TRAVELING WAVE SOLUTIONS FOR THE PAINLEVÉ-INTEGRABLE COUPLED KDV EQUATIONS

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Abstract. We study the traveling wave solutions for a system of coupled KdV equations derived by Lou et al [11]. In that paper, they found 5 types of Painlevé integrable systems for the coupled KdV system. We show that each of them can be reduced to a partially or completely uncoupled system, through which the dynamical behavior of traveling wave solutions can be determined. In some parameter regions, exact formulas for periodic and solitary waves can be obtained while in other cases, bounded traveling wave solution are discussed.

1. Introduction

The KdV equation is an important model for dispersive waves [1, 14]. There has been some interest in coupled KdV systems [4, 5, 6, 9, 12, 13]. In this paper we consider the coupled KdV system,

$$\begin{aligned} A_{1T} + \alpha_1 A_2 A_{1X} + (\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 A_{1XX} + \alpha_5 A_1^2)_X = 0, \\ A_{2T} + \delta_1 A_2 A_{1X} + (\delta_2 A_1^2 + \delta_3 A_1 A_2 + \delta_4 A_{2XX} + \delta_5 A_2^2)_X = 0, \end{aligned} \tag{1.0}$$

where the ten constants $\{\alpha_i,\ \delta_i,\ i = 1,2,3,4,5\}$ are arbitrary. This system is derived by Lou et al in 2006 [11] from a two-layer fluid model which is used to describe the atmospheric and oceanic phenomena such as the atmospheric blockings, the interactions between the atmosphere and ocean. Under the condition $\alpha_4 = \delta_4 = 1$, they obtained five types of Painlevé-integrable coupled KdV systems:

P-integrable model 1

$$\begin{aligned} A_{1T} + [A_{1XX} - (c_0+3)(c_0+6)A_1^2 - c_0^2 A_2^2]_X + 2c_0[(c_0+6)A_{1X}A_2 + (c_0+3)A_1 A_{2X}] = 0, \\ A_{2T} + [A_{2XX} - c_0(c_0-3)A_2^2 - (c_0+3)^2 A_1^2]_X + 2(c_0+3)[c_0 A_2 A_{1X} + (c_0-3)A_1 A_{2X}] = 0. \end{aligned} \tag{1.1}$$

P-integrable model 2

$$\begin{aligned} A_{1T} + (A_{1XX} + \frac{1}{2}(c_2 - c_1 - c_1 c_2)A_1^2 + c_1 A_1 A_2 - \frac{1}{2}A_2^2)_X = 0, \\ A_{2T} + (A_{2XX} + \frac{1}{2}(c_1 - c_2 - 1)A_2^2 + c_2 A_1 A_2 - \frac{1}{2}c_1 c_2 A_1^2)_X = 0. \end{aligned} \tag{1.2}$$

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P-integrable model 3

$$A_{1T} + (A_{1XX} + A_1^2 + A_1A_2)_X = 0, \quad A_{2T} + (A_{2XX} + A_2^2 + A_1A_2)_X = 0. \tag{1.3}$$

P-integrable model 4

$$A_{1T} + [A_{1XX} + (A_1 + A_2)^2]_X = 0, \quad A_{2T} + [A_{2XX} + (A_1 + A_2)^2]_X = 0. \tag{1.4}$$

P-integrable model 5

$$A_{1T} + [A_{1XX} + A_1^2]_X + 2A_2A_{1X} = 0, \quad A_{2T} + [A_{2XX} + A_2^2]_X + 2A_1A_{2X} = 0. \tag{1.5}$$

In this paper we are interested in the existence and exact expression of the traveling wave solutions of (1.1) and some dynamical behavior of these solutions such as whether the solutions are solitary, periodic or bounded solutions.

Note that the way to write (1.0) is not unique. Instead of A_2A_{1X}, one can leave A_1A_{2X} terms outside of the divergence forms. With $\alpha_4 = \delta_4 = 1$, we will use an equivalent form to (1.0):

$$\begin{aligned} A_{1T} + A_{1XXX} + a_1A_1A_{1X} + a_2A_1A_{2X} + a_3A_2A_{1X} + a_4A_2A_{2X} = 0, \\ A_{2T} + A_{2XXX} + b_1A_1A_{1X} + b_2A_1A_{2X} + b_3A_2A_{1X} + b_4A_2A_{2X} = 0. \end{aligned} \tag{1.6}$$

If we set

$$U = (A_1, A_2)^\tau, \quad Q_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},$$

where τ denote the transpose of a vector, then the nonlinear terms of the equations can be written as bilinear forms,

$$U^\tau Q_1 U_X, \quad U^\tau Q_2 U_X.$$

In the case that the matrices Q_1 and Q_2 are symmetric, we can express the bilinear forms as divergence of quadratic forms:

$$\frac{1}{2}(U^\tau Q_1 U)_X, \quad \frac{1}{2}(U^\tau Q_2 U)_X.$$

There are many results concerning simultaneously co-diagonalize symmetric matrices, see [8], that will be used in this paper to further simplify the quadratic forms.

