JOURNAL OF DIFFERENTIAL EQUATIONS **65**, 175–202 (1986)

Heteroclinic Orbits for Retarded Functional Differential Equations*

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Received March 20, 1985

Suppose Γ is a heteroclinic orbit of a C^k functional differential equation $\dot{x}(t)=f(x_t)$ with α-limit set $\alpha(\Gamma)$ and ω-limit set $\omega(\Gamma)$ being either hyperbolic equilibrium points or periodic orbits. Necessary and sufficient conditions are given for the existence of an $\bar{f}$ close to f in C^k with the property that $\dot{x}(t)=\bar{f}(x_t)$ has a heteroclinic orbit $\bar{\Gamma}$ close to Γ. The orbits $\bar{\Gamma}$ are obtained from the zeros of a finite number of bifurcation functions $G(\beta, \bar{f}) \in \mathbb{R}^{d^*}$, $\beta \in \mathbb{R}^{d+1}$. Transversality of Γ is characterized by the nondegeneracy of the derivative $D_\beta G$. It is also shown that the $\bar{f}$ which have heteroclinic orbits near Γ are on a C^k submanifold of finite codimension $=\max\{0, -\operatorname{ind}\Gamma\}$ or on the closure of it, where $\operatorname{ind}\Gamma$ is the index of Γ.

1. INTRODUCTION

Let $C[-r, 0]$ be the Banach space of continuous functions from $[-r, 0]$ into R^n with the supremum norm. Suppose x is any continuous function from R into R^n, $x_t(\theta)=x(t+\theta)$, $-r \leqslant \theta \leqslant 0$ is an element in $C[-r, 0]$. Let D be a bounded open ball in $C[-r, 0]$, and let $\chi^k=\{f \mid f \in C^k, f\colon \bar{D} \to R^n\}$ be the Banach space with the usual C^k-norm $\|\cdot\|_k$. For a given $f \in \chi^k$, suppose the autonomous retarded functional differential equation

$$\dot{x}(t)=f(x_t) \tag{1.1}$$

has a heteroclinic orbit $\Gamma \subset C[-r, 0]$ with α-limit set $\alpha(\Gamma)$ and ω-limit set $\omega(\Gamma)$ being hyperbolic periodic orbits or equilibrium points, $\Gamma \cup \omega(\Gamma) \cup \alpha(\Gamma) \subset D$.

Suppose X is a Banach space, the parameter space, $g \in C^k(\bar{D} \times X, \mathbb{R}^n)$,

* This research was supported in part by the Air Force Office of Scientific Research under Grant AF-AFOSR 84-0376; in part by the Army Research Office under Contract DAAG-29-83-K-0029; and in part by the National Science Foundation under Grant DMS 8205355.

0022-0396/86 $3.00

with $g(\cdot, \mu) \in \chi^k$, $\| g(\cdot, \mu) \|_k = O(|\mu|)$ as $\mu \to 0$ and consider the perturbation of (1.1) given by

$$\dot{x}(t) = f(x_t) + g(x_t, \mu). \tag{1.2}$$

The purpose of this paper is to determine conditions for the occurrence of a heteroclinic orbit Γ^μ of (1.2) in a neighborhood of Γ for μ in a neighborhood of zero. We also want to specify these conditions in terms of computable quantities which can be used to determine either the transversality or the order of nontransversality of the heteroclinic orbit.

In order to be specific about the results, let us assume first that $\alpha(\Gamma) = \gamma_1$, $\omega(\Gamma) = \gamma_2$, where γ_1, γ_2 are periodic orbits of periods ω_1, ω_2, respectively. Let $W^u(\gamma_j)$, $W^s(\gamma_j)$ be, respectively, the unstable and stable sets for γ_j, $j = 1, 2$. We refer to these as manifolds, even though it may not always be true that they are manifolds globally. The local unstable and stable manifolds $W^u_{\mathrm{loc}}(\gamma_j)$, $W^s_{\mathrm{loc}}(\gamma_j)$ near γ_j are C^k-manifolds.

Let $\hat{T}(t)$: $C[-r, 0] \to C[-r, 0]$, $t \geqslant 0$, be the C^k-semigroup generated by (1.1); that is, $\hat{T}(t)\phi$ is the solution through ϕ at $t = 0$. In the following, we let $\Gamma = \bigcup_{t \in \mathbb{R}} \{q_t\}$, $\gamma_1 = \bigcup_{t \in \mathbb{R}} \{p_{1,t}\}$, $\gamma_2 = \bigcup_{t \in \mathbb{R}} \{p_{2,t}\}$, where q, p_1, p_2 are solutions of (1.1).

DEFINITION 1.1. $\Gamma \subset W^u(\gamma_1) \cap W^s(\gamma_2)$ is said to be a *transverse heteroclinic orbit* if for $s, t > 0$ large enough such that $q_{-s} \in W^u_{\mathrm{loc}}(\gamma_1)$ and $q_t \in W^s_{\mathrm{loc}}(\gamma_2)$ then $\hat{T}(t+s)$ sends a disc in $W^u_{\mathrm{loc}}(\gamma_1)$ containing q_{-s} transverse to $W^s_{\mathrm{loc}}(\gamma_2)$ at q_t.

The important concept of general position will play an important role in the study of nontransversality.

DEFINITION 1.2. $\Gamma \subset W^u(\gamma_1) \cap W^s(\gamma_2)$ is said to be in *general position* if Γ is either transverse or, if, for any $s, t > 0$ large enough such that $q_{-s} \in W^u_{\mathrm{loc}}(\gamma_1)$ and $q_t \in W^s_{\mathrm{loc}}(\gamma_2)$, then $\hat{T}(t+s)$ sends a disc in $W^u_{\mathrm{loc}}(\gamma_1)$ containing q_{-s} diffeomorphically onto its image and $\dot{q}_t$ is the only tangent vector in $[\hat{T}(t+s)\, W^u_{\mathrm{loc}}(\gamma_1)] \cap W^s_{\mathrm{loc}}(\gamma_2)$ at q_t.

DEFINITION 1.3. The index of $\Gamma \subset W^u(\gamma_1) \cap W^s(\gamma_2)$ is $\operatorname{ind} \Gamma = \dim W^u(\gamma_1) - \dim W^u(\gamma_2)$.

If $\operatorname{ind} \Gamma = -1$, the concept of general position has been referred to as quasitransversal in the study of diffeomorphisms (see Sotomayor [12], Newhouse and Palis [10]).

For μ small, there is a family of hyperbolic periodic orbits $\gamma_j^\mu = \bigcup_{t \in \mathbb{R}} \{p^\mu_{j,t}\}$, $\gamma_j^0 = \gamma_j$ with $W^s_{\mathrm{loc}}(\gamma_j^\mu)$, $W^u_{\mathrm{loc}}(\gamma_j^\mu)$ being C^k in μ, $j = 1, 2$ (see [3]).

One of the main results of the paper is the following.

THEOREM. *If* $\Gamma \subset W^u(\gamma_1) \cap W^s(\gamma_2)$, $I = \max\{0, -\operatorname{ind} \Gamma)$ *then, for* μ *small, there are* C^k *submanifolds* $M(I) \subset \chi^k$ *of codimension* I *such that* $f + g(\cdot, \mu) \in M(I)$ *if and only if* (1.2) *has an orbit* $\Gamma^\mu \subset W^u(\gamma_1^\mu) \cap W^s(\gamma_2^\mu)$ *close to* Γ *and in general position. Furthermore,* $f \in \operatorname{Cl} M(I)$; *that is, if* Γ *is not in general position, then there is a perturbation* $\tilde{g}$ *of* f *such that* $f + \tilde{g} \in M(I)$.

In particular, this result implies that there are I linearly independent perturbations to break the heteroclinic orbit Γ if $I > 0$. This result is a local version of the genericity of transversal intersection of stable and unstable manifolds of γ_1 and γ_2. If $\operatorname{ind} \Gamma \geqslant 0$, a small perturbation can make it transverse; if $\operatorname{ind} \Gamma < 0$, a small perturbation can break it and there are $-\operatorname{ind} \Gamma$ ways to do it. For a more complete discussion of generic properties of functional differential equations, see [6, 8, 9].

Similar results hold when $\alpha(\Gamma)$, $\omega(\Gamma)$ contain equilibria if we define the index of Γ as $\operatorname{ind} \Gamma = \dim W^u_{\mathrm{loc}}(\alpha(\Gamma)) - \dim W^u_{\mathrm{loc}}(\omega(\Gamma)) + \beta$, where $\beta = -1$ if $\omega(\Gamma)$ is a point and $\beta = 0$ if $\omega(\Gamma)$ is a periodic orbit. Roughly speaking, the cause of the difference in the two cases is that for $\gamma = \omega(\Gamma)$ being periodic orbits, $\operatorname{codim} W^s_{\mathrm{loc}}(\gamma) = \dim W^u_{\mathrm{loc}}(\gamma) - 1$, while for γ being equilibria, $\operatorname{codim} W^s_{\mathrm{loc}}(\gamma) = \dim W^u_{\mathrm{loc}}(\gamma)$.

The proof of the above result uses the method of Liapunov-Schmidt to determine a set of bifurcation functions $G(\beta, \mu) \in \mathbb{R}^{d^*}$, $\beta \in \mathbb{R}^{d+1}$, such that there is a heteroclinic orbit Γ^μ if and only if there is a $\beta(\mu)$ such that $G(\beta(\mu), \mu) = 0$. Furthermore, the transversality of Γ^μ is equivalent to saying that $D_\beta G$ is onto. The degree of nontransversality of Γ^μ is measured by the rank of $D_\beta G$. The constant d is the number of linearly independent solutions which approach zero exponentially as $t \to \pm\infty$ of the linear variational equation about the solution defining Γ and d^* is the number of linearly independent solutions of the adjoint of this equation that are bounded on $(-\infty, \infty)$.

The manner in which the method of Liapunov-Schmidt is employed follows in the spirit of the investigations of Chow, Hale, and Mallet-Paret [1], Palmer [11], and Lin [7] for the determination of heteroclinic orbits for periodically perturbed autonomous systems. The case where the orbits γ_j are periodic and the perturbation is autonomous introduces additional technical difficulties. First, the linear variational equation

$$\dot{x}(t) = f'(q_t)\, x_t \tag{1.3}$$

around Γ has the bounded solution $\dot{q}(t)$ which does not approach zero either as $t \to +\infty$ or as $t \to -\infty$. This implies that (1.3) does not have an exponential dichotomy. Second, since the period of γ_j^μ changes with μ and the time that it takes to go from a transversal of γ_1^μ to a transversal of γ_2^μ is also changing with μ, these quantities must be determined in some way. This involves several careful time scalings.

We now give a brief outline of the contents of each section. Section 2 is a recollection of known results on stable and unstable manifold theory. Section 3 is devoted to the development of the theory of exponential trichotomies, generalizing the concept of exponential dichomoties to fit our needs. Section 4 is devoted to more details about exponential trichotomies including the roughness theorem. Also, it is shown that the linear variational operator around Γ defines a Fredholm operator in the Banach space of continuous bounded functions in $\mathbb{R}$ weighted by a factor $e^{\gamma t}$ for $t<0$ and $e^{-\gamma t}$ for $t>0$. In Section 5, we derive the bifurcation functions G and deduce various geometric consequences of them. In Section 6, we construct perturbations $g(\cdot, \mu)$, showing the manifold structure of $M(I)$, and that $\operatorname{Cl} M(I)$ contains all the vector fields having Γ^μ near Γ as a heteroclinic orbit.

2. Hyperbolic Equilibria, Periodic Orbits

Suppose (1.1) has an equilibrium point $x_0 \in \mathbb{R}^n$ and let $\tilde{x}_0 \in C$ be defined by $\tilde{x}_0(\theta) = x_0$, $-r \leqslant \theta \leqslant 0$. The linear variational equation about x_0 is

$$\dot{x}(t) = L(\tilde{x}_0)x_t, \qquad L(\tilde{x}_0) = f'(\tilde{x}_0). \tag{2.1}$$

The solution x_0 of (1.1) is hyperbolic if all eigenvalues of the characteristic equation of (2.1) have nonzero real parts. Let

$$W^s(x_0) = \{\phi \in C \colon \hat{T}(t)\phi \to \tilde{x}_0 \text{ as } t \to \infty\}$$
$$W^u(x_0) = \{\phi \in C \colon \hat{T}(t)\phi \text{ is defined for } t \leqslant 0,$$
$$\hat{T}(t)\phi \to x_0 \text{ as } t \to -\infty\}.$$

The following theorem may be found in [3, p. 230].

THEOREM 2.1. *If $f \in C^k(C, \mathbb{R}^n)$, $k \geqslant 1$, and x_0 is a hyperbolic equilibrium point of* (1.1), *then there is a neighborhood U of $\tilde{x}_0$ such that*

$$W^u_{\text{loc}}(x_0) = \{\phi \in W^u(x_0),\ \hat{T}(t)\phi \in U,\ t \leqslant 0\}$$
$$W^s_{\text{loc}}(x_0) = \{\phi \in W^s(x_0),\ \hat{T}(t)\phi \in U,\ t \geqslant 0\}$$

are C^k-manifolds. The approach of solutions to $\tilde{x}_0$ as $t \to +\infty$ (or $t \to -\infty$) is exponential.

Suppose $p(t)$ is a periodic solution of (1.1) of minimal period ω and let $\gamma = \bigcup_{t \in \mathbb{R}} \{p_t\} \subset C$ be the corresponding periodic orbit. Then necessarily

$p \in C^k(\mathbb{R}, \mathbb{R}^n)$ and $\dot{p}_t \neq 0$ for all $t \in \mathbb{R}$. The linear variational equation about the periodic solution p is

$$\dot{x}(t) = L_p(t) x_t \tag{2.2}$$

$$L_p(t) = f'(p_t) \tag{2.3}$$

and $\dot{p}(t)$ is a solution of (2.2).

Let $T(t, s)$: $C \to C$ be the solution operator for (2.2); that is, $T(t, s)\phi$ is the solution of (2.2) which coincides with ϕ at $t = s$. The characteristic multipliers of (2.2) are the eigenvalues of the operator $T(\omega, 0)$. The fact that $\dot{p}_t \neq 0$ for all $t \in \mathbb{R}$ satisfies (2.2) implies that 1 is a multiplier of (2.2). The orbit γ is said to be hyperbolic if

(i) 1 is a simple multiplier,

(ii) $[\sigma(T(\omega, 0)) \backslash \{1\}] \cap \{z \in \mathbb{C}\colon |z| = 1\} = \varnothing$.

