TRAVELING WAVE SOLUTIONS IN COUPLED CHUA'S CIRCUITS, PART II: CHAOTIC SOLUTIONS

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ABSTRACT. This paper studies a singularly perturbed system of partial differential equations which is an approximation for a system of coupled Chua's circuits. The PDE system is a natural generalization of the FitzHugh-Nagumo's equation and exhibits more complicated traveling wave solutions than the FN equation. First, we show that in the traveling coordinate, the system can have a pair of heteroclinic orbits that form a closed loop connecting two equilibrium points. The dominant eigenvalues of the equilibrium points are complex numbers. Using the idea of Silnikov [33], we show that in a neighborhood of the heteroclinic loop, all the solutions are one-to-one correspond to to two sequence of symbols. Thus there are infinitely many homoclinic, heteroclinic, periodic and chaotic orbits nearby.

First asymptotic method is used in the singularly perturbed system to construct solutions in singular and regular layers. Then dynamical systems method is used to obtain the exact solutions near the approximations obtained by the formal method. Moreover, we obtain chaotic solutions for this system based on a pair of heteroclinic solutions by a similar method as I obtain the periodic solutions.

1. Introduction and Preliminary

Complex dynamical networks are everywhere, such as the Internet, power networks, neural networks, literature search networks, etc. With the development of the engineering and biological sciences, it becomes more and more important to understand the complicated behavior of the dynamics of the coupled networks.

Chua's circuit is a simple electronic circuit that can have sophisticated behaviors like traveling wave solutions, periodic and chaotic solutions. This circuit consists of a nonlinear resistor N_R, linear inductor L, resistor R, and capacitors C_1, C_2. See Figure 1.1.

The coupled Chua's Circuit is an example of CNNs, as described by Chua in his book [6]. We notice that our system is one of the simplest generalizations of FitzHugh-Nagumo equation, which is a second order bistable PDE coupled with linear first order ODE and has periodic solutions in certain situation. The slow system we consider has two complex eigenvalues while in FitzHugh-Nagumo's system the one-dimensional slow system has only one real eigenvalue.

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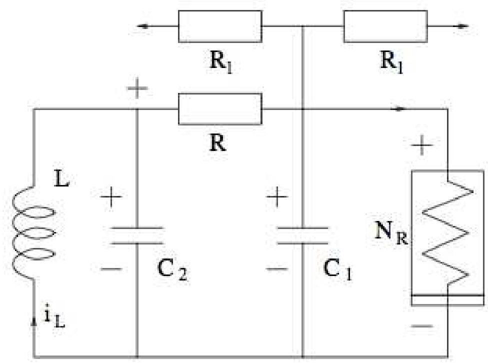

FIGURE 1.1. An 1D array of Chua's Circuit

In this paper, we consider an array of Chua's circuits connected by resistors R_1. We use k as the index for the kth circuit, we have a system of equations as:

$$\begin{aligned} C_1\frac{dV^k_{C_1}}{dt} &= (V^k_{C_2} - V^k_{C_1})/R - G(V^k_{C_1}) - (V^{k-1}_{C_1} - 2V^k_{C_1} + V^{k+1}_{C_1})/R_1 \\ C_2\frac{dV^k_{C_2}}{dt} &= (V^k_{C_1} - V^k_{C_2})/R + i^k_L \\ L\frac{di^k_L}{dt} &= -V^k_{C_2} \end{aligned}$$

where G is the conductance of Chua's diode. By change of variables, the above system can be transformed into the following dimensionless form, which we rewrite for each circuit cell k as:

$$\begin{aligned} \dot{u}_k &= \alpha(y_k - h(u_k)) + \bar{D}(u_{k-1} - 2u_k + u_{k+1}) \\ \dot{y}_k &= u_k - y_k + z_k \\ \dot{z}_k &= -\beta y_k \end{aligned} \tag{1.1}$$

where $k = 1, 2 \cdots l$, $u_0(t) = u_1(t), u_l(t) = u_{l+1}(t)$. h is defined as the nonlinear function $h(u) = mu(u+c)(u-c), c > 0, m > 0$ in this paper. We mainly consider the original system with the same scaling as P.P.C. in this paper, see [31] Let $\epsilon = 1/\alpha, \delta x = \sqrt{\epsilon}, u_k(\bar{t}) = u(\bar{t}, k\Delta x)$. We approximate (1.1) by the following PDE:

$$\begin{aligned} \epsilon u_{\bar{t}} &= (y - h(u)) + \epsilon^2 D u_{xx} \\ y_{\bar{t}} &= u - y + z \\ z_{\bar{t}} &= -\beta y \end{aligned} \tag{1.2}$$

Let the traveling wave solution be $w = w(x - s\bar{t}) = w(t)$, where $w = (u, v, y, z)$ and $\dot{w} = dw/dt$, we have the so called slow system, in which t is the slow time scale:

$$\begin{aligned} \epsilon\dot{u} &= v/D \\ \epsilon\dot{v} &= h(u) - sv/D - y \\ -s\dot{y} &= u - y + z \\ -s\dot{z} &= -\beta y \end{aligned} \tag{1.3}$$

Let $\tau = t/\epsilon$ and $w' = dw/d\tau$, then we have the so called fast system, in which τ is the fast variable:

$$\begin{aligned} u' &= v/D \\ v' &= h(u) - sv/D - y \\ sy' &= \epsilon(-u + y - z) \\ sz' &= \epsilon\beta y \end{aligned} \tag{1.4}$$

with equilibria $P_\pm = (u, v, y, z) = (u_\pm, 0, 0, -u_\pm)$, $P_0 = (0, 0, 0, 0)$.

In order to analyze the 0th order asymptotic solutions, we obtain the reduced limiting problem of the dynamics of (1.3), (1.4) when $\epsilon = 0$ on the slow and fast time scales, respectively:

$$\begin{aligned} 0 &= v \\ 0 &= h(u) - y \\ -s\dot{y} &= u - y + z \\ -s\dot{z} &= -\beta y \end{aligned} \tag{1.5}$$

$$\begin{aligned} u' &= v/D \\ v' &= h(u) - sv/D - y \\ y' &= 0 \\ z' &= 0 \end{aligned} \tag{1.6}$$

Acosta considered the case when $\beta < 0$ in [1], in which saddle points on the slow manifolds are studied, while in reality β should be positive as we study here. We can show that for a suitable wave speed s and β value, there exist unique traveling wave solution connecting the equilibria $P_\pm$. The idea of such internal layer solution is not new in singular perturbation methods [25], [21], [20], [17], [8]. They are based on singular perturbation and heteroclinic bifurcations. There are also geometric singular perturbation methods, such as: [9], [10], [16]. There are works about the periodic orbits and aperiodic orbits for a single uncoupled circuit. For example, in [13] the Brouwer's fixed-point theorem is applied to prove the existence of periodic solution for a Chua's circuit with smooth nonlinearity. In this paper, we use analytical method in singular perturbation and obtain different types of solutions for the systems (1.3), (1.4).

1.1. **Exponential dichotomies.** We introduce some properties of exponential dichotomies of linear systems of differential equations. Consider the linear homogeneous differential equation:

$$\dot{x} = A(t)x \tag{1.7}$$

Here $A : I \to R^{n\times n}$ is continuous, where $I \in R$ is a finite or infinite interval. Let $\Phi(t, s)$ be the principal matrix solution of (1.7).

Definition 1.1. We say that (1.7) has an exponential dichotomy on I if there exist positive constants K, α and projections $P_s(t) + P_u(t) = I_n$ such that for $t, s \in I$ we have:
(i)$\Phi(t, s)P_s(s) = P_s(t)\Phi(t, s)$
(ii)$|\Phi(t, s)P_s(s)| \le Ke^{-\alpha}(t - s), s \le t$
(iii)$|\Phi(t, s)P_u(s)| \le Ke^{-\alpha}(s - t), t \le s$

2. Formal traveling wave solutions

We can see that the first two equations of (1.5) $v = 0, u = h^{-1}(y)$ give us the three slow manifolds:

$$\begin{aligned} S_- &= \{w : v = 0, y < y_M, u = h_-^{-1}(y)\} \\ S_0 &= \{w : v = 0, y_m < y < y_M, u = h_0^{-1}(y)\} \\ S_+ &= \{w : v = 0, y > y_m, u = h_+^{-1}(y)\} \end{aligned}$$

which consists of the equilibrium points of the fast system (1.4). On these manifolds, we have the equations for the y-z variables:

$$\begin{aligned} -s\dot{y} &= h^{-1}(y) - y + z \\ -s\dot{z} &= -\beta y \end{aligned}$$

Let $h^{-1}(y) - y = ky + c, k(y) = 1/h'(u) - 1 = 1/h'(h^{-1}(y)) - 1$, where c is a constant, so that we have:

$$\begin{aligned} \dot{y} &= -(ky + c + z)/s \\ \dot{z} &= \beta y/s \end{aligned}$$

with equilibria $P_{Y\pm} = (0, \mp c)$. The characteristic polynomial for y is:

$$s^2r^2 + ksr + \beta = 0.$$

Notice that the product of two roots of the characteristic polynomial is $\beta/s^2 > 0$, $h'(u) > 0$ on $S_\pm$. Therefore, $k > 0$ if $0 < h'(u) < 1$ and $-1 < k < 0$ if $h'(u) > 1$. Thus we have:
Case 1: $P_\pm$ are stable spirals if $4\beta > k^2, sk > 0$.
Case 2: $P_\pm$ are unstable spirals if $4\beta > k^2, sk < 0$.
Case 3: $P_\pm$ are stable nodes if $4\beta < k^2, sk > 0$.
Case 4: $P_\pm$ are unstable nodes if $4\beta < k^2, sk < 0$.