If the coupled system of KdVs $U_T + U_{XXX} + F(U, U_X)$, $\quad U = (A_1, A_2)^\tau$ has a traveling wave solution with the wave speed c, then in the traveling coordinate $\xi = X - cT$, $U = U(\xi)$ and satisfies a system of ODEs:

$$-cU'(\xi) + U'''(\xi) + F(U, U') = 0. \tag{1.7}$$

If U is a traveling periodic or solitary wave of the PDE system, then $U(\xi)$ is a periodic or homoclinic solution of the corresponding ODE system. Throughout this paper, the higher order system (1.7) is associated to a first order system by introducing auxiliary variables (U, U', U'') in the standard way. We say U_0 is an equilibrium for (1.7) if $(U_0, 0, 0)$ is an equilibrium for the associated first order system. We say $U(\xi)$ is a homoclinic solution to (1.7) if $(U(\xi), U'(\xi), U''(\xi))$ is a homoclinic solution to the associated first order system, etc. This convention also applies to any coupled second order system of equations.

In Section 2, we treat the general coupled KdV system (CKdV) and P-integrable model 1. Following Lou et al [11], we identify an invariant subspace on which the system reduces to a single KDV equation. For the P-integrable mode 1, we show that the system can be partially decoupled. The reduced system is equivalent to the reduced system of the P-integrable models 3 and 5. Detailed description of the traveling waves are deferred to section 4 where the P-integrable models 3 and 5 are discussed.

In section 3, we treat the P-integrable mode 2 which is in the divergence form. The corresponding bilinear forms are symmetric. Using standard matrix algorithms, we introduce a method that can remove the non-diagonal terms of the quadratic forms. For the P-integrable model 2, the reduced system consists of two uncoupled equations. The method may be used on non-P-integrable system as long as the original system (1.0) is in divergence form.

The P-integrable models 3, 4 and 5 can be simplified by some change of variables and are treated in section 4. We show that the P-integrable model 4 can be completely decoupled while the models 3 and 5 can be partially decoupled. In some cases, we find bounded traveling wave solutions rather than traveling periodic or solitary waves.

In $(u, u')-$phase plane, the second order equation

$$u'' = cu + \beta u^2,\ c \neq 0,\ \beta \neq 0, \tag{1.8}$$

has a Hamiltonian $H(u, u')$ of which each orbit corresponds to a unique level curve

$$H(u, u') = \frac{(u')^2}{2} - c\frac{u^2}{2} - \frac{\beta}{3}u^3 = h, \quad h \in \mathbb{R}.$$

Bounded solutions of (1.8) can be classified by the following lemma:

Lemma 1.1. *Assume that $c \neq 0, \beta \neq 0$. In the phase plane (u, u'),* (1.8) *has two equilibrium points $O(0,0)$ and $E(-c/\beta, 0)$.*

(I) If $c > 0$ then O is a saddle and E is a center. If $c < 0$ then O is a center and E a saddle.

(II) There is a unique homoclinic orbit Γ asymptotic to the saddle and encircling the center. There is also a family of periodic orbits encircling the center and filling up the interior of the homoclinic loop Γ.

(III) Up to a shift in ξ, the homoclinic orbit Γ is parametrized by a homoclinic solution $u = q(\xi, c, \beta)$ to (1.8).

$$q(\xi,c,\beta) := \begin{cases} -\frac{3c}{2\beta} sech^2\left(\frac{\sqrt{c}}{2}\xi\right), & c>0, \\ \frac{|c|}{\beta}\left(1-\frac{3}{2}sech^2\left(\frac{\sqrt{|c|}}{2}\xi\right)\right), & c<0. \end{cases} \tag{1.9}$$

(IV) Each periodic orbit corresponds to a unique $h \in (-\frac{c^3}{6\beta^2}, 0), c>0$ or $h \in (0, -\frac{c^3}{6\beta^2}), c<0$. Up to a shift in ξ, the family of periodic orbits is parametrized by periodic solutions $p(\xi, c, \beta, h)$ of (1.8). *Depending on $\beta<0$ or $\beta>0$, using elliptic functions, the periodic solution can be expressed as:*

$$p(\xi,c,\beta,h) := \begin{cases} r_1-(r_1-r_2)sn^2\left(\Omega\xi,k_1\right), & \beta<0, \\ r_3+(r_2-r_3)sn^2\left(\Omega\xi,k_2\right), & \beta>0. \end{cases} \tag{1.10}$$