The stable set $W^s(\gamma)$ for γ and the unstable set $W^u(\gamma)$ of γ are defined as

$$W^s(\gamma) = \{\phi \in C\colon \hat{T}(t)\phi \to \gamma \text{ as } t \to \infty\}$$

$$W^u(\gamma) = \{\phi \in C\colon \hat{T}(t)\phi \text{ is defined for } t \leqslant 0 \text{ and } \to \gamma \text{ as } t \to -\infty\}.$$

For any $\alpha \geqslant 0$, define

$$W^s(\gamma, \alpha) = \{\phi \in W^s(\gamma)\colon \hat{T}(t)\phi - p_{t+\alpha} \to 0 \text{ as } t \to \infty\}$$

$$W^u(\gamma, \alpha) = \{\phi \in W^u(\gamma)\colon \hat{T}(t)\phi - p_{t+\alpha} \to 0 \text{ as } t \to -\infty\}.$$

The sets $W^s(\gamma, \alpha)$, $W^u(\gamma, \alpha)$ are points respectively on the stable and unstable sets which are synchronized in time with $p_{t+\alpha}$.

For any neighborhood U of γ, we define

$$W^s_{\text{loc}}(\gamma, \alpha) = \{\phi \in W^s(\gamma, \alpha)\colon \hat{T}(t)\phi \in U, t \geqslant 0\}$$

$$W^u_{\text{loc}}(\gamma, \alpha) = \{\phi \in W^u(\gamma, \alpha)\colon \hat{T}(t)\phi \in U, t \leqslant 0\}.$$

The following theorem may be found in [3, p. 242; 4].

THEOREM 2.2. *If $f \in C^k(C, \mathbb{R}^n)$, $k \geqslant 1$, and γ is a hyperbolic periodic orbit of (1.1), then there is a neighborhood U of γ such that $W^s_{\text{loc}}(\gamma, \alpha)$, $W^u_{\text{loc}}(\gamma, \alpha)$ are C^k-manifolds and*

$$W^s_{\text{loc}}(\gamma) = \bigcup_{0 \leqslant \alpha < \omega} W^s_{\text{loc}}(\gamma, \alpha), \qquad W^u_{\text{loc}}(\gamma) = \bigcup_{0 \leqslant \alpha < \omega} W^u_{\text{loc}}(\gamma, \alpha)$$

are C^k-manifolds. The approach of solutions to γ either as $t \to +\infty$ or as $t \to -\infty$ is exponential.

3. Exponential Trichotomies

For $t \geqslant s$ in some interval J, let $T(t, s)$ be a strongly continuous non-autonomous semigroup of linear bounded operators in a Banach space X; that is, $T(t, s)$ is strongly continuous in t, s, $T(s, s) = I$ (the identity), $T(t, \tau)\, T(\tau, s) = T(t, s)$, $t \geqslant \tau \geqslant s$. It is said that $T(t, s)$ has an exponential trichotomy on J if there exist projections $P_{\mathrm{u}}(t)$, $P_{\mathrm{s}}(t)$ and $P_{\mathrm{c}}(t) = I - P_{\mathrm{u}}(t) - P_{\mathrm{s}}(t)$, $t \in J$, strongly continuous in t, and

$$T(t, s)\, P_{\mathrm{s}}(s) = P_{\mathrm{s}}(t)\, T(t, s),$$

$$T(t, s)\, P_{\mathrm{u}}(s) = P_{\mathrm{u}}(t)\, T(t, s),$$

$$T(t, s)\, P_{\mathrm{c}}(s) = P_{\mathrm{c}}(t)\, T(t, s),$$

for $t \geqslant s$ in J. We also assume that $T(t, s)$: $\mathscr{R}P_{\mathrm{u}}(s) \to \mathscr{R}P_{\mathrm{u}}(t)$ and $T(t, s)$: $\mathscr{R}P_{\mathrm{c}}(s) \to \mathscr{R}P_{\mathrm{c}}(t)$, $t \geqslant s$, in J are isomorphisms and $T(s, t) = (T(t, s))^{-1}$, $t \geqslant s$, is defined from $\mathscr{R}P_{\mathrm{u}}(t)$ onto $\mathscr{R}P_{\mathrm{u}}(s)$ and from $\mathscr{R}P_{\mathrm{c}}(t)$ onto $\mathscr{R}P_{\mathrm{c}}(s)$. Furthermore, there exist constants $\alpha < \nu - \varepsilon < \nu + \varepsilon < \beta$, called the exponents of the trichotomy, and $K > 0$ such that

$$|T(t, s)\, P_{\mathrm{s}}(s)| \leqslant K e^{\alpha(t-s)},$$

$$|T(s, t)\, P_{\mathrm{u}}(t)| \leqslant K e^{-\beta(t-s)},$$

$$|T(t, s)\, P_{\mathrm{c}}(s)| \leqslant K e^{(\nu+\varepsilon)(t-s)},$$

$$|T(s, t)\, P_{\mathrm{c}}(t)| \leqslant K e^{(-\nu+\varepsilon)(t-s)}, \qquad t \geqslant s \in J.$$

We shall assume $\mathscr{R}P_{\mathrm{u}}(t)$ and $\mathscr{R}P_{\mathrm{c}}(t)$ are finite dimensional.

The adjoint operator $T^*(s, t)$ of $T(t, s)$ is a weak* continuous semigroup in X^*. If $T(t, s)$ has an exponential trichotomy on J, then $T^*(s, t)$ has an exponential trichotomy on J with projections $P_u^*(t)$, $P_s^*(t)$ and $P_c^*(t)$, weak* continuous with respect to $t \in J$, where * denotes the adjoint of a continuous operator. Obviously, $\dim \mathscr{R}P_u^*(t) = \dim \mathscr{R}P_{\mathrm{u}}(t)$ and $\dim \mathscr{R}P_{\mathrm{c}}^*(t) = \dim \mathscr{R}P_{\mathrm{c}}(t)$. It is also true that $T^*(s, t)$: $\mathscr{R}P_{\mathrm{u}}^*(t) \to \mathscr{R}P_{\mathrm{u}}^*(s)$ and $T^*(s, t)$: $\mathscr{R}P_{\mathrm{c}}^*(t) \to \mathscr{R}P_{\mathrm{c}}^*(s)$ are isomorphisms and the inverses $T^*(t, s) = (T^*(s, t))^{-1} = (T(s, t))^*$ are properly defined. See [7].

We call $\mathscr{R}P_{\mathrm{u}}(t)$, $\mathscr{R}P_{\mathrm{s}}(t)$, and $\mathscr{R}P_{\mathrm{c}}(t)$ the unstable space, stable space, and center space, since in most applications, $\beta > 0$, $\nu = 0$, and $\alpha < 0$. The case of $P_{\mathrm{c}}(t) \equiv 0$, $t \in J$, is also called a *shifted exponential dichotomy* if the splitting is not made at $\nu = 0$.

If (1.1) has a hyperbolic equilibrium point, then the solution map $T(t, s) = D_\phi \hat{T}(t-s)\tilde{x}_0$ of (2.1) has an exponential dichotomy for all $t \geqslant s$ in $\mathbb{R}$. This is a special case of an exponential trichotomy with the dimension of the center space equal to zero and $\alpha < 0 < \beta$. For a proof, see [3, p. 181].

If (1.1) has a hyperbolic periodic orbit $\gamma = \bigcup_{t \in \mathbb{R}} \{p_t\}$, then the solution map $T(t, s) = D_\phi \hat{T}(t-s)\, p_s$ of (2.2) has an exponential trichotomy for all $t \geq s$ in $\mathbb{R}$. This is a consequence of the decomposition theory of linear periodic systems in [3, Chap. 8]. In terms of the notation in [3, p. 203], the decomposition according to the multipliers with moduli greater than one yields projections P_u and $P_s + P_c$. With $\varepsilon > 0$ sufficiently small, the decomposition according to the multipliers with moduli greater than $1-\varepsilon$ yields projections $P_u + P_c$ and P_s. The adjoint system of (2.2) is then used to obtain the projections P_u, P_c, P_s.

The proofs of Lemmas 3.1, 3.2, 3.3, below, will not be given here, since they are similar to the case of exponential dichotomies of flows generated by ordinary differential equations. See [2]. The technical treatment of the additional difficulty caused by the noninvertibility of $T(t, s)$ can be found in [7].

LEMMA 3.1. *Let $T(t, s)$, $t \geq s$ have exponential trichotomies in R^- and R^+, with projections $P_u^\pm(t)$, $P_s^\pm(t)$, $P_c^\pm(t)$, $t \in R^\pm$. Suppose that the exponents in R^- and R^+ are the same, and the unstable spaces in R^- and R^+ and the center spaces in R^-* and *R^+ have the same dimensions, $\mathscr{R}P_u^-(0) \cap \{\mathscr{R}P_c^+(0) \oplus \mathscr{R}P_s^+(0)\} = \varnothing$, and $\{\mathscr{R}P_u^-(0) \oplus \mathscr{R}P_c^-(0)\} \cap \mathscr{R}P_s^+(0) = \varnothing$. Then $T(t, s)$ has an exponential trichotomy in $R = R^- \cup R^+$.*

LEMMA 3.2. *Let $T(t, s)$ be defined in $(-\infty, t_0]$ and have an exponential trichotomy in $(-\infty, \tau]$, $t_0 > \tau$. Suppose that $T(t_0, \tau)(\phi_1 + \phi_2) \neq 0$ for $\phi_1 \in \mathscr{R}P_u(\tau)$, $\phi_2 \in \mathscr{R}P_c(\tau)$, and $\phi_1 + \phi_2 \neq 0$. Then $T(t, s)$ has an exponential trichotomy in $(-\infty, t_0]$ with the same exponents, and the projections $\tilde{P}_u(t)$, $\tilde{P}_s(t)$, and $\tilde{P}_c(t)$ approach $P_u(t)$, $P_s(t)$, and $P_c(t)$ exponentially as $t \to -\infty$.*

LEMMA 3.3. *Let $T(t, s)$ be defined in $[t_0, +\infty)$ and have an exponential trichotomy in $[\tau, +\infty)$, $\tau > t_0$. Suppose that $T^*(t_0, \tau)(\psi_1 + \psi_2) \neq 0$ for $\psi_1 \in \mathscr{R}P_u^*(\tau)$, $\psi_2 \in \mathscr{R}P_c^*(\tau)$, and $\psi_1 + \psi_2 \neq 0$. Then $T(t, s)$ has an exponential trichotomy in $[t_0, +\infty)$ with the same exponents, and the projections $\tilde{P}_u(t)$, $\tilde{P}_s(t)$, and $\tilde{P}_c(t)$ approach $P_u(t)$, $P_s(t)$, and $P_c(t)$ exponentially as $t \to +\infty$.*

4. THE LINEAR VARIATIONAL OPERATOR

In this section, we give more details about exponential trichotomies for the linear variational operator for a heteroclinic orbit Γ of (1.1).

Let $\mathscr{U}$ be the Banach space of all the linear continuous functions L: $C([-r, 0], R^n) \to R^n$ with the operator norm. Let $C^k(R, \mathscr{U})$ be the space of C^k maps from R to $\mathscr{U}$ with the C^k uniform topology. Let $T(t, s)$ be the solution operator for the linear functional differential equation

$$\dot{x}(t) = L(t) x_t \tag{4.1}$$

with $L(\cdot) \in C^k(R, \mathscr{U})$. Let $L(t)\phi = \int_{-r}^{0} d\eta(t, \theta)\, \phi(\theta)$ for $\phi \in C[-r, 0]$. For each t, $\eta(t, \theta)$ is an $n \times n$ matrix function normalized so that $\eta(t, \theta) = 0$ for $\theta \geqslant 0$, $\eta(t, \theta) = \eta(t, -r)$ for $\theta \leqslant -r$, continuous from the left in θ on $(-r, 0)$ for each t and has bounded variation on $\theta \in [-r, 0]$ for each t. Such matrices constitute a Banach space $\mathscr{B}_0$ with $\|\eta(t, \cdot)\| = \max_{1 \leqslant i \leqslant n} [\sum_{j=1}^{n} \operatorname{Var} \eta_{ij}(t, \cdot)]$. Each $L(\cdot) \in C^k(R, \mathscr{U})$ is associated with a unique $\eta(\cdot, \cdot) \in C^k(R, \mathscr{B}_0)$ and the relation is an isomorphism from $C^k(R, \mathscr{U})$ to $C^k(R, \mathscr{B}_0)$.

The formal adjoint equation for (4.1) is

$$y(s) + \int_s^t y(\alpha)\, \eta(\alpha, s - \alpha)\, d\alpha = \text{const}, \qquad s \leqslant t - r. \tag{4.2}$$

Let $B_0([-r, 0], R^{n^*})$ be the space of functions from $[-r, 0]$ to R^{n^*} which have bounded variation on $[-r, 0]$ and are continuous from the left with $|\psi| = \max \sum_{j=1}^{n} \operatorname{Var} \psi_j$. The solution operator of (4.2) defines a semigroup $\tilde{T}(s, t)$, $s \leqslant t$, mapping $y_t \in B_0([-r, 0), R^{n^*})$ to $y_s \in B_0([-r, 0], R^{n^*})$. See [3]. From (4.2), it is clear that $y(s)$ is absolutely continuous for $s \leqslant t - r$. If $\eta \in C^k(R, \mathscr{B}_0)$, $k \geqslant 1$, we have that

$$\dot{y}(s) + \int_s^{t-r} y(\alpha)\, \eta_t(\alpha, s - \alpha)\, d\alpha$$
$$+ \int_s^{t-r} \dot{y}(\alpha)\, \eta(\alpha, s - \alpha)\, d\alpha = 0, \qquad s \leqslant t - 2r.$$

So $\dot{y}(s)$ is absolutely continuous for $s \leqslant t - 2r$. Therefore $y(s) \in C^k$ for $s \leqslant t - (k+1)r$, by induction.

