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Exponential Dichotomies in Intermediate Spaces with Applications to a Diffusively Perturbed Predator–Prey Model

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Intermediate spaces are useful for obtaining maximal regularity when studying reaction–diffusion equations. We give a definition of exponential dichotomies in these spaces. Roughness of the exponential dichotomies and other basic properties are also proved. We then use these notions to investigate the stability of a periodic orbit in a predator–prey model studied by Conway and Smoller when diffusion terms are added. © 1994 Academic Press, Inc.

1. INTRODUCTION

This paper is motivated by the study of the reaction–diffusion system

$$\begin{aligned} u_t &= d_1 u_{xx} + uM(u, v), \\ v_t &= d_2 v_{xx} + vN(u, v), \qquad 0 < x < 1, \end{aligned} \tag{1.1}$$

where either Neumann or periodic boundary conditions are imposed. When $d_1 = d_2 = 0$, the ODE system

$$\begin{aligned} u' &= uM(u, v), \\ v' &= vN(u, v) \end{aligned} \tag{1.2}$$

is a predator–prey model. Each solution of (1.2) becomes a spatially homogeneous solution of (1.1). It is known that for large d_1 and d_2, all the solutions for (1.1) approach solutions for (1.2) as $t \to \infty$. See Hale (7) and Conway *et al.* (3). However, as d_1 and d_2 decrease, stable spatially non-homogeneous solutions may bifurcate from the solutions of (1.2). Let the diffusion coefficients be $d_1 + \Delta d_1$ and $d_2 + \Delta d_2$. A useful approach for

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studying bifurcations caused by varying Δd_1 and Δd_2 is to use the integral equation

$$U(t) = T(t, \tau)\, U(\tau) + \int_\tau^t T(t, s)\, F(U(s), \Delta d_1, \Delta d_2)\, ds. \tag{1.3}$$

Here $U = (u, v)^\tau$ and $F = (\Delta d_1 u_{xx} + uM(u, v), \Delta d_2 v_{xx} + vN(u, v))^\tau$. We thus need results concerning the maximal regularity of the integral equation; i.e., $U'(t)$ and U_{xx} must have the same regularity as $F(U(s), \Delta d_1, \Delta d_2)$. The results in Henry (8) concerning semilinear parabolic equations are not useful here. We use results from Da Prato and Grisvard (6) etc., concerning so called fully nonlinear equations, though our problem is caused by the presence of the linear terms u_{xx} and v_{xx} in the perturbation terms.

The concept of exponential dichotomy and related estimates are very useful for studying dynamical systems generated by ODEs. See Coppel (5). Henry (8) discussed exponential dichotomies for semilinear parabolic equations. Lunardi (11, 12) used exponential dichotomies to study fully nonlinear periodic equations. In the first part of this paper we give a definition of exponential dichotomies for abstract parabolic equations in the spaces that allow the solutions to have maximal space regularity. Basic properties such as the roughness of the exponential dichotomies, see Coppel (5), are also proved.

In the second part of this paper we use the exponential dichotomy to study a diffusively perturbed predator–prey model. The ODE model was studied by Conway and Smoller (4). It is proved in their paper that a heteroclinic cycle exists when a parameter $\gamma = \gamma_0$. When γ increases from γ_0, a stable periodic solution $p(t, \gamma)$ bifurcates from the heteroclinic cycle. Corresponding to $p(t, \gamma)$ there is a spatial homogeneous solution for (1.1), still denoted by $p(t, \gamma)$. The stability of $p(t, \gamma)$ is known if d_1 and d_2 are both large. When d_1 and d_2 decrease, complicated bifurcations can occur at equilibria of the heteroclinic cycle. We want to know how those bifurcations affect the stability of $p(t, \gamma)$ in the function space used by Da Prato and Grisvard (6). In this paper (Theorem 4.5) we show that decreasing d_1 or d_2 alone does not destroy the stability of $p(t, \gamma)$. That orbit remains stable if one of the diffusion coefficients is large. More complete analysis of bifurcations near $p(t, \gamma)$ will appear in another paper.

