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Exponential Dichotomies and Homoclinic Orbits in Functional Differential Equations*

XIAO-BIAO LIN

Lefschetz Center for Dynamical Systems,
Division of Applied Mathematics, Brown University,
Providence, Rhode Island 02912

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Suppose an autonomous functional differential equation has an orbit Γ which is homoclinic to a hyperbolic equilibrium point. The purpose of this paper is to give a procedure for determining the behavior of the solutions near Γ of a functional differential equation which is a nonautonomous periodic perturbation of the original one. The procedure uses exponential dichotomies and the Fredholm alternative. It is also shown that any smooth function $p(t)$ defined on the reals which approaches zero monotonically as $t \to \pm\infty$ is the solution of a scalar functional differential equation and generates an orbit homoclinic to zero. Examples illustrating the results are also given.

1. INTRODUCTION

Exponential dichotomies have played an important role in both the theory and applications of ordinary differential equations. For example, under some very mild conditions, a linear system admits an exponential dichotomy on $[0, \infty)$ if and only if the nonhomogeneous linear system has a bounded solution on $[0, \infty)$ for every bounded forcing function (see, for example, Coppel [6]). The extension of this result to functional differential equations was given by Coffman and Schäffer [5], Pecelli [17]. If a linear system has an exponential dichotomy on some interval, then small perturbations of the linear system will not destroy the exponential dichotomy (see Coppel [6]).

Exponential dichotomies also arise as the linear variational equation near a homoclinic orbit of an autonomous equation. This observation was recently made by Palmer [16] for ordinary differential equations extending an idea used by Chow, Hale and Mallet-Paret [4] for a special case. He then exploited this property to discuss the behavior of the solutions of periodically perturbed systems near the homoclinic orbit. The procedure

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uses the Fredholm alternative and the method of Liapunov-Schmidt in bifurcation theory. It is related to but not the same as the method of Mel'nikov [14].

In Section 2 of this paper, we give the basic results on perturbations of systems which possess exponential dichotomies using the definition of dichotomy given by Henry [12] and in a setting which will be applicable to functional differential equations. In Section 3, we extend the above-mentioned work of Palmer on the Fredholm alternative to functional differential equations. In Section 4, we show that any smooth positive function $p(t)$ defined on the reals which approaches zero monotonically as $t \to \pm\infty$ is the solution of a scalar functional differential equation and generates an orbit homoclinic to zero. Section 5 is devoted to the application of the results to periodic perturbations of autonomous systems.

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2. Exponential Dichotomies and Linear Semiflows

In this section, we shall consider the following delay differential equation

$$\begin{aligned} x(t) &= L(t)\, x_t, \qquad t \geqslant s, \\ x_s &= \phi, \end{aligned} \tag{2.1}$$

where x_t, $\phi \in C([-r, 0], R^n)$, $x_t(\theta) = x(t+\theta)$, $-r \leqslant \theta \leqslant 0$, $x(t)$, is defined on some interval $J \subset R$ and takes value in R^n. $L(t) \in \mathscr{L}(C[-r, 0], R^n)$ and is continuous with respect to $t \in J$ in the operator norm. The solution operator $T(t, s)$: $C[-r, 0] \to C[-r, 0]$, $T(t, s)\,\phi = x_t$, is a semigroup for $t \geqslant s$, is linear and continuous in $\phi \in C[-r, 0]$, and is strongly continuous in t and s. It is said that $T(t, s)$ has an exponential dichotomy on an interval J with constants $K \geqslant 0$, $\alpha > 0$ if there are projections $P(s)$, $Q(s) = I - P(s)$, $s \in J$, strongly continuous in s, and

(i) $T(t, s)\, P(s) = P(t)\, T(t, s)$, $t \geqslant s$, in J;

(ii) $T(t, s) \mid \mathscr{R}Q(s)$, $t \geqslant s$, is an isomorphism of $\mathscr{R}Q(s)$ onto $\mathscr{R}Q(t)$ and $T(s, t)$: $\mathscr{R}Q(t) \to \mathscr{R}Q(s)$ is defined as the inverse of $T(t, s) \mid \mathscr{R}Q(s)$;

(iii) $|T(t, s)\, P(s)| \leqslant Ke^{-\alpha(t-s)}$, $t \geqslant s$, in J;

(iv) $|T(s, t)\, Q(t)| \leqslant Ke^{-\alpha(t-s)}$, $t \geqslant s$, in J.

$\mathscr{R}P(t)$ and $\mathscr{R}Q(t)$, $t \in J$ are called the stable and unstable subspaces of $T(t, s)$. We assume that $\mathscr{R}Q(t)$ has finite dimension. This is not an unreasonable assumption since $T(t, s)$ is compact for $t > s + r$.

Let C denote $C([-r,0], R^n)$, a Banach space with supremum norm. The adjoint of C, C^*, is identified with $B_0([-r,0], R^{n^*})$, the Banach space of all the functions ψ: $[-r,0] \to R^{n^*}$ of bounded variation with $\psi(0)=0$ and continuous from the left in $(-r,0)$. Let $T^*(s,t)$ be the adjoint of $T(t,s)$

$$\langle \psi, T(t,s)\,\phi \rangle = \langle T^*(s,t)\,\psi, \phi \rangle, \qquad \phi \in C, \psi \in C^*.$$

$T^*(s,t)$ is a linear semigroup for $s \leqslant t \in J$, and is weak* continuous with respect to $s \leqslant t \in J$. The integral of $T^*(s,t)\,\psi$ with respect to s or t are defined in the weak* sense. If $P^*(s)$ is the adjoint of $P(s)$, it is a projection operator, which is weak* continuous in $s \in J$. We claim that $T^*(s,t)$ has an exponential dichotomy on J with the projections $P^*(s)$ and $Q^*(s)$, except that the strong continuity is replaced by weak* continuity. Obviously, we have

(i)′ $T^*(s,t)\,P^*(t) = P^*(s)\,T^*(s,t)$, $s \leqslant t$, in J;

(iii)′ $|T^*(s,t)\,P^*(t)| \leqslant Ke^{-\alpha(t-s)}$, $s \leqslant t$, in J;

It is also true that

(iv)′ $|T^*(t,s)\,Q^*(s)| \leqslant Ke^{-\alpha(t-s)}$, $s \leqslant t$, in J;

provided that

(ii)′ $T^*(s,t)|\mathscr{R}Q^*(t)$, $s \leqslant t$, is an isomorphism of $\mathscr{R}Q^*(t)$ onto $\mathscr{R}Q^*(s)$ and $T^*(t,s)$: $\mathscr{R}Q^*(s) \to \mathscr{R}Q^*(t)$ is defined as the inverse of $T^*(s,t)|_{\mathscr{R}Q^*(t)}$ (or the adjoint of $T(s,t)$: $\mathscr{R}Q(t) \to \mathscr{R}Q(s)$).

Proof of (ii). For any $\phi \in C$, we write $\phi = (x,y)_s$, where $x \in \mathscr{R}P(s)$ and $y \in \mathscr{R}Q(s)$. By the invariance of $T(t,s)$ on $\mathscr{R}P(s)$ and $\mathscr{R}Q(s)$, we write $T(t,s) = T_1(t,s) \times T_2(t,s)$, $T_1(t,s)$: $\mathscr{R}P(s) \to \mathscr{R}P(t)$, $T_2(t,s)$: $\mathscr{R}Q(s) \to \mathscr{R}Q(t)$, such that $T(t,s)(x,y)_s = (T_1(t,s)\,x, T_2(t,s)\,y)_t$. Accordingly, for any $\psi \in C^*$, we write $\psi = (x^*, y^*)_t$, where $x^* \in (\mathscr{R}P(t))^*$ and $y^* \in (\mathscr{R}Q(t))^*$. $T^*(s,t) = T_1^*(s,t) \times T_2^*(s,t)$ and $T^*(s,t)(x^*,y^*)_t = (T_1^*(s,t)\,x^*, T_2^*(s,t)\,y^*)_s$. Since $T_2(t,s)$; $\mathscr{R}Q(s) \to \mathscr{R}Q(t)$ is one-to-one and onto, a general theorem in Functional Analysis implies that $T_2^*(s,t)$: $(\mathscr{R}Q(t))^* \to (\mathscr{R}Q(s))^*$ is an isomorphism. It is easy to see that $\mathscr{R}Q^*(t) = (\mathscr{N}Q(t))^0 = (\mathscr{R}P(t))^0$, where M^0 indicates the annihilator of a subset $M \subset C$. We then obtain a natural isomorphism between $\mathscr{R}Q^*(t)$ and $(\mathscr{R}Q(t))^*$; that is,

$$\mathscr{R}Q^*(t) = \{(0, y^*) \mid y^* \in (\mathscr{R}Q(t))^*\},$$

and $T^*(s,t)(0,y^*) = (0, T_2^*(s,t)\,y^*)$. Thus, $T^*(s,t)|_{\mathscr{R}Q^*(t)}$ is one-to-one and onto $\mathscr{R}Q^*(s)$. The continuity of $T^*(t,s)|_{\mathscr{R}Q^*(s)}$ follows from the open mapping theorem in Banach spaces.

The proof of the following lemma is the same as that of ordinary differential equations which can be found in [8].

LEMMA 2.1. *Let J be either $R+$, $R-$ or R. If $\dot{x} = L(t)\,x_t$ has an exponential dichotomy on J and $B(t)$ is a linear continuous operator for $t \in J$ and continuous with respect to t in the operator norm, $|B(t)| \leqslant \delta$ for all $t \in J$, then*

$$\dot{x}(t) = (L(t) + B(t))\,x_t$$

has an exponential dichotomy on J if δ is sufficiently small.

COROLLARY 2.2. *If $\dot{x}(t) = L^{\pm} x_t$ has zero as an hyperbolic equilibrium point, and if $L(t) \to L^{+}(L^{-})$ as $t \to +\infty$ $(t \to -\infty)$, then* (2.1) *has an exponential dichotomy on $[\tau, +\infty)((-\infty, -\tau])$ for some constant $\tau > 0$, with the projection $P(t) \to P^{+}(P^{-})$ as $t \to +\infty$ $(t \to -\infty)$, where $P^{\pm}$ is the projection associated with the autonomous equation.*

We now present some results concerning the extension of domains where the exponential dichotomy is valid. We first observe that the projections $P(t)$ and $Q(t)$ are uniquely defined if their ranges $X_1 = \mathscr{R}P(t)$ and $X_2 = \mathscr{R}Q(t)$, two closed subspaces in C, are given and $C = X_1 \oplus X_2$. We also observe that $\mathscr{R}P(t)$ is unique for $t \in J = [\tau, +\infty)$ whereas $\mathscr{R}Q(t)$ is unique for $t \in J = (-\infty, -\tau]$.

Let $\phi(t)$, $t \in J_1 = (-\infty, t_0]$, be in the unstable subspace of $T(t, s)$, such that $\phi(t)$ approaches 0 exponentially as $t \to -\infty$. Then $\phi(t)$ restricted to $J = (-\infty, -\tau]$, $-\tau < t_0$, is in the unstable subspace of $T(t, s)$ for $t \in J$. Let the dimension of the unstable subspace for $t \in J(J_1)$ be $M_J(M_{J_1})$, we have $M_{J_1} \leqslant M_J$. A necessary condition for the exponential dichotomy on J being extendable to J_1 is that $M_{J_1} = M_J$. We prove that this is also a sufficient condition.

