-(\gamma-\delta,X)}\leqslant\varepsilon\,|\xi_1-\xi_2|$. A simple computation implies that

$$I_1\leqslant 2\,\mathrm{Lip}_uFM_2\int_t^N e^{(\gamma-\delta)t-\alpha(t-s)}|u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda)|_X\,ds$$

$$\leqslant 2\,\mathrm{Lip}_uFM_2\int_t^N e^{(\gamma-\delta)t-\alpha(t-s)-(\gamma-2\delta)s}\,ds\,|u(\cdot;\xi_1,\lambda)$$

$$-u(\cdot;\xi_1,\lambda)|_{E_\tau^-(\gamma-2\delta,X)}$$

$$\leqslant\frac{2\,\mathrm{Lip}_uFM_1M_2e^{\gamma\tau}}{1-K(\beta-\gamma+2\delta,\gamma-2\delta-\alpha,\rho)\,\mathrm{Lip}_uF}(\gamma-\delta-\alpha)^{-1}\,e^{\delta N}\,|\xi_1-\xi_2|_X.$$

Choose $-N$ so large that

$$\frac{2\,\mathrm{Lip}_uFM_1M_2e^{\gamma\tau}}{1-K(\beta-\gamma+2\delta,\gamma-2\delta-\alpha,\rho)\,\mathrm{Lip}_uF}(\gamma-\delta-\alpha)^{-1}\,e^{\delta N}\leqslant\frac{1}{4}\varepsilon.$$

Hence for such N we have that

$$\sup_{t \leqslant \tau} I_1 \leqslant \tfrac{1}{4}\varepsilon \, |\xi_1 - \xi_2|_X.$$

Fix such N; for I_2 we have that

$$\begin{aligned}
I_2 \leqslant{}& M_2 \int_\tau^N e^{(\gamma-\delta)t-\alpha(t-s)} \int_0^1 |D_u F(s, \theta u(s; \xi_1, \lambda) + (1-\theta)\, u(s; \xi_2, \lambda), \lambda) \\
&- D_u F(s, u(s; \xi_2, \lambda), \lambda)|_Y \, d\theta \, |u(s; \xi_1, \lambda) - u(s; \xi_2, \lambda)|_X \, ds \\
\leqslant{}& \frac{M_1 M_2 e^{\gamma\tau}}{1 - K(\beta - \gamma + \delta, \gamma - \delta - \alpha, \rho)\, \mathrm{Lip}_u F} \int_\tau^N e^{(\gamma-\delta-\alpha)(\tau - s)} \\
&\times \int_0^1 |D_u F(s, \theta u(s; \xi_1, \lambda) + (1-\theta)\, u(s; \xi_2, \lambda), \lambda) \\
&- D_u F(s, u(s; \xi_2, \lambda), \lambda)|_Y \, d\theta \, ds \, |\xi_1 - \xi_2|_X.
\end{aligned}$$

The last integral is on the compact interval $[N, \tau]$. By the continuity of $u(s; \xi, \lambda)$, we have that there is a $\sigma_1 > 0$ such that if $|\xi_1 - \xi_2| \leqslant \sigma_1$, then

$$\sup_{t \leqslant \tau} I_2 \leqslant \tfrac{1}{4}\varepsilon \, |\xi_1 - \xi_2|_X.$$

Therefore if $|\xi_1 - \xi_2| \leqslant \sigma_1$, then

$$\sup_{t \leqslant \tau} I_1 + \sup_{t \leqslant \tau} I_2 \leqslant \tfrac{1}{2}\varepsilon \, |\xi_1 - \xi_2|_X.$$

Similarly, there exists $\sigma_2 > 0$ such that if $|\xi_1 - \xi_2| \leqslant \sigma_2$, then

$$\sup_{t \leqslant \tau} I_3 + \sup_{t \leqslant \tau} I_4 \leqslant \tfrac{1}{2}\varepsilon |\xi_1 - \xi_2|_X.$$

Taking $\sigma = \min\{\sigma_1, \sigma_2\}$, we have that if $|\xi_1 - \xi_2|_X \leqslant \sigma$, then

$$|I|_{E_\tau^-(\gamma - \delta, X)} \leqslant \varepsilon \, |\xi_1 - \xi_2|_X.$$

Therefore $|I|_{E_\tau^-(\gamma-\delta, X)} = o(|\xi_1 - \xi_2|_X)$ as $\xi_1 \to \xi_2$. We now prove that $D_\xi u(\cdot; \cdot, \lambda)$ is continuous from X to $E_\tau^-(\gamma, X)$. For $\xi_1, \xi_2 \in X$ let

$$\begin{aligned}
\bar{I} ={}& \int_\tau^t T(t, s)\, P(s)[D_u F(s, u(s; \xi_1, \lambda), \lambda) \\
&- D_u F(s, u(s; \xi_2, \lambda), \lambda)]\, D_\xi u(s; \xi_2, \lambda))]\, ds \\
&+ \int_{-\infty}^t T(t, s)\, Q(s)[D_u F(s, u(s; \xi_1, \lambda), \lambda) \\
&- D_u F(s, u(s; \xi_2, \lambda), \lambda)]\, D_\xi u(s; \xi_2, \lambda))]\, ds.
\end{aligned}$$

We claim that $|\bar{I}|_{E_\tau^-(\gamma,X)} = o(1)$ as $\xi_1 \to \xi_2$. Using this claim, from (3.17)′ and (3.16), we have that

$$\begin{aligned}
&|D_\xi u(\cdot;\xi_1,\lambda) - D_\xi u(\cdot;\xi_2,\lambda)|_{E_\tau^-(\gamma,X)} \\
&\quad \leqslant \frac{1}{1 - K(\beta-\gamma, \gamma-\alpha, \rho)\,\mathrm{Lip}_u F} \\
&\quad (M_1 e^{\gamma\tau} |\xi_1 - \xi_2|_X + |I|_{E_\tau^-(\gamma,X)}) \to 0, \qquad \text{as } \xi_1 \to \xi_2.
\end{aligned}$$