2. Linear Parabolic Equations with Maximal Space Regularity

There are basically three types of results concerning maximal regularity. The first type concerns the maximal time regularity; it shows that U_t and U_{xx} are in a function space of Hölder continuous functions if $f(t)$ is in the same space. The second type concerns the maximal space regularity; it shows that U_t and U_{xx} are in an intermediate space between X and D_A if $f(t)$ is in the same space for all t. Here D_A is the domain of an unbounded operator A. Other results showing that space (or time) regularity implies time (or space) regularity are also available. At the present time it is difficult even to quote the many existing results. To simplify the analysis, we will only use space regularity. The proof of Theorem 2.2 was given by Da Prato and Grisvard (6) and Sinestrari (15). Theorem 2.3 is due to Buttu (2); it is a major tool to construct evolution operators for time dependent parabolic systems.

Let A be a densely defined closed linear operator in a Banach space X. Assume that the resolvent set $\rho(A)$ contains the sector

$$S_{\omega,\phi} = \{z \in \mathbb{C}, z \neq \omega, |\arg(z-\omega)| \leqslant \phi\}, \tag{2.1}$$

where $\omega \in \mathbb{R}$ and $\pi/2 < \phi < \pi$. Assume that for $\lambda \in S_{\omega,\phi}$,

$$\|R(\lambda, A)\|_{L(X)} \leqslant M\,|\lambda - \omega|^{-1} \tag{2.2}$$

Then A is the infinitesimal generator of a C_0 analytic semigroup e^{At} in X.

DEFINITION 2.1. For each $0<\theta<1$, define a subspace $D_A(\theta)$ of X to be

$$D_A(\theta) = \{x \in X \mid \lim_{t \to 0} t^{1-\theta} A e^{At} x = 0\}.$$

Also define $D_A(\theta+1) = \{x \in D_A \mid Ax \in D_A(\theta)\}$. $D_A(\theta)$ and $D_A(\theta+1)$ are Banach spaces with norms

$$\|x\|_\theta = \sup_{0 \leqslant t \leqslant 1} \|t^{1-\theta} A e^{At} x\|_X + \|x\|_X,$$

$$\|x\|_{\theta+1} = \|Ax\|_\theta + \|x\|_X.$$

Definition 2.1 was used by Sinestrari (15) who also studied the case where D_A was not dense in X. Several equivalent definitions of $D_A(\theta)$ can be found in Berens and Butzer (1) or Da Prato and Grisvard (6).

We use $D_A(\theta)$ and $D_A(\theta+1)$ to study the following abstract parabolic equation:

$$\begin{aligned} u'(t) &= Au(t) + f(t), \\ u(a) &= u_0 \end{aligned} \tag{2.3}$$

THEOREM 2.2. *For each* $u_0 \in D_A(\theta+1)$ *and* $f \in C([a,b];\ D_A(\theta))$, *Eq.* (2.3) *admits a unique solution* $u(t) \in C^1([a,b]; D_A(\theta)) \cap C([a,b]; D_A(\theta+1))$, *with*

$$\sup_{a \leqslant t \leqslant b} \{\|u'(t)\|_\theta + \|u(t)\|_{\theta+1}\} \leqslant C\{\|u_0\|_{\theta+1} + \sup_{a \leqslant t \leqslant b} \|f(t)\|_\theta\}.$$

Moreover, the solution can be written as

$$u(t) = e^{A(t-a)} u_0 + \int_a^t e^{A(t-s)} f(s)\, ds, \qquad a \leqslant t \leqslant b. \tag{2.4}$$

The proof of Theorem 2.1 can be found in Da Prato and Grisvard (6) or Sinestrari (15). Solutions described in Theorem 2.2 will be called strict solutions.

Theorem 2.2 can be use to construct evolution operator $T(t,s)$ on $D_A(\theta+1)$ for time dependent systems. To construct $T(t,s)$ on $D_A(\theta)$ we need to consider nonstrict solutions to (2.3). Let U be any Banach space. Let $0 \leqslant \alpha \leqslant 1$, and $0 < \beta < 2$. Define the following Banach spaces:

$$B_1((a,b]; D_A(\beta)) = \{\phi : (a,b] \to x \mid \phi(t) \in D_A(\beta) \text{ for all } t \in (a,b];\ \langle \phi \rangle_\beta = \sup_{a < t \leqslant b} (t-a)\, \|\phi(t)\|_\beta < \infty\};$$