LEMMA 2.3. *Let $J = (-\infty, -\tau]$ and suppose* (2.1) *has an exponential dichotomy on J. Let $t_0 > -\tau$, and $T(t_0, -\tau)\,\phi \neq 0$ for any $0 \neq \phi \in \mathscr{R}Q(-\tau)$. Then* (2.1) *has an exponential dichotomy on $(-\infty, t_0]$ with the projections $\tilde{P}(t)$, $\tilde{Q}(t) \to P(t)$, $Q(t)$ exponentially as $t \to -\infty$.*

Proof. Let $X_1(t) = \mathscr{R}Q(t)$ for $t \leqslant -\tau$ and

$$X_1(t) = \{T(t, -\tau)\,\phi \mid \phi \in \mathscr{R}Q(-\tau)\} \qquad \text{for} \quad -\tau \leqslant t \leqslant t_0$$

Thus defined, $\dim X_1(t)$ is constant for $t \leqslant t_0$ and $T(t, s)$ is an isomorphism from $X_1(s)$ onto $X_1(t)$, $t_0 \geqslant t \geqslant s$. Let $X_2(t_0)$ be a closed subspace of C complementary to $X_1(t_0)$, i.e.,

$$X_1(t_0) \oplus X_2(t_0) = C.$$

Let

$$X_2(t)=\{\phi \mid T(t_0,t)\,\phi\in X_2(t_0)\},\qquad t\leqslant t_0.$$

$X_2(t)$ is a linear closed subspace of C and we claim that

$$X_1(t)\oplus X_2(t)=C,\qquad t\leqslant t_0. \tag{2.2}$$

This is proved as follows.

If $\phi\in X_1(t)\cap X_2(t)$, then $T(t_0,t)\,\phi\in X_1(t_0)\cap X_2(t_0)$. Thus, $T(t_0,t)\,\phi=0$, implying that $\phi=0$ for $T(t_0,t)$ is an isomorphism from $X_1(t)$ to $X_1(t_0)$. Now let $\phi\in C$, $T(t_0,t)\,\phi=\phi_1+\phi_2$, where $\phi_1\in X_1(t_0)$ and $\phi_2\in X_2(t_0)$. There exists $\tilde{\phi}_1\in X_1(t)$ such that $\phi_1=T(t_0,t)\,\tilde{\phi}_1$. Therefore $T(t_0,t)(\phi-\tilde{\phi}_1)=\phi_2\in X_2(t_0)$, implying that $\phi-\tilde{\phi}_1\in X_2(t)$. The proof of (2.2) is completed.

Define $\mathscr{R}\tilde{Q}(t)=X_1(t)$ and $\mathscr{R}\tilde{P}(t)=X_2(t)$ for $t\leqslant t_0$. Accordingly, the projections $\tilde{P}(t)$ and $\tilde{Q}(t)$ are defined and $\tilde{P}(t)+\tilde{Q}(t)=I$. Obviously, $\mathscr{R}\tilde{P}(s)$ and $\mathscr{R}\tilde{Q}(s)$ are invariant under $T(t,s)$, $t\geqslant s$ in $(-\infty,t_0]$ for $X_1(s)$ and $X_2(s)$ are. Consequently,

$$\begin{aligned}
&T(t,s)\,\tilde{P}(s)=\tilde{P}(t)\,T(t,s),\\
&T(t,s)\,\tilde{Q}(s)=\tilde{Q}(t)\,T(t,s),\qquad t\geqslant s,\ \text{in }(-\infty,t_0].
\end{aligned}$$

Therefore, $\tilde{Q}(t)=T(t,t_0)\,\tilde{Q}(t_0)\,T(t_0,t)$ for $t\leqslant t_0$, implying that $\tilde{Q}(t)$ is strongly continuous and bounded in norm for t in any compact subset of $(-\infty,t_0]$. So is $\tilde{P}(t)$.

If $s\leqslant -\tau$, then

$$\begin{aligned}
\tilde{Q}(s)&=\tilde{Q}(s)(Q(s)+P(s))\\
&=Q(s)+\tilde{Q}(s)\,P(s)\\
&=Q(s)+T(s,-\tau)\,\tilde{Q}(-\tau)\,T(-\tau,s)\,P(s)\\
&=Q(s)+T(s,-\tau)\,Q(-\tau)\,\tilde{Q}(-\tau)\,T(-\tau,s)\,P(s)
\end{aligned}$$

since $\mathscr{R}\tilde{Q}(s)=\mathscr{R}Q(s)$ implies that $Q(-\tau)\,\tilde{Q}(-\tau)=\tilde{Q}(-\tau)$ and $\tilde{Q}(s)\,Q(s)=Q(s)$. Therefore, $\tilde{Q}(s)\to Q(s)$ exponentially in norm as $s\to-\infty$, since $T(-\tau,s)\,P(s)$ and $T(s,-\tau)\,Q(-\tau)$ approach zero exponentially in norm as $s\to-\infty$. Thus, $\tilde{P}(s)=I-\tilde{Q}(s)$ approaches $P(s)$ exponentially in norm as $s\to-\infty$. Therefore, $\tilde{P}(s)$, $\tilde{Q}(s)$ are uniformly bounded in norm for $s\in(-\infty,t_0]$; say, $|\tilde{P}(s)|$, $|\tilde{Q}(s)|\leqslant K_1$ for $s\leqslant t_0$. If $-\tau\geqslant t\geqslant s$, then

$$\begin{aligned}
|T(s,t)\,\tilde{Q}(t)|&\leqslant |T(s,t)\,Q(t)\,\tilde{Q}(t)|\\
&\leqslant |T(s,t)\,Q(t)|\ |\tilde{Q}(t)|\\
&\leqslant KK_1e^{-\alpha(t-s)}.
\end{aligned}$$

By $\tilde{Q}(t)\,Q(t)=Q(t)$, we have $\tilde{P}(t)=\tilde{P}(t)(P(t)+Q(t))=\tilde{P}(t)\,P(t)+\tilde{P}(t)\,\tilde{Q}(t)\,Q(t)=\tilde{P}(t)\,P(t)$. Therefore

$$\begin{aligned}|T(t,s)\,\tilde{P}(s)| &= |\tilde{P}(t)\,T(t,s)|\\ &= |\tilde{P}(t)\,P(t)\,T(t,s)|\\ &\leqslant KK_1\,e^{-\alpha(t-s)}.\end{aligned}$$

We have shown that (2.1) has exponential dichotomy relevant to the projections $\tilde{P}(t)$ and $\tilde{Q}(t)$ in $(-\infty,-\tau]$. Dichotomy estimates hold true in $[-\tau, t_0]$ for $T(t,s)$ is nondegenerate on $\mathscr{R}Q(s)$. The proof of Lemma 2.3 is completed by combining the results on the two intervals.

LEMMA 2.4. *Let $J=[\tau,\infty)$ and suppose (2.1) has an exponential dichotomy on J. Let $t_0<\tau$ and*

$$X_1(t_0)=\{\phi\mid T(\tau,t_0)\,\phi\in\mathscr{R}P(\tau)\}.$$

Suppose that $X_1(t_0)$ has the same codimension as $\mathscr{R}P(\tau)$. Then (2.1) has an exponential dichotomy on $[t_0,+\infty)$ with the projections $\tilde{P}(t)$, $\tilde{Q}(t)\to P(t)$, $Q(t)$ exponentially as $t\to+\infty$.

Proof. Let

$$X_1(t)=\mathscr{R}P(t),\qquad t\geqslant\tau$$

and

$$X_1(t)=\{\phi\mid T(\tau,t)\,\phi\in\mathscr{R}P(\tau)\},\qquad t_0\leqslant t\leqslant\tau.$$

Let $m=\text{codimension } X_1(\tau)=\text{codimension } X_1(t_0)$. Choose an m-dimensional subspace $X_2(t_0)$ such that

$$X_1(t_0)\oplus X_2(t_0)=C.$$

Let $X_2(t)=T(t,t_0)\,X_2(t_0)$, $t\geqslant t_0$. We now show that

$$X_1(t)\oplus X_2(t)=C,\qquad t\geqslant t_0. \tag{2.3}$$

First, we show that $X_1(t)\cap X_2(t)=\{0\}$, $t\geqslant t_0$. In fact, if $\phi_t\in X_1(t)\cap X_2(t)$, then there exists $\phi_{t_0}\in X_2(t_0)$ such that $\phi_t=T(t,t_0)\,\phi_{t_0}$.

(1) By the definition of $X_1(t)$, if $t\leqslant\tau$, then $T(\tau,t)\,\phi_t\in X_1(\tau)$.
This in turn implies that $\phi_{t_0}\in X_1(t_0)$. Therefore, $\phi_{t_0}=0$ and $\phi_t=0$.

(2) If $t>\tau$, $T(t,\tau)[T(\tau,t_0)\,\phi_{t_0}]=T(t,t_0)\,\phi_{t_0}=\phi_t\in X_1(t)=\mathscr{R}P(t)$.
Then $T(\tau,t_0)\,\phi_{t_0}\in\mathscr{R}P(\tau)$, implying that $\phi_{t_0}\in X_1(t_0)$. Therefore, $\phi_{t_0}=\phi_t=0$.

Next, we show that $X_1(t) \oplus X_2(t) = C$, $t \geqslant t_0$. For any $0 \neq \phi_{t_0} \in X_2(t_0)$, we observe that $T(t, t_0)\,\phi_{t_0} \neq 0$, $t \geqslant t_0$. Otherwise, $\phi_{t_0} \in X_1(t_0)$ and also $X_1(t_0) \cap X_2(t_0) = \{0\}$, which is a contradiction. $T(t, t_0)\, X_2(t_0) = X_2(t)$ is an m-dimensional subspace for $t \geqslant t_0$. For $t \geqslant \tau$, the codimension of $X_1(t) = \mathscr{R}P(t)$ is known to be m. Therefore, (2.3) is valid for $t \geqslant \tau$. The proof of $X_1(t) + X_2(t) = C$, $t_0 \leqslant t < \tau$, is similar to that of (2.2) in Lemma 2.3.

The decomposition $X_1(t) \oplus X_2(t) = C$ of C defines projection operators $\tilde{P}(t)$, $\tilde{Q}(t) = I - \tilde{P}(t)$ such that $\mathscr{R}\tilde{P}(t) = X_1(t)$ and $\mathscr{R}\tilde{Q}(t) = X_2(t)$ for $t \geqslant t_0$. The invariance of $\mathscr{R}\tilde{P}(s)$ and $\mathscr{R}\tilde{Q}(s)$ under $T(t, s)$ follows from the invariance of $X_1(t)$ and $X_2(t)$. Thus,

$$\begin{aligned} &T(t, s)\, \tilde{P}(s) = \tilde{P}(t)\, T(t, s) \\ &T(t, s)\, \tilde{Q}(s) = \tilde{Q}(t)\, T(t, s), \qquad t \geqslant s, \text{ in } [t_0, +\infty). \end{aligned}$$

Let $t \geqslant \tau$. Since $\mathscr{R}P(t) = \mathscr{R}\tilde{P}(t)$, $P\tilde{P} = \tilde{P}$ and $\tilde{P}P = P$.

$$\begin{aligned} \tilde{P}(t) &= \tilde{P}(t)\, P(t) + \tilde{P}(t)\, Q(t) \\ &= P(t) + \tilde{P}(t)\, T(t, \tau)\, T(\tau, t)\, Q(t) \\ &= P(t) + T(t, \tau)\, \tilde{P}(\tau)\, T(\tau, t)\, Q(t) \\ &= P(t) + T(t, \tau)\, P(\tau)\, \tilde{P}(\tau)\, T(\tau, t)\, Q(t). \end{aligned}$$

Therefore $\tilde{P}(t)$ is strongly continuous for $t \geqslant \tau$ and $\tilde{P}(t) \to P(t)$ exponentially in norm as $t \to +\infty$, since $T(\tau, t)\, Q(t)$ and $T(t, \tau)\, P(\tau)$ approach zero exponentially in norm as $t \to +\infty$. Thus, $\tilde{Q}(t) = I - \tilde{P}(t)$ is strongly continuous and approaches $Q(t)$ exponentially as $t \to +\infty$. Furthermore, $|\tilde{P}(t)|$ and $|\tilde{Q}(t)|$ are uniformly bounded, say by $K_1 > 0$, for $t \geqslant \tau$.