Hence $D_\xi u(\cdot;\cdot,\lambda)$ is continuous from X to $E_\tau^-(\gamma, X)$. The proof of this claim is similar to that of the last claim. We omit it. Using the same arguments, we can show that $u(\cdot;\xi,\cdot)$ is C^1 from Λ to $E_\tau^-(\gamma, X)$. Now we show that u is C^k from $X\times\Lambda$ to $E_\tau^-(k\gamma, X)$ by induction. By the induction assumption, we know that u is C^{k-1} from $X\times\Lambda$ to $E_\tau^-((k-1)\gamma, X)$. Let us first look at $D_\xi^{k-1}u(t;\xi,\lambda)$. It satisfies the equation

$$\begin{aligned}
D_\xi^{k-1}u = &\int_\tau^t T(t,s)\,P(s)\,D_uF(s,u,\lambda)\,D_\xi^{k-1}u\,ds \\
&+ \int_{-\infty}^t T(t,s)\,Q(s)\,D_uF(s,u,\lambda)\,D_\xi^{k-1}u\,ds \\
&+ \int_\tau^t T(t,s)\,P(s)\,R_{k-1}(s,\xi,\lambda)\,ds \\
&+ \int_{-\infty}^t T(t,s)\,Q(s)\,R_{k-1}(s,\xi,\lambda)\,ds, \qquad (3.18)
\end{aligned}$$

where

$$R_{k-1}(s,\xi,\lambda) = \sum_{i=0}^{k-3} \binom{k-2}{i} D_\xi^{k-2-i}(D_uF(s,u(s;\xi,\lambda),\lambda))\,D_\xi^{i+1}u(s;\xi,\lambda).$$

We note that $D_\xi^i u \in E_\tau^-(i\gamma, X)$ for $i = 1, ..., k-1$. A simple computation implies that $R_{k-1}(\cdot,\xi,\lambda) \in L^{k-1}(X, E_\tau^-((k-1)\gamma, X))$ and is C^1 in ξ. In order to ensure that the above integrals are well defined one has to require that $\alpha < (k-1)\gamma < \beta$. This is why we need the gap condition. By the assumption of Theorem 3.3, we have that $K(\beta - k\gamma, k\gamma - \alpha, \rho)\,\mathrm{Lip}_u F < 1$. Using this fact and the same argument which we used in the case $k = 1$, we can show that $D_\xi^{k-1}u(\cdot;\cdot,\lambda)$ is C^1 from X to $L^k(X, E_\tau^-(k\gamma, X))$. Similarly, we can show that u is C^k from $X\times\Lambda$ to $E_\tau^-(k\gamma, X)$. This completes the proof. ∎

In the following we present a theorem which is used to get the stable foliations for evolution equations in the next section. We assume that F

satisfies Hypothesis A and the evolution operator has a pseudo-dichotomy with $J = [\tau, \infty)$. In addition, we assume that there exist constants ω, $M_5 > 0$, and $M_6 > 0$ such that

$$|T(t, s)\, x|_X \leqslant M_5 e^{\omega(t-s)} |x|_X, \qquad \text{for } t \geqslant s \tag{3.19}$$

$$|T(t, s)\, y|_X \leqslant M_6 (t-s)^{-\rho}\, e^{\omega(t-s)}\, |y|_Y, \qquad \text{for } t > s. \tag{3.20}$$

Fix λ and let $v(t, \eta, \lambda)$ be the solution of the integral equation

$$v = T(t, \tau)\, \eta + \int_\tau^t T(t, s)\, F(s, v, \lambda)\, ds.$$

The condition on F implies that $v(t, \eta, \lambda)$ is C^k in η. By using Lemma 7.1.1 in Henry [20], we have that

$$|D_\eta^i v(t, \eta, \lambda)|_{L^i(X,X)} \leqslant C(i, \rho, F, M_5, M_6)\, e^{(i\omega + (2i-1)\mu)(t-\tau)}, \qquad \text{for } i = 1, \ldots, k, \tag{3.21}$$

where $C(i, \rho, F, M_5, M_6)$ is a positive constant,

$$\mu = (M_6 \operatorname{Lip}_u F \Gamma(1-\rho))^{1/(1-\rho)},$$

and $\Gamma(s)$ is the gamma function.

Consider the integral equation for $t \geqslant \tau$,

$$\begin{aligned} u(t) = {} & T(t, \tau)\, Q(\tau)\, \xi + \int_\tau^t T(t, s)\, Q(s)[F(s, u(s) + v(s, \eta, \lambda), \lambda) \\ & - F(s, v(s, \eta, \lambda), \lambda)]\, ds \\ & + \int_\infty^t T(t, s)\, P(s)[F(s, u(s) + v(s, \eta, \lambda), \lambda) \\ & - F(s, v(s, \eta, \lambda), \lambda)]\, ds, \end{aligned} \tag{3.22}$$

where $\xi \in X$ and $u \in E_\tau^+(\gamma, X)$, $\alpha < \gamma < \beta$.

THEOREM 3.4. *Assume that the evolution operator $T(t, s)$ has a pseudo-dichotomy on Z and F satisfies Hypothesis* A *with $J = [\tau, \infty)$. If there exist $\gamma > 0$, $\alpha < \gamma \leqslant k\gamma < \beta$, and $0 < r \leqslant k - 1$ such that*

$$K(\beta - k\gamma, \gamma - a, \rho) \operatorname{Lip}_u F < 1, \tag{3.23}$$

$$\gamma - (r\omega + (2r-1)\, \mu) > 0, \tag{3.24}$$

then for each (ξ, η, λ), (3.22) *has a unique solution* $u(\cdot\,; \xi, \eta, \lambda) \in E_\tau^+(\gamma, X)$ *which has the following properties*:

(i) *u is* C^k *from* X *to* $E_\tau^+(\gamma, X)$ *with respect to* ξ *and* $D_\xi^i u$, $i = 1, ..., k$, *are continuous in all variables.*

(ii) *u is* C^r *from* X *to* $E_\tau^+(\gamma, X)$ *with respect to* η *and* $D_\eta^i u$, $i = 1, ..., \tau$, *are continuous in all variables.*

(iii) $D_\eta^i u$ *is* C^{k-1-i} *with respect to* ξ, $0 \leqslant i \leqslant r$.