$$B([a,b]; D_A(\theta)) = \{\phi : [a,b] \to D_A(\theta),\ \sup_{a \leqslant t \leqslant b} \|\phi(t)\|_\theta < \infty\};$$

$$C_{1-\alpha}((a,b]; U) = \{\phi \in C((a,b]; U) \mid [\phi]_\alpha = \sup_{a < t \leqslant b} (t-a)^{1-\alpha}\, \|\phi(t)\|_U < \infty\};$$

$$W[a,b] = C^\theta([a,b]; X) \cap B([a,b]; D_A(\theta));$$

$$Z(a,b) = C_{1-\theta}((a,b]; X) \cap B_1((a,b]; D_A(\theta));$$

$$Y(a,b) = C_{1-\theta}((a,b]; D(A)) \cap B_1((a,b]; D_A(\theta+1)).$$

Here C^θ denotes Hölder continuous functions. The norms in these spaces are defined in obvious ways. The following theorem is proved by Buttu (2).

THEOREM 2.3. *Let* $f \in Z(a,b)$, *and* $x \in D_A(\theta)$, *then the function* u *defined in* (2.4) *is the unique solution of* (2.3) *with* $u \in Y(a,b) \cap W[a,b]$. *Moreover,*

$$\|u\|_{Y(a,b)} + \|u\|_{W[a,b]} \leqslant C\{\|x\|_\theta + \|f\|_{Z(a,b)}\}. \tag{2.5}$$

Evolution operators for time dependent systems were constructed by Sobolevskii (16), Tanabe (17), and Lunardi (10). Our construction of $T(t, s)$ on $D_A(\theta)$ is adapted from Buttu (2). Let D be a continuously embedded dense subspace of X. Let $A(t): D \to X$, $a \leqslant t \leqslant b$, be a family of linear operators such that the graph norm of $A(t)$ is equivalent to the norm of D. Assume that

(i) for each $t \in [a, b]$, $A(t): D \to X$ satisfies (2.1) and (2.2);

(ii) $D_{A(t)}(\theta+1) = D_{A(\tau)}(\theta+1)$ with equivalent norms for all t, $\tau \in [a, b]$;

(iii) the function $t \to A(t)$ belongs to $C([a, b]; L(D, X)) \cap C([a, b]; L(D_A(\theta+1), D_A(\theta)))$.

Let $A = A(t_0)$, $t_0 \in [a, b]$. We will write $D_A(\theta+k)$ for $D_{A(t_0)}(\theta+k)$. Consider the initial value problem

$$\begin{aligned} u'(t) &= A(t)\, u(t) + f(t), \\ u(s) &= x, \qquad a \leqslant s < t \leqslant b. \end{aligned} \tag{2.6}$$

THEOREM 2.4. (i) *For each* $x \in D_A(\theta+1)$ *and* $f \in C([s, b]; D_A(\theta))$, *Eq.* (2.6) *has a unique solution* $u \in C^1([s, b]; D_A(\theta)) \cap C([s, b]; D_A(\theta+1))$, *with*

$$\sup_{s \leqslant t \leqslant b} \{ \|u'(t)\|_\theta + \|u(t)\|_{\theta+1} \} \leqslant C\{ \|x\|_{\theta+1} + \sup_{s \leqslant t \leqslant b} \|f(t)\|_\theta \}.$$

(ii) *Let* $x \in D_A(\theta)$, $f \in Z(s, b)$. *Then* (2.6) *has a unique solution* $u(t)$ *with* $u \in Y(s, b)$. *Moreover,* $u \in W[a, b]$ *and*

$$\|u\|_{Y(s,b)} + \|u\|_{W[s,b]} \leqslant C\{ \|x\|_\theta + \|f\|_{Z(s,b)} \}. \tag{2.7}$$

(iii) *Let the solution of* (2.6) *be denoted by* $T(t, s)x = u(t)$ *when* $x \in D_A(\theta+k)$, $k = 0, 1$, *and* $f = 0$. *Then* $(s, t) \to T(t, s)x$ *is a continuous function with domain* $a \leqslant s \leqslant t \leqslant b$ *and range* $D_A(\theta+k)$. *We have*

$$T(s, s) = I, \qquad T(t, \tau)\, T(\tau, s) = T(t, s), \qquad a \leqslant s \leqslant \tau \leqslant t \leqslant b. \tag{2.8}$$

$$\|T(t, s)x\|_{\theta+k} \leqslant C\, \|x\|_{\theta+k}, \qquad k = 0, 1. \tag{2.9}$$