We construct a fast heteroclinic solution in between $P_{U\pm} = (\pm c, 0)$ when $\epsilon = 0$. Consider the fast flow on the u-v plane in (1.6), where we have constants y, z. Plug the constant y,z values into the first two equations of (1.6), we have a system for the fast heteroclinic flow:

$$\begin{aligned} u_0' &= v_0/D \\ v_0' &= -sv_0/D - \bar{y}_0 + h(u_0) \end{aligned} \tag{2.1}$$

which can be rewritten as a second order differential equation of u:

$$Du'' + su' - h(u) + y = 0$$

with the characteristic equation $Dr^2 + sr - h'(u) = 0$, whose determinant $\Delta = s^2 + 4Dh'(u)$ is positive on $S_\pm$. So we have two eigenvalues with opposite signs, i.e. we have fast heteroclinic solution from saddle to saddle.

Lemma 2.1. *There is a $s_0 > 0$ such that for $0 \le s \le s_0$, system (2.1) has a unique heteroclinic solution A $(\hat{u}, \hat{v})$ connecting $(u_0^-(y), 0)$ to $(u_0^+(y), 0)$ for $y = y_0^-(s)$, and $\int_{u_0^-(y_0^-(0))}^{u_0^+(y_0^-(0))} [h(u) - y_0^-(0)] = 0$. Also system (2.1) has a unique heteroclinic solution B connecting $(u_0^+(y), 0)$ to $(u_0^-(y), 0)$ for $y = y_0^+(s)$.*

Proof. When $s = 0$, integrate the Hamiltonian system (**??**) so that $\int_{u_0(-\infty)}^{u_0(\infty)} [h(u_0) - \bar{y}_0] = 0$ and we can solve for a unique heteroclinic solution $(\hat{u}, \hat{v})$ connecting $(u_0(-\infty), 0)$ and $(u_0(\infty), 0)$. Similarly, there is a unique heteroclinic solution B connecting $(u_0(\infty), 0)$ and $(u_0(-\infty), 0)$. We can verify that $(\hat{u}', \hat{v}')$ is a solution of the linear variational system of the u-v equation of (1.6):

$$\begin{aligned} \Phi_1' &= \Phi_2/D \\ \Phi_2' &= h'(\hat{u})\Phi_1 - s\Phi_2/D \end{aligned} \tag{2.2}$$

We can show that (s, y) has to satisfy a bifurcation function $g(s, y) = 0$, whose solution near $(\hat{s}, \hat{y})$ corresponds to a unique heteroclinic solution near $(\hat{u}, \hat{v})$. Moreover, we can compute that:

$$\frac{\partial g(s,y)}{\partial s} = \int_{-\infty}^{\infty} [\hat{v}(\tau)]\Psi_2(\tau) d\tau \tag{2.3}$$

where (Ψ_1, Ψ_2) is the unique bounded solution to the adjoint system of (2.2). Also we can find out:

$$\frac{\partial g(s,y)}{\partial s} = \int_{-\infty}^{\infty} [\hat{v}(\tau)]^2 e^{-s\tau} d\tau > 0 \tag{2.4}$$

Therefore, $g(s, y) = 0$ has a solution $y = y_0(s)$ locally, with $\frac{\partial y_0}{\partial s} < 0$ on heteroclinic solution A, and $\frac{\partial y_0}{\partial s} < 0$ on heteroclinic solution B when $s > 0$, we can complete the proof of the Lemma by the continuous dependence of homotopy continuation, starting from the case when $s = 0$. □

2.1. **Formal expansion of ϵ^0th order.** Based on the analysis of the limiting slow and fast system, we look for a formal solution q for (1.5), (1.6) from equilibrium to equilibrium when $\epsilon = 0$, in the form of "Slow $\xrightarrow{\text{Fast}}$ Slow" type. Here $\longrightarrow$ means the heteroclinic solution on fast flow. For example:

2.1.1. *β large case($4\beta > k^2$), $s > 0$.* P_-(stable spiral) $\xrightarrow{\text{saddle to saddle}}$ P_+ (stable spiral). There exists a unique traveling wave solution if $4\beta > k^2, k > 0$. See Figure 2.1. In

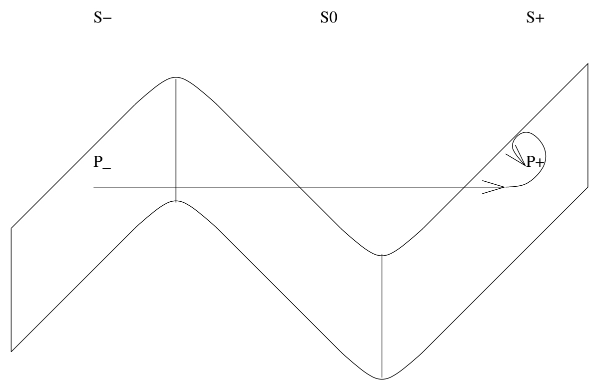

FIGURE 2.1. Traveling wave solution with positive wave speed and stable spirals on $S_\pm$

singular layer, $q_i(\tau) = (U_i(\tau), Y_i(\epsilon\tau))$, where $\tau \in [-\infty, \infty], i = 1$;
In regular layer, $q_i(t) = (U_i(t/\epsilon), Y_i(t))$, where $t \in [\alpha_i, \beta_i] = [0, \infty], i = 2$.
$q_i(\tau)$ satisfies the boundary conditions:

$$\lim_{\tau\to-\infty} q_1(\tau) = P_-, \lim_{\tau\to\infty} q_2(t) = P_+$$

and the matching conditions for the two solutions connected at the intermediate point P_m:

$$\lim_{\tau\to\infty} q_1(\tau) = q_2(0) = P_m$$

Notice that $Y_1(\tau)$ is constant and because of the above matching condition, we have $Y_2(0) = (0, 3)$ as the initial condition for $Y_2(t)$. Because the two equilibrium points for the slow system are the stable spirals around $P_\pm$ if $4\beta > k^2, sk > 0$, we have $Y_2(0) \in W^s((0, -3))$, the boundary condition is satisfied. The existence of the fast solution U_1 that connects the slow flows and the other equilibrium point is guaranteed as in Lemma 2.1. There are three other similar cases of the formal solutions from equilibrium to equilibrium when $\epsilon = 0$.

3. Construction of Approximated solution for Traveling Wave Solution

We divide the entire domain as the following to investigate the traveling wave solution w_j when ϵ gets sufficiently small. Let $I_i = \{t | t \in [\alpha_i, \beta_i]\}$, I_{2l+1} are singular layers, I_{2l} are regular layers. For any $m \geq 0, 0 < \lambda < 1$(in this paper, we have $m = 0$). We define the approximation when $\epsilon > 0$ of order ϵ^m to be:

$$w_{ap} = \begin{cases} \sum_{j=0}^m \epsilon^j w_j^{out}(t) & t \in [\alpha_i, \beta_i] = [\epsilon^\lambda, \infty], i = 2 \\ \sum_{j=0}^m \epsilon^j w_j^{in}(\tau) & \tau \in [\alpha_i, \beta_i] = [-\infty, \epsilon^{\lambda-1}], i = 1 \end{cases}$$

3.1. **Jump errors.** For any $m \geq 0$, estimates for the jump error $J_{wi} = w_{ap}(\alpha_{i+1}, \epsilon) - w_{ap}(\beta_i, \epsilon) = O(\epsilon^{(m+1)\lambda})$ can be obtained by comparing outer and inner approximations with the inner expansion of outer layers.

3.2. **Residual errors.** In the outer layers, the residual errors $RE^i(t), i = 2$ is defined by:

$$RE^i(t) = \begin{pmatrix} \epsilon \dot{u}_{ap}(t) - v_{ap}(t)/D \\ \epsilon \dot{v}_{ap}(t) - s v_{ap}(t)/D + y_{ap}(t) - h(u_{ap}(t)) \\ -s\dot{y}_{ap}(t) - u_{ap}(t) + y_{ap}(t) - z_{ap}(t) \\ -s\dot{z}(t) + \beta y_{ap}(t) \end{pmatrix} = \begin{pmatrix} O(\epsilon^{m+1}) \\ O(\epsilon^{m+1}) \\ 0 \\ 0 \end{pmatrix}$$

In the inner layers, the residual errors $RE^i(\tau), i = 1$ is defined by:

$$RE^i(\tau) = \begin{pmatrix} u'_{ap}(\tau) - v_{ap}(\tau)/D \\ v'_{ap}(\tau) - s v_{ap}(t)/D + y_{ap}(t) - h(u_{ap}(t)) \\ s y'_{ap}(\tau) - \epsilon[-u_{ap}(\tau) + y_{ap}(\tau) - z_{ap}(\tau)] \\ s z'_{ap}(\tau) + \epsilon\beta y_{ap}(\tau) \end{pmatrix} = \begin{pmatrix} O(\epsilon^{(m+1)\lambda}) \\ O(\epsilon^{(m+1)\lambda}) \\ O(\epsilon^{m\lambda+1}) \\ O(\epsilon^{m\lambda+1}) \end{pmatrix}$$

4. Existence of exact traveling wave solutions

4.1. **Linear variational system for the correction function.** In order to find the linear variational system for the correction function w, such that $w^{ex} = w^{ap} + w$, we define the residual error:

$$p^i(\tau) = U^{iap'} - F(U^{iap}, Y^{iap}, s)$$
$$q^i(t) = \dot{Y}^{iap} - G(U^{iap}, Y^{iap}, s)$$

with estimates:

$$(4.1) \qquad \begin{aligned} p^i(\tau) &= O(\epsilon^\lambda) \quad i = 1, 3, & p^i(\tau) &= O(\epsilon), \quad i = 2, 4 \\ q^i(t) &= O(\epsilon^0) \quad i = 1, 3, & q^i(t) &= O(\epsilon), \quad i = 2, 4 \end{aligned}$$