Proof. Let $\mathscr{G}$ be the right-hand side of (3.22); i.e.,

$$\begin{aligned}\mathscr{G}(u, \xi, \eta, \lambda) = T(t, \tau)\, Q(\tau)\, \xi &+ \int_\tau^t T(t, s)\, Q(s)[F(s, u(s) + v(s, \eta, \lambda), \lambda) \\ &- F(s, v(s, \eta, \lambda), \lambda)]\, ds \\ &+ \int_\infty^t T(t, s)\, P(s)[F(s, u(s) + v(s, \eta, \lambda), \lambda) \\ &- F(s, v(s, \eta, \lambda), \lambda)]\, ds.\end{aligned}$$

From the definition of the pseudo-dichotomy and the condition on F we have that $\mathscr{G}$ is well defined from $E_\tau^+(\gamma, X) \times X \times X \times \Lambda$ to $E_\tau^+(\gamma, X)$. And for each $\xi_1, \xi_2 \in X$, we have that

$$|\mathscr{G}(u, \xi_1, \eta, \lambda) - \mathscr{G}(u, \xi_2, \eta, \lambda)|_{E_\tau^+(\gamma, X)} \leqslant M_3 e^{\gamma\tau}\, |\xi_1 - \xi_2|_X.$$

For each $u, \bar{u} \in E_\tau^+(\gamma, X)$ we have that

$$\begin{aligned}&|\mathscr{G}(u, \xi, \eta, \lambda) - \mathscr{G}(\bar{u}, \xi, \eta, \lambda)|_{E_\tau^+(\gamma, X)} \\ &\qquad \leqslant K(\beta - \gamma, \gamma - \alpha, \rho)\, \mathrm{Lip}_u F\, |u - \bar{u}|_{E_\tau^+(\gamma, X)}.\end{aligned}$$

Condition (3.23) implies that $\mathscr{G}$ is a uniform contraction with respect to the parameters ξ, η and λ. By using the unform contraction mapping theorem, we have that $\mathscr{G}$ has a unique fixed point $u(\cdot\,; \xi, \eta, \lambda) \in E_\tau^+(\gamma, X)$ which is Lipschitz continuous with respect to ξ. Furthermore, we have that

$$\begin{aligned}&|u(\cdot\,; \xi_1, \eta, \lambda) - u(\cdot\,; \xi_2, \eta, \lambda)|_{E_\tau^+(\gamma, X)} \\ &\qquad \leqslant \frac{M_3 e^{\gamma\tau}}{1 - K(\beta - \gamma, \gamma - \alpha, \rho)\, \mathrm{Lip}_u F}\, |\xi_1 - \xi_2|_X.\end{aligned}$$

To see that u is continuous in η and λ, by (3.23), we choose a small $\delta > 0$ such that $\alpha < \gamma + \delta < \beta$ and

$$K(\beta - (\gamma + \delta), (\gamma + \delta) - \alpha, \rho)\, \mathrm{Lip}_u F < 1.$$

We have that for each u, $\bar{u} \in E_\tau^+(\gamma+\delta, X)$

$$|\mathscr{G}(u, \xi, \eta, \lambda) - \mathscr{G}(\bar{u}, \xi, \eta, \lambda)|_{E_\tau^+(\gamma+\delta, X)}$$
$$\leqslant K(\beta-(\gamma+\delta), (\gamma+\delta)-\alpha, \rho)\, \mathrm{Lip}_u F\, |u-\bar{u}|_{E_\tau^+(\gamma-\delta, X)}.$$

By the uniform contraction mapping theorem, we have that for each $(\xi, \eta, \lambda) \in X \times X \times \Lambda \mathscr{G}(\cdot, \xi, \eta, \lambda)$ has a unique fixed point $u_\delta(\cdot; \xi, \eta, \lambda) \in E_\tau^+(\gamma+\delta, X)$. Since $E_\tau^+(\gamma+\delta, X) \subset E_\tau^+(\gamma, X)$, by the uniqueness of solutions, we have that $u(\cdot; \xi, \eta, \lambda) = u_\delta(\cdot; \xi, \eta, \lambda)$. Hence $u(\cdot; \xi, \eta, \lambda) \in E_\tau^+(\gamma+\delta, X)$. In other words, the solution $u(t; \xi, \eta, \lambda)$ decays much faster than $e^{-\gamma t}$ as t goes to ∞. Using this property and the similar arguments used in the proof of Theorem 3.3, we have that u is continuous in η and λ.

Now we study the smoothness of the solution u. By (3.23), we have that there is a positive number δ such that $k\gamma + k\delta < \beta$ and

$$K(\beta-(k\gamma+k\delta), \gamma-\alpha, \rho)\, \mathrm{Lip}_u F < 1.$$

Using the contraction mapping principle, we have that $u \in E_\tau^+(k\gamma+k\delta, X)$. First let us look at the smoothness of u with respect to ξ. Using the same arguments as we used in Theorem 3.3, we have that u is C^k from X to $E_\tau^+(k\gamma, X)$ in ξ. Next we consider the smoothness of the solution u of (3.22) with respect to η. We claim that $u\colon X \to E_\tau^+((k-i)\gamma+(k-i)\delta, X)$ is C^i in η, for $i \leqslant r \leqslant k-1$. The idea of the proof of this claim is the same as in Theorem 3.3. We shall only give some hint of the proof. First let us consider the case $r = 1$. Formally differentiating u in (3.22) with respect to η, we have that

$$\begin{aligned} D_\eta u(t; \xi, \eta, \lambda) = {} & \int_\tau^t T(t,s)\, Q(s)\, D_u F(s, u+v, \lambda)\, D_\eta u(s; \xi, \eta, \lambda)\, ds \\ & + \int_\infty^t T(t,s)\, P(s)\, D_u F(s, u+v, \lambda)\, D_\eta u(s; \xi, \eta, \lambda)\, ds \\ & + \int_\tau^t T(t,s)\, Q(s)(D_u F(s, u(s; \xi, \eta, \lambda) + v(s, \eta, \lambda), \lambda) \\ & - D_u F(s, v(s, \eta, \lambda), \lambda))\, D_\eta v(s, \eta, \lambda)\, ds \\ & + \int_\infty^t T(t,s)\, P(s)(D_u F(s, u(s; \xi, \eta, \lambda) + v(s, \eta, \lambda), \lambda) \\ & - D_u F(s, v(s, \eta, \lambda), \lambda))\, D_\eta v(s, \eta, \lambda)\, ds. \end{aligned} \tag{3.25}$$