Also, for $x \in D_A(\theta+1)$,

$$\frac{\partial}{\partial s} T(t, s)x = -T(t, s)\, A(s)x. \tag{2.10}$$

$$\|A(t)\, T(t, s)\|_{L(D_A(\theta))} \leqslant C\{ (t-s)^{-1} + 1 \}, \qquad a \leqslant s < t \leqslant b. \tag{2.11}$$

Finally, if $x \in D_A(\theta+1)$ *and* $f \in ([s, b]; D_A(\theta))$, *then the solution of* (2.6) *can be written as*

$$u(t) = T(t, s)x + \int_s^t T(t, \xi) f(\xi)\, d\xi, \qquad a \leqslant s < t \leqslant b. \tag{2.12}$$

$$\left\| \int_s^t T(t, \xi) f(\xi)\, d\xi \right\|_{\theta+1} \leqslant C \sup_{s \leqslant \xi \leqslant t} \|f(\xi)\|_\theta \tag{2.13}$$

Proof. (i) For $x \in D_A(\theta+1)$ and $f \in C([a, b]; D_A(\theta))$, $s \leqslant t \leqslant s+h$, the equations

$$\begin{aligned} u'(t) &= A(s)\, u(t) + (A(t) - A(s))\, u(t) + f(t), \\ u(s) &= x, \end{aligned} \tag{2.14}$$

can be solved using the contraction mapping principle on $C([s, s+h]; D_A(\theta+1))$ and using Theorem 2.2. Here we need the fact that $\|A(\xi) - A(s)\|_{L(D_A(\theta+1);\, D_A(\theta))}$ is small if h is small. The smallness of h can be removed by continuing the solution to $[s, b]$.

(ii) Let $u \in Y(s, s+h)$. Then $(A(\cdot) - A(s))\, u(\cdot) \in Z(s, s+h)$ and

$$\|(A(\cdot) - A(s))\, u(\cdot)\|_{Z(s,s+h)} \leqslant C\delta\, \|u\|_{Y(s,s+h)},$$

where $\delta = \sup_{s \leqslant t \leqslant s+h} \{ \|A(t) - A(s)\|_{L(D_A,\, X)} + \|A(t) - A(s)\|_{L(D_A(\theta+1),\, D_A(\theta))} \}$. If $h > 0$ is sufficiently small, then $C\delta < 1$. Using Theorem 2.3, (2.14) can be solved by the contraction mapping principle to have a unique solution $u \in Y(s, s+h)$. The solution $u(t)$ can be continued to $t = b$. Details can be found in Buttu (2). Estimate (2.7) also follows from Theorem2.3.

(iii) Property (2.8) follows from the definition of $T(t, s)$. Estimate (2.9) has also been proved.

Our construction show that $t \to T(t, s)x$ is continuous from $[s, b]$ to $D_A(\theta+1)$ if $x \in D_A(\theta+1)$. Let $x \in D_A(\theta)$, $y_n \in D_A(\theta+1)$, $n \in \mathbb{Z}$, and $y_n \to x$ in $D_A(\theta)$ as $n \to \infty$. Then $t \to T(t, s)\, y_n$ is continuous in $D_A(\theta)$ and approaches $T(t, s)x$ uniformly in $D_A(\theta)$ for $t \in [s, s+b]$ due to (2.9). Thus $t \to T(t, s)x$ is continuous from $[s, b]$ to $D_A(\theta)$.