The parameters $(r_1, r_2, r_3, k_1, k_2)$, with $r_1 > r_2 > r_3$, are defined by $(u')^2 = 2h + cu^2 + \frac{2}{3}\beta u^3 = \frac{2}{3}|\beta|(r_1-u)(u-r_2)(u-r_3)$, $k_1^2 = \frac{r_1-r_2}{r_1-r_3}$ if $\beta<0$. While for $\beta>0$, they are defined by $(u')^2 = 2h + cu^2 + \frac{2}{3}\beta u^3 = \frac{2}{3}\beta(r_1-u)(r_2-u)(u-r_3)$, $k_2^2 = \frac{r_2-r_3}{r_1-r_3}$. $\Omega = \frac{\sqrt{|\beta|(r_1-r_3)}}{6}$.

2. General coupled KDV and the P-integrable mode 1

To find traveling wave wave solutions, let $\xi = X - cT$ be the traveling coordinate. From (CKdV) we obtain the traveling wave system

$$\begin{aligned} -cA_1' + A_1''' + a_1A_1A_1' + a_2A_1A_2' + a_3A_2A_1' + a_4A_2A_2' = 0, \\ -cA_2' + A_2''' + b_1A_1A_1' + b_2A_1A_2' + b_3A_2A_1' + b_4A_2A_2' = 0. \end{aligned} \tag{2.1}$$

Following Lou et al [11], we look for solutions that satisfy $A_1 = \omega A_2, \omega \neq 0$. Substituting $A_1 = \omega A_2$ into (2.1), integrating (2.1) and taking the integral constants as zero, we obtain

$$\begin{aligned} A_2'' &= cA_2 - \frac{1}{2}\left(a_1\omega + (a_2+a_3) + \frac{a_4}{\omega}\right)A_2^2, \\ A_2'' &= cA_2 - \frac{1}{2}\left(b_1\omega^2 + (b_2+b_3)\omega + b_4\right)A_2^2. \end{aligned} \tag{2.2}$$

The two equations of system (2.2) are the same if and only if ω is a non-zero real root of the cubic algebraic equation

$$b_1\omega^3 + (b_2+b_3-a_1)\omega^2 + (b_4-a_2-a_3)\omega - a_4 = 0. \tag{2.3}$$

We now assume that ω satisfies (2.3) and denote

$$B = \frac{1}{2}(b_1\omega^2 + (b_2+b_3)\omega + b_4). \tag{2.4}$$

System (2.2) is reduced to

$$A_2'' = cA_2 - BA_2^2. \tag{2.5}$$

This is the same as (1.8) with $\beta = -B$. In the phase plane (A_2, A_2'), (2.5) has two equilibrium points $O(0,0)$ and $E(c/B, 0)$. It is easy to see that when $c > 0$ (< 0), O is a saddle point (a center); E is a center (a saddle point).

Using Lemma 1.1, we obtain the following results.

Theorem 2.1. *Let ω be a real root of* (2.3) *and B be as in* (2.4)*.*

(1) If $c > 0$, then the origin O is a saddle and E a center. If $c < 0$, then O is a center and E a saddle.

(2) (CKdV) has a family of periodic wave solutions encircling the center parameterized by $h \in (-\frac{c^3}{6B^2}, 0)$ if $c > 0$ or $h \in (0, -\frac{c^3}{6B^2})$ if $c < 0$:

$$A_2(\xi) = p(\xi, c, -B, h), \quad A_1(\xi) = \omega A_2(\xi). \tag{2.6}$$

System (CKdV) also has a solitary wave solutions of peak type asymptotic to the saddle point:

$$A_2(\xi) = q(\xi, c, -B), \quad A_1(\xi) = \omega A_2(\xi). \tag{2.7}$$

To find traveling wave solutions for the P-integrable model 1, let $\xi = X - cT,\ u = A_1(\xi),\ v = A_2(\xi)$. From (1.1),

$$\begin{aligned} -cu' + u''' - [(c_0+3)(c_0+6)u^2 + c_0^2 v^2]_\xi + 2c_0[(c_0+6)u_\xi v + (c_0+3)uv_\xi] = 0, \\ -cv' + v''' - [(c_0+3)^2 u^2 + c_0(c_0-3)v^2]_\xi + 2(c_0+3)[c_0 v u_\xi + (c_0-3)uv_\xi] = 0, \end{aligned} \tag{2.8}$$