By (3.21), we have

$$|D_\eta v(t, \eta, \lambda)|_{L(X,X)} \leqslant C(1, \rho, F, M_5, M_6)\, e^{(\omega+\mu)(t-\tau)}.$$

The condition $\gamma-(\omega+\mu)>0$ implies that the final two integrals in (3.25) are convergent. On the other hand, by (3.23) and the uniform contraction principle, the equation

$$\begin{aligned} U(t;\xi,\eta,\lambda)=&\int_\tau^t T(t,s)\,Q(s)\,D_uF(s,u+v,\lambda)\,U(s;\xi,\eta,\lambda)\,ds\\ &+\int_\infty^t T(t,s)\,P(s)\,D_uF(s,u+v,\lambda)\,U(s;\xi,\eta,\lambda)\,ds\\ &+\int_\tau^t T(t,s)\,Q(s)(D_uF(s,u(s;\xi,\eta,\eta,\lambda)+v(s,\eta,\lambda),\lambda)\\ &-D_uF(s,v(s,\eta,\lambda),\lambda))\,D_\eta v(s,\eta,\lambda)\,ds\\ &+\int_\infty^t T(t,s)\,P(s)(D_uF(s,u(s;\xi,\eta,\lambda)+v(s,\eta,\lambda),\lambda)\\ &-D_uF(s,v(s,\eta,\lambda),\lambda))\,D_\eta v(s,\eta,\lambda)\,ds \end{aligned}$$

has a unique solution $U\in L(X,E_\tau^+((k-1)\gamma+(k-1)\delta,X))$ which can be shown to be the derivative of u. Now we consider $D_\eta^r u$, $r>1$. Formally differentiating u r-times in (3.22) with respect to η, we have that

$$\begin{aligned} D_\eta^r u(t;\eta,\lambda)=&\int_\tau^t T(t,s)\,Q(s)\,D_uF(s,u+v,\lambda)\,D_\eta^r u(s;\xi,\eta,\lambda)\,ds\\ &+\int_\infty^t T(t,s)\,P(s)\,D_uF(s,u+v,\lambda)\,D_\eta^r u(s;\xi,\eta,\lambda)\,ds\\ &+\int_\tau^t T(t,s)\,Q(s)\,\bar{R}_r(s,\xi,\eta,\lambda)\,ds\\ &+\int_\infty^t T(t,s)\,P(s)\,\bar{R}_r(s,\xi,\eta,\lambda)\,ds, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} \bar{R}_r(s,\xi,\eta,\lambda)=&\sum_{j=0}^{r-2}\binom{r-1}{j}D_\eta^{r-1-j}(D_uF(s,u(s;\xi,\eta,\lambda)\\ &+v(s,\eta,\lambda),\lambda))\,D_\eta^{j+1}u(s;\xi,\lambda)\,D_\eta^{r-1}[(D_uF(s,u(s;\xi,\eta,\lambda)\\ &+v(s,\eta,\lambda),\lambda)-D_uF(s,v(s,\eta,\lambda),\lambda))\,D_\eta v(s,\eta,\lambda)]. \end{aligned}$$

Since $D_\eta^i u\in L^i(X,E_\tau^+((k-i)\gamma+(k-i)\delta,X))$ for $i=1,\ldots,r-1$, and

$$\begin{aligned} &|D_\eta^i v(t,\eta,\lambda)|_{L^i(X,X)}\\ &\qquad\leqslant C(i,\rho,F,M_5,M_6)\,e^{(i\omega+(2i-1)\mu)(t-\tau)},\qquad \text{for } i=1,\ldots,k, \end{aligned}$$

a simple computation implies that $R_r(\cdot, \xi, \eta, \lambda) \in L^r(X, E_\tau^+((k-r)\gamma + (k-r)\delta, X))$. This implies that final two integrals in (3.26) are convergent provided $r \leqslant k-1$. On the other hand, by (2.23), (2.24), and the contraction mapping theorem, we have that the equation

$$\begin{aligned} U(t; \xi, \eta, \lambda) = \int_\tau^t & T(t, s)\, Q(s)\, D_u F(s, u+v, \lambda)\, U(s; \xi, \eta, \lambda)\, ds \\ & + \int_\infty^t T(t, s)\, P(s)\, D_u F(s, u+v, \lambda U(s; \xi, \eta, \lambda)\, ds \\ & + \int_\tau^t T(t, s)\, Q(s)\, \bar{R}_r(s, \xi, \eta, \lambda)\, ds \\ & + \int_\infty^t T(t, s)\, P(s)\, \bar{R}_r(s, \xi, \eta, \lambda)\, ds \end{aligned}$$

has a unique solution $U \in L^r(X, E_\tau^+((k-r)\gamma + (k-r)\delta, X))$ which can be shown to be the derivative of $D_\eta^{r-1} u$. Similarly we can prove (iii). This completes the proof. ▮

4. Applications

In this section, we discuss some direct applications of Theorems 3.3 and 3.4. Many differential equations in infinite dimensional spaces such as parabolic equations, hyperbolic equations, and delay equations can be written as integral equations by using the variation of constants formula. This observation is the key to use Theorems 3.3 and 3.4.