Moreover $|q^i(t)|_{L_1(-\epsilon^\lambda, \epsilon^\lambda)} = O(\epsilon^\lambda), i = 1, 3$. We obtain the system of equations that the correction functions $w = (U, Y)$ satisfy:

$$U'(\tau) = F(U^{ex}, Y^{ex}, s) - [F(U^{ap}, Y^{ap}, s) + p(\tau)]$$
$$\dot{Y}(t) = G(U^{ex}, Y^{ex}, s) - [G(U^{ap}, Y^{ap}, s) + q(t)]$$

which can be rewritten as:
(i) For $i = 2l$,

$$(4.2) \qquad \begin{aligned} U'_i(\tau) &= F_U U_i + F_Y Y_i - p_i(\tau) + (F^{ex} - F^{ap} - F_U U_i - F_Y Y_i) \\ &= F_U U_i + F_Y Y_i + \bar{P}_i = F_U V_i + \bar{P}_i \\ \dot{Y}_i(t) &= G_U U_i + G_Y Y_i - q_i(t) + (G^{ex} - G^{ap} - G_U U_i - G_Y Y_i) \\ &= G_U U_i + G_Y Y_i + \bar{Q}_i \end{aligned}$$

After change of variable: $V_i(\tau) = U_i(\tau) + F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau)Y_i(\epsilon\tau), i = 1, 2, 3, 4$, where $F_U(\epsilon\tau) = F_U(U_{ap}(\epsilon\tau), Y_{ap}(\epsilon\tau))$. We further reduce the above equations to be:

$$
\begin{aligned}
V_i'(\tau) &= F_U(\epsilon\tau)V_i(\tau) + \epsilon\frac{d}{dt}[F_U^{-1}F_Y Y] + \bar{P}_i = F_U(\epsilon\tau)V_i(\tau) + \bar{\bar{P}}_i \\
\dot{Y}_i &= (G_Y - G_U F_U^{-1}F_Y)Y_i + G_U V_i + \bar{Q}_i
\end{aligned} \tag{4.3}
$$

where

$$
\begin{aligned}
\bar{Q}_i &= -q_i(t) + (G^{ex} - G^{ap} - G_U U - G_Y Y) \\
&= G(U^{ex}, Y^{ex}, s) - G(U^{ap}, Y^{ap}, s) - D_w G(U^{ap}, Y^{ap}, s)w(t) \\
&+ [D_w G(U^{ap}, Y^{ap}, s, \epsilon) - D_w G(U^{ap}, Y^{ap}, s, \epsilon = 0)]w(t) - q_i(t) \\
&= O(|w_i|^2 + \epsilon|w_i| + |q_i(t)|)
\end{aligned}
$$

Similarly, $\bar{P}_i = -p^i(\tau) + (F^{ex} - F^{ap} - F_U U - F_Y Y) = O(|w_i|^2 + \epsilon|w_i| + |\epsilon q^i(t)| + |p^i|)$

$$
\begin{aligned}
\bar{\bar{P}}_i &= \bar{P}_i + \epsilon\frac{d}{dt}[F_U^{-1}F_Y Y] \\
&= \bar{P}_i + \epsilon(F_U^{-1}(\epsilon\dot{\tau})F_Y(\epsilon\tau))Y + \epsilon(F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau))\dot{Y} \\
&= \bar{P}_i + \epsilon|w| + \epsilon O(G_U U + G_Y Y + \bar{Q}_i) \\
&= \bar{P}_i + \epsilon|w| + \epsilon O(\bar{Q}_i) = \bar{P}_i + \epsilon\bar{Q}_i = O(\bar{P}_i)
\end{aligned}
$$

Theorem 4.1. *Let $\bar{P}_i, \bar{Q}_i$ be continuous and bounded in I^i satisfying:*

$$\bar{P}^1(-\infty) = \bar{Q}^1(-\infty) = 0, \bar{P}^2(\infty) = \bar{Q}^2(\infty) = 0$$

Consider system 4.2, with jump conditions $\bar{J}_{wi} = (\bar{J}_{Ui}, \bar{J}_{Yi}), i = 1$:

$$Y_2(\epsilon^\lambda) - Y_1(\epsilon^\lambda) = -\bar{J}_{Y1} \tag{4.4}$$

$$U_2(\epsilon^{\lambda-1}) - U_1(\epsilon^{\lambda-1}) = -\bar{J}_{U1} \tag{4.5}$$

and boundary conditions:

$$U^1(-\infty) = Y^1(-\infty) = 0, U^2(\infty) = Y^2(\infty) = 0 \tag{4.6}$$

Then there exists unique traveling wave solutions (U_i, Y_i) in $I^i, i = 1, 2$, to the linear variational system (4.2)with estimate:

$$\Sigma_{i=1}^2|U_i| + \Sigma_{i=1}^2|Y_i| \le C(|\bar{P}_1| + |\bar{P}_2| + |\bar{Q}_2| + |\bar{Q}_1|_{L_1} + |\bar{J}_{w1}|)) \tag{4.7}$$

We outline the proof of the above theorem. We use superposition principle to find the solution as the sum of two solutions, one with no non-homogeneous terms $\bar{P}_i, \bar{Q}_i$, the other one with no boundary or jump conditions. We use iteration method to deal with the residual/jump errors in obtaining the two solutions. For technical details, please refer to [15].

Theorem 4.2. *Suppose (U_i^{ap}, Y_i^{ap}) is given as the approximation solution that satisfies (1.5), (1.6), and boundary condition* (4.6). *Then there exists a unique exact solution w^{ex} of (1.3), (1.4) with boundary condition* (4.6) *such that*

$$|U^{ex} - U^{ap}| + |Y^{ex} - Y^{ap}| = O(\epsilon^\lambda). \tag{4.8}$$

We can prove this theorem by contraction mapping. For technical details, please refer to [15].

5. Chaotic Solutions

In this chapter we construct a solution near a pair of heteroclinic solutions, the solution spirals around the two equilibrium points with prescribed number of rotations on $S_\pm$ on the z-y plane.

5.1. **A pair of heteroclinic solutions when $\epsilon = 0$.** Based on the heteroclinic solution q we obtained in section 3.1, we have a pair of heteroclinic solutions when $\epsilon = 0$ by symmetry. See Figure 5.1. If we project the solution in Figure 5.1 onto the

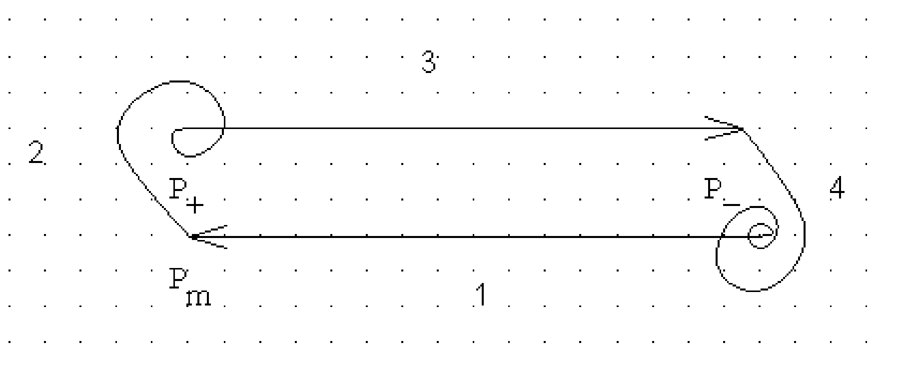

FIGURE 5.1. A pair of heteroclinic solutions when $\epsilon = 0$

y-z plane, we see that the stable spirals start from one equilibrium and approach the other one as $t \to \infty$. Recall that we only consider the two slow flows on $S_\pm$ within the two foldlines $y = y_m$ and $y = y_M$. We observe from Figure 5.2 that there are solutions starting at a point within the two foldlines(say P_-), but do not approach equilibria points $P_\pm$(say P_+) as $t \to \pm\infty$. In order to address this issue, we consider the *domain of influence* of $P_\pm$(or *domain of attraction* of $P_\pm$), that contains all the initial points (y_0, z_0) passing which the solutions exist for all $t \leq 0$(or for all $t \geq 0$), and will approach $P_\pm$ as $t \to -\infty$(or as $t \to \infty$). See Figure 5.2, the domain of attraction of stable spirals on $S_\pm$ is bounded by the two foldlines.

Theorem 5.1. *For different values of β, equilibrium $P_\pm$ are stable spirals if $4\beta > k^2, sk > 0$(unstable spirals if $4\beta > k^2, sk < 0$). Moreover,*
for $s > 0, t > 0$, the contour of domain of attraction of stable spirals on $S_\pm$ is the solution with initial point at $A_\pm = (\pm(2\sqrt{3}c^3m/9 + c/\sqrt{3}), \pm(2\sqrt{3}c^3m/9))$ in the z-y plane;
for $s < 0, t < 0$, the contour of domain of influence of unstable spirals on $S_\pm$ is the solution with initial point at $A_\pm = (\pm(2\sqrt{3}c^3m/9 + c/\sqrt{3}), \pm(2\sqrt{3}c^3m/9))$ in the z-y plane.