Let $t \geqslant s \geqslant \tau$,

$$\begin{aligned} |T(t, s)\, \tilde{P}(s)| &= |T(t, s)\, P(s)\, \tilde{P}(s)| \\ &\leqslant K_1 K e^{-\alpha(t-s)}. \end{aligned}$$

We shall prove that

$$T(s, t)\, \tilde{Q}(t) = \tilde{Q}(s)\, T(s, t)\, Q(t)\, \tilde{Q}(t). \tag{2.4}$$

This would imply that

$$\begin{aligned} |T(s, t)\, \tilde{Q}(t)| &= |\tilde{Q}(s)\, T(s, t)\, Q(t)\, \tilde{Q}(t)| \\ &\leqslant K_1^2 K e^{-\alpha(t-s)}, \qquad t \geqslant s \geqslant \tau. \end{aligned}$$

Note that the operators $T(s, t)$ in the two sides of (2.4) have different domains. However, when applied to $\phi \in C$, both sides of (2.4) obtain elements in $\mathscr{R}\tilde{Q}(s)$. And

$$\begin{aligned} T(t, s)[\tilde{Q}(s)\, T(s, t)\, Q(t)\, \tilde{Q}(t)\, \phi] &= \tilde{Q}(t)\, T(t, s)\, T(s, t)\, Q(t)\, \tilde{Q}(t)\, \phi \\ &= \tilde{Q}(t)\, Q(t)\, \tilde{Q}(t)\, \phi \\ &= \tilde{Q}(t)\, \phi = T(t, s)[T(s, t)\, \tilde{Q}(t)\, \phi] \end{aligned}$$

since $\tilde{Q}(t) = \tilde{Q}(t)\, Q(t) + \tilde{Q}(t)\, P(t) = \tilde{Q}(t)\, Q(t)$. But, $T(t, s)$ is an isomorphism from $\mathscr{R}\tilde{Q}(s)$ onto $\mathscr{R}\tilde{Q}(t)$, (2.4) is valid.

We now turn to $t \in [t_0, \tau]$. Since

$$\tilde{Q}(t) = T(t, \tau)\, \tilde{Q}(\tau)\, T(\tau, t),$$

the operators $\tilde{Q}(t)$ and $\tilde{P}(t)$ are strongly continuous and uniformly bounded in $[t_0, \tau]$. Dichotomy estimates hold in $t \in [t_0, \tau]$ for $T(t, s)$ is non-degenerate on $\mathscr{R}Q(s)$, $t_0 \leqslant s \leqslant t \leqslant \tau$. The proof of Lemma 2.4 is fulfilled by combining the results on the two intervals. Q.E.D.

The hypotheses in Lemma 2.4 are also necessary. However, a direct computation of $X_1(t_0)$ in Lemma 2.4 is hardly feasible. An alternative approach is to consider the adjoint system.

LEMMA 2.5. *Assume that* (2.1) *has an exponential dichotomy on* $J = [\tau, +\infty)$. *Let* $t_0 < \tau$. *Suppose that* $T^*(t_0, \tau)\, \psi \neq 0$ *for any* $0 \neq \psi \in \mathscr{R}Q^*(\tau)$. *Then* $T^*(s, t)$ *has an exponential dichotomy on* $[t_0, +\infty)$ *and the hypothesis in Lemma* 2.4 *is satisfied.*

Proof. Obviously, $T^*(s, t)$ has an exponential dichotomy on J, $\mathscr{R}Q^*(t)$ is a finite dimensional subspace and $\dim \mathscr{R}Q^*(t) = \dim \mathscr{R}Q(t) = m$. Proceeding as in the proof of Lemma 2.3, we see that $T^*(s, t)$ has an exponential dichotomy on $[t_0, +\infty)$. Let $\tilde{P}^*(t)$, $\tilde{Q}^*(t)$ be the relevant projections on $[t_0, +\infty)$. The uniqueness of the stable subspace implies that $\mathscr{R}\tilde{Q}^*(t) = \mathscr{R}Q^*(t)$, $t \geqslant \tau$ and $\dim \mathscr{R}\tilde{Q}^*(t) = m$ for all $t \in [t_0, +\infty)$. Define the annihilators E^0 and ${}^0F$ as

$$E^0 = \{x^* \in X^* \mid \langle x^*, x \rangle = 0, \forall x \in E\},$$

$${}^0F = \{x \in X \mid \langle x^*, x \rangle = 0, \forall x^* \in F\},$$

where X is a Banach space, X^* its adjoint space, $E \subset X$ and $F \subset X^*$. Obviously $\mathscr{R}\tilde{Q}^*(\tau) = [\mathscr{R}P(\tau)]^0$. Let $X_1(t_0)$ be defined as in Lemma 2.4. We show that ${}^0[\mathscr{R}\tilde{Q}^*(t_0)] = X_1(t_0)$.

First, for any $\phi \in X_1(t_0)$, let $\psi \in \mathscr{R}\tilde{Q}^*(\tau) = [\mathscr{R}P(\tau)]^0$. Then

$\langle \psi, T(\tau, t_0)\,\phi \rangle = 0$, since $T(\tau, t_0)\,\phi \in \mathscr{R}P(\tau)$. Hence $\langle T^*(t_0, \tau)\,\psi, \phi \rangle = 0$ for any $\psi \in \mathscr{R}\tilde{Q}^*(\tau)$. Also, $T^*(t_0, \tau)\colon \mathscr{R}\tilde{Q}^*(\tau) \to \mathscr{R}\tilde{Q}^*(t_0)$ is onto. Thus, $\phi \in {}^0[\mathscr{R}\tilde{Q}^*(t_0)]$.

Conversely, if $\phi \in {}^0[\mathscr{R}\tilde{Q}^*(t_0)]$, the above procedure can be reversed to show that $T(\tau, t_0)\,\phi \in {}^0[\mathscr{R}\tilde{Q}^*(\tau)] = \mathscr{R}P(\tau)$. Therefore, $\phi \in X_1(t_0)$.

Finally, we obtain that $\operatorname{codim} X_1(t_0) = \dim \mathscr{R}\tilde{Q}^*(t_0) = m = \operatorname{codim} \mathscr{R}P(\tau)$.

Remark 2.6. Lemmas 2.3, 2.4 and 2.5 are valid for general non-autonomous linear semigroups, not only the one generated by delay differential equations, provided that $\mathscr{R}Q(t)$ is of finite dimension. Hypothesis in Lemma 2.5 can be checked by considering the formal adjoint system for (2.1), see Theorem 3.3 in Section 3.

3. A Fredholm Alternative for Functional Differential Equations

In this section, we shall generalize a Fredholm operator defined for ordinary differential equations in [8] to the one defined for (2.1). To perform the splitting induced by the exponential dichotomy in the variation of constants formula, we have to consider the matrix valued jump function

$$\begin{aligned} X_0(\theta) &= 0, \qquad -r \leqslant \theta < 0, \\ &= I, \qquad \theta = 0. \end{aligned} \tag{3.1}$$

The function X_0 plays an important role since the solution of the non-homogeneous system

$$\begin{aligned} \dot{x}(t) &= L(t)\,x_t + h(t), \\ x_\sigma &= \phi, \end{aligned} \tag{3.2}$$

where $h\colon R \to R^n$ is continuous, can be written as

$$x_t = T(t, \sigma)\,\phi + \int_\sigma^t T(t, s)\,X_0 h(s)\,ds. \tag{3.3}$$

Since X_0 is not continuous, we must either extend our phase space to include jump functions or check the following formulas for X_0:

(1) $T(t, s)\,X_0$;

(2) $\int_a^b T(t, s)\,X_0 h(s)\,ds$;

(3) $\langle \psi, T(t, s)\,X_0 \rangle = \langle T^*(s, t)\,\psi, X_0 \rangle$;

(4) $\langle \psi, \int_a^b T(t, s)\,X_0 h(s)\,ds \rangle = \int_a^b \langle \psi, T(t, s)\,X_0 \rangle\, h(s)\,ds$;

(5) $P(s)\,X_0, Q(s)\,X_0$;

(6) $T(t, s)\,P(s)\,X_0 = P(t)\,T(t, s)\,X_0$;

(7) $\langle \psi, T(t, s)\,P(s)\,X_0 \rangle = \langle T^*(s, t)\,P^*(t)\,\psi, X_0 \rangle$;

(8) $\langle \psi, T(s, t)\,Q(t)\,X_0 \rangle = \langle T^*(t, s)\,Q^*(s)\,\psi, X_C \rangle$;

where $\psi \in C^*$ and $t \geqslant s$ in J where the exponential dichotomy of (2.1) holds with projection operators $P(t)$ and $Q(t)$. The first problem is how to interpret product. Recall that $\psi \in C^*$ is identified with $\psi \in B_0[-r, 0]$. Therefore, $\langle \psi, \phi \rangle = \int_{-r}^{0} d\psi(\theta)\,\phi(\theta)$ is defined even though the function ϕ has some jumps. Thus defined, (1)–(3) are discussed in [9]. For example, (3) is precisely (4.10) on page 153. Formula (4) is merely a consequence of Fubini's theorem. Therefore, our discussion starts from (5).

An invariant basis $\{q_i(t)\}$, $i = 1, ..., m$, is a basis of $\mathscr{R}Q(t)$ for each $t \in J$ and $q_i(t) = T(t, s)\,q_i(s)$ for $t, s \in J$. An invariant basis $\{q_i^*(t)\}$ in $\mathscr{R}Q^*(t)$ is defined similarly. We assume that $\langle q_i^*(t), q_j(t) \rangle = \delta_{ij}$. This is true if we choose $\langle q_i^*(t_0), q_j(t_0) \rangle = \delta_{ij}$ for any $t_0 \in J$. Then

$$\begin{aligned}\langle q_i^*(t), q_j(t) \rangle &= \langle q_i^*(t), T(t, t_0)\,q_j(t_0) \rangle \\ &= \langle T^*(t_0, t)\,q_i^*(t), q_j(t_0) \rangle \\ &= \langle q_i^*(t_0), q_j(t_0) \rangle \\ &= \delta_{ij}.\end{aligned}$$

Let $\Phi(t) = (q_1(t), ..., q_m(t))$ and $\Psi(t) = (q_1^*(t), ..., q_m^*(t))^\tau$, where τ is the transpose. Then $\langle \psi(t), \Phi(t) \rangle = I_{m \times m}$, $t \in J$. We readily see that $Q(t)\,\phi = \Phi(t) \langle \psi(t), \phi \rangle$ and $P(t)\,\phi = \phi - Q(t)\,\phi$ for $\phi \in C$.

DEFINITION 3.1. Define

$$\begin{aligned} Q(t)\,X_0 &= \Phi(t) \langle \Psi(t), X_0 \rangle, \qquad t \in J, \\ P(t)\,X_0 &= X_0 - Q(t)\,X_0, \qquad t \in J. \end{aligned}$$

The functions $Q(t)\,T(t, s)\,X_0$ and $P(t)\,T(t, s)\,X_0$ are defined similarly for $t \geqslant s$ in J. With this definition, we observe that

$$\begin{aligned} T(t, s)\,Q(s)\,X_0 &= T(t, s)\,\Phi(s) \cdot \langle \Psi(s), X_0 \rangle \\ &= \Phi(t) \langle T^*(s, t)\,\Psi(t), X_0 \rangle \\ &= \Phi(t) \langle \Psi(t), T(t, s)\,X_0 \rangle \\ &= Q(t) \cdot T(t, s)\,X_0. \end{aligned}$$

Consequently, $T(t, s)\,P(s)\,X_0 = P(t)\,T(t, s)\,X_0$ and (6) is justified.