Let Z and Λ be Banach spaces. Consider the semilinear evolutionary equation

$$\frac{du}{dt} + Au = F(u, \lambda), \tag{4.1}$$

where $u \in Z$, the parameter $\lambda \in \Lambda$. We assume that the linear operator A satisfies

Hypothesis B. *A is a sectorial operator and*

$$\sigma(A) = \sigma_1(A) \cup \sigma_2(A), \tag{4.2}$$

$$-\omega < \inf_{\nu \in \sigma_1(A)} \operatorname{Re} \nu \leqslant \sup_{\nu \in \sigma_1(A)} \operatorname{Re} \nu < \alpha < \beta < \inf_{\nu \in \sigma_2(A)} \operatorname{Re} \nu, \tag{4.3}$$

where ω, α, and β, $\beta > 0$ are constants and $\sigma_1(A)$ is bounded.

It is known that there is a positive number a such that the fractional powers of $(A+aI)$ are well defined, which we denote by $(A+aI)^{\theta}$, for all $\theta \in \mathbb{R}$. See Henry [20], for example. The domain of $(A+aI)^{\theta}$, which we denote by Z^{θ}, is a Banach space under the graph norm $|\cdot|_{\theta}$. Furthermore $-A$ is the infinitesimal generator of an analytic semigroup, which we denote by e^{-At}.

The nonlinear term F is assumed to satisfy

HYPOTHESIS C. *There exist nonnegative constants $\theta_1 \leqslant 1$ and $\theta_2 \leqslant 1$, $0 \leqslant \theta_1 - \theta_2 < 1$, such that F is a continuous mapping from Z^{θ_1} to Z^{θ_2}.*

We note that many differential equations such as reaction–diffusion equations, Cahn–Hilliard equations, Kuramoto–Sivashinsky equations, and Navier–Stokes equations can be written in the form of (4.1). Also Hypotheses B and C are satisfied.

By using Lemma 1.4.3 in [20] we have the following lemma

LEMMA 4.1. *There are positive constants M_i, $i = 1, ..., 6$, and projection P corresponding to $\sigma_1(A)$ such that*

$$|e^{-At}P|_{L(Z^{\theta_1}, Z^{\theta_1})} \leqslant M_1 e^{-\alpha t}, \qquad \text{for } t \leqslant 0, \tag{4.4}$$

$$|e^{-At}P|_{L(Z^{\theta_2}, Z^{\theta_1})} \leqslant M_2 e^{-\alpha t}, \qquad \text{for } t \leqslant 0, \tag{4.5}$$

$$|e^{-At}(I-P)|_{L(Z^{\theta_1}, Z^{\theta_1})} \leqslant M_3 e^{-\beta t}, \qquad \text{for } t \geqslant 0, \tag{4.6}$$

$$|e^{-At}(I-P)|_{L(Z^{\theta_2}, Z^{\theta_1})} \leqslant M_4 t^{\theta_2 - \theta_1} e^{-\beta t}, \qquad \text{for } t > 0, \tag{4.7}$$

$$|e^{-At}|_{L(Z^{\theta_1}, Z^{\theta_1})} \leqslant M_5 e^{\omega t}, \qquad \text{for } t \geqslant 0, \tag{4.8}$$

$$|e^{-At}|_{L(Z^{\theta_2}, Z^{\theta_1})} \leqslant M_6 t^{\theta_2 - \theta_1} e^{\omega t}, \qquad \text{for } t > 0. \tag{4.9}$$

As an application of Theorem 3.3, we give the following invariant manifold theorem which generalizes the usual center-unstable manifold theorem (see, for example, [7]):

THEOREM 4.2. *Assume that Hypotheses* B *and* C *are satisfied, $F \in C^k(Z_{\theta_1}, Z_{\theta_2})$ for some integer $k \geqslant 1$, and there is a $\gamma > 0$ such that*

$$\alpha < \gamma \leqslant k\gamma < \beta, \tag{4.10}$$

$$K(\beta - k\gamma, \gamma - \alpha, \theta_1 - \theta_2) \operatorname{Lip}_u F < 1, \tag{4.11}$$

where K is given in Lemma 3.1. Then there exists a unique C^k γ-unstable manifold $\mathscr{W}_{\gamma}^{u}$ for (4.1),

$$\mathscr{W}_{\gamma}^{u} = \{u_0 \mid u(t, u_0) \text{ exists for } t \leqslant 0, u \in E_0^{-}(\gamma, Z^{\theta_1})\}, \tag{4.12}$$

which is the graph of a C^k mapping h: $PZ^{\theta_1} \times \Lambda \to (I-P) Z^{\theta_1}$.

Remark. The gap condition (4.10) always holds if $\alpha \leqslant 0$. Then the γ-unstable manifold becomes the unstable manifold or center-unstable manifold. If the condition (4.10) fails, then the examples given in [9] show that the invariant manifolds lose smoothness even if the nonlinearities are analytic.

Proof. From the definition of γ-unstable manifold we have

$$\mathscr{W}^{\mathrm{u}}_{\gamma} = \{u_0 \mid u(t, u_0) \text{ exists for } t \leqslant 0, u \in E_0^-(\gamma, Z^{\theta_1})\},$$

where $u(t, u_0)$ is a solution of (4.1) with the initial data $u(0, u_0) = u_0$. It is clear that $\mathscr{W}^{\mathrm{u}}_{\gamma}$ is invariant under the flows of (4.1). We want to show that $\mathscr{W}^{\mathrm{u}}_{\gamma}$ is given by the graph of a C^k function over PZ^{θ_1}. First we claim

Claim. $u^0 \in \mathscr{W}^{\mathrm{u}}_{\gamma} \Leftrightarrow u(\cdot) \in E_0^-(\gamma, Z^{\theta_1})$ with $u(0) = u^0$ and satisfies

$$u = e^{-APt}\xi + \int_0^t e^{-AP(t-s)} PF(u, \lambda)\, ds$$
$$+ \int_{-\infty}^t e^{-AQ(t-s)} QF(u, \lambda)\, ds, \tag{4.13}$$

where $Q = I - P$ and $\xi = Pu_0$.

This claim can be easily verified by using the variation of constants formula.