Consider Eq. (4.1) in some interval $J \subset R$. For any $L(\cdot) \in C(J, \mathscr{U})$, let $\tilde{A}(t, L) = \eta(t, -r^+) - \eta(t, -r)$ be the jump of $\eta(t, \tau)$ at $-r$. The function $L(\cdot)$ is said to satisfy the H–O property if for any compact set $K \subset J$, there exists an ε, $0 < \varepsilon < r$, such that

$$\int_{-r}^{-r+\varepsilon} d\eta(t, \theta)\, \phi(\theta) = \tilde{A}(t)\, \phi(-r), \qquad t \in K,$$

and the set $\{t \mid \det \tilde{A}(t) = 0,\ t \in J\}$ contains only isolated points.

Lemma 4.1 (Hale and Oliva [5]). *The solution operator of* (4.1) *is one-to-one if L satisfies the H–O property. Furthermore, the class of L satisfying the H–O property is dense in $C^k(J, \mathscr{U})$ if J is compact and $k \geqslant 1$.*

Lemma 4.2. *Suppose that L satisfies the H–O property in J. Then the solution operator $\tilde{T}(s, t)$, $s \leqslant t$, for the formal adjoint equation* (4.2) *is one-to-one for all $s < t$ for which $[s - r, t] \subset J$.*

Proof. Suppose that $y(\alpha)$ is a solution of (4.2) and there exists a constant t, $[t-r, t] \subset J$, such that $y(\alpha)=0$ for $\alpha \leqslant t$. We want to show that there exists $\rho>0$ such that $y(\alpha)=0$ for $\alpha \leqslant t+\rho$.

Let $\varepsilon>0$ be the constant in defining the H–O property. For $s \leqslant t+\varepsilon-r$, $y(s)$ satisfies the equation

$$y(s)+\int_{s}^{t+\varepsilon} y(\alpha)\, \eta(s, s-\alpha)\, d\alpha=\text{const.}$$

Since $y(s)=0$ for $s \leqslant t$

$$\int_{t}^{t+\varepsilon} y(\alpha)\, \eta(\alpha, s-\alpha)\, d\alpha=\text{const}, \qquad s \leqslant t+\varepsilon-r. \tag{4.3}$$

Let $s=t+\varepsilon-r$ in (4.3). Since $t \leqslant \alpha \leqslant t+\varepsilon$, $-r \leqslant s-\alpha \leqslant \varepsilon-r$, we know, by H–O property, $\eta(\alpha, s-\alpha)=\eta(\alpha, \varepsilon-r)$ for $t \leqslant \alpha \leqslant t+\varepsilon$; therefore

$$\text{constant}=\int_{t}^{t+\varepsilon} y(\alpha)\, \eta(\alpha, \varepsilon-r)\, d\alpha.$$

If $t-r<s<t+\varepsilon-r$ in (4.3), we have

$$\int_{t}^{s+r} y(\alpha)[\eta(\alpha, s-\alpha)-\eta(\alpha, \varepsilon-r)]\, d\alpha + \int_{s+r}^{t+\varepsilon} y(\alpha)[\eta(\alpha, s-\alpha)-\eta(\alpha, \varepsilon-r)]\, d\alpha=0.$$

But, for $t<\alpha<s+r$, $\eta(\alpha, s-\alpha)=\eta(\alpha, \varepsilon-r)$, and, for $s+r<\alpha$, $\eta(\alpha, s-\alpha)=\eta(\alpha, -r)$ and so

$$\int_{s+r}^{t+\varepsilon} y(\alpha)[\eta(\alpha, -r)-\eta(\alpha, \varepsilon-r)]\, d\alpha=0, \qquad t-r<s<t+\varepsilon-r.$$

Differentiating with respect to s, we have $y(s+r) \cdot \tilde{A}(s+r)=0$ for $t<s+r<t+\varepsilon$. There exists $0<\rho<\varepsilon$ such that $\tilde{A}(s+r)$ is nonsingular for $t<s+r<t+\rho$. Thus $y(s+r)=0$ for $t<s+r \leqslant t+\rho$. This proves the lemma.

If we suppose that $T(t, s)$ has an exponential trichotomy in J, then so does $T^{*}(s, t)$, $s \leqslant t$. If $J=(-\infty, +\infty)$ or $[0, +\infty)$, the relation between the true adjoint operator and the formal adjoint operator (see [3, pp. 152 ff.]) implies that $\tilde{T}(s, t)$ also has an exponential trichotomy in J, with the same exponents $\alpha<v-\varepsilon<v+\varepsilon<\beta$.

LEMMA 4.3. *Suppose that* (4.1) *has an exponential trichotomy in* $J=(-\infty, 0]$ *or* $[0, +\infty)$ *or* $(-\infty, +\infty)$ *with projections* $P_{\mathrm{u}}(t)$, $P_{\mathrm{s}}(t)$, *and*

$P_c(t)$, and exponents $\alpha < \nu - \varepsilon < \nu + \varepsilon < \beta$. Assume that $\delta = \sup_{t \in J} \| B(t) \|$, where $B(\cdot) \in C^k(J, \mathscr{U})$. Then the functional differential equation

$$\dot{x}(t) = L(t)x_t + B(t)x_t \tag{4.4}$$

has an exponential trichotomy in J, with projections $\tilde{P}_u(t)$, $\tilde{P}_s(t)$, and $\tilde{P}_c(t)$, and exponents $\tilde{\alpha} < \tilde{\nu} - \tilde{\varepsilon} < \tilde{\nu} + \tilde{\varepsilon} < \tilde{\beta}$, provided that $|\delta| < \delta_0$ for some constant $\delta_0 > 0$. Furthermore, $\tilde{P}_u(t) \to P_u(t)$, $\tilde{P}_s(t) \to P_s(t)$, and $\tilde{P}_c(t) \to P_c(t)$ uniformly in t and $\tilde{\alpha}$, $\tilde{\nu}$, $\tilde{\beta}$, $\tilde{\varepsilon} \to \alpha$, ν, β, ε as $\delta \to 0$.

Under the same hypotheses on (4.1) and $J = (-\infty, 0]$ (or $[0, \infty)$) and $\| B(t) \| \to 0$ as $t \to -\infty$ (or $t \to \infty$), there is a $\tau > 0$ such that (4.4) has an exponential trichotomy on $(-\infty, -\tau]$ (or $[\tau, \infty)$) and $\tilde{P}_u(t) - P_u(t) \to 0$, $\tilde{P}_s(t) - P_s(t) \to 0$, $\tilde{P}_c(t) - P_c(t) \to 0$ as $t \to -\infty$ (or $t \to \infty$).

Proof. We observe that, if (4.1) has an exponential dichotomy in $J = (-\infty, 0]$ or $[0, +\infty)$ with projections $P_u(t)$ and $P_s(t)$, exponents $\alpha < \beta$, and if δ is small, then (4.4) has an exponential dichotomy in J with projections $\tilde{P}_u(t)$ and $\tilde{P}_s(t)$ and exponents $\tilde{\alpha} < \tilde{\beta}$. Furthermore $\tilde{P}_u(t) \to P_u(t)$, $\tilde{P}_s(t) \to P_s(t)$ uniformly in $t \in J$ and $\tilde{\alpha}$, $\tilde{\beta} \to \alpha$, β as $\delta \to 0$. The proof of these facts is similar to the roughness of exponential dichotomies in the ordinary differential equation case, and can be found in [2], although necessary changes have to be made to avoid using the inverse of the solution map too arbitrarily—it is only defined on the unstable spaces and center spaces.

Now from the exponential trichotomy of (4.1), two exponential dichotomies can be defined. One is defined by $P_u^1 = P_u + P_c$, $P_s^1 = P_s$, and with the exponents $\alpha < \nu - \varepsilon$. Another is defined by $P_u^2 = P_u$, $P_s^2 = P_c + P_s$ with exponents $\nu + \varepsilon < \beta$. From our previous observation, for small δ, (4.4) has two exponential dichotomies. One is defined by $\tilde{P}_u^1$, $\tilde{P}_s^1$ with exponents $\tilde{\alpha} < \tilde{\nu} - \tilde{\varepsilon}$. Another is defined by $\tilde{P}_u^2$, $\tilde{P}_s^2$ with exponents $\tilde{\nu} + \tilde{\varepsilon} < \tilde{\beta}$. Also, $\tilde{P}_u^i$, $\tilde{P}_s^i$ are close to P_u^i, P_s^i and $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\nu}$, $\tilde{\varepsilon}$ are close to α, β, ν, ε if δ is small. There are three cases to be considered.

(1) $J = [0, +\infty)$. In this case, $\mathscr{R}\tilde{P}_s^1$ and $\mathscr{R}\tilde{P}_s^2$ are uniquely determined and $\mathscr{R}\tilde{P}_s^1 \subset \mathscr{R}\tilde{P}_s^2$. The difference of their codimensions is equal to $\dim \mathscr{R}P_c$. We see that $\tilde{P}_s(t) = \tilde{P}_s^1(t)$, $\tilde{P}_u(t) = \tilde{P}_u^2(t)$, and $\tilde{P}_c(t)$ equals the operation of $\tilde{P}_s^2(t)$ followed by a projection from $\mathscr{R}\tilde{P}_s^2(t)$ onto the invariant subspaces complementary to $\mathscr{R}\tilde{P}_s^1(t)$ in $\mathscr{R}\tilde{P}_s^2(t)$.

(2) $J = (-\infty, 0]$. In this case, $\tilde{P}_u^1$ and $\tilde{P}_u^2$ are uniquely determined and $\tilde{P}_u^2 \subset \tilde{P}_u^1$. The difference of their dimensions is equal to $\dim \mathscr{R}P_c$. We see that $\tilde{P}_u(t) = \tilde{P}_u^2(t)$, $\tilde{P}_s(t) = \tilde{P}_s^1(t)$, and $\tilde{P}_c(t)$ equals the operation of $\tilde{P}_u^1$ followed by a projection from $\tilde{P}_u^1(t)$ onto the invariant subspace complementary to $\mathscr{R}\tilde{P}_u^2(t)$ in $\mathscr{R}\tilde{P}_u^1(t)$.

(3) $J=(-\infty, \infty)$. We use (1) and (2) and Lemma 3.1. Note that

$$\mathscr{R}\tilde{P}_u^-(0) \cap \{\mathscr{R}\tilde{P}_c^+(0) \oplus \mathscr{R}\tilde{P}_s^+(0)\} = \varnothing$$

and

$$\{\mathscr{R}\tilde{P}_u^-(0) \oplus \mathscr{R}\tilde{P}_c^-(0)\} \cap \mathscr{R}\tilde{P}_s^+(0) = \varnothing$$

for small δ, since these two equalities are rough under small perturbations. To show this, one needs that $\dim \mathscr{R}P_u$ and $\dim \mathscr{R}P_c$ are finite.

The proof of the last part of the Lemma follows as in the ordinary differential equations case (see, for example, Palmer [11]). An immediate consequence of Lemma 4.3 is the following.

THEOREM 4.4. *Let $\Gamma = \bigcup_{t \in \mathbb{R}} \{q_t\}$ be a heteroclinic orbit with $\alpha(\Gamma)$ and $\omega(\Gamma)$ hyperbolic equilibria or periodic orbits. If $T(t, s)$, $t \geqslant s$, is the solution map for $\dot{x}(t) = f'(q_t)x_t$, then $T(t, s)$ has exponential trichotomies in $(-\infty, -\tau]$ and $[\tau, +\infty)$, $\tau > 0$. The orbit Γ is transverse if and only if*

$$T(\tau, -\tau)\, \mathscr{R}P_u^-(-\tau) + (\mathscr{R}P_c^+(\tau) \oplus \mathscr{R}P_s^+(\tau)) = X$$

or, equivalently,

$$T(\tau, -\tau)(\mathscr{R}P_u^-(-\tau) \oplus \mathscr{R}P_c^-(-\tau)) + \mathscr{R}P_s^+(\tau) = X.$$

Γ is in general position if and only if Γ is transverse or

$$T(\tau, -\tau)\{\mathscr{R}P_u(-\tau) \oplus \mathscr{R}P_c(-\tau)\} \cap \mathscr{R}P_s(\tau) = \{0\}.$$

When applying Theorem 4.4 to the special case that Γ is a homoclinic orbit and $\alpha(\Gamma) = \omega(\Gamma)$ is a hyperbolic periodic orbit, we have that Γ is transverse if and only if $T(t, s)$ has an exponential trichotomy in R. This can be seen from Lemmas 3.1, 3.2, and 3.3.

Let γ_1 and γ_2 be two real constants. Let $C^0(\gamma_1, \gamma_2)$ be the Banach space of all the continuous functions $x(t)$ defined from R into R^n such that $|x(t)| \leqslant Ke^{\gamma_1 t}$, $t \leqslant 0$, and $|x(t)| \leqslant Ke^{\gamma_2 t}$, $t \geqslant 0$, for some constant $K > 0$. The norm in $C^0(\gamma_1, \gamma_2)$ is defined as

$$\|x\|_{C^0(\gamma_1, \gamma_2)} = \sup_{t \geqslant 0} \{|x(t)|\, e^{-\gamma_2 t}, |x(-t)|\, e^{\gamma_1 t}\}.$$

Let $C^k(\gamma_1, \gamma_2)$ be the Banach space of all the C^k functions $x(t)$ such that $x^{(i)}(t) \in C^0(\gamma_1, \gamma_2)$, $i = 0, 1, \ldots, k$, with the norm

$$\|x\|_{C^k(\gamma_1, \gamma_2)} = \sum_{i=0}^{k} \|x^{(i)}\|_{C^0(\gamma_1, \gamma_2)}.$$

For $L(\cdot) \in C^k(R, \mathscr{U})$, $k \geqslant 0$, the linear operator F_L: $C^{k+1}(\gamma_1, \gamma_2) \to C^k(\gamma_1, \gamma_2)$ is defined as F_L: $x \mapsto h$, $h(t) = dx(t)/dt - L(t)x_t$. We write $F_L(\gamma_1, \gamma_2)$ to indicate the space $C^k(\gamma_1, \gamma_2)$ under consideration.