Corresponding to (2.8), the parameters of (2.1) has the special values:

$$\begin{aligned} &a_1 = -2(c_0+3)(c_0+6), \quad a_2 = 2c_0(c_0+3), \quad a_3 = 2c_0(c_0+6), \quad a_4 = -2c_0^2, \\ &b_1 = -2(c_0+3)^2, \quad b_2 = 2(c_0+3)(c_0-3), \quad b_3 = 2c_0(c_0+3), \quad b_4 = -2c_0(c_0-3). \end{aligned}$$

The quibic equation (2.3) becomes

$$\begin{aligned} &(c_0+3)^2\omega^3 - 3(c_0+3)(c_0+1)\omega^2 + 3c_0(c_0+2)\omega - c_0^2 \\ =&((c_0+3)\omega - c_0)^2(\omega - 1) = 0. \end{aligned} \tag{2.9}$$

The roots of (2.9) are $\omega = c_0/(c_0+3)$ and $\omega = 1$. This suggests the change of variables $X = (c_0+3)u - c_0 v,\ Y = u - v$, or $u = \frac{1}{3}X - \frac{c_0}{3}Y, v = \frac{1}{3}X - \frac{c_0+3}{3}Y$. The result is a partially uncoupled system of equations,

$$X''' = cX' + 12XX', \tag{2.10}$$

$$Y''' = cY' + 6XY'. \tag{2.11}$$

We can recover (u, v) by

$$\begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} X \\ Y \end{pmatrix}, \quad M = \frac{1}{3}\begin{pmatrix} 1 & -c_0 \\ 1 & -(c_0+3) \end{pmatrix}. \tag{2.12}$$

Integrating once and taking the integration constant to be zero, we have

$$X'' = cX + 6X^2,$$

$$Z'' = cZ + 6XZ,$$

where $Z = Y'$ and $Y = \int Z d\xi$.

Theorem 2.2. *For the P-integrable model 1, we have*

(1) on the plane $(c_0 + 3)u - c_0 v = 0$, or $X = 0$, the P-integrable model 1 reduces to $Y''' = cY'$. The only bounded solutions are harmonic periodic waves oscillating around the mean value $A_1 = K/c$, $A_2 = (c_0 + 3)K/(cc_0)$. They occur only if $c < 0$.

(2) On the plane $u - v = 0$ or $Y = 0$, model 1 reduces to $X'' = cX + 6X^2$, the same as (1.8) *with $\beta = 6$. The only bounded solutions are solitary waves $X = q(\xi, c, 6)$ and periodic waves $X = p(\xi, c, 6, h)$. The traveling waves in (A_1, A_2) can be expressed as $(A_1, A_2)^\tau = M(X, 0)^\tau$.*

Apart from the particular solutions described in Theorem 2.2, much richer dynamical behavior of the system can be found if we consider bounded traveling wave solutions of X from (2.10) first then plug them into (2.11) for Y. Discussion of such solutions will be deferred to Section 4 while similar cases from P-integrable models 3 and 5 are considered.

3. Traveling wave solutions of the P-integrable model 2

The traveling wave solutions of (1.2) in traveling coordinate satisfy

$$\begin{aligned} A_{1\xi\xi} &= cA_1 + \frac{1}{2}(c_1 - c_2 + c_1c_2)A_1^2 - c_1A_1A_2 + \frac{1}{2}A_2^2, \\ A_{2\xi\xi} &= cA_2 + \frac{1}{2}c_1c_2A_1^2 - c_2A_1A_2 + \frac{1}{2}(c_2 - c_1 + 1)A_2^2. \end{aligned} \tag{3.1}$$

The quadratic forms in (3.1) can be expressed as

$$(A_1, A_2)Q_1(c_1, c_2)(A_1, A_2)^\tau, \quad (A_1, A_2)Q_2(c_1, c_2)(A_1, A_2)^\tau,$$

where

$$Q_1(c_1, c_2) = \begin{pmatrix} (c_1 - c_2 + c_1c_2)/2 & -c_1/2 \\ -c_1/2 & 1/2 \end{pmatrix}, \qquad Q_2(c_1, c_2) = \begin{pmatrix} c_1c_2/2 & -c_2/2 \\ -c_2/2 & (c_2 - c_1 + 1)/2 \end{pmatrix}.$$