Let $X = Z^{\theta_1}$, $Y = Z^{\theta_2}$, $T(t, s) = e^{-A(t-s)}$, $\tau = 0$, and $P(t) = P$. Then the integral equation (4.13) has the same form as (3.12). It is easy to see that the conditions of Theorem 4.2 are the same as those in Theorem 3.3 in this case. By using Theorem 3.3, we have that for every $(\xi, \lambda) \in PZ^{\theta_1} \times \Lambda$ the integral equation (4.13) has a unique solution $u(\cdot; \xi, \lambda) \in E_0^-(\gamma, Z^{\theta_1})$ which has the property that

$$u\colon X \times \Lambda \to E_0^-(k\gamma, X)$$

is a C^k mapping. Let

$$h(\xi, \lambda) = Qu(0; \xi, \lambda)$$
$$= \int_{-\infty}^0 e^{-AQ(t-s)} QF(u(s; \xi, \lambda), \lambda)\, ds.$$

Then $h\colon PZ^{\theta_1} \times \Lambda \to (I - P)\, Z^{\theta_1}$ is a C^k mapping and satisfies

$$|h(\xi_1, \lambda) - h(\xi_2, \lambda)|_{Z^{\theta_1}}$$
$$\leqslant \frac{M_1 K(\beta - \gamma, \gamma - \alpha, \theta_1 - \theta_2)\, \mathrm{Lip}_u F}{1 - K(\beta - \gamma, \gamma - \alpha, \theta_1 - \theta_2)\, \mathrm{Lip}_u F} |\xi_1 - \xi_2|_{Z^{\theta_1}}. \tag{4.14}$$

Finally, we have that

$$\mathscr{W}^{\mathrm{u}}_{\gamma} = \{\xi + h(\xi, \lambda) \mid \xi \in PZ^{\theta_1}\}.$$

This completes the proof. ▮

As an application of Theorem 3.4, we prove a theorem on invariant foliations of space Z^{θ_1} in such a way that the leaves of the foliation are transverse to the invariant manifold $\mathscr{W}^{\mathrm{u}}_{\gamma}$ in Theorem 4.2. If $F(0, \lambda) = 0$, then the unique leaf that passes through 0 is the stable manifold of (4.1). We also show that this invariant foliation gives us exponential attractivity. Hence, if $\mathscr{W}^{\mathrm{u}}_{\gamma}$ is finite dimensional, then this implies the inertial manifold theorem given in [16].

Let $u(t, u_0)$, $t \geq 0$, be the solution of (4.1) with the initial data u_0. By using Lemma 7.1.1 in [20], we have that

$$|D^i_\eta v(t, \eta)|_{L^i(X,X)} \leq C(i, \rho, F, M_5, M_6)\, e^{(i\omega + (2i-1)\mu)t}, \quad \text{for } i = 1, ..., k, \tag{4.15}$$

where $C(i, \rho, F, M_5, M_6)$ is a positive constant,

$$\mu = (M_6 \operatorname{Lip}_u F\Gamma(1-\rho))^{1/(1-\rho)},$$

and $\Gamma(s)$ is the gamma function.

THEOREM 4.3. *Assume that all the conditions in Theorem* 4.2 *are satisfied. In addition, we assume that there exist* γ, $\max\{0, \alpha\} < \gamma < \beta$, *and* r, $0 < r \leq k-1$, *such that*

$$(\max\{M_1, M_2\} + 1)\, K(\beta - \gamma, \gamma - \alpha, \theta_1 - \theta_2) \operatorname{Lip}_u F < 1, \tag{4.16}$$

$$\gamma - (r\omega + (2r-1)\mu) > 0. \tag{4.17}$$

Then there exists a unique invariant foliation of Z^{θ_1} *whose leaves are* γ*-stable. Moreover, each leaf is given by*

$$\mathscr{W}^{\mathrm{s}}_{\gamma}(\xi + h(\xi, \lambda)) = \{\zeta + h(\xi, \lambda) + \phi(\xi, \zeta, \lambda) \mid \zeta \in QZ^{\theta_1}\},$$

where $\xi \in PZ^{\theta_1}$ *(regarded as an index set),* $\mathscr{W}^{\mathrm{s}}_{\gamma}(\xi + h(\xi, \lambda))$ *is the leaf that passes through* $\xi + h(\xi, \lambda)$, *and* $\phi\colon PZ^{\theta_1} \times QZ^{\theta_1} \times \Lambda \to PZ^{\theta_1}$ *satisfies the following*:

(i) $\phi(\xi, \cdot, \lambda)\colon QZ^{\theta_1} \to PZ^{\theta_1}$ *is* C^k *and* $D^k_\zeta \phi$ *is continuous.*

(ii) $\phi(\cdot, \zeta, \lambda)\colon PZ^{\theta_1} \to PZ^{\theta_1}$ *is* C^r *and* $D^r_\xi \phi$ *is continuous.*

(iii) $D^i_\xi \phi$ *is* C^{k-i-1} *differentiable in* ζ *for* $i \leq r$.

(iv) $\mathscr{W}^{\mathrm{s}}_{\gamma}(\xi + h(\xi, \lambda))$ *intersects* $\mathscr{W}^{\mathrm{u}}_{\gamma}$ *transversely at a unique point.*

Remark. We note that the condition (4.17) holds for $r=k-1$ if the positive real parts of eigenvalues and the Lipschitz constant of the nonlinearity are small enough.