LEMMA 4.5. *Suppose that $L(\cdot) \in C^k(R, \mathscr{U})$ and that (4.1) has shifted dichotomies in R^- and R^+ with exponents $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, respectively. Let $\alpha_1 < \gamma_1 < \beta_1$ and $\alpha_2 < \gamma_2 < \beta_2$. Then F_L: $C^{k+1}(\gamma_1, \gamma_2) \to C^k(\gamma_1, \gamma_2)$ is Fredholm of index $I(F_L)$ with*

$$I(F_L) = \dim \mathscr{R}P_{\mathrm{u}}^-(0) - \dim \mathscr{R}P_{\mathrm{u}}^+(0)$$

$$\mathscr{N}(F_L) = \{(T(t, 0)\phi)(0) \colon \phi \in \mathscr{R}P_{\mathrm{u}}^-(0) \cap \mathscr{R}P_{\mathrm{s}}^+(0),\ t \in R\}$$

$\mathscr{R}(F_L) = \{h$: $h \in C^k(\gamma_1, \gamma_2)$, $\int_{-\infty}^{\infty} y(t)\, h(t) = 0$ *for all the solutions of the formal adjoint equation $y(t)$ such that* $|y(t)| \leqslant Ke^{-\beta_2 t}$, $t \geqslant 0$; $|y(t)| \leqslant Ke^{-\alpha_1 t}$, $t \leqslant 0\}$.

Proof. To discuss the solutions of $F_L x = h$, let $u(t) = e^{-\gamma t}x_t$ and $g(t) = e^{-\gamma t}h(t)$, where $\gamma = \gamma_1$ for $t \leqslant 0$ and $\gamma = \gamma_2$ for $t \geqslant 0$, respectively. The function u: $R \to C[-r, 0]$ does not satisfy any delay equation, but, from the variation of constraints formula (see [3]),

$$x_t = T(t, s)x_s + \int_s^t T(t, v)\, X_0 h(v)\, dv, \qquad s \leqslant t, \tag{4.5}$$

we have

$$u(t) = T_\gamma(t, s)\, u(s) + \int_s^t T_\gamma(t, v)\, X_0\, g(v)\, dv, \qquad s \leqslant t, \tag{4.6}$$

where $T_\gamma(t, s) = T(t, s)e^{-\gamma(t-s)}$ and $\gamma = \gamma_1$ or γ_2 depending on $s \leqslant t \leqslant 0$ or $0 \leqslant s \leqslant t$, and $X_0(\theta) = 0$ for $\theta < 0$, $X_0(0) = I$, the identity. The operator $T_\gamma(t, s)$ has the usual exponential dichotomies on R^- and R^+ with projections $P_{\mathrm{u}\gamma}^{\pm}(t) = P_{\mathrm{u}}^{\pm}(t)$ and $P_{\mathrm{s}\gamma}^{\pm}(t) = P_{\mathrm{s}}^{\pm}(t)$. Discussion of the usual exponential dichotomy case can be found in [7], where we proved that the bounded solutions for (4.6), when $g \equiv 0$, are

$$\{u(t) = (T_\gamma(t, 0)\phi)(0) \mid \phi \in \mathscr{R}P_{\mathrm{u}\gamma}^-(0) \cap \mathscr{R}P_{\mathrm{s}\gamma}^+(0)\}.$$

Also, the set of the bounded functions $g(t)$ such that (4.6) admits a bounded solution $u(t)$ is (the symbol $\langle , \rangle$ is the dual pairing)

$$\{g(t) \text{ bounded} : \int_{-\infty}^{\infty} \xi(t)\, g(t) = 0,$$

$$\xi(t) = \langle T_\gamma^*(t, 0)\zeta, X_0\rangle, \zeta \in \mathscr{R}P_{\mathrm{u}\gamma}^{+\,*}(0) \cap \mathscr{R}P_{\mathrm{s}\gamma}^{-\,*}(0)\}.$$

Returning to (4.5) and observing that $T^*_\gamma(s,t)=T^*(s,t)\,e^{-\gamma(t-s)}$, one easily obtains the desired results in the lemma.

LEMMA 4.6. *Assume all of the hypotheses of Lemma 4.5 except that the shifted dichotomies are valid only in $(-\infty,-\tau]$ and $[\tau,+\infty)$, $\tau>0$. Then all the results in Lemma 4.5 are valid except that*

$$I(F_L)=\dim \mathscr{R}P_u^-(-\tau)-\dim \mathscr{R}P_u^+(\tau), \tag{4.7}$$

$$\mathscr{N}(F_L)=\{(T(t,0)\phi)(0)\colon t\in R,\ \phi\in\mathscr{R}P_u^-(-\tau)$$
$$\text{and } T(\tau,-\tau)\phi\in\mathscr{R}P_s^+(\tau)\}.$$

Proof. We first assume that there is a function $A\in C^k(R,\mathscr{U})$, with compact support in $(-\tau-1,\tau+1)$ and $\dot{x}(t)=(L(t)+A(t))x_t$ has shifted exponential dichotomies in R^- and R^+. The existence of such an A shall be discussed later. It is clear that $z(y)(t)=A(t)\,y_t$ is a compact operator as a map from $C^{k+1}(\gamma_1,\gamma_2)$ to $C^k(\gamma_1,\gamma_2)$. From Lemma 4.5, F_{L+A} is Fredholm. From the perturbation theorem of Fredholm operators, F_L is Fredholm and $\operatorname{ind} F_L=\operatorname{ind} F_{L+A}$. This proves (4.7).

The characterization of $\mathscr{N}(F_L)$ is obvious. Let $y(t)$ be a solution of the formal adjoint equation for (4.1), and $|y(t)|\leqslant Ke^{-\beta_2 t}$, $t\geqslant 0$, $|y(t)|\leqslant Ke^{-\alpha_1 t}$, $t\leqslant 0$. Such solutions $\{y(\cdot)\}$ form a finite-dimensional linear space Ψ. If $h\in\mathscr{R}F_L$, then $\int_{-\infty}^{\infty} y(t)\,h(t)=0$ for all $y\in\Psi$. Therefore, $\mathscr{R}F_L\subset\{h\colon \int_{-\infty}^{\infty} y(t)\,h(t)\,dt=0,$ for all $y\in\Psi\}$. One can show that $\dim\mathscr{N}F_L-\dim\Psi=\dim\mathscr{R}P_u^-(-\tau)-\dim\mathscr{R}P_u^+(\tau)$. The proof is omitted since it is similar to standard arguments relating an operator to its adjoint (see [7]). Now, from the definition $\operatorname{ind} F_L=\dim\mathscr{N}F_L-\operatorname{codim}\mathscr{R}F_L$, we have $\dim\Psi=\operatorname{codim}\mathscr{R}F_L$, proving the characterization for $\mathscr{R}F_L$.

It remains to show the existence of $A\colon R\to\mathscr{U}$. First, we assume that $k\geqslant 1$. By Lemmas 4.1 and 4.2, we can find $B_2\in C^k(R,\mathscr{U})$, sufficiently small and with compact support in $(-\tau-\frac{1}{2},\tau+\frac{1}{2})$, such that $\dot{x}(t)=(L(t)+B_2(t))x_t$ is H–O in $[-\tau,\tau]$. Thus, $T(t,s)$ and $\tilde{T}(s,t)$ are one-to-one in $[-\tau,\tau]$. The perturbed system has exponential dichotomies in $(-\infty,-\tau]$ and $[\tau,+\infty)$ by Lemma 4.3, and in R^- and R^+ by Lemmas 3.2 and 3.3. If $k=0$, we can use mollifiers to find $B_1(t)\in C^0(R,\mathscr{U})$, with compact support in $(-\tau-1,\tau+1)$ so that $\tilde{L}(t)=L(t)+B_1(t)\in C^1([-\tau-\frac{1}{2},\ \tau+\frac{1}{2}],\mathscr{U})$. Then $A(t)=B_1(t)+\tilde{B}_2(t)$ is the desired perturbation where $\tilde{B}_2(t)$ is constructed from $\tilde{L}(t)$ as above.

5. Bifurcation Functions

In this section, we obtain bifurcation functions whose zeros will be in one-to-one correspondence to heteroclinic orbits Γ^μ of (1.2). These functions will also be used to characterize the transversality or degree of nontransversality of Γ^μ.

The linear variational equation around $\Gamma = \bigcup_{t \in \mathbb{R}} \{q_t\}$ is

$$\dot{x}(t) = f'(q_t) x_t = L_q(t) x_t = \int_{-r}^{0} d\eta(t, \theta)\, x(t+\theta) \tag{5.1}$$

with the formal adjoint equation being

$$y(s) + \int_s^t y(\alpha)\, \eta(\alpha, s-\alpha)\, d\alpha = \text{constant}, \qquad s \leqslant t - r \tag{5.2}$$

$$y_t = \psi.$$

Since $q_t \to \alpha(\Gamma) = \gamma_1$ as $t \to -\infty$, $q_t \to \omega(\Gamma) = \gamma_2$ as $t \to +\infty$ with asymptotic phase, we may assume that $\gamma_1 = \bigcup_{t \in \mathbb{R}} p_{1,t}$, $\gamma_2 = \bigcup_{t \in \mathbb{R}} p_{2,t}$, where $p_1(t)$, $p_2(t)$ are periodic solutions of (1.1) and $q_t - p_{1,t} \to 0$ as $t \to -\infty$, $q_t - p_{2,t} \to 0$ as $t \to \infty$. Thus,

$$\begin{aligned} \| D_\phi f(q_t) - D_\phi f(p_{1,t}) \| &\to 0 \qquad \text{as} \quad t \to -\infty \\ \| D_\phi f(q_t) - D_\phi f(p_{2,t}) \| &\to 0 \qquad \text{as} \quad t \to \infty. \end{aligned} \tag{5.3}$$

We have already remarked in Section 2 that $\dot{x}(t) = L_{p_1}(t) x_t$ and $\dot{x}(t) = L_{p_2}(t) x_t$ have exponential trichotomies on $\mathbb{R}$. This fact, together with (5.3) and Lemma 4.3, implies that there is a $\tau > 0$ such that (5.1) has an exponential trichotomy on $(-\infty, -\tau]$ and $[\tau, \infty)$ with exponents $\alpha_1 < 0 < \beta_1$ and $\alpha_2 < 0 < \beta_2$, respectively. Let $\gamma > 0$ be a small constant such that $0 < \gamma < \min\{|\alpha_1|, |\alpha_2|, \beta_1, \beta_2\}$.

For μ small, let $\gamma_1^\mu = \bigcup_{t \in \mathbb{R}} \{p_{1,t}^\mu\}$, $\gamma_2^\mu = \bigcup_{t \in \mathbb{R}} \{p_{2,t}^\mu\}$ be the hyperbolic periodic orbits of (1.2) with $p_1^0 = p_1$, $p_2^0 = p_2$. As remarked earlier, we wish to determine those solutions $x^\mu(t)$ of (1.2) whose orbits Γ^μ are close to Γ and have $\alpha(\Gamma^\mu) = \gamma_1^\mu$, $\omega(\Gamma^\mu) = \gamma_2^\mu$. We also want to do this by considering x^μ as a small variation from the function q that describes Γ. To do this, extreme care must be exercised in order to have x^μ as a small perturbation of q uniformly in t, and approaches 0 as $t \to \pm\infty$. Several time scalings are involved and that is the reason for so much of the following cumbersome notation.

Let $\tilde{\beta}$: $\mathbb{R} \to \mathbb{R}^+$ be a C^∞-function with $\tilde{\beta}(t) = 0$ for $t \leqslant -1$, $\tilde{\beta}(t) = 1$ for $t \geqslant 1$. Let $\zeta_2(t)$ be a C^∞-function such that $\zeta_2(t) = 0$ for $t \leqslant r+1$, $\zeta_2(t) = 1$ for $t \geqslant r+2$ and let $\zeta_1(t) = \zeta_2(-t)$. Let $\omega_j(\mu)$ be the period of $p_{j,\mu}(t)$ and $\omega_j(\mu)/\omega_j(0) = 1 + \beta_j(\mu)$, $j = 1, 2$. Since the perturbed periodic solution $p_{j,\mu}(t)$ does not have the same period as $p_j(t)$, $j = 1, 2$, and $x^\mu(t) \to p_{j,\mu}(t)$ as $t \to \pm\infty$, while $q(t) \to p_j(t)$ as $t \to \pm\infty$, $j = 1, 2$, respectively, it is necessary to rescale time $t \to (1 + \beta(\mu)) t$ near $\pm\infty$ so that $p_{j,\mu}(t(1+\beta))$ has the same period as $p_j(t)$, $j = 1, 2$. The bridge function $\tilde{\beta}(t)$ is introduced to make the scaling a smooth function in $t \in R$. Moreover, one can choose a phase shift so that $x^\mu((1+\beta)t \to p_1^\mu((1+\beta)t)$ as $t \to -\infty$, but another parameter α has to be introduced such that $x^\mu((1+\beta)t) \to p_2((1+\beta)t+\alpha)$ as $t \to \infty$. With the help of the bridge functions $\xi_1(t)$ and $\xi_2(t)$, a further correction term $\omega(t)$ is to be subtracted from $x^\mu((1+\beta)t)$ to make it approach $q(t)$ as $t \to \pm\infty$. For $\alpha \in \mathbb{R}$ and μ small define

$$\begin{aligned} \beta &= \beta(t, \mu) = \tilde{\beta}(t)\, \beta_2(\mu) + (1 - \tilde{\beta}(t))\, \beta_1(\mu) \\ \omega(t) &= \omega(\alpha, \mu)(t) = \zeta_1(t)[p_1^\mu((1+\beta)t) - p_1(t)] \\ &\quad + \zeta_2(t)[p_2^\mu((1+\beta)t+\alpha) - p_2(t)]. \end{aligned} \tag{5.4}$$

Since

$$|\, p_i^\mu((1+\beta_i(\mu))t) - p_i(t)| = O(|\mu|) \qquad \text{as} \quad \mu \to 0 \tag{5.5}$$

it follows that

$$\omega(t) = \omega(\alpha, \mu)(t) = O(|\alpha| + |\mu|) \qquad \text{as} \quad (\alpha, \mu) \to (0, 0). \tag{5.6}$$

We need one other observation. For $-r \leqslant \theta \leqslant 0$, consider the equation for ζ,

$$(1 + \beta(t+\zeta, \mu))\zeta + t(\beta(t+\zeta, \mu) - \beta(t, \mu)) = \theta.$$

By the Implicit Function Theorem, there is a solution $\zeta = \zeta(\theta, t, \mu) = \theta + O(|\mu|)$ as $\mu \to 0$. In particular, $\zeta = \theta(1+\beta_1(\mu))^{-1}$ for $t \leqslant -1$, $\zeta = \theta(1+\beta_2(\mu))^{-1}$ for $t \geqslant 1$. For any function $x\colon \mathbb{R} \to \mathbb{R}^n$, we define $x_{t,\beta}$ from $\mathbb{R}$ to $C([-r, 0], \mathbb{R}^n)$ by the relation $x_{t,\beta}(\theta) = x(t + \zeta(\theta, t, \mu))$, $-r \leqslant \theta \leqslant 0$.