Let $\psi \in C^*$. Obviously, $Q^*(t)\,\psi = \langle \psi, \Phi(t)\rangle \cdot \Psi(t)$. Thus,

$$\begin{aligned}\langle \psi, Q(t)\,X_0\rangle &= \langle \psi, \Phi(t)\langle \Psi(t), X_0\rangle\rangle \\ &= \langle \psi, \Phi(t)\rangle \cdot \langle \Psi(t), X_0\rangle \\ &= \langle \langle \psi, \Phi(t)\rangle\, \Psi(t), X_0\rangle \\ &= \langle Q^*(t)\,\psi, X_0\rangle.\end{aligned} \tag{3.4}$$

For $t \geqslant s$ in J,

$$\langle \psi, T(t,s)\,Q(s)\,X_0\rangle = \langle T^*(s,t)\,\psi, Q(s)\,X_0\rangle$$

since $Q(s)\,X_0 \in C$. By (3.4), $\langle \psi, T(t,s)\,Q(s)\,X_0\rangle = \langle Q^*(s)\,T^*(s,t)\,\psi, X_0\rangle$. Hence,

$$\begin{aligned}\langle \psi, T(t,s)\,P(s)\,X_0\rangle &= \langle P^*(s)\,T^*(s,t)\,\psi, X_0\rangle \\ &= \langle T^*(s,t)\,P^*(t)\,\psi, X_0\rangle, \qquad t \geqslant s, \text{ in } J,\end{aligned}$$

and (7) is justified. For $t \geqslant s$ in J, one similarly obtains

$$\begin{aligned}\langle \psi, T(s,t)\,Q(t)\,X_0\rangle &= \langle \psi, Q(s)\,T(s,t)\,Q(t)\,X_0\rangle \\ &= \langle Q^*(t)\,T^*(t,s)\,Q^*(s)\,\psi, X_0\rangle \\ &= \langle T^*(t,s)\,Q^*(s)\,\psi, X_0\rangle.\end{aligned}$$

Thus, (8) is justified.

We give the estimates for $T(t,s)\,P(s)\,X_0$ and $T(s,t)\,Q(t)\,X_0$, $t \geqslant s$, in J. Let $\psi \in C^* = B_0[-r,0]$ and $\|\psi\|_{B_0} = 1$. Then

$$\begin{aligned}|\langle \psi, T(t,s)\,P(s)\,X_0\rangle| &= |\langle T^*(s,t)\,P^*(t)\,\psi, X_0\rangle| \\ &\leqslant \|T^*(s,t)\,P^*(t)\,\psi\|_{B_0} \\ &\leqslant Ke^{-\alpha(t-s)}. \\ |\langle \psi, T(t,s)\,Q(t)\,X_0\rangle| &= |\langle T^*(t,s)\,Q^*(s)\,\psi, X_0\rangle| \\ &\leqslant \|T^*(t,s)\,Q^*(s)\,\psi\|_{B_0} \\ &\leqslant Ke^{-\alpha(t-s)}.\end{aligned}$$

Therefore,

$$\begin{aligned}|T(s,t)\,Q(t)\,X_0|_C &\leqslant Ke^{-\alpha(t-s)} \\ \sup_{-r \leqslant \theta \leqslant 0} |(T(t,s)\,P(s)\,X_0)(\theta)| &\leqslant Ke^{-\alpha(t-s)}, \qquad t \geqslant s, \text{ in } J.\end{aligned} \tag{3.5}$$

The second inequality of (3.5) shall be written as $|T(t,s)\,P(s)\,X_0| \leqslant Ke^{-\alpha(t-s)}$ although $T(t,s)\,P(s)\,X_0$ is not an element in C for $s \leqslant t \leqslant s+r$.

We now define the formal adjoint system of (2.1). Let (2.1) be written as

$$\dot{x}(t) = \int_{-r}^{0} d\eta(t,\theta)\,x(t+\theta), \tag{3.6}$$

where $\eta(t,\cdot) \in B_0[-r,0]$ and is a continuous map from $t \in \mathbb{R}$ to $B_0[-r,0]$. Let $\xi \in B[-r,0] = \{$bounded, measurable functions on $[-r,0]\}$. A function $y\colon [\sigma,t] \to R^{n^*}$, $t \geqslant \sigma + r$, is said to satisfy the formal adjoint system with initial data $\xi \in B[-r,0]$ at t if

$$\begin{gathered} y(s) + \int_s^t y(\alpha)\,\eta(\alpha, s-\alpha)\,d\alpha = \text{constant}, \qquad \sigma \leqslant s \leqslant t-r, \\ y_t = \xi, \end{gathered} \tag{3.7}$$

where $y_t(\theta) = y(t+\theta)$, $-r \leqslant \theta \leqslant 0$, $\eta(\alpha, s-\alpha) = 0$ if $s \geqslant \alpha$ and $\eta(\alpha, s-\alpha) = \eta(\alpha, -r)$ if $s \leqslant \alpha - r$.

The existence and uniqueness of solutions of (3.7) are known [9]. The solution operator $\tilde{T}(s,t)\colon B[-r,0] \to B[-r,0]$,

$$y_s = \tilde{T}(s,t)\,\xi = \tilde{T}(s,t)\,y_t, \qquad \sigma + r \leqslant s \leqslant t,$$

is a semigroup for $s \leqslant t$.

(1) $\tilde{T}(s_1,s_2)\,\tilde{T}(s_2,t) = \tilde{T}(s_1,t)$, $s_1 \leqslant s_2 \leqslant t$;

(2) $\tilde{T}(t,t) = I$.

When $J = (-\infty, +\infty)$ or $[\sigma, +\infty)$, $y\colon J \to R^{n^*}$ is said to be a solution of the formal adjoint equation if for any $s \leqslant t$ in J, $y_s = \tilde{T}(s,t)\,y_t$.

Let $x\colon [\sigma - r, t) \to R^n$ be a solution of (3.2) and $y\colon [\sigma, t+r] \to R^{n^*}$ be a solution of (3.7), $t \geqslant \sigma$. The following identity is proved in [1].

$$\begin{aligned} y(t)\,x(t) &+ \int_{t-r}^{t} d_s\left[\int_t^{t+r} y(\alpha)\,\eta(\alpha, s-\alpha)\,d\alpha\right] x(s) \\ &= y(\sigma)\,x(\sigma) + \int_{\sigma-r}^{\sigma} d_s\left[\int_\sigma^{\sigma+r} y(\alpha)\,\eta(\alpha, s-\alpha)\,d\alpha\right] x(s) \\ &\quad + \int_\sigma^t y(\alpha)\,h(\alpha)\,d\alpha, \qquad \sigma \leqslant t. \end{aligned} \tag{3.8}$$

We define an operator $\tilde{\Psi}\colon C([\sigma, t+r], R^{n^*}) \to C([\sigma, t], B_0[-r, 0])$, $\psi(u, \cdot) = \tilde{\Psi}(y)(u)$, (allow σ or t to be $-\infty$ or $+\infty$), as

$$\begin{aligned} &\psi(u, 0) = 0, \\ &\psi(u, s-u) = \int_u^{u+r} y(\alpha)\, \eta(\alpha, s-\alpha)\, d\alpha - \int_u^{u+r} y(\alpha)\, \eta(\alpha, u-\alpha)\, d\alpha - y(u). \end{aligned} \tag{3.9}$$

Thus, when $h = 0$, (3.8) can be written as

$$\langle \psi(t, \cdot), x_t \rangle = \langle \psi(\sigma, \cdot), x_\sigma \rangle.$$

Hence, $\psi(s, \cdot) = T^*(s, t)\, \psi(t, \cdot)$. Unfortunately, the operator $\tilde{\Psi}$ is not always invertible. Given an element $\psi \in B_0[-r, 0]$, we can not always find $y(\alpha)$, $\alpha \in [u, u+r]$ such that $\psi = \psi(u, \cdot)$ with $\psi(u, \cdot)$ defined as in (3.9). Nevertheless, if $\psi(u, \cdot) = T^*(u, u+r)\, \psi(u+r, \cdot) \in B_0[-r, 0]$, for some $\psi(u+r, \cdot) \in B_0[-r, 0]$, one can find $y(\alpha)$, $\alpha \in [u, u+r]$ such that (3.9) reproduces $\psi(u, \cdot)$; that is,

$$y(\alpha) = -\psi(\alpha, 0-), \qquad \alpha \in [u, u+r], \tag{3.10}$$

where $\psi(\alpha, \cdot) = T^*(\alpha, u+r)\, \psi(u+r, \cdot)$ and $\psi(\alpha, 0-) = \lim_{\theta \to 0-} \psi(\alpha, \theta)$. This can be shown by the relation of the true adjoint operators and the formal adjoint operators, see [9, 11]. Details shall be omitted here. We obtain that, when $\psi(t, \cdot)$, $t \in [\sigma, +\infty)$ or $(-\infty, +\infty)$ is a trajectory of the adjoint system, then (3.10) defines an inverse of $\tilde{\Psi}$. We have the following lemma.

LEMMA 3.2. *Let $J = [t_0, +\infty)$ or $(-\infty, +\infty)$. Define*

$$Y_1 = \{ y \mid y(t) \text{ satisfies the formal adjoint system (3.7) on } J, \text{ i.e., } y_s = \tilde{T}(s, t)\, y_t,\ s \leqslant t \text{ in } J \}$$

and

$$\Psi_1 = \{ \psi(\cdot, \theta),\ -r \leqslant \theta \leqslant 0 \mid \psi(t, \cdot) \text{ satisfies the true adjoint system on } J, \text{ i.e., } \psi(s, \cdot) = T^*(s, t)\, \psi(t, \cdot),\ s \leqslant t \text{ in } J \}.$$

Then $\tilde{\Psi}\colon Y_1 \to \Psi_1$, $u \in J$, defined as (3.9) *is a linear isomorphism with the inverse $\tilde{\Psi}^{-1}\colon \Psi_1 \to Y_1$, $u \in J$, defined as* (3.10). *Furthermore, let $Y_{1b} \subset Y_1$ be the subspace of the bounded solutions for* (3.7) *on J with the norm*

$$|y|_{Y_{1b}} = \sup_{s \in J} |y(s)|_{R^{n^*}},$$

and let $\Psi_{1b} \subset \Psi_1$ be the subspace of the bounded solutions for the true adjoint system on J with the norm

$$|\psi(\cdot, \cdot)|_{\psi_{1b}} = \sup_{s \in J} |\psi(s, \cdot)|_{B_0},$$

then $\tilde{\Psi}$ is also a topological isomorphism of Y_{1b} onto Ψ_{1b} if $\int_u^{u+r} |L(\alpha)|\, d\alpha \leqslant M$, $u \in J$ for some constant $M > 0$.

Proof. Obviously, Ψ^{-1} is continuous. Conversely,

$$\begin{aligned}
|\psi(u, \cdot)|_{B_0} &= |y(u)| + V_u^{u+r}\left(\int_u^{u+r} y(\alpha)\, \eta(\alpha, s-\alpha)\, d\alpha\right) \\
&\leqslant |y(u)| + \int_u^{u+r} |y(\alpha)| \cdot |\eta(\alpha, s-\alpha)|_{B_0}\, d\alpha \\
&\leqslant |y|_{Y_{1b}} + |y|_{Y_{1b}} \int_u^{u+r} |L(\alpha)|\, d\alpha \leqslant (M+1)\, |y|_{Y_{1b}},
\end{aligned}$$

where we use V_u^{u+r} to denote the total variation.

It is now easy to see that if (2.1) has exponential dichotomy on $J = [\tau, +\infty)$, and if $\int_u^{u+r} |L(\alpha)|\, d\alpha \leqslant M$, $u \in J$ for some constant $M > 0$, then the solution operator $\tilde{T}(s, t)$ of (3.7) also has exponential dichotomy on J, and vice versa.