Proof. For any $\eta\in Z^{\theta_1}$ let $u(t,\eta)$ be the solution of (4.1) with the initial data $u(0,\eta)=\eta$. We are looking for all solutions $u(t,\bar\eta)$ of (4.1) which are asymptotically equivalent to $u(t,\eta)$ in the sense that $u(\cdot,\bar\eta)-u(\cdot,\eta)\in E_0^+(\gamma,X)$. In other words, we are looking for

$$\mathscr{W}_\gamma^s(\eta)=\{\bar\eta\in Z^{\theta_1}\mid u(\cdot,\bar\eta)-u(\cdot,\eta)\in E_0^+(\gamma,X)\}.$$

Set $w(t)=u(t,\bar\eta)-u(t,\eta)$. Then w satisfies the equation

$$\frac{dw}{dt}+Aw=F(w+u(t,\eta),\lambda)-F(u(t,\eta),\lambda). \tag{4.18}$$

Using the same arguments as in Theorem 4.4, we have that $w(t,w_0)$, $t\geqslant 0$, is a solution of (4.18), which belongs to $E_0^+(\gamma,Z^{\theta_1})$, if and only if $w(\cdot)\in Z^{\theta_1}$ with $w(0)=w_0=\bar\eta-\eta$ and satisfies the integral equation

$$\begin{aligned} w=e^{-AQt}\zeta&+\int_0^t e^{-AQ(t-s)}Q[F(w+u(s,\eta),\lambda)-F(u(s,\eta),\lambda)]\,ds\\ &+\int_\infty^t e^{-AP(t-s)}P[F(w+u(s,\eta),\lambda)-F(u(s,\eta),\lambda)]\,ds, \end{aligned} \tag{4.19}$$

where $\zeta=Qw(0)$.

Let $X=Z^{\theta_1}$, $Y=Z^{\theta_2}$, $T(t,s)=e^{-A(t-s)}$, $\tau=0$, and $P(t)=P$. Then the integral equation (4.19) has the same form as (3.22). By Theorem 3.4, we have that for every $(\zeta,\eta,\lambda)\in QZ^{\theta_1}\times Z^{\theta_1}\times\Lambda$ the integral equation (4.19) has a unique solution $w(\cdot;\zeta,\eta,\lambda)\in E_0^+(\gamma,Z^{\theta_1})$ which has the following properties:

(i) w is C^k from QZ^{θ_1} to $E_0^+(\gamma,Z^{\theta_1})$ with respect to ζ and $D_\zeta^i w$, $i=1,\ldots,k$, are continuous in all variables.

(ii) w is C^r from Z^{θ_1} to $E_0^+(\gamma,Z^{\theta_1})$ with respect to η and $D_\eta^i w$, $i=1,\ldots,r$, are continuous in all variables.

(iii) $D_\eta^i w$ is C^{k-i-1} in ζ for $i\leqslant r$. Moreover, let

$$\psi(\zeta,\eta,\lambda)=P(w(0;\zeta,\eta,\lambda)+u(0,\eta)). \tag{4.20}$$

Then $\psi(\zeta,\eta,\lambda)$ satisfies

$$\begin{aligned} &|\psi(\zeta_1,\eta,\lambda)-\psi(\zeta_2,\eta,\lambda)|_{\theta_1}\\ &\quad\leqslant\frac{M_3K(\beta-\gamma,\gamma-\alpha,\theta_2-\theta_1)\operatorname{Lip}_uF}{1-K(\beta-\gamma,\gamma-\alpha,\theta_2-\theta_1)\operatorname{Lip}_uF}|\zeta_1-\zeta_2|_{\theta_1}. \end{aligned} \tag{4.21}$$

Since $\bar{\eta} = w(0; \zeta, \eta, \lambda) + u(0, \eta) = P(w(0; \zeta, \eta, \lambda) + u(0, \eta)) + Q(w(0; \zeta, \eta, \lambda) + u(0, \eta)) = \psi(\zeta, \eta, \lambda)$, we have that

$$\mathscr{W}_\gamma^{\mathrm{s}}(\eta) = \{\bar{\eta} \mid \bar{\eta} = \psi(\zeta, \eta, \lambda) + \zeta + Q\eta, \zeta \in QZ^{\theta_1}\}.$$

Furthermore, $u(t, \mathscr{W}_\gamma^{\mathrm{s}}(\eta)) \subset \mathscr{W}_\gamma^{\mathrm{s}}(u(t, \eta))$ for $t \geqslant 0$. This implies that $\mathscr{W}_\gamma^{\mathrm{s}}(\eta)$ gives an invariant foliation of Z^{θ_1}. We claim that $W_\gamma^{\mathrm{s}}(\eta)$ transversely intersects W_γ^{u} at a unique point $\xi + h(\xi, \lambda)$ for some $\xi \in PZ^{\theta_1}$. First if $\bar{\eta} \in \mathscr{W}_\gamma^{\mathrm{s}}(\eta) \cap \mathscr{W}_\gamma^{\mathrm{u}}$, then there exist ζ_0 and ξ_0 such that

$$\bar{\eta} = \psi(\zeta_0, \eta, \lambda) + \zeta_0 + Q\eta = \xi_0 + h(\xi_0, \lambda).$$

This implies that ξ_0 is a solution of $\xi_0 = \psi(h(\xi_0, \lambda) - Q\eta, \eta, \lambda)$. On the other hand, let $g(\xi, \eta, \lambda) = \psi(h(\xi, \lambda) - Q\eta, \eta, \lambda)$. By (4.14) and (4.21), the condition (4.16) implies that

$$\mathrm{Lip}_\xi\, g < 1.$$

Namely, g is a contraction in ξ. By the contraction mapping theorem, we have that $\xi_0 = \psi(h(\xi_0, \lambda) - Q\eta, \eta, \lambda)$ has a unique fixed point ξ_0. This implies that $\mathscr{W}_\gamma^{\mathrm{s}}(\eta)$ transversely intersects $\mathscr{W}_\gamma^{\mathrm{u}}$ at $\xi_0 + h(\xi_0, \lambda)$. It is easy to see that $\mathscr{W}_\gamma^{\mathrm{s}}(\eta) = \mathscr{W}_\gamma^{\mathrm{s}}(\xi_0 + h(\xi_0, \lambda))$. It suffices to consider leaves passing through $\eta = \xi + h(\xi, \lambda)$, indexed by $\xi \in PZ^{\theta_1}$. Let

$$\phi(\xi, \zeta, \lambda) = \psi(\zeta, \xi + h(\xi, \lambda), \lambda)).$$

We have, since $Q\eta = h(\xi, \lambda)$, that

$$\mathscr{W}_\gamma^{\mathrm{s}}(\xi + h(\xi, \lambda)) = \{\phi(\xi, \zeta, \lambda) + \zeta + h(\xi, \lambda) \mid \zeta \in QZ^{\theta_1}\}.$$

This completes the proof. ∎

If the condition $\gamma - (r\omega + (2r-1)\mu) > 0$ is not valid for any $r \geqslant 1$, then we can still show that $\phi(\xi, \zeta, \lambda)$ is Hölder continuous in ξ with a small Hölder exponent.