With the above notation, let us make the transformation $x((1+\beta)t) = q(t) + \omega(\alpha, \mu)(t) + z(t)$. The equation for z is

$$F(z)(t) = N(z, \mu, \alpha, t), \tag{5.7}$$

where

$$\begin{aligned} F(z)(t) &= \dot{z}(t) - L_q(t) z_t \\ N(z, \mu, \alpha, t) &= M(z, \mu, \alpha, t) - F(\omega) \\ M(z, \mu, \alpha, t) &= \left(1 + \beta + t\frac{\partial \beta}{\partial t}\right) [f(q_{t,\beta} + \omega_{t,\beta} + z_{t,\beta}) \\ &\quad + g(q_{t,\beta} + \omega_{t,\beta} + z_{t,\beta}, \mu)] - f(q_t) - L_q(t)\omega_t - L_q(t) z_t. \end{aligned} \tag{5.8}$$

Any solution $x^\mu(t)$ of (1.2) with α-limit set γ_1^μ and ω-limit set γ_2^μ must satisfy (5.7), (5.8). If $0 < \gamma < \min\{|\alpha_1|, |\alpha_2|, \beta_1, \beta_2\}$, where $\alpha_1 < 0 < \beta_1$,

$\alpha_2 < 0 < \beta_2$ are respectively the exponents for the trichotomy of (5.1) on $(-\infty, -\tau]$, $[\tau, \infty)$, then it follows that, probably after a time shift in $x^\mu(t)$, $z^\mu(t) = x^\mu((1+\beta)t) - q(t) - \omega(\alpha, \mu)(t)$ must approach zero as $t \to -\infty$ like $e^{\gamma t}$ and must approach zero as $t \to \infty$ like $e^{-\gamma t}$. Therefore, it is sufficient to consider only the solutions of (5.7), (5.8) in $C^1(\gamma, -\gamma)$.

The map F: $C^1(\gamma, -\gamma) \to C^0(\gamma, -\gamma)$ is Fredholm by Lemma 4.6. To estimate $N(z, \mu, \alpha, \cdot)$ as a map from $C^1(\gamma, -\gamma)$ to $C^0(\gamma, -\gamma)$, we need the following observation.

For $|t|$ sufficiently large, one can use the definition (5.4) and show that

$$F(\omega)(t) = (1+\beta)[f(p_{t,\beta} + \omega_{t,\beta}) + g(p_{t,\beta} + \omega_{t,\beta}, \mu) \\ - f(p_t) - L_q(t)\omega_t], \tag{5.9}$$

where $\beta = \beta_2$, $p = p_2$ if t is large and positive and $\beta = \beta_1$, $p = p_1$ if t is large and negative.

Using (5.8), (5.9), (5.6), (5.5) and the fact that $z_{t,\beta} - z_t = O(|\mu|)$, one can show that

$$|N(z, \mu, \alpha, \cdot)|_{C^0(\gamma, -\gamma)} = O(|\mu| + |\alpha| + |z|_{C^1(\gamma, -\gamma)})$$

as $(\mu, \alpha, z) \to (0, 0, 0)$.

Let E_1 be a projection from $C^1(\gamma, -\gamma)$ onto $\mathcal{N}(F)$ and E_2 a projection from $C^0(\gamma, -\gamma)$ onto $\mathcal{R}(F)$. Then (5.7) is equivalent to

$$F(z) = E_2 N(z, \mu, \alpha, \cdot) \tag{5.10}$$

$$0 = (I - E_2)\, N(z, \mu, \alpha, \cdot). \tag{5.11}$$

If $\mathcal{K}$: $\mathcal{R}(E_2) \to \mathcal{R}(I - E_1)$ is a right inverse of F, then $\mathcal{K}$ is bounded since F is Fredholm. If $\{(y^i, i = 1, 2, ..., d\}$ is a basis for $\mathcal{N}(F)$ and $z = z^* + \sum_{i=1}^d k_i y^i$, $k = (k_1, ..., k_d) \in \mathbb{R}^d$, $z^* \in \mathcal{R}(I - E_1)$, then (5.10) is equivalent to

$$z^* = \mathcal{K} E_2 N\left(z^* + \sum_{i=1}^d k_i y^i, \mu, \alpha, \cdot\right). \tag{5.12}$$

Using the contraction mapping principle, one can show there are constants $\bar{\alpha} > 0$, $\bar{\mu} > 0$, $\bar{k} > 0$ such that (5.12) has a unique solution $z^* = z^*(\alpha, k, \mu) \in C^1(\gamma, -\gamma)$ for $|\alpha| < \bar{\alpha}$, $|k| < \bar{k}$, $|\mu| < \bar{\mu}$, $z^*(0, 0, 0) = 0$. By induction, one can actually show that $z^*(\alpha, k, \mu) \in C^k(\gamma, -\gamma)$. If we consider z^* as a map from $\mathbb{R} \times \mathbb{R}^d \times X$ into $C^0(\gamma, -\gamma)$ and use an argument similar to the one for the proof of Lemma 2.2, Chapter 10 of [3], then one can show that z^*: $\mathbb{R} \times \mathbb{R}^d \times X \to C^0(\gamma, -\gamma)$ is C^k.

Let $\Psi=\{\psi^1,..., \psi^{d^*}\}$ be a basis for the bounded solutions of the formal adjoint equation (5.2). By Lemma 4.6, Eqs. (5.10), (5.11) are equivalent to

$$G^j(\alpha, k, \mu) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi^j(t)\, N(z^*(\alpha, k, \mu)(t) + \sum_{i=1}^{d} k_i y^i, \mu, \alpha, t)\, dt = 0, \qquad j = 1, 2,..., d^*. \tag{5.13}$$

The functions G^j are called the bifurcation functions and the perturbed equation has a heteroclinic solution in a neighborhood of $\Gamma \cup \alpha(\Gamma) \cup \omega(\Gamma)$ if and only if $G^j(\alpha, k, \mu)=0$, $j=1,..., d^*$, for some $|\alpha| < \bar{\alpha}$, $|k| < \bar{k}$ and $|\mu| < \bar{\mu}$. The heteroclinic solution is, up to a phase shift,

$$x^\mu((1+\beta)t) = q(t) + \omega(\alpha, \mu)(t) + z^*(\alpha, k, \mu)(t) + \sum_{i=1}^{d} k_i y^i(t). \tag{5.14}$$

Further discussion of the bifurcation function needs the following lemma in which $\alpha(\Gamma)$ and $\omega(\Gamma)$ are hyperbolic periodic orbits or equilibria.

LEMMA 5.1. *The formal adjoint equation* (5.2) *has a bounded solution* $\psi \notin C^1(\gamma, -\gamma) \cup C^1(-\gamma, -\gamma) \cup C^1(\gamma, \gamma)$ *if and only if*

(H) *both* $\alpha(\Gamma)$ *and* $\omega(\Gamma)$ *are hyperbolic periodic orbits and* $\dot{q}$ *is the only bounded solution of* (5.1) *not in* $C^1(\gamma, -\gamma)$.

In all the other cases, bounded solutions of the formal adjoint equation (5.2) *are in* $C^1(\gamma, -\gamma)$.

Proof. It is obvious that all the bounded solutions ψ of (5.2) are in $C^1(\gamma, -\gamma)$ if $\alpha(\Gamma)$ and $\omega(\Gamma)$ are equilibria.

Suppose that $\alpha(\Gamma)$ is an equilibrium and $\omega(\Gamma)$ is a periodic orbit. If $F(\alpha, \beta) = F$ restricted to $C^1(\alpha, \beta)$, then $\operatorname{ind} F(-\gamma, \gamma) = \operatorname{ind} F(\gamma, -\gamma) + 1$, and $\dim \mathcal{N}F(-\gamma, \gamma) = \dim \mathcal{N}F(\gamma, -\gamma) + 1$. Therefore, $\operatorname{codim} \mathcal{R}F(-\gamma, \gamma) = \operatorname{codim} \mathcal{R}F(\gamma, -\gamma)$. This shows that all the bounded solutions of (5.2) are in $C^1(\gamma, -\gamma)$. Similarly, we can prove that all the bounded solutions of (5.3) are in $C^1(\gamma, -\gamma)$ if $\alpha(\Gamma)$ is a periodic orbit and $\omega(\Gamma)$ an equilibrium.

There are two cases when $\alpha(\Gamma)$ and $\omega(\Gamma)$ are both periodic orbits.

Case I. There are two linearly independent bounded solutions of (5.1); one is $\dot{q}(t)$, another one approaches zero as $t \to -\infty$, and approaches $\dot{q}(t)$ as $t \to +\infty$, exponentially. In this case, $\operatorname{ind} F(-\gamma, \gamma) = \operatorname{ind} F(\gamma, -\gamma) + 2$, and $\dim \mathcal{N}F(-\gamma, \gamma) = \dim \mathcal{N}F(\gamma, -\gamma) + 2$. Thus, all the bounded solutions of (5.2) are in $C^1(\gamma, -\gamma)$.

Case II. Suppose (H) is satisfied; that is, there is only one bounded solution of (5.1), $\dot{q}(t)$, up to the linear combination of solutions in $C^1(\gamma, -\gamma)$. In this case, $\operatorname{ind} F(-\gamma, \gamma) = \operatorname{ind} F(\gamma, -\gamma) + 2$, and $\dim \mathcal{N}F(-\gamma, \gamma) = \dim \mathcal{N}F(\gamma, -\gamma) + 1$. Thus, $\operatorname{codim} \mathcal{R}F(-\gamma, \gamma) = \operatorname{codim} \mathcal{R}F(\gamma, -\gamma) - 1$. This shows that there is a bounded solution ψ^1 of (5.2), $\psi^1 \notin C^1(\gamma, -\gamma)$. By comparing $F(\gamma, -\gamma)$ with $F(\gamma, \gamma)$ and also $F(\gamma, -\gamma)$ with $F(-\gamma, -\gamma)$, one shows that $\psi \notin C^1(-\gamma, -\gamma)$ and $\psi \notin C^1(\gamma, \gamma)$. This completes the proof of the lemma.

To study the bifurcation functions $G^j(\alpha, k, \mu)$ in (5.13) in more detail, we need the bilinear form associated with (5.1), (5.2). For any ψ satisfying (5.2), $\phi \in C([-r, 0], \mathbb{R}^n)$, let

$$(\psi^t, \phi)_t = \psi(t)\,\phi(0) + \int_{-r}^{0} d_\theta \left[\int_t^{t+r} \psi(\alpha)\,\eta(\alpha, \theta + t - \alpha)\, d\alpha \right] \phi(\theta),$$

where $\psi^t(s) = \psi(t+s)$, $0 \leqslant s \leqslant r$.

If then ψ^j in (5.13) belong to $C^1(\gamma, -\gamma)$ and ω is defined in (5.4), then

$$\int_{-\infty}^{\infty} \psi^j(t)\, F(\omega)(t)\, dt = (\psi^{j,t}, \omega_t)_t \Big|_{-\infty}^{\infty} = 0.$$

Therefore, with the exception of Case (H) of Lemma 5.1, we may replace N in (5.13) by M defined in (5.8); that is, drop the term $F(\omega)$ in N.

LEMMA 5.2. *Assume* (H) *of Lemma* 5.1 *and let* $\psi^1(t)$ *be the bounded solution of* (5.2) *not in* $C^1(\gamma, -\gamma) \cup C^1(-\gamma, -\gamma) \cup C^1(\gamma, \gamma)$. *Then* $\partial G^1(0, 0, 0)/\partial\alpha \neq 0$.

Remark 5.3. In case (H) of Lemma 5.1, Lemma 5.2 says that we can determine the variation of the transition time from a cross section of $\alpha(\Gamma)$ to another one of $\omega(\Gamma)$. In the case of ordinary differential equations, it is not hard to construct an example with $\dim W^u_{\mathrm{loc}}(\alpha(\Gamma)) = 2$ and $\dim W^u_{\mathrm{loc}}(\omega(\Gamma)) = 1$, and there is a continuum of heteroclinic orbits from $W^u_{\mathrm{loc}}(\alpha(\Gamma))$ hitting a cross section of $\omega(\Gamma)$ at a continuum of transition times. Thus, the variation of transition time α cannot always be determined as a function of $k \in \mathbb{R}^d$ if (H) is not valid.

Proof of Lemma 5.2. Since $z^* = 0$, $\omega = 0$ for $\alpha = 0$, $\mu = 0$, $k = 0$, it follows from (5.8) that $\partial M(z, \mu, \alpha, \cdot)/\partial z = 0$ for $\mu = 0$, $\alpha = 0$, $k = 0$. Therefore, (5.13) implies that $\partial G^1(0, 0, 0)/\partial\alpha = -\int_{-\infty}^{\infty} \psi^1(t)\, F(\partial\omega(0, 0)/\partial\alpha)(t)\, dt = -(\psi^{1,t}, (\partial\omega(0, 0)/\partial\alpha)_t)_t \,|_{-\infty}^{+\infty}$. It is easy to see that $\partial\omega(0, 0)/\partial\alpha = \zeta_2(t)\,\dot{p}_2(t)$, and so $(\psi^{1,t}, (\partial\omega(0, 0)/\partial\alpha)_t)_t \to 0$ as $t \to -\infty$. For solutions $\psi(t)$ of (5.2) and $\phi \in C[-r, 0]$, the bilinear form $(\psi^{1,t}, \phi)_t$ defines an element

$\psi^*(t) \in C^*[-r, 0]$, $(\psi^{1,t}, \phi)_t = \langle \psi^*(t), \phi \rangle$. $\psi^*(t)$ is a trajectory of $T^*(s, t)$ which has exponential trichotomies in $(-\infty, -\tau]$ and $[\tau, +\infty)$. The hypothesis on $\psi^1(t)$ implies that $\psi^*(\tau) \in \mathscr{R}P_u^*(\tau) \oplus \mathscr{R}P_c^*(\tau)$, with $P_c^*(\tau)\psi^*(\tau) \neq 0$. We also know that $\dot{x}(t) - f'(p_{2,t})x_t = 0$ has exponential trichotomy with projections $\tilde{P}_u$, $\tilde{P}_s$, and $\tilde{P}_c$. Lemma 4.3 implies that $P_c(t) \to \tilde{P}_c(t)$ exponentially as $t \to +\infty$. Therefore, $P_c^*(t) \to \tilde{P}_c^*(t)$ exponentially as $t \to +\infty$. Thus $\psi^*(t) = T^*(t, \tau)\psi^*(\tau) = T^*(t, \tau) P_c^*(\tau)\psi^*(\tau) + T^*(t, \tau) P_u^*(\tau)\psi^*(\tau) \to T^*(t, \tau) P_c^*(\tau)\psi^*(\tau)$. Therefore, $\tilde{P}_c^*(t)\psi^*(t) \to T^*(t, \tau) P_c^*(\tau)\psi^*(\tau)$, as $t \to +\infty$. Now, clearly, $\liminf_{t \to \infty} |\tilde{P}_c^*(t)\psi^*(t)| > 0$. For large t, $\zeta_2(t) = 1$. Therefore, $(\psi^{1,t}, (\zeta_2 \dot{p}_2)_t)_t = \langle \psi^*(t), \dot{p}_{2,t} \rangle = \langle \tilde{P}_c^*(t)\psi^*(t), \dot{p}_{2,t} \rangle$. Since $\dot{p}_{2,t}$ spans the eigenspace for the simple multiplier one of the linear variational equations about p_2, the latter quantity is nonzero. This proves the lemma.