Under the conditions $c_1 \neq 1, c_2 \neq 1$ and $c_2 \neq c_1$, the matrices A and B satisfies a condition of simultaneous diagonilization by nonsingular real matrices [8]. Our calculation shows that only $c_2 \neq 1$ is required in the co-diagonalization. Setting

$$M = \frac{1}{1 - c_2}\begin{pmatrix} 1 & -1 \\ 1 & -c_2 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} -c_2 & 1 \\ -1 & 1 \end{pmatrix}, \quad c_2 \neq 1$$

we have

$$M^\tau Q_1 M = \frac{1}{2(1-c_2)}\begin{pmatrix} 1-c_1 & 0 \\ 0 & c_1-c_2 \end{pmatrix} \quad M^\tau Q_2 M = \frac{1}{2(1-c_2)}\begin{pmatrix} 1-c_1 & 0 \\ 0 & c_2(c_1-c_2) \end{pmatrix}.$$

By the change of variables

$$(A_1, A_2)^\tau = M \cdot (u, v)^\tau, \tag{3.2}$$

the non-diagonal terms in the quadratic forms of (3.1) can removed. This leads to

$$\begin{aligned} A_1'' &= cA_1 + \frac{1-c_1}{2(1-c_2)}u^2 + \frac{c_1-c_2}{2(1-c_2)}v^2, \\ A_2'' &= cA_2 + \frac{1-c_1}{2(1-c_2)}u^2 + \frac{c_2(c_1-c_2)}{2(1-c_2)}v^2. \end{aligned} \tag{3.3}$$

Applying the inverse transform of (3.2), $u = A_2 - c_2 A_1$, $v = A_2 - A_1$ to (3.3), the reduced system should have no uv term. What unexpected is that the result is a completely uncoupled system of two equations.

$$u_{\xi\xi} = cu + \frac{1-c_1}{2}u^2, \tag{3.4}$$

$$v_{\xi\xi} = cv + \frac{c_2-c_1}{2}v^2. \tag{3.5}$$

Equation (3.4) has two equilibria $U_0 = 0, U_1 = 2c/(c_1-1)$ while (3.5) has two equilibria $V_0 = 0, V_1 = 2c/(c_1-c_2)$.

Lemma 3.1. *Assume that $c_1 \neq 1, c_2 \neq 1$ and $c_1 \neq c_2$. Then*
(I) If $c > 0$, then for (3.4), U_0 is a saddle with eigenvalues $\pm\sqrt{|c|}$, and U_1 is a center with eigenvalues $\pm\sqrt{|c|}i$. For (3.5), V_0 is a saddle with eigenvalues $\pm\sqrt{|c|}$, and V_1 is a center with eigenvalues $\pm\sqrt{|c|}i$.

(II) If $c < 0$, then similar properties for (3.4))(or (3.5)) still hold if we switch U_0 with U_1 (or V_0 with V_1).

Define

$$e_1 = -\frac{2c}{1-c_1-c_2+c_1c_2}, \; e_2 = \frac{2c}{c_2-c_1-c_2^2+c_1c_2}, \; e_3 = -\frac{2c}{c_1-c_2-c_1^2+c_1c_2}.$$

It is now clear that (3.1) has four equilibrium points corresponding to the combinations of equilibrium points of (3.4) and (3.5):

$$(U_0, V_0) \Leftrightarrow E_0 : \{(A_1, A_2) = (0,0)\}, \qquad (U_1, V_0) \Leftrightarrow E_1 : \{(A_1, A_2) = (e_1, e_1)\},$$

$$(U_0, V_1) \Leftrightarrow E_2 : \{(A_1, A_2) = (e_2, c_2 e_2)\}, \qquad (U_1, V_1) \Leftrightarrow E_3 : \{(A_1, A_2) = (e_3, c_1 e_3)\}.$$

From Lemma 3.1, we have the following results about equilibria E_0 to E_3 of (3.1).

Lemma 3.2. *Assume that $c_1 \neq 1, c_2 \neq 1$ and $c_1 \neq c_2$. Then for* (3.1),

(I) if $c > 0$, E_0 is a saddle with eigenvalues $\pm\sqrt{|c|}$ while E_3 is a center with eigenvalues $\pm\sqrt{|c|}i$. Both the algebraic and geometric multiplicities of these eigenvalues are equal to 2. (semi-simple eigenvalues). E_1 and E_2 are center-saddle points with eigenvalues $\pm\sqrt{|c|}$ and $\pm\sqrt{|c|}i$.

(II) If $c < 0$, then the properties on E_1 and E_2 remain unchanged but properties on E_0 and E_3 must be switched.