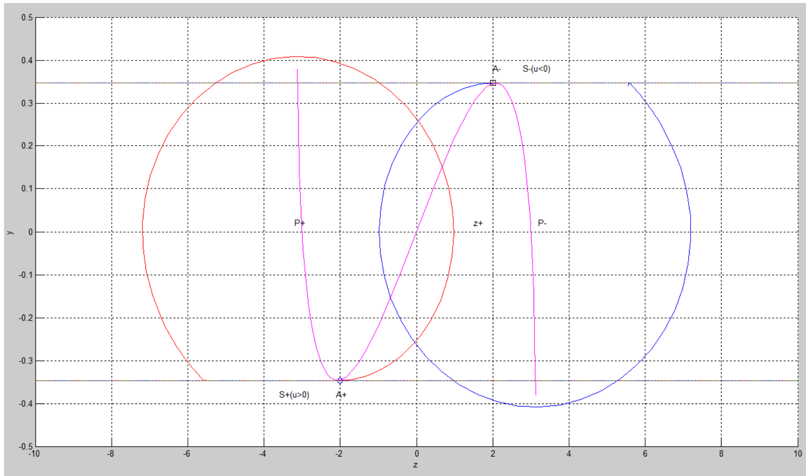

FIGURE 5.2. Domain of attraction of stable spirals on $S_\pm$

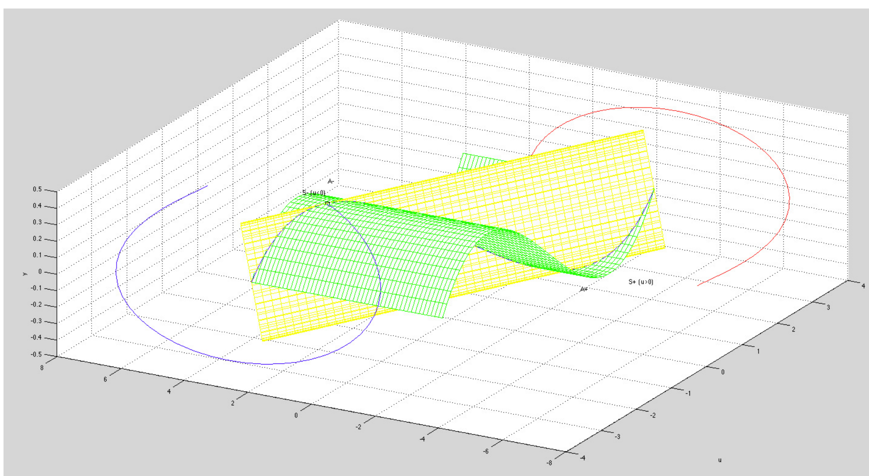

FIGURE 5.3. Domain of influence of unstable spirals on $S_\pm$ in 3D

Proof. We observe from Figure 5.2 that the vector field is horizontal on the initial points $A_\pm$, where the solution is tangent to the foldlines. See Figure 5.3 for 3D view. Take a point to the left of B_+ on the foldline $y = y_m$, the vector field points left and up. But the solution starts there will go back down and hit the foldline $y = y_m$, it can't go across the foldline, therefore will not be able to go towards P_+. To the right of A_+ on the foldline $y = y_m$, the vector field points left and down. In fact, when $u = -c/\sqrt{3}, y = -2\sqrt{3}c^3m/9, z > -2\sqrt{3}c^3m/9 + c/\sqrt{3}$, we have $-s\dot{y} = u - y + z > 0$, therefore $\dot{y} < 0$ for $s > 0$, so the solutions starting there will go below the foldlines when $t > 0$ and can't approach P_+ as $t \to \infty$. These indicate that the solutions start on $A_\pm$ are the contours of domain we want.

□

Notice that it is possible that P_- is not contained in the domain of attraction of P_+. Therefore, we need the following theorem to guarantee the existence of $q(\tau)$.

Theorem 5.2. *The existence of the heteroclinic solution $q(\tau)$ from an equilibrium point P_- connected by fast flow to the stable spirals towards P_+ is guaranteed, if and only if the equilibrium point P_- is contained in the domain of attraction on S_+ in between the foldlines when projected in y-z plane. There is a critical value $\beta = \beta_0$ when the contours of domain of attraction(stable spirals) exactly hit $P_\pm$, See figure 5.5. Also when $\beta \geq \beta_0$ one equilibrium point is contained in the domain of attraction of the other equilibrium point. See Figure 5.4.*

Proof. Consider $\beta > k^2/4, sk > 0$ where we have stable spirals on $S_\pm$, also consider $\frac{dy}{dz} = \frac{u-y+z}{-\beta y}$ in the region $D = \{(z,y)|u-y+z > 0, y < 0\}$ such that $\frac{dy}{dz}$ decreases when β increases on the point (z,y). Therefore we compare the differential equations $\frac{dy_i}{dz} = f_i(\beta_i, z), i = 1, 2$ with different vector fields, by comparison theorem in [2] we have: if $\beta_2 > \beta_1$, then $f_2(\beta_2, z) < f_1(\beta_1, z)$ and $y_2 < y_1$. So if the domain of attraction doesn't contain P_-, we can increase β value continuously so that the contour of the domain of attraction around P_+ hits the z-axis exactly at P_- when $\beta = \bar{\beta}$. Now we take $\beta_0 = max\{\bar{\beta}, k^2/4\}$, then when $\beta \geq \beta_0$ one equilibrium point is contained in the domain of attraction of the other equilibrium point. □

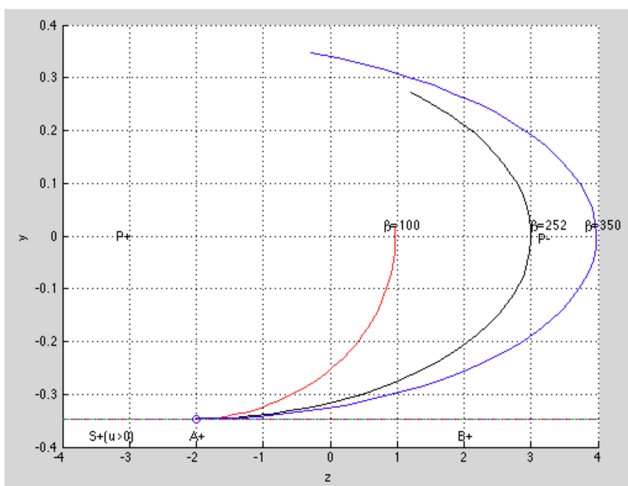

FIGURE 5.4. The domain of attraction becomes larger as β increases

Now that we have the pair of heteroclinic solutions as in Figure 5.1, we can obtain an ordered sequence of heteroclinic solutions $\bar{q}_k(\tau), k \in Z$ in between $P_\pm$ by repeating the pair of heteroclinic solutions.

6. EXISTENCE OF EXACT CHAOTIC SOLUTION WHEN $\epsilon > 0$

6.1. **Set up and Jump conditions.** We first define the counting surface to be Z_c, the z-axis, in order to keep track of the intersections of the solutions with Z_c around $P_\pm$. Based on the heteroclinic solution $\bar{q}_k(\tau)$ obtained from the previous section when

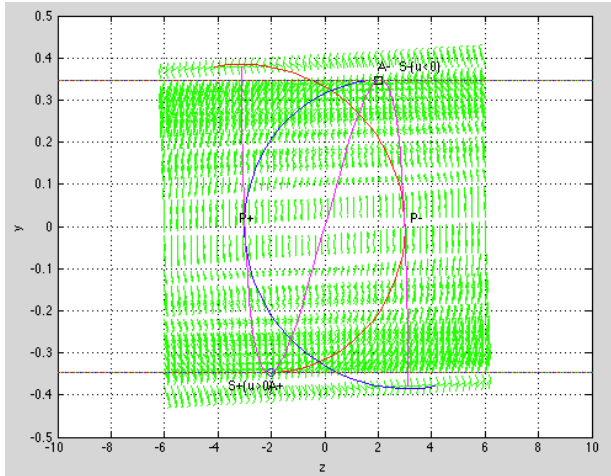

FIGURE 5.5. Stable spirals around one equilibrium that exactly hit the other equilibrium at $P_\pm$

$\epsilon = 0$, we define $Q_i(Q_j), i \geq \bar{i}, j \geq \bar{j}$ to be the points, where the heteroclinic solution $\bar{q}_k(\tau)$ intersects with Z_c for the i-th time, and are close enough to $P_-(P_+)$. See Figure 6.1.

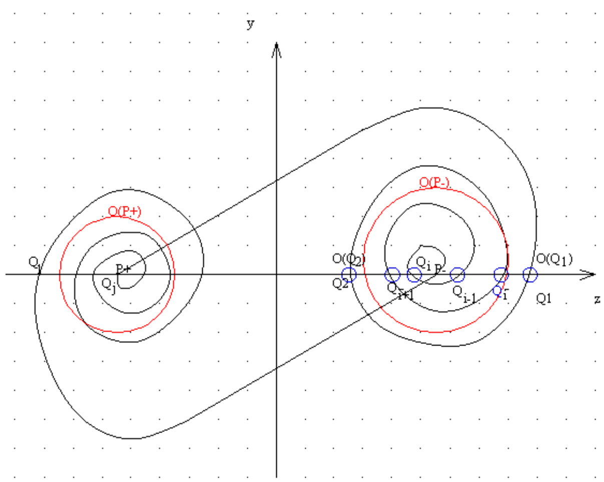

FIGURE 6.1. The heteroclinic solution $\bar{q}_k$ intersects with the Z_c at points Q_i on S_- for the i-th time

Given symbol (j, i), we define a family of approximated solutions $q_{ji}(t)$ that intersect with the Z_c exactly i times, $q_{ji}(t), t \in [-\gamma_i, \gamma_i]$ starts at $q_i(-\gamma_i) = Q_j$ on S_+ and ends at $L_i = q_i(\gamma_i) \in O(Q_i)$, a neighborhood of Q_i on S_-. Also notice that $q_{ji}(t)$ intersects with Z_c i times on S_- as the perturbation of the heteroclinic solution $\bar{q}_k(\tau)$ for $\epsilon = 0$. See Figure 6.2.