We now state the main result of this section:

THEOREM 5.4. *Let $\Gamma = \bigcup_{t \in \mathbb{R}} \{q_t\}$ be a heteroclinic orbit with $\alpha(\Gamma) = \bigcup_{t \in \mathbb{R}} \{p_{1,t}\}$ and $\omega(\Gamma) = \bigcup_{t \in \mathbb{R}} \{p_{2,t}\}$ hyperbolic periodic orbits. Then there is a heteroclinic orbit $\Gamma^\mu = \bigcup_{t \in \mathbb{R}} \{x_t^\mu\}$ in a neighborhood of $\Gamma \cup \alpha(\Gamma) \cup \omega(\Gamma)$, with x^μ as in* (5.14), *if and only if $G^j(\alpha, k, \mu) = 0$, $j = 1, ..., d^*$, $k \in R^d$, $|\alpha| < \bar{\alpha}$, $|k| < \bar{k}$, $|\mu| < \bar{\mu}$ and G^j is given in* (5.13). *If $\psi^j \in C^1(\gamma, -\gamma)$, then N in* (5.13) *can be replaced by M in* (5.8). *The only situation in which there is a $\psi^1 \notin C^1(\gamma, -\gamma)$ is when* (H) *of Lemma* 5.1 *is satisfied. In this case, $\partial G^1(0, 0, 0)/\partial\alpha \neq 0$. Moreover, $d - d^* = \operatorname{ind}\Gamma - 1$. If $G^j(\alpha^0, k^0, \mu^0) = 0$, $j = 1, ..., d^*$, then the heteroclinic orbit Γ^{μ_0} defined by α^0, k^0, μ^0 in* (5.14) *is transverse if and only if the rank of the following matrix is d^*.*

$$\left\{ \frac{\partial G^j(\alpha^0, k^0, \mu^0)}{\partial \alpha}, \frac{\partial G^j(\alpha^0, k^0, \mu^0)}{\partial k} \right\}, \qquad j = 1, 2, ..., d^*. \tag{5.15}$$

Proof. Only the transversality needs a proof. This will be postponed until the end of the next section since it involves special types of perturbations of the vector field.

We end this section with some formulas for the derivatives of G^j. It is easy to show $\partial G^j(0, 0, 0)/\partial k_i = 0$. Also $\partial G^j(0, 0, 0)/\partial\alpha = 0$ except when (H) of Lemma 5.1 is satisfied. It is not hard to show that

$$\partial^2 G^j(0, 0, 0)/\partial k_l \partial k_m = \int_{-\infty}^{\infty} \psi^j(t) f''(q_t)(y_t^l, y_t^m)(t)\, dt$$

$$\partial^2 G^j/\partial\alpha\partial k_i = \int_{-\infty}^{\infty} \psi^j(t) f''(q_t)((\zeta_2 \dot{p}_2)_t, y_t^i)(t)\, dt.$$

However, the formulas for $\partial G^j/\partial\mu$, $\partial^2 G^j/\partial\mu\partial k_i$, and $\partial^2 G^j/\partial\mu\partial\alpha$ are difficult to compute for general perturbations $g(\phi, \mu)$. We therefore consider only specific perturbations $g(\phi, \mu)$ such that $g(p_{i,t}, \mu) = 0$, $i = 1, 2$. We then have

$$\partial G^j(0, 0, 0)/\partial\mu = \int_{-\infty}^{\infty} \psi^j(t)[\partial g(q_t, 0)/\partial\mu](t)\, dt \tag{5.16}$$

$$\begin{aligned}\partial^2 G^j(0, 0, 0)/\partial\mu\partial k_i = \int_{-\infty}^{\infty} &\psi^j(t)[f''(q_t)((\partial z^*/\partial\mu)_t, y_t^i)\\ &+ (\partial^2 g(q_t, 0)/\partial\mu\partial\phi)(y_t^i)](t)\, dt\end{aligned} \tag{5.17}$$

$$\begin{aligned}\partial^2 G^j(0, 0, 0)/\partial\mu\partial\alpha = \int_{-\infty}^{\infty} &\psi^j(t)[f''(q_t)((\partial z^*/\partial\mu)_t, (\zeta_2 \dot{p}_2)_t)\\ &+ (\partial^2 g(q_t, 0)/\partial\mu\partial\phi)(\zeta_2 \dot{p}_2)_t](t)\, dt,\end{aligned} \tag{5.18}$$

where $\partial z^*/\partial\mu = \partial z^*(0, 0, 0)/\partial\mu$.

6. Perturbations to Heteroclinic Orbits

For $f \in X^{k+1}$, $k \geqslant 1$, in (1.1), with Γ as a heteroclinic orbit, we want to show first that there exists a $g \in X^k$, arbitrarily small such that (1.2) has Γ as a heterorclinic orbit in general position. Assume that (5.1) has exponential trichotomies in $(-\infty, -t_0]$ and $[t_0, +\infty)$. Without loss of generality, we assume that the orbit segment $\{x_t = q_t,\ t \in [-t_0 - \varepsilon,\ t_0 + \varepsilon)\}$ has no intersection with $\alpha(\Gamma)$ and $\omega(\Gamma)$, and $t_0 > (k+2)r/2$.

First, we need a lemma for the perturbation of linear equations. Suppose that the linear functional differential equation (4.1), $L(\cdot) \in C^k(R, \mathscr{U})$, $k \geqslant 0$, has shifted exponential dichotomies in $J_1 = (-\infty, t_0]$ and $J_2 = [-t_0, +\infty)$, where $t_0 > ((k+2)/2)r$ is a constant, with projection $P_{\mathrm{u}}^-(t)$, $P_{\mathrm{s}}^-(t)$ ($(P_{\mathrm{u}}^+(t)$, $P_{\mathrm{s}}^+(t))$ and exponents $\alpha_1 < \beta_1$ ($\alpha_2 < \beta_2$) for $t \in J_1$ ($t \in J_2$). Let γ_1 and γ_2 be two real constants, $\alpha_1 < \gamma_1 < \beta_1$ and $\alpha_2 < \gamma_2 < \beta_2$, $F = F_L$: $C^{k+1}(\gamma_1, \gamma_2) \to C^k(\gamma_1, \gamma_2)$ be defined as in Section 4, $F_L(h)(t) = dh(t)/dt - L(t)h_t$. Assume that $\dim\{\mathscr{R}P_{\mathrm{u}}^-(0) \cap \mathscr{R}P_{\mathrm{s}}^+(0)\} = b$, $\dim \mathscr{R}P_{\mathrm{u}}^-(0) = b + c$, and $\dim \mathscr{R}P_{\mathrm{u}}^+(0) = e + c$, where $b > 0$, $e > 0$, $c \geqslant 0$ are integers. If $T(t, s)$ is the solution operator of (4.1), then, for any $y_0 \in \mathscr{R}P_{\mathrm{u}}^-(0)$, $y_t = T(t, 0)\, y_0$ is defined for all $t \in R$. Also, it is clear that $y_t \in \mathscr{R}P_{\mathrm{u}}^-(t)$ for $t \in J_1$. We shall use $[\phi_1, ..., \phi_m]$ to denote the linear space spanned by $\phi_1, ..., \phi_m$.

Lemma 6.1. *Assume that all the above are satisfied. Let a be an integer, $0 < a \leqslant \min(b, e)$. Take any basis $\{y_0^1, ..., y_0^a, y_0^{a+1}, ..., y_0^b\}$ in $\mathscr{R}P_{\mathrm{u}}^-(0) \cap \mathscr{R}P_{\mathrm{s}}^+(0)$ and let y^i; $\mathbb{R} \to \mathbb{R}^n$ be the solution of* (4.1) *through y_0^i at zero, $y_t^i = T(t, 0)\, y_0^i$.*

Consider the perturbed equation

$$\dot{x}(t) = L(t)x_t + \varepsilon B(t)x_t, \tag{6.1}$$

and the operator $F_{L+\varepsilon B}$: $C^{k+1}(\gamma_1, \gamma_2) \to C^k(\gamma_1, \gamma_2)$, *where* ε *is a real parameter. Then thre exists an* $\varepsilon_0 > 0$ *and a* $B(\cdot) \in C^k(R, \mathscr{U})$ *with compact support in* $(-t_0, t_0)$ *and* $B(t)\phi = 0$ *if* $\phi \in [y_t^{a+1},..., y_t^b]$ *such that, for* $|\varepsilon| \leqslant \varepsilon_0$, $\mathscr{N}(F_{L+\varepsilon B}) = \{0\}$ *if* $a = b$, $\mathscr{N}(F_{L+\varepsilon B}) = [y^{a+1},..., y^b]$ *if* $a < b$.

Proof. By Lemma 4.5, F_L is Fredholm with Index $= (b+c) - (c+e) = b - e$. Since $\dim \mathscr{N}(F_L) = b$, we have codim $\mathscr{R}(F_L) = e$. Let Ψ be the set of functions from $\mathbb{R}$ to $\mathbb{R}^{n*}$ corresponding to the linear space of the solutions of the formal adjoint equation of (4.1) defined in R, $\psi \in \Psi$ if $|\psi(t)| \leqslant Ke^{-\beta_2 t}$, $t \geqslant 0$, $|\psi(t)| \leqslant Ke^{-\alpha_1 t}$, $t \leqslant 0$. Then Ψ is of dimension e and $\Psi = [\psi^1,..., \psi^e]$ where $\psi^1,..., \psi^e$ are linearly independent.

Choose $\{y_0^{b+1},..., y_0^{b+c}\} \subset \mathscr{R}P_u^-(0)$ such that $\{y_0^j\}$, $j = 1,.., b+c$, form a basis in $\mathscr{R}P_u^-(0)$. Define y^j as the solution of (4.1) through y_0^j, $y_t^j = T(t, 0)\, y_0^j$, $t \in R$, $j = 1,..., b+c$. Obviously, $\{y_t^j\}$, $j = 1,..., b+c$, is a basis in $\mathscr{R}P_u(t)$, $t \in J_1$. Let $\Phi(t) = [y_t^1,..., y_t^a]$ and $\Phi = \{z\colon \mathbb{R} \to C[-r, 0]\colon z_t = \sum_{j=1}^a b_j y_t^j,\ b_j \in \mathbb{R},\ j = 1,..., a\}$. We now define $\tilde{B}(t)\colon C[-r, 0] \to R^n$, $t \in (-t_0, t_0)$ as

(i) $\tilde{B}(t) z_t = 0$ if $z_t \in [y_t^{a+1},..., y_t^{b+c}] \oplus \mathscr{R}P_s^-(t)$,

(ii) $\tilde{B}(t)\, y_t^i = (\psi^i(t))^\tau$, $i = 1, 2,..., a$, where τ denotes the transpose, and extend it linearly to $\Phi(t)$.

It is not hard to show that $\tilde{B}(t)$ is C^k for $t \in (-t_0, t_0 - (k+2)r)$. We proceed as follows. Let $\{\bar{\Psi}_t^i\}$, $i = 1,..., b+c$, be an invariant basis for $P_u^{-*}(t)$, $t \in (-\infty, t_0]$, i.e., $T^*(s, t)\bar{\Psi}_t^i = \bar{\Psi}_s^i$, $-\infty < s \leqslant t \leqslant t_0$. Assume that $\langle \bar{\Psi}_t^i, y_t^j \rangle = \delta_{ij}$, $1 \leqslant i, j \leqslant b+c$. Then

$$\tilde{B}(t)\phi = \sum_{i=1}^a (\psi^i(t))^\tau \langle \bar{\Psi}_t^i, \phi \rangle. \tag{6.2}$$

From the relation of the true adjoint and formal adjoint operators, we know that there exist functions $\tilde{\psi}^i(t)$, $t \leqslant t_0 - r$, $i = 1,..., b+c$, such that $\langle \bar{\Psi}_t^i, \phi \rangle = (\tilde{\psi}^{i,t}, \phi)_t$ for $t \leqslant t_0 - r$, where $\tilde{\psi}^i(t)$, $i = 1,..., b+c$, are solutions of the formal adjoint of (4.1) and

$$(\tilde{\psi}^{i,t}, \phi)_t = \tilde{\psi}^i(t)\, \phi(0) + \int_{-r}^0 d_\theta \left[\int_{-r}^0 \tilde{\psi}^i(t-\xi)\, \eta(t-\xi, \xi+\theta)\, d\xi \right] \phi(\theta).$$

From the comment after (4.2), $\tilde{\psi}^i$ are C^k functions for $t \in (-\infty,$

$t_0-(1+k)r$). Identifying $\tilde{\psi}_t^i$ with a function of bounded variation $\tilde{\psi}^i(t,\cdot)\in B_0=\mathscr{B}_0([-r,0],R^{n*})$ we have

$$\tilde{\psi}^i(t,0)=0,$$

$$\tilde{\psi}^i(t,\theta)=\int_{-r}^{0}\tilde{\psi}^i(t-\xi)\,\eta(t-\xi,\xi+\theta)\,d\xi$$

$$-\int_{-r}^{0}\tilde{\psi}^i(t-\xi)\,\eta(t-\xi,\xi)\,d\xi-\tilde{\psi}^i(t).$$

After a few computations and exploiting the fact that $\eta(\cdot,\cdot)\in C^k(R,\mathscr{B}_0)$, we see that $\tilde{\psi}^i(\cdot,\cdot)\in C^k((-\infty,t_0-(1+k)r),B_0)$. This implies that $\langle\tilde{\psi}_t^i,\cdot\rangle$ is $C^k((-\infty,t_0-(1+k)r),\mathscr{U})$. The C^k smoothness of $\tilde{B}(t)$ follows from (6.2).