Define

$$W_u(E_0) := \{(A_1, A_2) : A_2 - A_1 = 0\}, \quad W_v(E_0) := \{(A_1, A_2) : A_2 - c_2 A_1 = 0\}.$$

$$W_u(E_1) := \{(A_1, A_2) : A_2 - A_1 = 0\}, \quad W_v(E_1) := \{(A_1, A_2) : A_2 - c_2 A_1 = U_1\}.$$

$$W_u(E_2) := \{(A_1, A_2) : A_2 - A_1 = V_1\}, \quad W_v(E_2) := \{(A_1, A_2) : A_2 - c_2 A_1 = 0\}.$$

$$W_u(E_3) := \{(A_1, A_2) : A_2 - A_1 = V_1\}, \quad W_v(E_3) := \{(A_1, A_2) : A_2 - c_2 A_1 = U_1\}.$$

From (3.4) and (3.5), $W_u(E_j)$ and $W_v(_j)$ are invariant under the flow of (3.1) and $E_j \in W_u(E_j) \cap W_v(E_j)$. Using Lemma 1.8, we find all the traveling waves for the P-integrable model 2.

Theorem 3.3. *Assume that $c_1 \neq 1, c_2 \neq 1$ and $c_1 \neq c_2$. Then*
(I) if $c > 0$, then on $W_u(E_1)$ there is a family of traveling periodic solutions encircling E_1: $u = p(\xi, c, (1 - c_1)/2, h)$. On $W_v(E_1)$, exists a unique solitary wave solution $v = q(\xi, c, (c_2 - c_1)/2)$.

On $W_u(E2)$, there exists a unique solitary wave $u = q(\xi, c, (1 - c_1)/2$ asymptotic to E_2. On $W_v(E_2)$ there is a family of traveling periodic solutions encircling E_2: $v = p(\xi, c, (c_2 - c_1)/2, h)$.

(II) If $c < 0$ then the conclusions similar to part (I) hold if E_1 and E_2 get switched.

Theorem 3.4. *Assume that $c_1 \neq 1, c_2 \neq 1$ and $c_1 \neq c_2$. Then*
(I) if $c > 0$, then there exist solitary waves on $W_u(E_0)$: $u = q(\xi, c, (1 - c_1)/2)$ and on $W_v(E_0)$: $v = q(\xi, c, (c_2 - c_1)/2)$ asymptotic to E_0.

There exist families of traveling periodic waves on both $W_u(E_3)$ and $W_v(E_3)$ encircling E_3. They are $u = p(\xi, c, (1 - c_1)/2, h)$ and $v = p(\xi, c, (c_2 - c_1)/2, h)$.

(II) If $c < 0$ then similar conclusion hold if we switch E_0 with E_3.

Corollary 3.5. *The traveling wave solutions for P-integrable model 2 are*

$$(A_1(\xi), A_2(\xi))^\tau = M(u(\xi - \xi_1), v(\xi - \xi_2))^\tau$$

where (u, v) are traveling wave solutions as in Theorem 3.3 and Theorem 3.4 and ξ_1, ξ_2 are arbitrarily constants.

Remark 3.1. (1) If $c_1 = 1$, $c_2 \neq 1$, then the only equilibria are E_0 and E_2. If $c_1 \neq 1$, $c_2 = c_1$, then the only equilibria are E_0 and E_1. If $c_1 = c_2 = 1$, the only the equilibrium is E_0. In these special cases, (3.1) is much simpler and its traveling waves are easy to analyze. We will skip the details.

(2) The cubic equation (2.3) for P-integrable model 2 is

$$c_1c_2\omega^3 - (c_2 + c_1 + c_1c_2)\omega^2 + (c_1 + c2 + 1)\omega - 1 = (c_2\omega - 1)(c_1\omega - 1)(\omega - 1)) = 0,$$

with three distinct roots $\omega = 1, c_1, c_2$. Only $\omega = 1$ and c_2 are used in our change of variables. We have tried the variable $A2 - c_1A_1$ and found that (3.1) does not get simplifed.

We prefer matrices diagonalization since it provides definitive result. If after eliminating the non-diagonal terms the system does not decouple, then we can show that there does not exist a linear change of variable that can further decouple the system, unless the two original quadratic forms are linearly dependent. In this case, one of the decoupled equation is linear.