When we construct the exact chaotic solution, the points $W_i(W_j)$ where the chaotic solution intersects with the Z_c for the i-th(j-th) time on the z-y plane must be close to

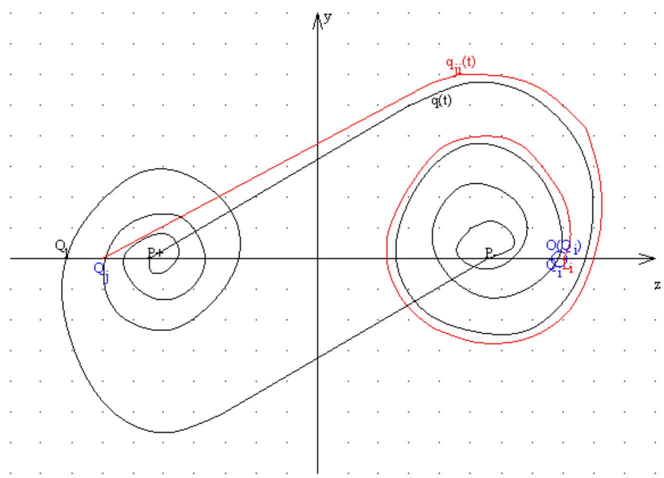

FIGURE 6.2. The approximated solution $q_{ji}(t)$ near heteroclinic solution $q(t)$ when $\epsilon = 0$

the equilibria $P_-(P_+)$. Therefore, we should only consider the $W_i(W_j)$ points within the neighborhoods $O(P_\pm) = \{Y | dist(Y, P_\pm) < \delta\}$ to be relevant for counting, see Figure 6.3. Next we define the extended neighborhood: $\mathcal{O}_- = \{O(Q_1), O(Q_2), O(Q_3), \ldots, O(Q_{\bar{i}-1}), O(P_-)\}$ such that $W_i \in O(Q_i) \subset O(P_-)$, for $i \geq \bar{i}$ on S_-. Similarly we define $\mathcal{O}_+$ for W_j on S_+.

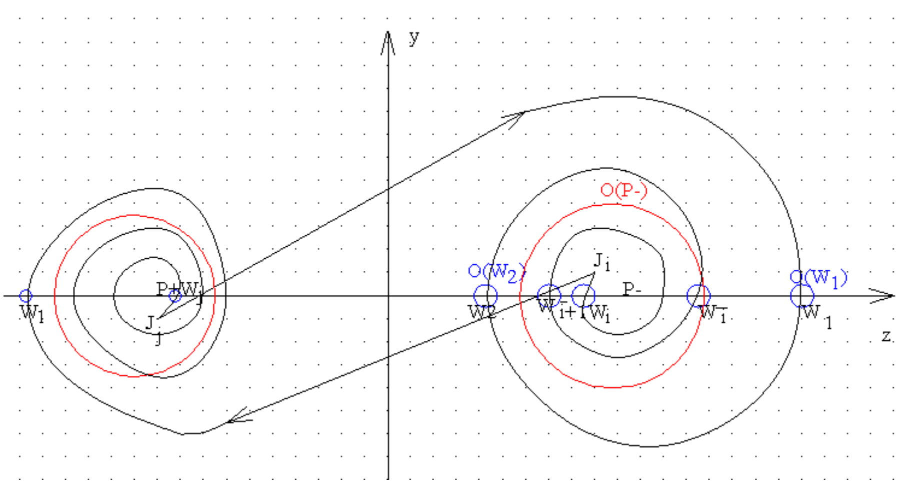

FIGURE 6.3. The chaotic solution intersects with the Z_c at points W_i on S_-.

Given $i(j)$, the number of intersections of the solution with Z_c on $S_-(S_+)$, we want to construct an exact chaotic solution w^{ex} when $\epsilon > 0$ based on the *ordered* approximated solutions $q_k^{ap}(t) = q_{ji}(t), t \in [-\gamma_k, \gamma_k], k \in Z$, such that $w_k^{ex} = q_k^{ap} + w_k$, where $w = (U, Y)$ is the correction solution. Notice that the approximation $q_{ji}(t)$

intersects with Z_c exactly i times, the exact solution near $q_{ji}(t)$ must intersect with Z_c exactly i times as well.

Before we obtain the correction solution, we figure out the jump conditions at the junction of two adjacent solutions for the correction solutions. First we define the jump conditions at the junction of two adjacent solutions for the approximated solutions:

$$U_k^{ap}(-\gamma_k/\epsilon) - U_{k-1}^{ap}(\gamma_{k-1}/\epsilon) = JU_k, \tag{6.1}$$

$$Y_k^{ap}(-\gamma_k) - Y_{k-1}^{ap}(\gamma_{k-1}) = JY_k \tag{6.2}$$

In order to obtain an exact solution where $Y_k^{ex}(-\gamma_k) = Y_{k-1}^{ex}(\bar{\gamma}_{k-1})$, we notice that:

$$\begin{aligned}
&Y_k^{ex}(-\gamma_k) - Y_k^{ap}(-\gamma_k) = Y_k(-\gamma_k) \\
&Y_{k-1}^{ex}(\bar{\gamma}_{k-1}) - Y_{k-1}^{ap}(\gamma_{k-1}) \\
&= [Y_{k-1}^{ex}(\bar{\gamma}_{k-1}) - Y_{k-1}^{ap}(\bar{\gamma}_{k-1})] + [Y_{k-1}^{ap}(\bar{\gamma}_{k-1}) - Y_{k-1}^{ap}(\gamma_{k-1})] \\
&= Y_{k-1}(\bar{\gamma}_{k-1}) + \delta Y_{k-1}(\gamma_{k-1})
\end{aligned}$$

Subtracting the above two equations gives us:

$$Y_k(-\gamma_k) - Y_{k-1}(\bar{\gamma}_{k-1}) = -JY_k + \delta Y_{k-1}(\gamma_{k-1}) = -\hat{J}Y_k \tag{6.3}$$

Similarly we have:

$$U_k(-\gamma_k/\epsilon) - U_{k-1}(\bar{\gamma}_{k-1}/\epsilon) = -JU_k + \delta U_{k-1}(\gamma_{k-1}/\epsilon) = -\hat{J}U_k \tag{6.4}$$

6.2. **Introduction of the Melnikov integral for the generalized solution.** Based on the approximated solution w_{ap}^i when $s = s_0$, we want to find a correction solution to the linearized variational system. Notice the homogeneous part of (6.5) has exponential dichotomy on $[-N, 0], [0, N]$ respectively, but not on $[-N, N]$ due to the non-transversal intersection on $RP_u^i(0-)$ and $RP_s^i(0+)$ at $\tau = 0$ for $i = 1, 3$. In fact, $RP_u^i(0-) + RP_s^i(0+) = R^1$. We need to take care of the non-transversal intersection issue for the $U = (u, v)$ equation by introducing the Melnikov integral.

Define operator $\mathcal{F} : U \to H(\tau)$ with $\tau \in I^{1,3} = [-N, N], N = \epsilon^{\lambda-1}$ and boundary value $U(\pm N)$ given.

$$H(\tau) = \mathcal{F}(U) = U'(\tau) - A(\tau)U(\tau), A(\tau) = D_U F(U_{ap}^i, Y_{ap}^i) \tag{6.5}$$

We have the following lemma for the existence of a generalized solution that allows a gap at $\tau = 0$:

Lemma 6.1. *Assume that $\mathcal{F}$ is of codimension one, $\dot{U}_{ap}^i(\tau), i = 1, 3$ is the unique nonzero bounded solution to the equation $U^{i'}(\tau) - D_U F^i(U_{ap}^i, Y_{ap}^i, s = s_0)U^i(\tau) = 0$, and ψ^i is the unique nonzero bounded solution to the adjoint equation of the previous homogeneous equation, then $\mathcal{F}$ is Fredholm with index being 0. The range of $\mathcal{F}$ is of codimension one and the kernel of $\mathcal{F}$ is one dimensional for each $H^i(\tau)$. There exists a unique generalized solution U^i for system (6.5) such that $U^i(0) \perp Ker\mathcal{F}$, i.e. $U^i(0) \perp \dot{U}_{ap}^i(0)$ and U^i has a gap at $\tau = 0$ along the given direction d^i:*

$$U^i(0+) - U^i(0-) = g^i d^i$$

Also we have estimate for g^i, U^i as:

$$|g^i| \le C(|P_s^i(-N)U^i(-N)|e^{-\alpha N} + |P_u^i(N)U^i(N)|e^{-\alpha N} + |H^i|) \tag{6.6}$$

$$|U^i| \le C(|\phi_1^i| + |\phi_2^i| + |H^i|) \tag{6.7}$$

Proof. Here we introduce the Melnikov function.
Let $S^i(t,s)$ be the principal matrix solution of the homogeneous part of equation $U^{i'} = AU^i + H^i$, which has exponential dichotomies on $(-N,0]$, and $[0,N)$, but no exponential dichotomy on $(-N,N)$ because:

$$RP_u^i(0-) + RP_s^i(0+) = R^{n-1}$$

Let $\psi^i(0) \perp RP_u^i(0-) + RP_s^i(0+), M^i = \{x| < \psi^i(0), x >= 0\}, \dim M^i = n-1$.
We define $d^i = \psi^i(0)/(||\psi^i(0)||^2) \in X, \phi_1^i = P_s^i(-N)U^i(-N), \phi_2^i = P_u^i(N)U^i(N)$ such that $< d^i, \psi^i(0) >= 1, span\{d^i\} \bigoplus M^i = R^n$ and we define the following solution:

$$\begin{aligned} U^i(\tau) &= S^i(\tau,-N)P_s^i(-N)U^i(-N) + \int_{-N}^{\tau} S^i(\tau,s)P_s^i(s)H^i(s)ds \\ &\quad + S^i(\tau,0-)P_u^i(0-)U^i(0-) + \int_0^{\tau} S^i(\tau,s)P_u^i(s)H^i(s)ds, \quad \tau < 0 \end{aligned} \tag{6.8}$$

$$\begin{aligned} U^i(\tau) &= S^i(\tau,N)P_u^i(N)U^i(N) + \int_{N}^{\tau} S^i(\tau,s)P_u^i(s)H^i(s)ds \\ &\quad + S^i(\tau,0+)P_s^i(0+)U^i(0+) + \int_0^{\tau} S^i(\tau,s)P_s^i(s)H^i(s)ds, \quad \tau > 0 \end{aligned} \tag{6.9}$$