We observe that $\tilde{B}(t)$ sends Φ injectively into Ψ. We also observe that, if $\psi\in\Psi$, $\psi(t)\not\equiv 0$ for $t\in R$, then $\psi|_{[\tau,\tau+r]}\not\equiv 0$ restricted to some interval $[\tau,\tau+r]\in(-t_0,\ t_0-(k+1)r)$. Otherwise, since $\tilde{T}(s,t)$ has a shifted exponential dichotomy in $[-t_0,+\infty)$, the assertion $\psi|_{[\tau,\tau+r]}\equiv 0$ together with the exponential estimate for elements in Ψ implies that $\psi(t)\equiv 0$ for all $t\in R$.

Let $B(t)=\xi(t)\,\tilde{B}(t)$, where $\xi\colon R\to R$ is C^∞, $\xi(t)=1$ on $[\tau,\tau+r]$, $\xi(t)$ has compact support in $(-t_0,t_0-(k+1)r)$, and $\xi(t)\geqslant 0$ for $t\in R$. If we extend $B(t)\equiv 0$ outside $(-t_0,t_0)$, then $B(\cdot)\in C^k(R,\mathscr{U})$. It is easy to see that, for any $z\in\Phi$, $z_t\not\equiv 0$, $t\in\mathbb{R}$, there exists at least one $\psi\in\Psi$ such that

$$\int_{-\infty}^{\infty}\psi(t)\,B(t)z_t\,dt\neq 0. \tag{6.3}$$

For example, we can choose $\psi(t)=\tilde{B}(t)z_t$.

We have to show that $B(t)$ is the desired perturbation. Solutions of (6.1) are denoted by $y(t,\varepsilon)$ with $y(t,\varepsilon)=y(t,0)$ for $t\leqslant t_0$. If $u(t)=\partial y(t,\varepsilon)/\partial\varepsilon|_{\varepsilon=0}$, then $u(t)$ satisfies the system

$$\begin{aligned}\dot{u}(t)&=L(t)u_t+B(t)\,y_t,\\ u(t)&=0,\qquad t\leqslant -t_0.\end{aligned}\tag{6.4}$$

If $y\in\Phi$, $y_t\not\equiv 0$, $t\in\mathbb{R}$, we infer that $u\notin C^{k+1}(\gamma_1,\gamma_2)$ in (6.4). For otherwise, $B(t)\,y_t\in\mathscr{R}(F_L)$, which contradicts (6.3). Moreover, $u\notin C^k(\gamma_1,\gamma_2)$. For otherwise, (6.4) implies that $u\in C^{k+1}(\gamma_1,\gamma_2)$.

Let $u^j(t)$ be the solution of (6.4) corresponding to the forcing term $B(t)\,y_t^j$. We show that $\{u_t^1,\dots,u_t^a,y_t^{b+1},\dots,y_t^{b+c}\}$, $t\geqslant t_0$, are linearly independent and $[u_t^1,\dots,u_t^a,y_t^{b+1},\dots,y_t^{b+c}]\cap\mathscr{R}P_s^+(t)=\{0\}$, $t\geqslant t_0$. For this, suppose that there exist real constants $\{\alpha_j\}$, $j=1,\dots,b$, such that $\tilde{u}_t=\sum_{j=1}^{a}\alpha_j u_t^j+\sum_{j=b+1}^{b+c}\alpha_j y_t^j\in\mathscr{R}P_s^+(t)$, $t\geqslant t_0$. It is easy to see that $\tilde{u}(t)$ is a solution of (6.4) with the initial condition $\tilde{u}(t)=\sum_{j=b+1}^{b+c}\alpha_j y^j(t)$ for $t \leqslant -t_0$, corresponding to the unique forcing $B(t)\tilde{y}_t = B(t)\sum_{j=1}^{a}\alpha_j y_t^j$, $\tilde{y}\in\Phi$. But $\tilde{u}_t \in C^1(\gamma_1, \gamma_2)$, since $\tilde{u}_t \in \mathscr{R}P_s^+(t)$, $t \geqslant t_0$, and $\tilde{u}_t \in \mathscr{R}P_u^-(t)$, $t \leqslant -t_0$. This would be a contradiction to (6.3) unless $\alpha_j = 0$, $j=1,..., a$, $b+1,..., b+c$.

We now prove that $\{y_t^1(\cdot, \varepsilon),..., y_t^a(\cdot, \varepsilon), y_t^{b+1}(\cdot, \varepsilon),..., y_t^{b+c}(\cdot, \varepsilon)\}$ are linearly independent and $[y_t^1(\cdot, \varepsilon),..., y_t^a(\cdot, \varepsilon), y_t^{b+1}(\cdot, \varepsilon),..., y_t^{b+c}(\cdot, \varepsilon)] \cap \mathscr{R}P_s^+(t)=\{0\}$ for $t \leqslant t_0$, $0<|\varepsilon|<\varepsilon_0$, ε_0 is some small constant. It suffices to show that $[y_{t_0}^1(\cdot, \varepsilon)-y_{t_0}^1(\cdot, 0),..., y_{t_0}^a(\cdot, \varepsilon)-y_{t_0}^a(\cdot, 0), y_{t_0}^{b+1}(\cdot, \varepsilon),..., y_{t_0}^{b+c}(\cdot, \varepsilon)] \cap \mathscr{R}P_s^+(t_0)=\{0\}$ since $y_{t_0}^j(\cdot, 0) \in \mathscr{R}P_s^-(t_0)$, $j=1,..., a$. That is, $[\varepsilon u_{t_0}^1 + o(\varepsilon),..., \varepsilon u_{t_0}^a + o(\varepsilon), y_{t_0}^{b+1}+o(1),..., y_{t_0}^{b+c}+o(1)] \cap \mathscr{R}P_s^+(t_0)=\{0\}$. Dividing by ε in the first a vectors, we obtain $[u_{t_0}^1+o(1),..., u_{t_0}^a+o(1), y_{t_0}^{b+1}+o(1),..., y_{t_0}^{b+c}+o(1)] \cap \mathscr{R}P_s^+(t_0)=\{0\}$. Since the last equality is valid if $o(1)$'s are dropped, it is valid if ε is sufficiently small.

Finally, the proof of the lemma is completed by observing that $\mathscr{R}P_u^-(-t_0)$ and $\mathscr{R}P_s^+(t_0)$ are independent of ε.

Define δ_N: $C[-r, 0] \to R^{nN}$ by $\delta_N\phi = (\phi(0), \phi(-r/N),..., \phi(-r+r/N) = (w_0,..., w_{N-1})$, $w_j \in R^n$, $j=0,..., N-1$. For N sufficiently large, $\phi \to \delta_N\phi$ embeds the periodic orbits $\gamma_j = \bigcup_{t\in\mathbb{R}} p_{j,t}$, $j=1, 2$, and the segment of the heteroclinic orbit q_t, $t\in[-t_0-\varepsilon, t_0+\varepsilon]$ into R^{nN} with disjoint images in R^{nN} provided that $\gamma_j \cap \{q_t, t\in[-t_0-\varepsilon, t_0+\varepsilon]\}$ is empty, $j=1, 2$.

The proof of the above is similar to a lemma in [7] and shall be omitted.

LEMMA 6.2. *For $f\in\chi^{k+1}$, $k\geqslant 1$, there is an arbitrarily small $g(\cdot)\in\chi^k$ such that $g=0$ on $\Gamma\cup\alpha(\Gamma)\cup\omega(\Gamma)$ and $\dot{x}(t)=f'(q_t)x_t+g'(q_t)x_t$ has exponential trichotomies in $(-\infty, +t_0]$ and $[-t_0, +\infty)$.*

Proof. We first construct a linear perturbation to (5.1). There exists $B_1(\cdot)\in C^k(R, \mathscr{U})$ with compact support in $J=(-t_0-\varepsilon, t_0+\varepsilon)$ and arbitrarily small such that $L(t)+B_1(t)$ satisfies the H–O property on $[-t_0, t_0]$. This is seen from Lemma 4.1, followed by a multiplication of a C^∞ cutoff function. We claim that an additional perturbation $\Delta B_1(t)$ can be made such that $B_1+\Delta B_1(t)=B_2(t)$, $L(t)+B_2(t)$ satisfies the H–O property on $[-t_0, t_0]$ and $B_2(t)\dot{q}_t=0$ for all $t\in R$. To see this, consider the map $B_1(t)\to l(t)$, $C^k(J, \mathscr{U})\to C^k(J, R^n)$ given by $l(t)=B_1(t)\dot{q}_t$. This map has compact support in J. Let N be a large number such that δ_N embeds $\{q_t, t\in J\}$ into R^{nN}. We can find a finite set of integers $\{k_1,..., k_m\}$, $0\leqslant k_i\leqslant N-1$, and open intervals $\{I_1,..., I_m\}$ which cover J and $\|\dot{q}(t-k_ir/N)\|\geqslant\varepsilon_i>0$ in I_i. Also, in I_i, we can solve the equation $w_{k_i}=q(t-k_ir/N)$ for $t=\tilde{t}(w_{k_i})$. There is a C^∞ partition of unity $\{\xi_i\}$ on J subordinate to $\{I_i\}$, $i=1,..., m$, and $\sum_1^m \xi_i(t)=1$ for $t\in J$. Let $\Delta B_1(t)$: $C[-r, 0]\to R^n$ be defined as

$$\Delta B_1(t)\phi = -\sum_{i=1}^{m} \xi_i(t)\,\dot{q}(t-k_ir/N)\cdot\phi(-k_ir/N)\,l(t)/\|\dot{q}(t-k_ir/N)\|^2.$$

Then $\Delta B_1(\cdot) \in C^k(J, \mathscr{U})$ and depends continuously on $l(\cdot) \in C^k(J, R^n)$. Moreover, $\Delta B_1(t)\dot{q}_t = -l(t)$; hence, $(B_1(t) + \Delta B_1(t))\dot{q}_t = 0$ for all $t \in R$.

The support of $B_2(t) = B_1(t) + \Delta B_1(t)$ has some overlap with $(-\infty, -t_0]$ and $[t_0, +\infty)$. However, from Lemma 4.3, $\dot{x}(t) = (L(t) + B_2(t))x_t$ has exponential trichotomies in $(-\infty, -t_0]$ and $[t_0, +\infty)$ if $B_2(t)$ is small. Moreover, by Lemmas 3.2 and 3.3, the domain of the exponential trichotomies is extended to $(-\infty, t_0]$ and $[-t_0, +\infty)$.

The proof of Lemma 6.2 is fulfilled if we prove the following lemma. The same notation as above will be used in the proof.

LEMMA 6.3. *If $B(\cdot) \in C^k(R, \mathscr{U})$ with support in $J = (-t_0 - \varepsilon, t_0 + \varepsilon)$ and $B(t)\dot{q}_t = 0$, and the orbit segment $\{q_t, t \in [-t_0 - \varepsilon, t_0 + \varepsilon]\}$, has no intersection with $\alpha(\Gamma)$ and $\omega(\Gamma)$, then one can find $g(\cdot) \in \chi^k$ such that $g = 0$ on $\Gamma \cup \gamma_1 \cup \gamma_2$ and $g'(q_t) = B(t)$, $t \in R$.*

Proof. Let U_i, $i = 1, ..., m$, be an open covering of $\{\delta_N q_t : t \in J\}$ in R^{nN}, with $U_i \cap (\alpha(\Gamma) \cup \omega(\Gamma)) = \varnothing$. Let $U_i \cap \{\delta_N q_t : t \in R\} = \{\delta_N q_t : t \in I_i\}$. Let $B(t)\phi = \int_{-r}^0 d\eta_B(t, \theta)\, \phi(\theta)$. For $\delta_N \phi \in U_i$, we define

$$g_i(\phi) = \int_{-r}^0 d\eta_B(\tilde{t}(\pi_{k_i} \delta_N \phi), \theta)(\phi(\theta) - q(\tilde{t}(\pi_{k_i} \delta_N \phi) + \theta)),$$

where $t = \tilde{t}(w_{k_i})$ is the solution for $w_{k_i} = q(t - k_i r/N)$ and $\pi_{k_i} \delta_N \phi = \phi(-k_i r/N)$. Direct computations show that

$$g_i'(q_t)\phi = \int_{-r}^0 d\eta_B(t, \theta)\phi - \int_{-r}^0 d\eta_B(t, \theta)\, \dot{q}(t + \theta) \frac{d\tilde{t}}{dw_{k_i}} \pi_{k_i} \delta_N \phi.$$

The last term vanishes since $B(t)\dot{q}_t = 0$. Let $\xi_i(w)$, $i = 1, ..., m$, be a partition of unity in R^{nN} subordinate to $\{U_i\}$ such that $\sum_1^m \xi_i(w) = 1$ for w being in a neighborhood of $\{\delta_N q_t : t \in J\}$ in R^{nN}. Then $q(\phi) = \sum_1^m \xi_i(\delta_N \phi)\, g_i(\phi)$ is the desired perturbation.

THEOREM 6.4. Let $I = \max\{-\operatorname{ind} \Gamma, 0\}$, $f \in \chi^{k+1}$, $k \geq 1$. *If Γ is a heteroclinic orbit of* (1.1) *and is in general position, then $f \in M^{k+1}(I)$, a C^{k+1} submanifold of χ^{k+1} with* $\operatorname{codim} M^{k+1}(I) = I$. *The equation $\dot{x}(t) = \tilde{f}(x_t)$ has a heteroclinic orbit in a neighborhood of $\Gamma \cup \alpha(\Gamma) \cup \omega(\Gamma)$ and $\tilde{f}$ is close to f in χ^{k+1} if and only if $\tilde{f} \in M^{k+1}(I)$. If Γ is not in general position, there exists a perturbation $g \in \chi^k$, arbitrarily small, and Γ is a heteroclinic orbit in general position of the perturbed equation* $\dot{x}(t) = f(x_t) + g(x_t)$.