4. Traveling wave solutions for the P-integrable mode 3, 4 and 5

For the P-integrable models 3, 4 and 5, (see (1.3), (1.4) and (1.5)), we make the change of variables $A_1(\xi) + A_2(\xi) = u(\xi)$, $A_1(\xi) - A_2(\xi) = v(\xi)$, i.e., $A_1(\xi) = \frac{1}{2}(u + v)$, $A_2(\xi) = \frac{1}{2}(u - v)$. Then, the traveling wave solutions of (1.3) are determined by the system

$$u_{\xi\xi} - cu + u^2 = 0, \quad v_{\xi\xi} + (u - c)v = 0. \tag{4.1}$$

The traveling wave solutions of (1.4) are given by the system

$$u_{\xi\xi} - cu + 2u^2 = 0, \quad v_{\xi\xi} - cv = 0. \tag{4.2}$$

The traveling wave solutions of (1.5) are determined by the system

$$u_{\xi\xi} - cu + u^2 = 0, \quad A_{1\xi\xi\xi} + (2u - c)A_{1\xi} = 0.$$

Let $A_{1\xi} = w$. Then

$$u_{\xi\xi} - cu + u^2 = 0, \quad w_{\xi\xi} + (2u - c)w = 0. \tag{4.3}$$

Note that the change of variables is invertible: $A_1(\xi) = \int^\xi w(s)ds$, $A_2(\xi) = u(\xi) - A_1(\xi)$.

4.1. **The P-integrable model 4.** We first discuss system (4.2) which consists of two uncoupled equations. We are interested in the bounded solutions of (4.2). Therefore, we assume that $c < 0$. Using Lemma 1.1 with $\beta = -2$, we have the following conclusion.

Theorem 4.1. *System* (1.4) *has the following bounded exact traveling wave solutions:*

(i) Asymptotically periodic solutions:

$$A_1(\xi) = \frac{1}{2}\left[q(\xi, c, -2) + \gamma \cos\sqrt{|c|}\xi\right], \qquad (4.4)$$
$$A_2(\xi) = \frac{1}{2}\left[q(\xi, c, -2) - \gamma \cos\sqrt{|c|}\xi\right].$$

(ii) Quasi-periodic solutions, with $h \in (0, -c^3/24)$*:*

$$A_1(\xi) = \frac{1}{2}\left[p(\xi, c, -2, h) + \gamma \cos\sqrt{|c|}\xi\right], \qquad (4.5)$$
$$A_2(\xi) = \frac{1}{2}\left[p(\xi, c, -2, h) - \gamma \cos\sqrt{|c|}\xi\right].$$

4.2. **The P-integrable model 3 and 5.** We now consider systems (4.1) and (4.3). The first equations for the two systems are the same:

$$u'' = cu - u^2 \qquad (4.6)$$

Assume that $c < 0$. Equation (4.6) has two equilibrium points: center $O(0,0)$ and saddle point $E(c,0)$. By Lemma 1.1, with $\beta = -1$, we find that

(1) Equation (4.6) has a family of periodic orbits encircling O, parametrized by the periodic solutions

$$u = p(\xi, c, -1, h), \quad h \in (0, -c^3/6). \qquad (4.7)$$

(2) Equation (4.6) also has a unique homoclinic orbit asymptotic to E defined by the homoclinic solution:

$$u(\xi) = q(\xi, c, -1). \qquad (4.8)$$

Substituting (4.7) and (4.8) into (4.1), we find two possible equations for v:

$$v_{\xi\xi} + \left(|c| + r_1 - (r_1 - r_2)\mathrm{sn}^2(\Omega\xi, k)\right) v = 0, \qquad (4.9)$$

$$v_{\xi\xi} + \left(\frac{3|c|}{2}\mathrm{sech}^2\left(\frac{\sqrt{|c|}}{2}\xi\right)\right) v = 0 \qquad (4.10)$$

Substituting (4.7) and (4.8) into (4.3), we find two possible equations for w:

$$w_{\xi\xi} + \left(2r_1 + |c| - 2(r_1 - r_2)\mathrm{sn}^2(\Omega\xi, k)\right) w = 0, \qquad (4.11)$$

$$w_{\xi\xi} + \left(c + 3|c|\mathrm{sech}^2\left(\frac{\sqrt{|c|}}{2}\xi\right)\right) w = 0. \qquad (4.12)$$

Equations (4.9) and (4.11) are special forms of the Hill equation $x''+(a+\phi(t))x = x''+p(t)x = 0$ (see Cesari [1959]). Denote $p_1(\xi) = |c| + r_1 - (r_1 - r_2)\mathrm{sn}^2(\Omega\xi, k)$ and $p_2(\xi) = 2r_1 + |c| - 2(r_1 - r_2)\mathrm{sn}^2(\Omega\xi, k)$. It is easy to show that for $h \in \left(0, -\frac{1}{6}c^3\right)$, we have $p_1(\xi) > 0$,

$$p_{1m} \equiv \frac{\Omega}{2K(k)} \int_0^{\frac{2K(k)}{\Omega}} p_1(\xi)d\xi = |c| + r_3 + \frac{(r_1 - r_3)}{2}\frac{E(k)}{K(k)}$$

and when $2r_2 + |c| > 0,\ p_2(\xi) > 0$,

$$p_{2m} \equiv \frac{\Omega}{2K(k)} \int_0^{\frac{2K(k)}{\Omega}} p_2(\xi)d\xi = |c| + 2r_3 + (r_1 - r_3)\frac{E(k)}{K(k)}.$$