We project them onto the stable and unstable spaces, and there are exponential dichotomies on $[-N,0],[0,N]$ respectively:

$$P_s^i(0-)U^i(0-) = \int_{-N}^{0-} S^i(0-,s)P_s^i(s)H^i(s)ds + S^i(0-,-N)P_s^i(-N)U^i(-N)$$

$$P_u^i(0+)U^i(0+) = \int_{N}^{0+} S^i(0+,s)P_u^i(s)H^i(s)ds + S^i(0+,N)P_u^i(N)U^i(N)$$

define $\phi_3^i = P_u^i(0-)U^i(0-), \phi_4^i = P_s^i(0+)U^i(0+)$, $\psi^i(s) = T^\star(s,0)\psi^i(0)$, where $T^\star(s,t)$ is the adjoint of $T(t,s)$. After subtracting ϕ_3^i and ϕ_4^i we have:

$$\begin{aligned} \phi_4^i - \phi_3^i &= [I - P_u^i(0+)]U^i(0+) - [I - P_s^i(0-)]U^i(0-) \\ &= g^i(Y,N)d^i + P_s^i(0-)U^i(0-) - P_u^i(0+)U^i(0+) \\ &= g^i(y^1(0),y^3(0),N)d^i + [\int_{-N}^{0-} S^i(0-,s)P_s^i(s)h^i(s)ds + S^i(0-,-N)P_s^i(-N)U^i(-N)] \\ &\quad - [\int_{N}^{0+} S^i(0+,s)P_u^i(s)h^i(s)ds + S^i(0+,N)P_u^i(N)U^i(N)] \end{aligned}$$

which leads to the estimate:

$$|g^i| + |\phi_3^i| + |\phi_4^i| \le C(e^{-\alpha N}|\phi_1^i| + e^{-\alpha N}|\phi_2^i| + |H^i|e^{-\eta N}) \tag{6.10}$$

Based on (6.8), (6.9), (6.10), we obtain the estimate (6.7) for U^i. Since $\phi_4^i - \phi_3^i \in RP_u^i(0-) + RP_s^i(0+)$, thus $< \psi^i(0), \phi_4^i - \phi_3^i >= 0$. As a result,

(6.11)
$$g^i =< \psi^i, \int_{-N}^{0-} S^i(0-,s)P_s^i(s)H^i(s)ds > - < \psi^i, \int_N^{0+} S^i(0+,s)P_u^i(s)H^i(s)ds >$$
$$+ < \psi^i, S^i(0-,-N)P_s^i(-N)U^i(-N) > - < \psi^i, S^i(0+,N)P_u^i(N)U^i(N) >$$
$$= \int_{-N}^{N} < \psi^i(s), H^i(s) > ds + < \psi^i(-N), P_s^i(-N)U^i(-N) > - < \psi^i(N), P_u^i(N)U^i(N) >$$

Estimate (6.6) follows from (6.11) and $\psi^i(\tau) \leq e^{-\alpha|\tau|}$. □

Lemma 6.2. *Let* $\Omega_1 = \{y = (\dots y_{-1}, y_0, y_1 \dots) : |y_i - y_i(0)| \leq \delta y_i, i \in Z\}$ *be a rectangle in* R^∞. $H : \Omega_1 \to R^\infty$

$$H(y) = (\dots, g^{-1}(y), g^0(y), g^1(y), \dots)$$

is continuous with $g^i > 0 (< 0)$ *if* $y_i = y_i(0) + \delta y_i (y_i = y_i(0) - \delta y_i)$. *Then there exists a* $\hat{y} \in \Omega_1$ *with* $H(\hat{y}) = 0$.

Proof. Let $H_1(y) = (\dots, y_{-1} - \delta y_{-1} g^{-1}(y)/|g^{-1}|, y_0 - \delta y_0 g^0(y)/|g^0|, y_1 - \delta y_1 g^1(y)/|g^1|, \dots)$ be a mapping from Ω_1 to Ω_1. There exists a fixed point $\hat{y} \in \Omega_1$ for H_1 by Schauder fixed point theorem. Therefore, there exists a $\hat{y} \in \Omega_1$ with $H(\hat{y}) = 0$. □

Lemma 6.3. *In regular layers* $[\alpha_i, \beta_i], i = 2, 4$, *there exists a unique solution to the following system of equations:*

$$\dot{V}_i - F_U V_i = h_i(\tau) \tag{6.12}$$

with $S^i(t,s)$ *to be the principal matrix of the system above and the jump conditions:*

$$\begin{aligned} J_{V1} &= V_2(\alpha_2/\epsilon) - V_1(\beta_1), J_{V4} = V_1(\alpha_1) - V_4(\beta_4/\epsilon) \\ J_{V3} &= V_4(\alpha_4/\epsilon) - V_3(\beta_3), J_{V2} = V_3(\alpha_3) - V_2(\beta_2/\epsilon) \end{aligned} \tag{6.13}$$

and estimate:

$$|V_2| + |V_4| \leq C_1(|h_2| + |h_4| + \Sigma_{i=1}^4 |J_{Vi}|) \tag{6.14}$$

Proof. Because of the hyperbolicity of the coefficient matrix F_U, we know that the slow varying system has exponential dichotomy on $I^i, i = 2, 4$ with corresponding projections Q_s^i, Q_u^i. Also, we have the following decomposition:

$$RP_u^1(\beta_1) \oplus RP_s^2(\alpha_2/\epsilon) = R^2, RP_u^2(\beta_2/\epsilon) \oplus RP_s^3(\alpha_3) = R^2$$
$$RP_u^3(\beta_3) \oplus RP_s^4(\alpha_4/\epsilon) = R^2, RP_u^4(\beta_4/\epsilon) \oplus RP_s^1(\alpha_1) = R^2$$

Based on the decomposition above, we can split the jump conditions as following:

$$J_{V1} = Q_s^2 J_{V1} - (-Q_u^1 J_{V1}), J_{V4} = Q_s^1 J_{V4} - (-Q_u^4 J_{V4})$$
$$J_{V3} = Q_s^4 J_{V3} - (-Q_u^3 J_{V3}), J_{V2} = Q_s^3 J_{V2} - (-Q_u^2 J_{V2})$$

We give the stable component of each jump as the initial value for the solution after the jump, and the negated unstable component of the jump as the backward initial value for the solution before the jump, as in figure 6.4. That is to say, the solution between two jumps takes the negated unstable component of the latter jump as the backward initial value, and the stable component of the previous jump as the forward initial value. Therefore we define:

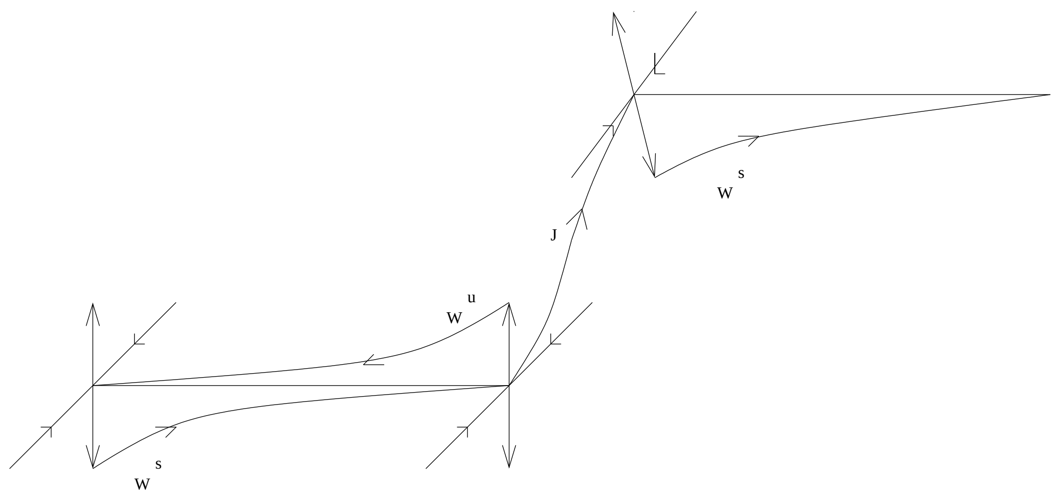

FIGURE 6.4. Decomposition of the jump errors for defining the solutions V_i

$$\begin{aligned} V_2^1(\tau) = S^2(\tau, \alpha_2/\epsilon) Q_s^2 J_{V1} + \int_{\alpha_2/\epsilon}^{\tau} S^2(\tau, s) Q_s^2(s) h^2(s) ds \\ -S^2(\tau, \beta_2/\epsilon) Q_u^2 J_{V2} + \int_{\beta_2/\epsilon}^{\tau} S^2(\tau, s) Q_u^2(s) h^2(s) ds \\ \tau \in [\alpha_2/\epsilon, \beta_2/\epsilon] \end{aligned}$$

The above solutions satisfy (6.12), but satisfy the jump conditions almost accurately, for example:

$$\begin{aligned} J_{V1}^1 &= V_2^1(\alpha_2/\epsilon) - V_1^1(\beta_1) \\ &= [Q_s^2 J_{V1} - S^2(\alpha_2/\epsilon, \beta_2/\epsilon) Q_u^2 J_{V2} + \int_{\beta_2/\epsilon}^{\alpha_2/\epsilon} S^2(\tau, s) Q_u^2(s) h^2(s) ds] \\ &- [S^1(\beta_1, \alpha_1) Q_s^1 J_{V4} + \int_{\alpha_1}^{\beta_1} S^1(\tau, s) Q_s^1(s) h^1(s) ds - Q_u^1 J_V(\Sigma_1+)] \\ &= J_{V1} + E^1(J_{V1}) \\ E^1(J_{V1}) &= -S^2(\alpha_2/\epsilon, \beta_2/\epsilon) Q_u^2 J_{V2} + \int_{\beta_2/\epsilon}^{\alpha_2/\epsilon} S^2(\alpha_2/\epsilon, s) Q_u^2(s) h^2(s) ds \\ &- S^1(\beta_1, \alpha_1) Q_s^1 J_{V4} + \int_{\alpha_1}^{\beta_1} S^1(\beta_1, s) Q_s^1(s) h^1(s) ds \end{aligned}$$

with the estimate:

$$
\begin{aligned}
|E^1(J_{V1})| \leq C_2(e^{\alpha(\alpha_2-\beta_2)/\epsilon}J_{V2} + \int_{\beta_2/\epsilon}^{\alpha_2/\epsilon} e^{\alpha(\alpha_2/\epsilon - s)}|h_2|ds) \\
+ C_1(e^{-\alpha(\beta_1-\alpha_1)}J_{V4} + \int_{\alpha_1}^{\beta_1} e^{-\alpha(\beta_1-s)}|h_1|ds)
\end{aligned}
$$

Notice that $e^{-\alpha(\beta_2-\alpha_2)/\epsilon}, e^{-\alpha(\beta_1-\alpha_1)}$ as well as the two integral terms added together are all of $O(\epsilon)$. Thus $|E^1(JV_1)| = O(\epsilon\sum_1^4|JV_i|)$ is small compared to the given jump conditions by multiplying ϵ in each iteration process. Therefore, we can define the solution $V_i^k(\tau), \tau\in[\alpha_i,\beta_i]$ recursively, with $-E^k(JV_i)$ as the jump condition in the next iteration. The jump condition will be satisfied by $V_i(\tau) = \Sigma_{k=1}^{\infty}V_i^k(\tau)$ after the iteration process.