THEOREM 3.3. *Let $L(t)$ in (2.1) be continuous for $t \in R$ and $L(t) \to L^{\pm}$ for $t \to \pm\infty$. Suppose $\dot{x}(t) = L^{\pm} X_t$ has zero as a hyperbolic equilibrium point with $P^{\pm}$ as the projection to the stable subspace. Suppose $T(t, s)\, \phi \neq 0$ for $s \leqslant t$ in R^-, $\phi \in C$, $\phi \neq 0$, and $\tilde{T}(s, t)\, \psi \neq 0$ for $s \leqslant t$ in $[-r, +\infty)$. $\psi \in B[-r, 0]$ and $\psi \neq 0$. Then (2.1) has exponential dichotomies in R^- and R^+ with the projections $P(t) \to P^{\pm}$ as $t \to \pm\infty$.*

Let $C_b^0(R, R^n)$ be the space of continuous bounded functions with supremum norm and let $C_b^1(R, R^n)$ be the space of functions in $C_b^0(R, R^n)$ with derivatives in $C_b^0(R, R^n)$, and $\|X\|_{C_b^1} = \|x\|_{C_b^0} + \|x'\|_{C_b^0}$ for $x \in C_b^1(R, R^n)$. Similar definitions are given with R^n replaced by R^{n*}. We now consider the operator F: $C_b^1(R, R^n) \to C_b^0(R, R^n)$, $h(t) = F(x)(t)$ defined as

$$h(t) = \frac{d}{dt} x(t) - L(t)\, x_t.$$

LEMMA 3.4. *Let $L(t)$ be continuous in the operator norm with respect to $t \in R$ and $|L(u)| < M$ for any $u \in R$. Suppose the equation (2.1) has an exponential dichotomy on $R+$ and $R-$ with projections $P^+(t)$ and $P^-(t)$,* *respectively. Then F is a Fredholm operator of index* $I(F) = \dim \mathscr{R}Q^-(0) - \dim \mathscr{R}Q^+(0)$. *Furthermore,*

$$\mathcal{N}(F) = \{(T(t,0)\,\phi)(0) \mid \phi \in \mathscr{R}P^+(0) \cap \mathscr{R}Q^-(0),\ t \in R\}.$$

$$\mathscr{R}(F) = \Big\{ h \mid h \in C_b^0(R, R^n), \int_{-\infty}^{\infty} y(t)\,h(t)\,dt = 0 \textit{ for all } y \in C_b^0(R, R^{n^*}) \textit{ satisfying the formal adjoint system } (3.7) \Big\}.$$

Proof. The characterization of $\mathcal{N}(F)$ is obvious. Let $y \in C_b^0(R, R^{n^*})$ satisfy the formal adjoint system in R. Then Lemma 3.2 implies $\tilde{\Psi}_t y$ is a bounded trajectory of $T^*(s,t)$ in R. Since $T^*(s,t)$ has an exponential dichotomy, $\tilde{\Psi}_t y \to 0$ exponentially as $t \to \pm\infty$. The relation (3.10) implies that $y(t) \to 0$ exponentially as $t \to \pm\infty$. Therefore, if $h \in \mathscr{R}(F)$, $\int_{-\infty}^{\infty} y(t)\,h(t)\,dt$ is convergent, and by (3.8) letting $\sigma \to -\infty$ and $t \to +\infty$, we obtain that

$$\int_{-\infty}^{\infty} y(t)\,h(t)\,dt = 0.$$

The sufficient conditions for $h \in \mathscr{R}(F)$ are that there exists $\phi \in C$ and a continuous map $\mathbb{R} \ni t \mapsto x_t \in C$ such that

$$x_t = T(t,0)\,P^+(0)\,\phi + \int_0^t T(t,s)\,P^+(s)\,X_0 h(s)\,ds - \int_t^{\infty} T(t,s)\,Q^+(s)\,X_0 h(s)\,ds, \qquad t \geqslant 0, \tag{3.11}$$

$$x_t = T(t,0)\,Q^-(0)\,\phi + \int_0^t T(t,s)\,Q^-(s)\,X_0 h(s)\,ds + \int_{-\infty}^{t} T(t,s)\,P^-(s)\,X_0 h(s)\,ds, \qquad t \leqslant 0. \tag{3.12}$$

In deriving (3.11) and (3.12), the exponential estimates of $T(t,s)\,P(s)\,X_0$ and $T(t,s)\,Q(s)\,X_0$ in (3.5) are employed. Relations (3.11) and (3.12) have a solution $\phi \in C$ if and only if

$$[P^+(0) - Q^-(0)]\,\phi = \int_{-\infty}^{0} T(0,s)\,P^-(s)\,X_0 h(s)\,ds + \int_0^{\infty} T(0,s)\,Q^+(s)\,X_0 h(s)\,ds. \tag{3.13}$$

Since $P^+(0)$ is Fredholm and $Q^-(0)$ is of finite rank and, thus, compact, it follows that $P^+(0)-Q^-(0)$ is Fredholm. Therefore, $\mathscr{R}[P^+(0)-Q^-(0)]$ is closed and (3.13) has a solution $\phi \in C$ if and only if $\langle \psi, \int_{-\infty}^0 + \int_0^\infty \rangle = 0$ for all $\psi \in \mathscr{N}(P^+(0)-Q^-(0))^* = \mathscr{N}(P^{+*}(0)-Q^{-*}(0))$. If $\psi \in \mathscr{N}(P^{+*}(0)-Q^{-*}(0))$, then

$$P^{-*}(0)\,\psi = Q^{+*}(0)\,\psi \overset{\text{def}}{=} \psi_0 \qquad \text{and} \qquad \psi_0 \in \mathscr{R}P^{-*}(0) \cap \mathscr{R}Q^{+*}(0).$$

The function $\bar{\psi}(t,\cdot) =^{\text{def}} T^*(t,0)\,\psi_0$, $t \in R$ is a trajectory of the adjoint system in B_0 and $\bar{\psi}(t,\cdot) \to 0$ exponentially as $t \to \pm\infty$. Also,

$$\begin{aligned}
\left\langle \psi, \int_{-\infty}^0 + \int_0^\infty \right\rangle &= \int_{-\infty}^0 \langle \psi, T(0,s)\,P^-(s)\,X_0\rangle\, h(s)\,ds \\
&\quad + \int_0^\infty \langle \psi, T(0,s)\,Q^*(s)\,X_0\rangle\, h(s)\,ds \\
&= \int_{-\infty}^0 \langle T^*(s,0)\,P^{-*}(0)\,\psi, X_0\rangle\, h(s)\,ds \\
&\quad + \int_0^\infty \langle T^*(s,0)\,Q^{+*}(0)\,\psi, X_0\rangle\, h(s)\,ds \\
&= \int_{-\infty}^0 \langle T^*(s,0)\,\psi_0, X_0\rangle\, h(s)\,ds \\
&\quad + \int_0^\infty \langle T^*(s,0)\,\psi_0, X_0\rangle\, h(s)\,ds \\
&= \int_{-\infty}^\infty \langle T^*(s,0)\,\psi_0, X_0\rangle\, h(s)\,ds \\
&= \int_{-\infty}^\infty \bar{\psi}(s, 0-)\, h(s)\,ds.
\end{aligned}$$

By Lemma 3.2, $\bar{\psi}(s, 0-)$, $s \in R$ is a bounded trajectory of the formal adjoint system in R. Therefore $\langle \psi, \int_{-\infty}^0 + \int_0^\infty \rangle = 0$ if $\int_{-\infty}^\infty y(t)\,h(t)\,dt = 0$ for all bounded trajectories of formal adjoint system $y(\cdot)$. The characterization of $\mathscr{R}(F)$ is justified. It is now easy to see that $\mathscr{N}(F)$ and $\mathscr{R}(F)$ are closed in $C_b^1(R, R^n)$ and $C_b^0(R, R^n)$, respectively. Furthermore, $\dim \mathscr{N}(F) = \dim[\mathscr{R}P^+(0) \cap \mathscr{R}Q^-(0)]$, $\operatorname{codim} \mathscr{R}(F) = \dim[\mathscr{R}P^{-*}(0) \cap \mathscr{R}Q^{+*}(0)]$, and both $\dim \mathscr{N}(F)$ and $\operatorname{codim} \mathscr{R}(F)$ are finite. Therefore, F is Fredholm and $I(F) = \dim[\mathscr{R}P^+(0) \cap \mathscr{R}Q^-(0)] - \dim[\mathscr{R}P^{-*}(0) \cap \mathscr{R}Q^{+*}(0)]$.

We now prove that $I(F) = \dim \mathscr{R}Q^-(0) - \dim \mathscr{R}Q^+(0)$. Let $\{q_1^-, ..., q_k^-\}$ be a basis in $\mathscr{R}Q^-(0) \cap \mathscr{R}P^+(0)$. We can add $\{q_{k+1}^-, ..., q_{m_1}^-\}$, where $m_1 =$

$\dim \mathscr{R}Q^-(0)$ to form a basis $\{q_1^-, ..., q_{m_1}^-\}$ in $\mathscr{R}Q^-(0)$. We also choose a basis $\{q_1^{+*}, ..., q_h^{+*}\}$ in $\mathscr{R}Q^{+*}(0) \cap \mathscr{R}P^{-*}(0)$ and add $\{q_{h+1}^{+*}, ..., q_{m_2}^{+*}\}$ to form a basis $\{q_1^{+*}, ..., q_{m_2}^{+*}\}$ in $\mathscr{R}Q^{+*}(0)$, where $m_2 = \dim \mathscr{R}Q^{+*}(0)$. Let $[e_1, ..., e_\mu]$ denote the linear space spanned by vectors $\{e_1, ..., e_\mu\}$. If $q^* \neq 0$, $q^* \in [q_{h+1}^{+*}, ..., q_{m_2}^{+*}]$, we claim that there is a $q \in [q_{k+1}^-, ..., q_{m_1}^-]$ such that $\langle q^*, q\rangle \neq 0$. Otherwise, $q^* \in [q_{k+1}^-, ..., q_{m_1}^-]^0$ and we already know that $q^* \in [q_1^-, ..., q_k^-]^0$; thus $q^* \in [\mathscr{R}Q^-(0)]^0 = \mathscr{R}P^{-*}(0)$ and $q^* \in \mathscr{R}P^{-*}(0) \cap \mathscr{R}Q^{+*}(0) = [q_1^{+*}, ..., q_h^{+*}]$, contradicting the fact that $[q_1^{+*}, ..., q_h^{+*}] \cap [q_{h+1}^{+*}, ..., q_{m_2}^{+*}] = \{0\}$.

The same type of argument shows that, for $q \neq 0$, $q \in [q_{k+1}^-, ..., q_{m_1}^-]$, there is a $q^* \in [q_{h+1}^{+*}, ..., q_{m_2}^{+*}]$ such that $\langle q^*, q\rangle \neq 0$. Therefore, we easily see that $\dim[q_{k+1}^-, ..., q_{m_1}^-] = \dim[q_{h+1}^{+*}, ..., q_{m_2}^{+*}]$, i.e., $m_1 - k = m_2 - h$. Finally, $I(F) = k - h = m_1 - m_2$. Q.E.D.

4. Homoclinic Trajectories and Quasi-linear Equations

In this section, we shall construct autonomous scalar delay equations with one delay and bearing a homoclinic trajectory asymptotic to the equilibrium. Homoclinic trajectories are rare. Generally, the unstable set $W^u(0)$ misses the stable set $W^s(0)$ and a small perturbation would destroy the homoclinic connection even if it does exist. However, in one parameter families of vector fields, one often enounters a homoclinic orbit for some value of the parameter. For Hamiltonian systems, $W^u(0)$ and $W^s(0)$ lie in a codimension one energy surface and they generically intersect transversely within this surface [4, 7, 15]. Here we shall take an alternative approach. For a given scalar function $x = p(t)$, $p(t) \to 0$ as $t \to \pm\infty$, we show that with some general conditions on $p(t)$, there exist $\alpha(x)$, $\beta(x)$ such that the quasi-linear scalar equation

$$\dot{x}(t) = \alpha(x(t))\, x(t) + \beta(x(t))\, x(t-1) \tag{4.1}$$

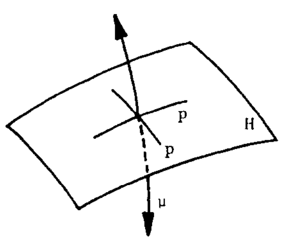

FIG. 4.1. Systems bearing homoclinic trajectories form a surface H. By changing μ we pass H at some $\mu = \mu_0$, or we consider H as parameterized by p.

has $x=p(t)$ as a solution. Thus, we obtain a family of equations parameterized by p. Although this approach seems upside down, in applications, we do have situations in which data are collected before the coefficients of an equation are determined. And (4.1) includes many interesting equations that are currently being studied extensively. Furthermore, we are close to saying that if there is a certain homoclinic trajectory of (4.1), the coefficients have to be determined uniquely in this way.