Moreover, if $\operatorname{ind} \Gamma \geq 0$ *and Γ is in general position (transverse) and H is the set of heteroclinic orbits of* (1.1) *near Γ, then $H \cap W^u_{\text{loc}}(\alpha(\Gamma))$ is an* $(\operatorname{ind} \Gamma + 1)$*-submanifold of $W^u_{\text{loc}}(\alpha(\Gamma))$. If, in addition the flow near Γ is one-one-one, then H is an immersed* $(\operatorname{ind} \Gamma + 1)$*-submanifold.*

Proof. We first observe that being in general position is rough for the perturbations that do not destroy the heteroclinic orbit; that is, if $\tilde{f}$ is close to f in χ^{k+1} and with a heteroclinic orbit $\tilde{\Gamma}$ close to Γ which is in general position, then $\tilde{\Gamma}$ is in general position. Nothing has to be proved if Γ is transverse. Suppose Γ is in general position and ind $\Gamma < 0$, $I > 0$. We want to show that $f \in M^{k+1}(I)$. In this case $\dot{q}(t)$ is the only bounded solution of (5.2) not in $C(\gamma, -\gamma)$ and there are no solutions of (5.1) in $C(\gamma, -\gamma)$. Thus, the bifurcation function $G^j(\alpha, \mu)$ in (5.13) will depend only on α, μ. By Lemma (5.1), there exists a bounded solution of (5.2), denoted by $\psi^1(t)$, not in $C^1(\gamma, -\gamma)$, and there are $d^* - 1$ bounded independent solutions $\psi^j(t)$, $j = 2,..., d^*$, in $C^1(\gamma, -\gamma)$, which, together with $\psi^1(t)$ form a basis of the bounded solutions of (5.2). From Lemma 5.2, $\partial G^1(0, 0)/\partial\alpha \neq 0$. Let $g(\varphi, \mu) = g(\phi) = \sum_{l=1}^{d^*} \mu_l g_l(\phi)$. Using the technique in [8], we can find C^k functions $\tilde{g}_l$: $R^{nN} \to R^n$ such that, if $g_l(\phi) = \tilde{g}_l(\delta_N \phi)$, then

$$\left\{ \int_{-\infty}^{+\infty} \psi^j(t)\, g_l(q_t)\, dt \right\}_{l=1,...,d^*}^{j=1,...,d^*}$$

is nonsingular. Moreover, $g_l(p_{i,t}) = 0$, $i = 1, 2$. Details are omitted. From (5.16), $\partial G^j/\partial\mu_l = \int_{-\infty}^{\infty} \psi^j(t)\, g_l(q_t)\, dt$. We solve $\alpha = \alpha(\mu_l)$ from $G^1(\alpha, \mu_l) = 0$. For $j = 2,..., d^*$, $\partial G^j(0, 0)/\partial\alpha = 0$. Therefore,

$$\begin{aligned} \frac{dG^j(0,0)}{d\mu_l} &= \frac{\partial G^j(0,0)}{\partial\mu_l} + \frac{\partial G^j(0,0)}{\partial\alpha}\frac{d\alpha}{d\mu_l} \\ &= \frac{\partial G^j(0,0)}{\partial\mu_l} = \int_{-\infty}^{\infty} \psi^j g_l(q_t)\, dt, \qquad j = 2,..., d^* \end{aligned}$$

and the matrix

$$\left\{ \frac{dG^j(0,0)}{d\mu_l} \right\}_{l=1,...,d^*}^{j=1,...,d^*}$$

has rank $d^* - 1 = I$. This shows that $f \in M^{k+1}(I)$.

Now suppose Γ is not in general position. By Lemma 6.2, we assume that $f \in \chi^k$, $f'(q_t) \in C^k(R, \mathscr{U})$, and (5.2) has exponential trichotomies in $(-\infty, t_0]$ and $[t_0, +\infty)$. We use Lemma 6.1 to prove that there is a perturbation $\varepsilon g(\phi)$ to make Γ a heteroclinic orbit in general position. For this, observe that $\tilde{P}_u^-(t) = P_u^-(t) + P_c^-(t)$, $P_s^-(t) = P_s^-(t)$, $t \in (-\infty, t_0]$ and $\tilde{P}_u^+ = P_u^+(t)$, $\tilde{P}_s^+(t) = P_c^+(t) + P_s^+(t)$ define shifted exponential trichotomies in $(-\infty, t_0]$ and $[-t_0, +\infty)$. In the notation of Lemma 6.1, nothing is to be proved if $\mathscr{R}\tilde{P}_u^-(0) \cap \mathscr{R}\tilde{P}_s^+(0)$ is of dimension $b = 1$, spanned by $\dot{q}(t)$. If not, let $b > 1$ and $y_0^b = \dot{q}_0$ and $\{y_0^1,..., y_0^{b-1}, y_0^b\}$ be a basis in $\mathscr{R}\tilde{P}_u^-(0) \cap \mathscr{R}\tilde{P}_s^+(0)$. It is clear that if $e = 0$, Γ is transverse. Thus, we assume

$e > 0$. Let $a = \min(b-1, e)$, and $\varepsilon B(t)$ be the perturbation determined by Lemma 6.1. Since $\varepsilon B(t)\dot{q}_t = 0$, by Lemma 6.3, we can find $g \in \chi^k$ such that $g = 0$ on $\Gamma \cup \alpha(\Gamma) \cup \omega(\Gamma)$ and $g'(q_t) = \varepsilon B(t)$. For the perturbed equation, Γ is clearly in general position. There are two cases. If $b-1 \geqslant e$, then $\operatorname{ind} \Gamma \geqslant 0$, and Γ is transverse with respect to the perturbed equation. If $b-1 < e$, then the perturbed equation has $\dot{q}(t)$ as the unique bounded solution for its linear variational equation. Thus, Γ is in general position after perturbation.

The last part of the theorem follows from (5.13) and (5.14). For, in that case, $d^* = 1$, $d = \operatorname{ind} \Gamma$, one can choose $|k| < \bar{k}$, $k \in R^d$ in an arbitrary manner and obtain α from (5.13) since $\partial G/\partial\alpha \neq 0$. This completes the proof of the theorem.

The bifurcation functions G^j and results similar to Theorem 6.4 are easier to obtain in the other three cases in which Γ is a heteroclinic orbit of (1.1) and $\alpha(\Gamma)$ and $\omega(\Gamma)$ are hyperbolic:

(1) $\alpha(\Gamma)$ and $\omega(\Gamma)$ are equilibria.

(2) $\alpha(\Gamma)$ is an equilibrium and $\omega(\Gamma)$ is a periodic orbit.

(3) $\alpha(\Gamma)$ is a periodic orbit and $\omega(\Gamma)$ is an equilibrium.

In case (1), exponential dichotomy is employed and no frequency β and phase variation α are needed. However, since $\dot{q}(t) \in \mathcal{N}F(\gamma, -\gamma)$, we let $x(t) = q(t) + z(t)$, with $z(t) \in \sum \oplus [y^1, ..., y^{d-1}]$, where $\{\dot{q}, y^1, ..., y^{d-1}\}$ is a basis of $\mathcal{N}F(\gamma, -\gamma)$ and $\sum \oplus \mathcal{N}F(\gamma, -\gamma) = C^{k+1}(\gamma, -\gamma)$. Then we assume $z = z^* + \sum_1^{d-1} k_i y^i$ with $z^* \in \Sigma$.

In cases (2) and (3), we need $\beta(t, \mu)$ for only one side and by a proper phase shift we assume that $x^\mu((1+\beta)t) \to p_i^\mu(t)$ as $t \to +\infty$ or $-\infty$ for $i = 1, 2$. No parameter α is needed.

The following is true for $\alpha(\Gamma)$ and $\omega(\Gamma)$ being hyperbolic periodic orbits or equilibria, with general position defined in an obvious way.

THEOREM 6.5. *Let* $\operatorname{ind} \Gamma = \dim W^{\mathrm{u}}_{\mathrm{loc}}(\alpha(\Gamma)) - \dim W^{\mathrm{u}}_{\mathrm{loc}}(\omega(\Gamma)) + \dim \omega(\Gamma) - 1$. *Then the results of Theorem* 6.4 *are valid with* $\alpha(\Gamma)$ *and* $\omega(\Gamma)$ *being hyperbolic periodic orbits or equilibria.*

Completion of Proof of Theorem 5.4. We owe the readers a proof of transversality in Theorem 5.4. If the heteroclinic orbit Γ^{μ^0} is transverse, then, for a small perturbation $\tilde{g}(\phi)$ to $g(\phi, \mu^0)$, there is a heteroclinic orbit $\Gamma^{\tilde{g}}$ which is within $O(|\tilde{g}|)$ of Γ^{μ^0} and the phase variation $\alpha(\tilde{g})$ is also within $O(|\tilde{g}|)$ to α^0. Conversely, if we denote $\Gamma^{\mu^0} = \bigcup_{t \in \mathbb{R}} x_t^{\mu^0}$, and if Γ^{μ^0} is not transverse, we can find a family of perturbations $\varepsilon\tilde{g}_1(\phi)$ to $g(\phi, \mu^0)$ such that trajectories starting from $W^{\mathrm{u}}_{\mathrm{loc}}(\alpha(\Gamma^{\mu^0}))$ are moved to a direction transverse to $TW^{\mathrm{u}}(\alpha(\Gamma^{\mu^0})) + TW^{\mathrm{s}}(\omega(\Gamma^{\mu^0}))$. Thus, we either eliminate the intersection of $W^{\mathrm{u}}(\alpha(\Gamma^{\mu^0}))$ and $W^{\mathrm{s}}(\omega(\Gamma^{\mu^0}))$ or move it to a distance $> 0(|\varepsilon g_1|)$.

To show the existence of such $\tilde{g}_1$, we use the technique in proving Theorem 6.4 to construct a $\tilde{g}_1 \in \chi^{k+1}$ such that $\tilde{g}_1 = 0$ in some neighborhoods of $\alpha(\Gamma)$ and $\omega(\Gamma)$, $\tilde{g}_1(x^{\mu^0}_{\cdot}) \notin \mathscr{R}F_{L+Dg(\cdot,\mu^0)}(-\gamma, \gamma)$. Let $t_0 > 0$ be sufficiently large and consider the solution $x(t, \varepsilon)$ of

$$\dot{x}(t) = f(x_t) + g(x_t, \mu^0) + \varepsilon \tilde{g}_1(x_t)$$
$$x(t) = x^{\mu^0}(t), \qquad t \leqslant -t_0.$$

It is not difficult to show that $(\partial x(t, \varepsilon)/\partial\varepsilon)_{t_0} \notin \mathscr{R}\tilde{P}_s(t_0) + (T(t_0, -t_0)\ \mathscr{R}\tilde{P}_u(-t_0))$, where $\tilde{P}_s$ and $\tilde{P}_u$ are projections associated with the shifted exponential dichotomies in $(-\infty, t_0]$ and $[t_0, +\infty)$ for the linearized equation around Γ^{μ^0}. Therefore, $\varepsilon\tilde{g}_1$ is the desired perturbation.

On the other hand, we consider the extended perturbations $g_1(\phi, \mu, \tilde{g}) = g(\phi, \mu) + \tilde{g}(\phi)$, with the parameters $(\mu, \tilde{g}) \in X \times \chi^{k+1}$. If the matrix in (5.15) has rank d^*, then, for small $\tilde{g}$, there exist $\alpha^0 + \delta\alpha$, $k^0 + \delta k$ such that $G^j(\alpha^0 + \delta\alpha, k^0 + \delta k, \mu^0, \tilde{g}) = 0$, $j = 1, ..., d^*$, and $\delta\alpha$, $\delta k = O(|\tilde{g}|)$. Therefore, there is a new heteroclinic orbit $\Gamma^{\tilde{g}}$, $O(|\tilde{g}|)$ near Γ^{μ^0} and with a phase variation $\alpha(\tilde{g})$, $O(|\tilde{g}|)$ near α^0. Conversely, if the matrix in (5.15) has rank $< d^*$, without loss of generality, let $\partial G^{j_0}(\alpha^0, k^0, \mu^0)/\partial k_i = 0$, $\partial G^{j_0}(\alpha^0, k^0, \mu^0)/\partial\alpha = 0$, $i = 1, ..., d$. Fot the extended family of perturbations, it is clear that $\partial G^{j_0}(\alpha^0, k^0, \mu^0, 0)/\partial\tilde{g} \neq 0$ from the proof of Theorem 6.4. Thus, there are small $\tilde{g}$ such that either we cannot find α, k near α^0, k^0 such that $G^{j_0}(\alpha, k, \mu^0, \tilde{g}) = 0$, or they are moved to a distance $> O(|\tilde{g}|)$ to α^0, k^0. The heteroclinic orbit $\Gamma^{\tilde{g}}$ is moved to a distance $> 0(|\tilde{g}|)$ in the latter case if we can show that $\partial z/\partial\alpha$, $\partial z/\partial k_i$, and $\partial x^{\mu^0}(t)/\partial t$ are linearly independent. It is obviously true when $\alpha^0 = \mu^0 = k^0 = 0$, for then $\partial z^*/\partial\alpha = \partial z^*/\partial k_i = 0$, and $\partial\omega/\partial\alpha = \zeta_2(t)\, \dot{p}_2(t)$ and

$$\frac{\partial}{\partial k_i}\left(\sum_1^d k_i y^i\right) = y^i(t), \qquad i = 1, ..., d.$$

The linear independence holds for α^0, μ^0, k^0 being small.

We have two characterizations by which the perturbation $\tilde{g}$ will not break the heteroclinic orbit Γ^{μ^0} and only move it to a distance $= O(|\tilde{g}|)$. By comparison we see that the transversality of Γ^{μ^0} is equivalent to the rank of the matrix in (5.15) being d^*.

ACKNOWLEDGMENT

The authors acknowledge useful conversations with John Mallet-Paret in the preparation of the paper.

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