Next we give an estimate of the solution, for example:

$$
\begin{aligned}
|V_2(\tau)| &\leq e^{-\alpha(\tau-\alpha_2/\epsilon)}J_{V1} + |h_2|\int_{\tau}^{\alpha_2/\epsilon} e^{-\alpha(\tau-s)}ds + e^{\alpha(\tau-\beta_2/\epsilon)}J_{V2} + |h_2|\int_{\beta_2/\epsilon}^{\tau} e^{\alpha(\tau-s)}ds \\
&\leq C_1(|h_2| + |J_{V1}| + |J_{V2}|)
\end{aligned}
$$

$$
\begin{aligned}
|V_1^1(\tau)| &\leq e^{-\alpha(\tau-\alpha_1)}J_{V4} + |h_1|\int_{\tau}^{\alpha_1} e^{-\alpha(\tau-s)}ds + e^{\alpha(\tau-\beta_1)}J_{V1} + |h_1|\int_{\beta_1}^{\tau} e^{\alpha(\tau-s)}ds \\
&\leq e^{-\alpha(\tau-\alpha_1)}J_{V4} + e^{\alpha(\tau-\beta_1)}J_{V1} + (e^{\alpha(\tau-\beta_1)} - e^{-\alpha(\tau-\alpha_1)})|h_1|/\alpha \\
&\leq C_2(|h_1| + |J_{V4}| + |J_{V1}|)
\end{aligned}
$$

These above estimates result in the estimates for the solution V^i. □

6.3. **Proof of existence of the solution to the linear variational system.** We give an outline of the proof. (1) We first obtain a generalized solution $w_k, k\in Z$, that allows a gap at $\tau = 0$ but satisfies the jump conditions by Lemma 6.3 and 6.1. During this step, we eliminate the residual error caused by dropping the $\epsilon\frac{d}{dt}[F_U^{-1}F_Y\bar{Y}]$ term with iteration method. (2) We use the Melnikov integral to eliminate the gap by shifting the y values in the gap function g^k by Lemma 6.2, the change of y value results in the updated domains for the solutions. Then we obtain solutions $\hat{U}_k$ on the updated domain as the exact solutions that satisfy the jump conditions JU_k exactly with no gap at $\tau = 0$. However, the JY_k on the updated domain is not satisfied exactly. After we define the Y_k^{i+1}, the difference of jump errors $E(JY_k^{i+1})$ is reduced by a multiple of a small number in the i-th iteration, due to the contraction caused by the stable spiral near the equilibrium points. Therefore, the exact solution can be obtained after iterations.

Theorem 6.4. *There exists unique solutions $w_k = (U_k(\tau), Y_k(t)), t\in[-\gamma_k,\bar{\gamma}_k]$ to the linear variational system (4.2) with jump conditions $J_k = (JU_k, JY_k)$ according to* (6.3), (6.4)*:*

$$U_k(-\gamma_k/\epsilon) - U_{k-1}(\bar{\gamma}_{k-1}/\epsilon) = -\hat{J}U_k, \tag{6.15}$$

$$Y_k(-\gamma_k) - Y_{k-1}(\bar{\gamma}_{k-1}) = -\hat{J}Y_k \tag{6.16}$$

and phase condition $\dot{q}_k^{ap}(0) \perp w_k(0)$. *We also have the estimate:*

$$|U_k| + |Y_k| \leq C(|\bar{P}_k| + |\bar{Q}_k| + |\hat{J}_k|)$$

Proof. The linear variational system of equations is autonomous. So if $w(t)$ is a solution then $w(t+k)$ is also a solution, where k is a constant. Without loss of generality, after a proper time shift we assume that at time t=0 the solution is in a cross section T_i that is transverse to the flow as in the phase condition $w_i(0) \in T_i$, where $T_i := \{x | \langle \dot{q}_i^{ap}(0), x\rangle = 0\}$.

(1) We want to solve for a generalized solution $(U_k(\tau), Y_k(t)), t \in [-\gamma_k, \gamma_k]$.

We first solve for $\bar{V}_k(\tau), \tau \in [-\gamma_k/\epsilon, \gamma_k/\epsilon]$ in (4.3) according to Lemma 6.3, with the $\epsilon \frac{d}{dt}[F_U^{-1} F_Y Y]$ term dropped and $H_k = \bar{P}_k, \bar{V}_k(-\gamma_k/\epsilon) - \bar{V}_{k-1}(\gamma_k/\epsilon) = -JV_k$, also we obtain estimates $|\bar{V}_k(\tau)| \leq C(JV_k + |\bar{P}_k|)$.

Next we solve for $\bar{Y}_k(t)$ with the initial condition $\bar{Y}_k(-\gamma_k) = \bar{Y}_{k-1}(\gamma_{k-1}) - JY_k$ after we plug in $\bar{V}_k(\tau)$ into the Y equation of (4.3). We also have the estimates $|\bar{Y}_k(t)| \leq C(|\bar{V}_k(\tau)| + JY_k + |\bar{Q}_k|)$.

Now we define $\bar{U}^k(\tau) = \bar{V}_k(\tau) - F_U^{-1}(\epsilon\tau) F_Y(\epsilon\tau) \bar{Y}^k(\epsilon\tau)$ which satisfies:

$$\bar{U}_k' = F_U \bar{U}_k + F_Y \bar{Y}_k + \bar{P}_k - \frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}] \tag{6.17}$$

Notice that the first equation of (4.2) is not satisfied by $\bar{U}^k$ because of the error term $\frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}]$ in (6.17), we have the estimate of the $\frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}]$ as:

$$|\frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}]| = \epsilon |\frac{d}{dt}[F_U^{-1} F_Y \bar{Y}]| = K\epsilon(|\bar{V}^k| + |\bar{Q}^k| + |JY^k|) \leq K\epsilon(|\bar{P}^k| + |\bar{Q}^k| + |J_k|)$$

Now we have $(\bar{U}^k, \bar{Y}^k)$ is a good approximation with residual errors $O(\epsilon(|\bar{P}^k| + |\bar{Q}^k| + |J_k|))$ in the U equation. Therefore, we can obtain the generalized solution $U_k(\tau), Y_k(t)$ to (4.2) with jump conditions (6.15) by iteration process. We have the estimates for $U_k(\tau)$ as:

$$|U_k| \leq C(|V_k| + |Y_k|) \leq C(|\bar{P}_k| + |\bar{Q}_k| + |J_k|)).$$

(2) However the above solution $U_k(\tau), Y_k(t)$ is still a generalized solution that allows a gap at $\tau = 0$ for U as:

$$U^i(0+) - U^i(0-) = g^i d^i$$

according to Lemma 6.1. Here the gap is defined as:

$$g^k(y(0)) = \int_{-\gamma_k}^{\gamma_k} < \psi^k(s), H^k(s) > ds$$
$$+ < \psi^k(-\gamma_k), P_s^k(-\gamma_k) U^k(-\gamma_k) > - < \psi^k(\gamma_k), P_u^k(\gamma_k) U^k(\gamma_k) >$$

where $y(0) = (\ldots y^{-1}(0), y^0(0), y^1(0) \ldots)$. Considering q_k is the perturbation of the heteroclinic solution $q(t)$, we have $G_y = \{\frac{\partial g^k}{\partial y_j}\}$ is almost a diagonal matrix. Therefore g^k mainly depends on $y^k(0)$. According to the higher dimensional Intermediate Value Theorem Lemma 6.2, we obtain $\hat{y}(0)$ with $\hat{y}^k(0) \in [y^k(0) - \delta y^0, y^k(0) + \delta y^0]$ such that $g^k(\hat{y}(0)) = 0, k \in Z$. Now we have eliminated the gaps for U at at $\tau = 0$ by shifting the equal gap surface and the y values.