Let $x=p(t)$, $t\in R$ be a smooth (C^∞) function satisfying the following conditions:

(1) $\dot{p}(t)>0$ for $t<0$, $\dot{p}(t)<0$ for $t>0$, $p(0)=\max_{t\in R} p(t)$ and $2a=\ddot{p}(0)<0$.

(2) There are smooth functions $G(z)$, $H(z)$, $G(0)=H(0)=0$, $G_1=\dot{G}(0)>0$, $H_1=\dot{H}(0)>0$, and constants $T>0$, $\lambda_1>0$, $\lambda_2<0$ such that

$$p(t)=G(e^{\lambda_1 t}), \qquad t<-T,$$
$$p(t)=H(e^{\lambda_2 t}), \qquad t>T.$$

LEMMA 4.1. *Assume* (1) *and* (2). *Then there are smooth functions* $\alpha(x)$ *and* $\beta(x)$ *uniquely defined on* $[0, p(0)]$ *such that* $x=p(t)$ *is a solution of* (4.1). *The functions* $\alpha(x)$, $\beta(x)$ *can be extended smoothly to* $x\in R$. *Furthermore,* $-\beta(x)\geqslant\eta>0$ *on* $[0, p(0)]$ *and,* $x_1=e^{\lambda_1 t}$, $x_2=e^{\lambda_2 t}$ *are solutions of* $\dot{x}(t)=\alpha(0)\,x(t)+\beta(0)\,x(t-1)$.

Proof. For $0<x<p(0)$, there correspond unique $t_1<0$ and $t_2>0$ such that $p(t_1)=p(t_2)=x$. By the Implicit Function Theorem, $t_1=t_1(x)$ and $t_2=t_2(x)$ are smooth functions of x. It is obvious that $p(t_1-1)<p(t_2-1)$. Solving

$$\dot{p}(t_1)=\alpha(x)\,x+\beta(x)\,p(t_1-1),$$
$$\dot{p}(t_2)=\alpha(x)\,x+\beta(x)\,p(t_2-1),$$

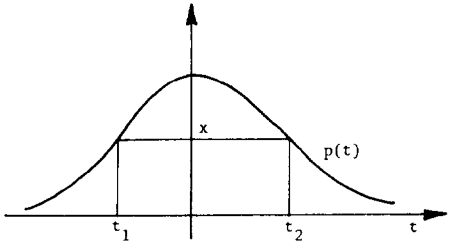

FIGURE 4.2

we obtain that

$$\alpha(x)=\frac{\dot{p}(t_1)\,p(t_2-1)-\dot{p}(t_2)\,p(t_1-1)}{x\cdot(p(t_2-1)-p(t_1-1))},$$
$$\beta(x)=\frac{\dot{p}(t_2)-\dot{p}(t_1)}{p(t_2-1)-p(t_1-1)}, \tag{4.2}$$

where t_1, t_2 are functions of x. Therefore, $\alpha(x)$, $\beta(x)$ are uniquely defined smooth functions on $x\in(0, p(0))$. Also $\beta(x)<0$ since $\dot{p}(t_2)-\dot{p}(t_1)<0$ and $p(t_2-1)-p(t_1-1)>0$.

Let $p(t)=p(0)+at^2(1+D(t))$ in a neighborhood of $t=0$, where $a<0$ and $D(t)$ is a smooth function, $D(0)=0$. If $y^2=(p(t)-p(0))/a$ and $\operatorname{sgn} y=\operatorname{sgn} t$, then $y=t(1+D(t))^{1/2}$, which is a smooth function of t in a neighborhood of $t=0$. By the Implicit Function Theorem, we have $t=t(y)$, $t(0)=0$, $dt/dy|_{y=0}=1$, where $t(y)$ is a smooth function defined in a neighborhood of $y=0$. Thus, we have $t_1=t(\pm y)$, $t_2=t(\mp y)$ if $y \lessgtr 0$. In any case, if $y\neq 0$,

$$x\cdot\alpha(x)=\frac{\dot{p}(t(-y))\,p(t(y)-1)-\dot{p}(t(y))\,p(t(-y)-1)}{p(t(y)-1)-p(t(-y)-1)}$$
$$=\frac{[\dot{p}(t(-y))\,p(t(y)-1)-\dot{p}(t(y))\,p(t(-y)-1)]/y}{[p(t(y)-1)-p(t(-y)-1)]/y}.$$

The denumerator and numerator are even functions of y and as $y\to 0$,

$$[p(t(y)-1)-p(t(-y)-1)]/y\to 2\dot{p}(-1)>0.$$

The numerator also has a limit $-2\ddot{p}(0)\cdot p(-1)>0$. Hence, $x\cdot\alpha(x)$ is a smooth even function for y in a neighborhood of 0. Substituting $y=\sqrt{(x-p(0))/a}$, for $x\in(p(0)-\varepsilon, p(0)]$, we obtain that $\alpha(x)$ is a smooth function in $(p(0)-\varepsilon, p(0)]$, where $\varepsilon>0$ is some constant. More precisely, $\alpha(x)\in C^\infty(p(0)-\varepsilon, p(0))$ and $\alpha^{(k)}(x)$ has a limit as $x\to p(0)$, $k=0, 1, 2,\ldots$.

It is a good exercise to show that $\alpha(x)$ can be extended smoothly to $x\in[p(0), +\infty)$ and we shall omit it here.

Similar arguments apply to $\beta(x)$, showing that $\beta(x)$ is smooth for $x\in(p(0)-\varepsilon, p(0)]$ and is extendable to $x\in[p(0), +\infty)$ smoothly.

$$\beta(p(0))=\lim_{y\to 0}\frac{[\dot{p}(t(y))-\dot{p}(t(-y))]/y}{[p(t(y)-1)-p(t(-y)-1]/y}$$
$$=\frac{2\ddot{p}(0)}{2p(-1)}<0.$$

We now study the behavior of $\alpha(x)$ and $\beta(x)$ when $x \to 0_+$. Since

$$\alpha(x) = \frac{[\dot{p}(t_1)\, p(t_2-1) - \dot{p}(t_2)\, p(t_1-1)]/x^2}{[p(t_2-1) - p(t_1-1)]/x},$$

to show $\alpha(x) \in C^\infty[0, \varepsilon)$, it suffices to show that $\dot{p}(t_1)/x$, $\dot{p}(t_2)/x$, $p(t_1-1)/x$, $p(t_2-1)/x$ have smooth extensions for $x \in (-\varepsilon, \varepsilon)$ and $\lim_{x \to 0^+}[p(t_2-1) - p(t_1-1)]/x \neq 0$.

Let $G(z) = G_1 z + G_2(z)$ and $H(z) = H_1 z + H_2(z)$, where G_1 and H_1 are positive constants and $G_2(z) = O(z^2)$, $H_2(z) = O(z^2)$. By the Implicit Function Theorem, $x = G(z)$ can be solved in a neighborhood of $x = 0$, $z = 0$, as $z = Z(x)$ with $Z(0) = 0$ and $\dot{Z}(0) = 1/G_1$. If $e^{\lambda_1 t} = z$, we have

$$\begin{aligned} \frac{dp(t_1)}{dt} &= G_1 \cdot \lambda_1 \cdot z + G_2'(z)\, \lambda_1 \cdot z \\ &= G_1 \cdot \lambda_1 \cdot Z(x) + G_2'(Z(x)) \cdot \lambda_1 \cdot Z(x). \end{aligned} \tag{4.3}$$

Since $Z(x)/x \in C^\infty(-\varepsilon, \varepsilon)$, $(dp(t_1)/dt)/x$ can be extended as a C^∞ function for $x \in (-\varepsilon, \varepsilon)$ by (4.3) and

$$\lim_{x \to 0^+} \dot{p}(t_1)/x = \lambda_1 \tag{4.4}$$

for $\lim_{x \to 0} Z(x)/x = 1/G_1$. Similarly, we can show that $\dot{p}(t_2)/x$ can be extended as a C^∞ function for $x \in (-\varepsilon, \varepsilon)$ and

$$\lim_{x \to 0^+} \dot{p}(t_2)/x = \lambda_2. \tag{4.5}$$

Since

$$\begin{aligned} p(t_1 - 1) &= G_1 z e^{-\lambda_1} + G_2(z e^{-\lambda_1}) \\ &= G_1 e^{-\lambda_1} \cdot Z(x) + G_2(e^{-\lambda_1} Z(x)) \end{aligned}$$

we obtain that $p(t_1-1)/x$ can be extended as a C^∞ function for $x \in (-\varepsilon, \varepsilon)$ and

$$\lim_{x \to 0^+} p(t_1-1)/x = e^{-\lambda_1}. \tag{4.6}$$

Similarly, $p(t_2-1)/x$ can be extended as a C^∞ function for $x \in (-\varepsilon, \varepsilon)$ and

$$\lim_{x \to 0^+} p(t_2-1)/x = e^{-\lambda_2} \tag{4.7}$$

$\lim_{x \to 0^+}[p(t_2-1) - p(t_1-1)]/x = e^{-\lambda_2} - e^{-\lambda_1} > 0$. Thus, we conclude that $\alpha(x)$ has a smooth extension on $(-\varepsilon, \varepsilon)$. Finally, it is easy to see that $\alpha(x)$

has a smooth extension on R. In the same manner, we show that $\beta(x)$ has a smooth extension on $(-\varepsilon, \varepsilon)$ and is extendible to $x \in R$ smoothly. From (4.2), (4.4), (4.5), (4.6) and (4.7),

$$\beta(0) = \frac{\lambda_2 - \lambda_1}{e^{-\lambda_2} - e^{-\lambda_1}} < 0.$$

Therefore, $\beta(x) < 0$ for $x \in [0, p(0)]$ and there is an $\eta > 0$ such that $-\beta(x) > \eta$ since $[0, p(0)]$ is compact.

In the equation

$$\frac{\dot{x}(t)}{x(t)} = \alpha(x(t)) + \beta(x(t))\, x(t-1)/x(t),$$

let $x = p(t)$ and $t \to -\infty$. We obtain $\lambda_1 = \alpha(0) + \beta(0)\, e^{-\lambda_1}$. Similarly, letting $t \to +\infty$, we have $\lambda_2 = \alpha(0) + \beta(0)\, e^{-\lambda_2}$. Therefore, $x = e^{\lambda_1 t}$ and $x = e^{\lambda_2 t}$ are solutions of $\dot{x}(t) = \alpha(0)\, x(t) + \beta(0)\, x(t-1)$. This proves Lemma 4.1.

We shall need the following lemma, the proof of which can be found in [2].

LEMMA 4.2. *Let $\lambda_1 \geqslant \lambda_2$ be two real characteristic values for*

$$\dot{x}(t) = Ax(t) + Bx(t-1),$$

where A and B are real constants. Then all the other characteristic values are complex and have real parts less than λ_2.

In our case, $\lambda_1 > 0$, $\lambda_2 < 0$. Therefore, 0 is hyperbolic equilibrium of (4.1) with the local unstable manifold $W^u(0)$ of dim 1 and the local stable manifold $W^s(0)$ of codim 1. The function $x = p(t)$ defines a homoclinic trajectory connecting $W^u(0)$ to $W^s(0)$.