We can show that the condition of Borg's theorem [2]

$$T\int_0^T |p_j(\xi)|d\xi = \left(\frac{2K(k)}{\Omega}\right)^2 |p_{jm}| \leq 4, \quad j = 1, 2 \tag{4.13}$$

cannot be satisfied. So we cannot use it to conclude that any solution of (4.9) and (4.11) is bounded or stable.

However conditions (4.13) are only sufficient conditions for the existence of bounded solutions of (4.9) and (4.11). By using Theorem 8.1 in Hale [7], there exist two real sequences of the number $|c|$: $\{c_0 < c_1 \leq c_2 \leq \cdots\}$ and $\{c_1^* \leq c_2^* \leq c_3^* \leq \cdots\}$, when $k \to \infty, c_k, c_k^* \to \infty$,

$$c_0 < c_1^* \leq c_2^* < c_1 \leq c_2 < c_3^* \leq c_4^* < c_3 \leq c_4 < \cdots$$

such that (4.9) and (4.11) have periodic solutions with period $\frac{2K(k)}{\Omega}$ (or $\frac{4K(k)}{\Omega}$), if and only if for some $k = 0, 1, 2, \cdots$, we have $|c| = c_k$ (or for some $k = 0, 1, 2, \cdots$, we have $|c| = c_k^*$). The solutions of (4.9) and (4.11) are stable in the intervals

$$(c_0, c_1^*), (c_2^*, c_1), (c_2, c_3^*), (c_4^*, c_3), \cdots. \tag{4.14}$$

And the solutions of (4.9) and (4.11) are unstable in the intervals

$$(-\infty, c_0], (c_1^*, c_2^*), (c_1, c_2), (c_3^*, c_4^*), (c_3, c_4), \cdots.$$

Therefore, (4.9) and (4.11) have bounded solutions when the parameter $|c|$ belongs to a stable interval in (4.14). We summarize our results in the following theorem.

Theorem 4.2. *Assume that $c < 0$ in (4.1) and (4.3). Then there are infinitely many pairs (c, h) where $h \in \left(0, -\frac{1}{6}c^3\right)$, $|c| = c_k, c_k^*$ or $|c|$ is in one of the intervals of (4.14). For such (c, h), (4.1) and (4.3) have solutions (u, v) and (u, w) where $u = p(\xi, c, -1, h)$ is periodic and $v(\xi)$ and $w(\xi)$ are bounded.*

(1) For the P-integrable model 3, the bounded traveling waves are

$$A_1 = \frac{1}{2}(u + v), \quad A_2 = \frac{1}{2}(u - v).$$

(2) For the P-integrable model 5, if $\int^{\xi} w(s)ds$ *is a bounded function on* $\mathbb{R}$*, then The bounded traveling wave solutions are*

$$A_1(\xi) = \int^{\xi} w(s)ds, \quad A_2(\xi) = u(\xi) - A_1(\xi).$$

In particular, for any constant γ*,* $(A_1, A_2) = (\gamma, u - \gamma)$ *is a periodic traveling wave solution.*

Remark 4.1. The condition for $\int w(\xi)d\xi$ to be a bounded function is rather complicated and better left to a separate paper.

If $c > 0$, there are periodic solutions $u = p(\xi, c, -1, h)$ oscillating around the center E. It is possible to plug these solutions into the equations for v and w and look for bounded solutions.

Finally, we consider equation (4.10) and (4.12). Let

$$p_3(\xi) = \frac{3|c|}{2}\mathrm{sech}^2\left(\frac{\sqrt{|c|}}{2}\xi\right), \quad p_4(\xi) = c + 3|c|\mathrm{sech}^2\left(\frac{\sqrt{|c|}}{2}\xi\right).$$

Because $\int_{-\infty}^{\infty} p_3(t)dt$ is convergent and $c < 0$, by using the results mentioned in Cesari [3], we find that the solutions of (4.10) and (4.12) are non-oscillating and unbounded.

Remark 4.2. A general coupled KdV system has been studied in [12] where the third order coefficients may not be equal. Apparently (2.10)–(2.11) from model 1 correspond to the case (ii) in [12], system (4.1) from model 3 corresponds to the case (vii) in [12], and system (4.3) corresponds to (vi) in [12]. Models 2 and 4 were not studied in [12].

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