(3) After the change of y values, $\hat{Y}_k(t)$ needs extra time Δt_k to get to the shifted equal gap surface, which can be used to update the domain of the solutions $\hat{w}_k(t)$,

$t \in [-\gamma_k, \bar{\gamma}_k], \bar{\gamma}_k = \gamma_k + \Delta\gamma_k$. We repeat the procedure on $[-\gamma_k, \bar{\gamma}_k]$ in step (1) to obtain $\hat{V}_k(\tau)$ on the updated domain. By the change of variables, we have $\hat{U}_k(\tau)$, which eliminates the gap at $\tau = 0$ and satisfies the jump condition:

$$-J\hat{V}_k = \hat{V}_k(-\gamma_k/\epsilon) - \hat{V}_{k-1}(\bar{\gamma}_{k-1}/\epsilon)$$

Next we want to obtain the $\hat{Y}_k$ solutions that satisfy the linear variational system and the jump condition by iteration method. First, we compare the jump errors of two adjacent Y^1 solutions Y_k^1, Y_{k-1}^1: $-JY_{k-1}^1 = Y_k^1(-\gamma_k) - Y_{k-1}^1(\gamma_{k-1})$ with the updated jump errors on the updating the domains:

$$-JY_{k-1}^2 = Y_k^1(-\gamma_k) - Y_{k-1}^2(\bar{\gamma}_{k-1})$$

We obtain the difference of the above jump errors caused by the extra time $\Delta\gamma_{k-1}$ to be

$$E(JY_{k-1}^2) = JY_{k-1}^2 - JY_{k-1}^1 = [Y_{k-1}^2(\bar{\gamma}_{k-1}) - Y_{k-1}^1(\gamma_{k-1})]$$

In order to reduce the difference of the jump errors $E(JY_{k-1}^2)$, we define the initial value $Y_k^2(-\gamma_k)$ for $Y_k^2(t)$ based on $Y_k^2(-\gamma_k) - Y_k^1(-\gamma_k) := E(JY_k^1) = Y_k^1(\bar{\gamma}_k) - Y_k^0(\gamma_k)$, where $Y_k^0(t)$ is the Y solution obtained in step (1).

$$Y_k^1(t) = Y_k^0(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \le t \le \bar{\gamma}_k$$

For the general iteration process, see Figure 6.5. We compare the jump errors in the i-th iteration of two adjacent Y^i solutions Y_k^i, Y_{k-1}^i: $-JY_{k-1}^i = Y_k^i(-\gamma_k) - Y_{k-1}^i(\gamma_{k-1}^i)(= BD)$ with the updated jump errors on the updating the domains:

$$-JY_{k-1}^{i+1} = Y_k^i(-\gamma_k) - Y_{k-1}^{i+1}(\bar{\gamma}_{k-1}^i)(= BC)$$

We define the difference of the above jump errors to be:

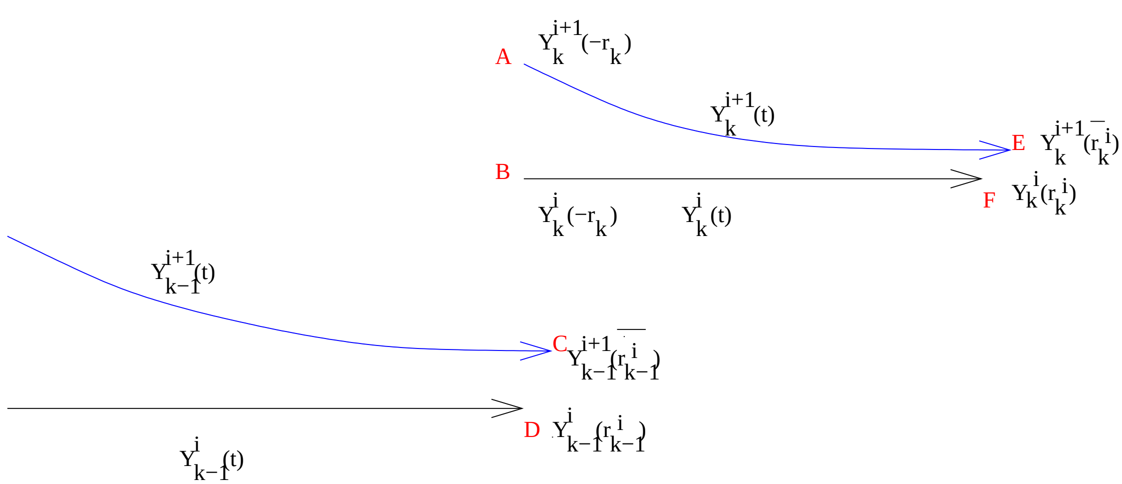

FIGURE 6.5. The jump errors $JY_{k-1}^i(= BD)$ and updated jump errors $JY_{k-1}^{i+1}(= BC)$ result in the difference of the jump errors $E(JY_{k-1}^{i+1})(= CD)$, which is to be reduced in the next iteration.

$$E(JY_{k-1}^{i+1}) = JY_{k-1}^{i+1} - JY_{k-1}^{i} = [Y_{k-1}^{i+1}(\bar{\gamma}_{k-1}^{i}) - Y_{k-1}^{i}(\gamma_{k-1}^{i})](= CD)$$

In order to reduce the difference of the jump errors $E(JY_k^{i+1})(= EF)$, we define the initial value $Y_k^{i+1}(-\gamma_k)$ for $Y_k^{i+1}(t)$ based on $AB = Y_k^{i+1}(-\gamma_k) - Y_k^i(-\gamma_k) := E(JY_k^i) = Y_k^i(\bar{\gamma}_k^i) - Y_k^{i-1}(\gamma_k^i)$, from the $E(JY_k^i)$ in the i-th iteration (previous ith EF).

$$Y_k^{i+1}(-\gamma_k) = Y_k^i(-\gamma_k) + E(JY_k^i)$$

Notice that the Y_k^i solutions are changed to

$$\bar{Y}_k^{i+1}(t) = Y_k^{i+1}(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \le t \le \gamma_k^i$$

Accordingly, the nonlinear term H in (6.5) that involves Y_k^i is changed. Therefore, we need to repeat step (2) to eliminate the gaps for the generalized solutions at $\tau = 0$ and update the domain as $-\gamma_k \le t \le \bar{\gamma}_k^i$, where $\bar{\gamma}_k^i = \gamma_k^i + \Delta\gamma_k^i$. Now we define for $i \ge 1$:

$$Y_k^{i+1}(t) = Y_k^{i+1}(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \le t \le \bar{\gamma}_k^i$$

when compared with:

$$Y_k^{i}(t) = Y_k^{i}(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \le t \le \gamma_k^i$$

We take the sup norm of the difference of the jump errors and define:

$$\Delta_{i+1} = \sup_{k\in Z} |E(JY_k^{i+1})| = \sup_{k\in Z} |Y_k^{i+1}(\bar{\gamma}_k^i) - Y_k^i(\gamma_k^i)|$$

We give an estimate for the difference of the jump error $E(JY_k^{i+1})(= |EF|$ in Figure (6.5)):

$$\begin{aligned}
|E(JY_k^{i+1})| &= [Y_k^{i+1}(\bar{\gamma}_k^i) - Y_k^i(\gamma_k^i)] \\
&= Y_k^i(-\gamma_k)[S_k(\bar{\gamma}_k^i, -\gamma_k) - S_k(\gamma_k^i, -\gamma_k)] + E(JY_k^i)S_k(\bar{\gamma}_k, -\gamma_k) \\
&+ \int_{-\gamma_k}^{\bar{\gamma}_k^i} S_k(\bar{\gamma}_k^i, -\gamma_k)H^k(s)ds - \int_{-\gamma_k}^{\gamma_k^i} S_k(\gamma_k^i, -\gamma_k)H^k(s)ds \\
&\le Ce^{-2\alpha\gamma_k}|E(JY_k^i)|
\end{aligned} \tag{6.18}$$

Here we use the fact that there are stable spirals near equilibria on $S_\pm$ with eigenvalues $-\alpha \pm i\beta, \alpha > 0$ for the linear variational system, so that the principal matrix solution $S_k(t, -\gamma_k) \le Ce^{-2\alpha(t+\gamma_k)}$ for large enough t.

The existence of an exact solution $\hat{w}_k, k \in Z$, follows by iteration method. In fact, after the i-th iteration, $E(JY_k^i)$ gets reduced by a multiple of an exponentially small number. Therefore, $\Delta_{i+1} = \sup_{k\in Z} |E(JY_k^{i+1})| \le C_i\Delta_i$ by (6.18), where $C_i << 1$. Moreover, Δ_i gets reduced by a multiple of an exponentially small number. Therefore, $\sum_i |Y_k^{i+1} - Y_k^i| \le \sum_i \Delta_i < \infty$, and we have $\lim_{i\to\infty} Y_k^i = \hat{Y}_k$. Now we obtain the correction solutions $(\hat{U}_k, \hat{Y}_k)$ to the linear variational system (4.2) with jump conditions (6.15) satisfied. Estimates of the correction solutions follow similarly to those in step (1).

□

Theorem 6.5. *In a small neighborhood of* q_k*, there exists a unique exact chaotic solution* w^{ex}*, which satisfies* (1.3), (1.4) *with estimates:*

$$|w_k^{ex} - q_k| \leq K\epsilon^{\lambda} \quad 0 < \lambda < 1$$

We can prove this theorem by contraction mapping.

7. Symbolic dynamics

We want to make correspondence of the solution w^{ex} to a sequence of symbols.

Theorem 7.1. *The chaotic solution* w^{ex} *that intersects with* Z_c *exactly i(j) times on* $S_-(S_+)$ *corresponds to a sequence of symbols* $\{(i,j)\}_{i\geq\bar{i},j\geq\bar{j}}, i,j$ *can be* ∞.

Proof. Given any chaotic solution near the heteroclinic solutions with $W_i \in \mathcal{O}_-(W_j \in \mathcal{O}_+)$, we set $y = 0$ and solve for the t values on which W_i points fall into $O(P_-)$. We can keep track of the symbols (i,j) by putting the W_i points in order according to the orientation of the spirals. For clockwise orientation, W_1 is defined to be the intersection point with the largest z value within $O(P_-)$; W_2 is defined to be the intersection point with the smallest z value within $O(P_-)$; W_3 is defined to be the intersection point with the second largest z value within $O(P_-)$; W_4 is defined to be the intersection point with the second smallest z value within $O(P_-)$, etc. We count up to $W_i(W_j)$.

On the other hand, given sequence of symbols $\{(i,j)\}_{i\geq\bar{i},j\geq\bar{j}}$, we want to obtain a chaotic solution with $W_i \in \mathcal{O}_-(W_j \in \mathcal{O}_+)$ which intersects with the Z_c for the prescribed number of times. We first construct the approximated solutions $q_{ji}(q_{ij})$ on $S_-(S_+)$ when $\epsilon = 0$, which intersect with the Z_c only i(j) times, then according to Theorem 6.5, we obtain the chaotic solution near the approximated solutions corresponding to the symbols.

□

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