A simple example constructed to have $x = (\operatorname{sh} 1) \cdot (\operatorname{ch} t)^{-1}$ as a homoclinic trajectory is

$$\dot{x}(t) = (\operatorname{ch} 1)(\operatorname{sh} 1)^{-1} \cdot x(t) - (\operatorname{sh} 1)^{-1}(1 + x^2(t))\, x(t-1). \qquad (4.8)$$

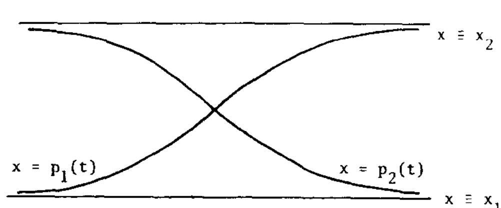

FIGURE 4.3

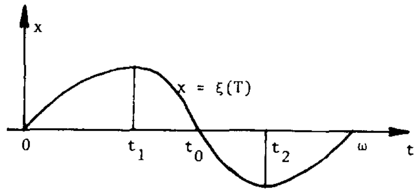

FIGURE 4.4

Remark 4.3. Our method can be employed to construct a scalar delay equation (4.1) with $x \equiv x_1$, $x \equiv x_2$ as two equilibria and $x = p_1(t)$, $x = p_2(t)$ as two heteroclinic trajectories joining $x \equiv x_1$ and $x \equiv x_2$ as $t \to \pm\infty$. See Fig. 4.3.

We can also construct a scalar delay equation (4.1) with any sinuous periodic solution $x = \xi(t)$, $\xi(t+\omega) = \xi(t)$ which oscillates slowly, i.e., $\xi(0) = \xi(t_0) = \xi(\omega) = 0$, $\xi > 0$ in $(0, t_0)$ and $\xi < 0$ in (t_0, ω), $\xi(t_1) = \max_t \xi(t) > 0$, $\xi(t_2) = \min_t \xi(t) < 0$, $t_1 \geqslant 1$ and $t_2 - t_0 \geqslant 1$. See Fig. 4.4.

Less obviously, our method can be refined to construct scalar delay equations with a homoclinic trajectory looking like Fig. 4.5, i.e., $p(t) \sim e^{\lambda_1 t}$, $t < -T$ and $p(t) \sim e^{-\zeta t} \cos(\omega t + \theta_0)$, $t > T$, for some constants ω, θ_0, $\lambda_1 > 0$, $\zeta > 0$ and $T > 0$. It is suggested by Hale that $\dot{x} = f(x(t), x(t-1), x(t-2))$ would be a good candidate for such an equation. We only give a few suggestions of how this can be accomplished. The orbit of $(p(t), p(t-1), p(t-2))$, $t \in R$, lies in a two-dimensional plane in R^3 for $p_1(t) = e^{-\zeta t} \cos(\omega t + \theta_0)$, while for $p_2(t) = e^{\lambda_1 t}$, it is a straight line in R^3. Assigning a linear functional on the plane and the line, we find $\dot{x} = Ax(t) + Bx(t-1) + Cx(t-2)$ will have $p_1(t)$ and $p_2(t)$ as solutions. The general situation can be handled by a smooth change of coordinates. It was first discovered by Sil'nikov [18] that three-dimensional ODEs having this type of homoclinic trajectory may have chaotic behaviors in a neighborhood of the homoclinic orbit. We shall show that this is the case for certain delay equations elsewhere.

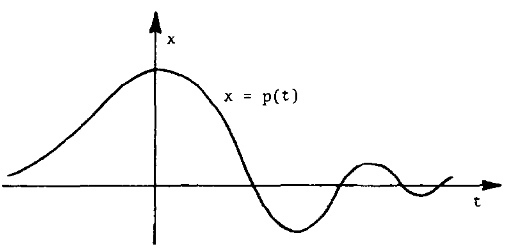

FIGURE 4.5

5. Perturbations of Homoclinic Orbits and Bifurcation Functions

We have constructed (4.1) such that $\alpha(x)$, $\beta(x) \in C^{\infty}(R)$, $x=0$, is a hyperbolic equilibrium with $\dim W^u(0)=1$ and $\operatorname{codim} W^s(0)=1$. The function $x=p(t)$ defines a homoclinic trajectory asymptotic to $x=0$ as $t \to \pm\infty$. Furthermore, $-\beta(x) \geqslant \eta > 0$ for $x \in [0, p(0)]$. In this section, we study the perturbed system of (4.1):

$$\dot{x}(t) = \alpha(x(t))\, x(t) + \beta(x(t))\, x(t-1) + h(t, x_t, \mu), \tag{5.1}$$

where $\mu \in X$, a Banach space and $h\colon R \times C \times X \to R$ is C^1 with bounded derivative, and $h(t+\omega, \phi, \mu) = h(t, \phi, \mu)$, $\omega > 0$, a constant. We assume that $h(t, \phi, \mu) = O(|\mu|)$ for μ close to 0, uniformly with respect to t, ϕ. Since zero is a hyperbolic equilibrium, from a standard result of perturbation theory [9], we know that for μ sufficiently small, (5.1) has a unique periodic solution $x = \tilde{x}_\mu(t)$ with period ω in a neighborhood of 0 and $|\tilde{x}_\mu| \to 0$ as $\mu \to 0$. The solution $\tilde{x}_\mu$ is hyperbolic and the local unstable manifold W^u_μ and the local stable manifold W^s_μ of $\tilde{x}_\mu$ are C^1 close to $W^u(0) \times R$ and $W^s(0) \times R$ when μ is small. We shall study the conditions that ensure the existence of some homoclinic trajectory $x = p_\mu(t)$ in a neighborhood of $x = p(t)$ and W^u_μ intersects W^s_μ transversally along the orbit of $x = p_\mu(t)$. The investigation uses the method of Liapunov and Schmidt and Lemma 3.4. This method was first used to treat a similar problem in ODEs [4, 8]; see also [14, 16]. The appearance of the transverse homoclinic orbit in delay equations is of great interest since it has been proved that trajectories near $x = p_\mu(t)$ are very complicated. Discussions using symbolic dynamics may be found in [10].

We first consider the linear variational equation of (4.1) with respect to $x = p(t)$.

$$\begin{aligned} &\dot{x}(t) = \mathscr{A}(t)\, x(t) + \mathscr{B}(t)\, x(t-1), \\ &\mathscr{A}(t) = \alpha(p(t)) + \dot{\alpha}(p(t))\, p(t) + \dot{\beta}(p(t))\, p(t-1), \\ &\mathscr{B}(t) = \beta(p(t)). \end{aligned} \tag{5.2}$$

Define the operator $L(t)\colon C \to R$, $t \in R$, as

$$L(t)\, \phi = \mathscr{A}(t)\, \phi(0) + \mathscr{B}(t)\, \phi(-1).$$

It is clear that $L(t)$ is continuous with respect to t and is uniformly bounded since $\mathscr{A}(t)$ and $\mathscr{B}(t)$ are uniformly bounded for $t \in R$. Then (5.2) can be written as $\dot{x}(t) = L(t)\, x_t$.

Since $\mathscr{A}(t) \to \alpha(0)$, $\mathscr{B}(t) \to \beta(0)$ as $t \to \pm\infty$, and $\dot{x} = \alpha(0)\, x(t) +$

$\beta(0)\, x(t-1)$ has an exponential dichotomy, by Corollary 2.2, there is a constant $\tau > 0$ such that (5.2) has an exponential dichotomy on $(-\infty, -\tau]$ and $[\tau, +\infty)$, with $\dim \mathscr{R}Q^{\pm}(t) = 1$. Since $-\beta(x) \geqslant \eta > 0$, $x \in [0, p(0)]$, we have $-\mathscr{B}(t) \geqslant \eta > 0$ for $t \in R$. Thus, $T(t, s)\, \phi \neq 0$ for any $\phi \in C$, $\phi \neq 0$ and $s \leqslant t$, since the uniqueness of the backward continuation [9] applies to (5.2) when $-\mathscr{B}(t) \geqslant \eta > 0$.

The formal adjoint equation of (5.2), in a form equivalent to that of Section 3, is

$$\dot{y}(t) = -\mathscr{A}(t)\, y(t) - \mathscr{B}(t+1)\, y(t+1). \tag{5.3}$$

Similar reasoning as above shows that the forward continuation (to the direction of the increasing t) of any solution of (5.3) is unique when $-\mathscr{B}(t) \geqslant \eta > 0$, $t \in R$. Therefore $\psi \in B[-r, 0]$, $\psi \neq 0$ implies that $\tilde{T}(s, t)\, \psi \neq 0$ for $s \leqslant t$. We can now apply Theorem 3.3 to conclude that (5.2) has exponential dichotomies in R^+ and R^- with $\dim \mathscr{R}Q^-(0) = \dim \mathscr{R}Q^+(0) = 1$. Hence, by Lemma 3.4, $F = d/dt - L(t)$ is Fredholm from $C_b^1(R)$ into $C_b^0(R)$ with $I(F) = \dim \mathscr{R}Q^-(0) - \dim \mathscr{R}Q^+(0) = 0$. Dim $\mathscr{N}(F) = \dim[\mathscr{R}P^+(0) \cap \mathscr{R}Q^-(0)] \leqslant 1$. Since $\dot{p}(t) \in \mathscr{N}(F)$, $\dim \mathscr{N}(F) = 1$ and $\mathscr{N}(F)$ is spanned by $x = \dot{p}(t)$. Codim $\mathscr{R}(F) = 1$ follows from $I(F) = 0$. It is now clear that (5.3) has a unique bounded solution $y = y_0(t) \not\equiv 0$, $t \in R$, up to a scalar multiplication and $y_0(t) \to 0$ exponentially as $t \to \pm\infty$. And

$$\mathscr{R}(F) = \left\{ h \in C_b^0(R) \,\middle|\, \int_{-\infty}^{\infty} y_0(t)\, h(t)\, dt = 0 \right\}.$$

We assume that $\int_{-\infty}^{\infty} y_0^2(t)\, dt = 1$. We choose $t_0 \in R$ such that $\dot{p}(t_0) \neq 0$ and, define $x_0(t) = \dot{p}(t)/\dot{p}(t_0)$. Thus, $x_0(t)$ spans $\mathscr{N}(F)$ and $x_0(t_0) = 1$.

Definition 5.1. Projections $E_1\colon C_b^1(R) \to \mathscr{N}(F)$ and $E_2\colon C_b^0(R) \to \mathscr{R}(F)$ are defined as $E_1\colon x \mapsto x_1$, $x_1(t) = x(t_0)\, x_0(t)$ and $E_2\colon f \mapsto f_1$, $f_1(t) = f(t) - y_0(t) \int_{-\infty}^{\infty} y_0(s)\, f(s)\, ds$.

Note that $x \in \mathscr{R}(I - E_1)$ if and only if $x \in C_b^1(R)$ and $x(t_0) = 0$. And $f \in \mathscr{R}(I - E_2)$ if and only if $f = k y_0(t)$ for some constant k.