THEOREM 4.4. *Assume that all the conditions in Theorem 4.2 are satisfied. In addition, assume that*

$$(\max\{M_1, M_3\} + 1)\, K(\beta - \gamma, \gamma - \alpha, \theta_1 - \theta_2)\, \mathrm{Lip}_u F < 1. \tag{4.22}$$

Then the mapping ϕ: $PZ^{\theta_1} \times QZ^{\theta_1} \times \Lambda \to PZ^{\theta_1}$ *satisfies the following*:

(i) $\phi(\xi, \cdot, \lambda)$: $QZ^{\theta_1} \to PZ^{\theta_1}$ *is* C^k *and* $D_\zeta^k \phi$ *is continuous.*

(ii) $\phi(\cdot, \zeta, \lambda)$: $PZ^{\theta_1} \to PZ^{\theta_1}$ *is* C^ε *(Hölder continuous), where* $\varepsilon > 0$ *is a small number.*

Proof. It suffices to show that the solution $w(\cdot; \zeta, \eta, \lambda)$ of (4.19) is ε-Hölder continuous from Z^{θ_1} to $E_0^+(\gamma, Z^{\theta_1})$ in η. Since $K(\beta - \gamma, \gamma - \alpha, \theta_1 - \theta_2) \operatorname{Lip}_u F < 1$, there is a $\delta > 0$ such that $\gamma + \delta < \beta$ and

$$K(\beta - \gamma - \delta, \gamma + \delta - \alpha, \theta_1 - \theta_2) \operatorname{Lip}_u F < 1.$$

Using the contraction mapping theorem, we have that $w(\cdot; \zeta, \eta, \lambda) \in E_0^+(\gamma + \delta, Z^{\theta_1})$. For each $\eta_0, \eta_1 \in Z^{\theta_1}$ we have that

$$\begin{aligned}
&|e^{\gamma t}(w(t; \zeta, \eta_1, \lambda) - w(t; \zeta, \eta_0, \lambda)|_{\theta_1} \\
&= \Bigg| e^{\gamma t} \bigg\{ \int_0^t e^{-AQ(t-s)} Q[F(w(s; \zeta, \eta_1, \lambda) + u(s, \eta_1), \lambda) - F(u(s, \eta_1), \lambda) \\
&\quad - F(w(s; \zeta, \eta_0, \lambda) + u(s, \eta_0), \lambda) + F(u(s, \eta_0), \lambda)]\, ds \\
&\quad + \int_\infty^t e^{-AP(t-s)} P[F(w(s; \zeta, \eta_1, \lambda) + u(s, \eta_1), \lambda) - F(u(s, \eta_1), \lambda) \\
&\quad - F(w(s; \zeta, \eta_0, \lambda) + u(s, \eta_0), \lambda) + F(u(s, \eta_0), \lambda)\, ds \bigg\} \Bigg|_{\theta_1} \\
&\leqslant M_4 \operatorname{Lip}_u F \int_0^t (t-s)^{\theta_2 - \theta_1} e^{\gamma t - \beta(t-s) - \gamma s} |w(\cdot; \zeta, \eta_1, \lambda) \\
&\quad - w(\cdot; \zeta, \eta_0, \lambda)|_{E_\tau^+(\gamma, Z^{\theta_1})}\, ds \\
&\quad + M_2 \operatorname{Lip}_u F \int_t^\infty e^{\gamma t - \alpha(t-s) - \gamma s} |w(\cdot; \zeta, \eta_1, \lambda) \\
&\quad - w(\cdot; \zeta, \eta_0, \lambda)|_{E_\tau^+(\gamma + \delta, Z^{\theta_1})}\, ds \\
&\quad + 2M_4 \operatorname{Lip}_u F \int_0^t (t-s)^{\theta_2 - \theta_1} e^{\gamma t - \beta(t-s) - (\gamma + \delta)s} \\
&\quad \times |w(\cdot; \zeta, \eta_0, \lambda)|_{E_\tau^+(\gamma, Z^{\theta_1})}^{1-\varepsilon} |u(s, \eta_1) - u(s, \eta_0)|_{\theta_1}^\varepsilon\, ds \\
&\quad + 2M_2 \operatorname{Lip}_u F \int_t^\infty e^{\gamma t - \alpha(t-s) - (\gamma + \delta)s} \\
&\quad \times |w(\cdot; \zeta, \eta_0, \lambda)|_{E_\tau^+(\gamma, Z^{\theta_1})}^{1-\varepsilon} |u(s, \eta_1) - u(s, \eta_0)|_{\theta_1}^\varepsilon\, ds.
\end{aligned}$$

Taking $\varepsilon > 0$ such that $\varepsilon(\gamma + \omega + \mu) \leqslant (1-\varepsilon)\,\delta$ and using (4.16), we have that

$$\begin{aligned} &|w(t; \zeta, \eta_1, \lambda) - w(\cdot; \zeta, \eta_0, \lambda)|_{E_\tau^+(\gamma, Z^{\theta_1})} \\ &\quad \leqslant \frac{C(1, \rho, F, M_5, M_6)\, K(\beta-\gamma, \gamma-\alpha, \theta_1-\theta_2)\, \mathrm{Lip}_u F}{1 - K(\beta-\gamma, \gamma-\alpha, \theta_1-\theta_2)\, \mathrm{Lip}_u F} \\ &\qquad \times |w(\cdot; \zeta, \eta_0, \lambda)|^{1-\varepsilon}_{E_\tau^+(\gamma, Z^{\theta_1})}\, |\eta_1 - \eta_0|^{\varepsilon}_{\theta_1}. \end{aligned}$$

This completes this proof. ▮

Remark. One can apply Theorems 3.3 and 3.4 to get invariant manifolds and invariant foliations around periodic orbits, homoclinic orbits, and heteroclinic orbits for evolutionary equations.

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