It is known that $F\colon \mathscr{R}(I - E_1) \to \mathscr{R}(E_2)$ is one-to-one and onto. Denote the inverse as $\mathscr{K}\colon \mathscr{R}(E_2) \to \mathscr{R}(I - E_1)$. $\mathscr{K}$ is continuous; that is, there is a constant $k > 0$ such that for $f \in \mathscr{R}(E_2) = \mathscr{R}(F)$,

$$\|\mathscr{K} f\|_{C_b^1} \leqslant k\, \|f\|_{C_b^0}. \tag{5.4}$$

Let $x = x(t)$ be any trajectory of (5.1) whose orbit is in the ε-neighborhood of the orbit of $x = p(t)$, when both are considered in the space $C[-1,0]$. We write $x(t)=p(t+\zeta)+z(t+\zeta)$. Since $\dot{p}(t_0)\neq 0$, we can choose ζ such that $z(t_0)=0$, if ε is sufficiently small. We have

$$z\in\mathscr{R}(I-E_1). \tag{5.5}$$

$z(t)$ satisfies the equation

$$F(z)(t)=f(z,\zeta,\mu)(t), \tag{5.6}$$

where f: $C_b^0(R)\times R\times X\to C_b^0(R)$ is defined as

$$\begin{aligned} f(z,\zeta,\mu)(t)=\alpha(p(t)&+z(t))(p(t)+z(t))\\ &+\beta(p(t)+z(t))(p(t-1)+z(t-1))\\ &-[\alpha(p(t))\,p(t)+\beta(p(t))\,p(t-1)]\\ &-[\mathscr{A}(t)\,z(t)+\mathscr{B}(t)\,z(t-1)]\\ &+h(t-\zeta,p_t+z_t,\mu), \end{aligned} \tag{5.7}$$

$f(0,\zeta,0)=0$, $f(z,\zeta,\mu)=O(|z|^2+|\mu|)$, $f_z(0,\zeta,0)=0$. By the Liapunov–Schmidt method, (5.6) is equivalent to

$$\begin{aligned} F(z)&=E_2 f(z,\zeta,\mu),\\ 0&=(I-E_2)\,f(z,\zeta,\mu). \end{aligned} \tag{5.8}$$

By (5.5), the first of (5.8) is equivalent to

$$z=\mathscr{K}E_2 f(z,\zeta,\mu). \tag{5.9}$$

Before applying the Implicit Function Theorem to (5.9), we mention that according to Irwin [13], f is not C^1 as a map from $C_b^0(R)\times R\times X$ to $C_b^0(R)$. However, for $|\mu|\leqslant\mu_0$, $\mu_0>0$, is a constant, we easily get an a priori estimate for $\|z\|_{C_b^1}$ from (5.4) and (5.9). Let $Y=\{z\mid z\in C_b^1(R),\ \|z\|_{C_b^1}\leqslant M\}$ for some positive constant M. $\{z_t\mid z\in Y, t\in R\}$ is a precompact subset of $C[-1,0]$. The argument in [13] can be modified to show that f: $Y_1\times R\times Z\to Y_0$ is C^1, where Y_1 in the domain of f is the set Y furnished with C_b^1 metric while Y_0 in the range of f is the set Y furnished with C_b^0 metric. From the Implicit Function Theorem, there are open balls $B_{Y_1}(\varepsilon_1)\subset Y_1$, $B_X(\varepsilon_2)\subset X$, with radii ε_1, ε_2, both are centered at zeros, and a unique C^1 map $z=z^*(\zeta,\mu)$: $R\times B_X(\varepsilon_2)\to B_{Y_1}(\varepsilon_1)$, such that $z=z^*(\zeta,\mu)$ is the solution of (5.9) for $(z,\zeta,\mu)\in B_{Y_1}(\varepsilon_1)\times R\times B_X(\varepsilon_2)$, and $z^*(\zeta,\mu)=O(|\mu|)$ for μ close to zero, uniformly with respect to $\zeta\in\mathbb{R}$. Note that to derive the above result, we need $f(z, \zeta+\omega, \mu)=f(z, \zeta, \mu)$. The bifurcation equation is obtained by substituting $z^*(\zeta, \mu)$ into the second of (5.8)

$$G(\zeta, \mu) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} y_0(t)\, f(z^*(\zeta, \mu), \zeta, \mu)(t)\, dt=0.$$

Obviously,

$$G(\zeta, \mu)=\int_{-\infty}^{\infty} y_0(t)\, h(t-\zeta, p_t, \mu)\, dt+o(|\mu|). \tag{5.10}$$

Since $\int_{-\infty}^{\infty} y_0(t)\cdot[g(t)]\, dt$: $C_b^0(R)\to R$ is a linear continuous functional, we readily see that $G(\zeta, \mu)$: $R\times X\to R$ is C^1. An analogous argument based on the perturbations of a vector field as in the case of ODEs [3, 4], shows that for $|\mu|<\mu_0$, $\zeta\in R$ and $G(\zeta, \mu)=0$, $x(t)=p(t+\zeta)+z^*(\zeta, \mu)(t+\zeta)$ is a transverse homoclinic orbit if and only if $(\partial/\partial\zeta)\, G(\zeta, \mu)\neq 0$. Details will appear in a later paper with more general cases being considered.

We have the following theorem.

THEOREM 5.2. *Let $h(t, \phi, \mu)$ satisfy all of the assumptions made previously after (5.1). Then there exists $\mu_0>0$ such that $x=x(t)$ is a transverse homoclinic trajectory of (5.1), $|\mu|\leqslant\mu_0$ with its orbit being in a neighborhood of the orbit of $x=p(t)$ if and only if $x(t)=p(t+\zeta)+z^*(\zeta, \mu)$ $(t+\zeta)$ with $z^*(\zeta, \mu)(t_0)=0$ and*

$$G(\zeta, \mu)=0,$$

$$\frac{\partial}{\partial\zeta} G(\zeta, \mu)\neq 0.$$

COROLLARY 5.3 (Palmer [16]). *Let $\mu\in X=R$. Assume further that h: $R\times C\times R\to R$ is C^2 with all derivatives up through second order being bounded. Let*

$$\Delta(\zeta) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} y_0(t)\, h_1(t-\zeta, p_t, 0)\, dt,$$

with

$$h_1(t, \phi, \mu)=\frac{1}{\mu} h(t, \phi, \mu), \qquad \mu\neq 0$$

$$=\frac{\partial}{\partial\mu} h(t, \phi, 0), \qquad \mu=0.$$

If $\Delta(\zeta_0)=0$ and $\Delta'(\zeta_0)\neq 0$ for some $\zeta_0 \in R$, then tere is a unique transverse homoclinic trajectory of (5.1) *near $x=p(t)$ for each μ, $|\mu| \leqslant \mu_0$. Furthermore, ζ is a C^1 function of μ, $\zeta=\zeta^*(\mu)$, $|\mu| \leqslant \mu_0$ and $\zeta_0=\zeta^*(0)$, and the homoclinic trajectory is*

$$x(t)=p(t+\zeta^*(\mu))+z^*(\mu, \zeta^*(\mu))(t+\zeta^*(\mu)).$$

Proof. Using the Implicit Function Theorem and (5.10).

A special case in which the condition of Corollary 5.3 is easily verified is presented in the following corollary.

COROLLARY 5.4. *Let $h(t, \phi, \mu)=\mu \cos t$, $\mu \in R$. Define*

$$\Delta(\zeta) \overset{\text{def}}{=} \int_{-\infty}^{\infty} y_0(t) \cos(t-\zeta)\, dt.$$

If $\Delta(\zeta) \not\equiv 0$ for all $\zeta \in R$, then the conditions of Corollary 5.3 are satisfied for some ζ_0.

EXAMPLE 5.5. Consider the equation

$$\dot{x}(t)=(\operatorname{ch} 1)(\operatorname{sh} 1)^{-1} x(t)-(\operatorname{sh} 1)^{-1}(1+x^2(t))\, x(t-1)+\mu \cos t, \qquad \mu \in R. \tag{5.11}$$

When $\mu=0$, the equation has a homoclinic solution $x=(\operatorname{sh} 1)(\operatorname{ch} t)^{-1}$. The formal adjoint system relevant to the linear variational equation with respect to $x=(\operatorname{sh} 1)(\operatorname{ch} t)^{-1}$ has a unique bounded solution $y=y_0(t)$, up to a scalar multiplier, and $y_0(t) \to 0$ exponentially as $t \to \pm\infty$. Although the explicit form of $y_0(t)$ is not known, the coefficients $\mathscr{A}(t)$ and $\mathscr{B}(t)$ in (5.3) are known. Therefore, it is not hard to compute $y_0(t)$ and $\Delta(\zeta)$ numerically. We summarize our numerical calculation as follows.

Take the initial data to be $y_0(t)=e^{-t}$ for $10 \leqslant t \leqslant 11$. Use Adams–Moulton scheme with the step size $h=1/128$. We compute the unique $y_0(t)$ (not normalized) from $y_0(10) \approx 4.54 \times 10^{-5}$ backwards to $y_0(-8) \approx 6.59 \times 10^{-5}$. There is only one maximum $y_0(1.58) \approx 0.118$, one minimum $y_0(-1.08) \approx -0.105$ and one zero $y_0(0.37) \approx 0$ in between. We claim that $|\Delta(\zeta)-\int_{-8}^{10} y_0(t) \cos(t-\zeta)\, dt| \leqslant 2 \times 10^{-4}$ for all $\zeta \in R$. Integrating from -8 to 10, we obtain $\Delta(-1.27) \approx -0.31$, $\Delta(0.30) \approx 0$ and $\Delta'(0.30) \approx 0.31$. The other extrema and zeros are obtained by considering $\Delta(\zeta+\pi)=-\Delta(\zeta)$, and $\Delta(\zeta) \approx 0.31 \sin(\zeta-0.30)$. Therefore, for small μ, (5.11) would have a unique transverse homoclinic trajectory in the neighborhood of the orbit of $x=(\operatorname{sh} 1)(\operatorname{ch} t)^{-1}$.

Remark 5.6. A similar technique can be applied to periodic perturbations of an autonomous retarded functional differential equation in a neighborhood of a heteroclinic orbit asymptotic to hyperbolic equilibria which may have unstable manifolds of more than one dimension. The bifurcation function $G(\zeta, \mu, k_i)$ will have some parameters k_i, which are the coordinates in the null space $\mathcal{N}(F)$. Also, Lemma 3.4 can be improved such that the exponential dichotomies are required only in $(-\infty, -\tau]$ and $[\tau, +\infty)$ for some $\tau > 0$.

REFERENCES

1. H. T. BANKS, The representation of solutions of linear functional differential equations, *J. Differential Equations* **5** (1969), 399–410.
2. R. BELLMAN AND K. COOKE, "Differential Difference Equations," Academic Press, New York/London, 1963.
3. S.-N. CHOW AND J. K. HALE, "Methods of Bifurcation Theory," Grund. Math. Wiss., Vol. 251, Springer-Verlag, Berlin/New York/Heidelberg, 1982.
4. S.-N. CHOW, J. K. HALE, AND J. MALLET-PARET, An example of bifurcation to homoclinic orbits, *J. Differential Equations* **37** (1980), 351–373.
5. C. V. COFFMAN AND J. J. SCHÄFFER, Linear differential equations with delays: Admissibility and conditional exponential stability, *J. Differential Equations* **9** (1971), 521–535.
6. W. A. COPPEL, "Dichotomies in Stability Theory," Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, Berlin/New York/Heidelberg, 1978.
7. R. L. DEVANEY, Homoclinic orbits to hyperbolic equilibria, *Ann. N.Y. Acad. Sci.* **316** (1979), 108–117.
8. J. K. HALE, "Introduction to Dynamical Bifurcation," LCDS Report No. 83-16, 1983.
9. J. K. HALE, "Theory of Functional Differential Equations," Springer-Verlag, Berlin/New York/Heidelberg, 1977.
10. J. K. HALE AND X.-B. LIN, "Symbolic Dynamics and Nonlinear Semiflows," LCDS Report No. 84-8, 1984.
11. D. HENRY, The adjoint of a linear functional differential equation and boundary value problems, *J. Differential Equations* **9** (1971), 55–66.
12. D. HENRY, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, Vol. 840, Springer-Verlag, Berlin/New York/Heidelberg, 1981.
13. M. C. IRWIN, On the smoothness of the composition map, *Quart. J. Math. Oxford Ser. 2* **23** (1972), 113–133.
14. V. K. MEL'NIKOV, On the stability of the center for time periodic perturbations, *Trans. Moscow Math. Soc.* **12**(1) (1964), 1–57.
15. S. E. NEWHOUSE, Asymptotic behavior and homoclinic points in nonlinear systems, *Ann. N.Y. Acad. Sci.* **357** (1980), 292–299.
16. K. J. PALMER, Exponential dichotomies and transversal homoclinic points, *J. Differential Equations* **55** (1984), 225–256.
17. G. PECELLI, Dichotomies for linear functional differential equations, *J. Differential Equations* **9** (1971), 555–579.
18. L. P. SIL'NIKOV, A case of the existence of a denumerable set of periodic motions, *Soviet Math. Dokl.* **6** (1965), 163–166.