Define $A(t) = A(b)$ for $t > b$. Let $h > 0$ be a small constant such that $\|A(t_1) - A(t_2)\|_{L(Y,\, Z)} < \delta$ whenever $|t_1 - t_2| \leqslant 2h$. Fix $s \geqslant a$ and let $s_1 \in [s, s+h]$. For each $w \in Y(s, s+h)$, consider the initial value problem

$$\begin{aligned} u'(t) &= A(s)\, u(t) + [(A(t + s_1 - s) - A(s)]\, w(t), \qquad s \leqslant t \leqslant s+h, \\ u(s) &= x. \end{aligned} \tag{2.15}$$

Let $g(t)=[A(t+s_1-s)-A(s)]\,w(t)$; then $\|g\|_{Z(s,s+h)}\leqslant\delta\,\|w\|_{Y(s,s+h)}$. Using Theorem 2.3, (2.15) has a unique solution $u=\Gamma(w,s_1)$. Γ is continuous from $Y(s,s+h)\times[s,s+h]$ to $Y(s,s+h)$. Also, $\|u\|_{Y(s,s+h)}\leqslant c\delta\,\|w\|_{Y(s,s+h)}+c\,\|x\|_\theta$. If h is so small that $c\delta<1$, we can use the uniform contraction principle to find a unique fixed point of Γ, denoted by $u(t)=U(t,s_1)x$. $s_1\to U(\cdot,s_1)x$ is continuous from $[s,s+h]$ to $Y(s,s+h)$. Thus, by Theorem 2.3, $s_1\to U(\cdot,s_1)x$ is continuous from $[s,s+h]$ to $W[s,s+h]\subset B([s,s+h];D_A(\theta))$. It is not hard to show that $T(t,s_1)x=U(t+s-s_1,s_1)x$, $s_1\leqslant t\leqslant s_1+h$. Thus, $T(t,s_1)x$ is continuous from $(s_1,t)\to D_A(\theta)$ if $s_1\in[s,s+h]$ and $s_1\leqslant t\leqslant s_1+h$. Such restrictions can easily be removed.

We have proved, for $x\in D_A(\theta)$, that $T(t,s)x$ is continuous from $(t,s)\to D_A(\theta)$. The result for $x\in D_A(\theta+1)$ can also be proved similarly.

We now prove (2.10). Let $B(\tau)=A(\tau)-A(s)$. For $x\in D_A(\theta+1)$, $t\to U(t,s)x=-T(t,s)\,A(s)x$ is a function in $Y(s,s+h)$ satisfying

$$U(t,s)x=-e^{A(s)(t-s)}A(s)x+\int_s^t e^{A(s)(t-\tau)}B(\tau)\,U(\tau,s)\,x\,d\tau.$$

Let $\varDelta\geqslant 0$. Then

$$T(t,s+\varDelta)x=e^{A(s)(t-s-\varDelta)}x+\int_{s+\varDelta}^t e^{A(s)(t-\tau)}B(\tau)\,T(\tau,s+\varDelta)\,x\,d\tau.$$

Let $\eta(t,s,\varDelta)x=\varDelta^{-1}(T(t,s+\varDelta)x-T(t,s))x$, $0<\varDelta\leqslant t-s\leqslant h$. Then we can verify that

$$\begin{aligned}
&\eta(t,s,\varDelta)x-U(t,s)x\\
&\quad=\int_{s+\varDelta}^t e^{A(s)(t-\tau)}B(\tau)[\eta(\tau,s,\varDelta)x-U(\tau,s)x]\,d\tau\\
&\qquad+[\varDelta^{-1}(e^{A(s)(t-s-\varDelta)}-e^{A(s)(t-s)})x+e^{A(s)(t-s)}A(s)x]\\
&\qquad-\varDelta^{-1}\int_s^{s+\varDelta} e^{A(s)(t-\tau)}B(\tau)\,T(\tau,s)\,x\,d\tau\\
&\qquad-\int_s^{s+\varDelta} e^{A(s)(t-\tau)}B(\tau,s)\,U(\tau,s)\,x\,d\tau\\
&\quad=J_1+J_2+J_3+J_4
\end{aligned}$$

Using $\sup_{s\leqslant\tau\leqslant s+h}\|B(\tau)\|_{L(Y,Z)}\leqslant\delta\to 0$ as $h\to 0$, we find that $\|J_1\|_{Y(s+\varDelta,s+h)}\leqslant\frac{1}{2}\cdot\|\eta-U\|_{Y(s+\varDelta,s+h)}$ if h is sufficiently small. Also, using the smallness of

$\sup_{s\leqslant\tau\leqslant s+\Delta}\|B(\tau)\|_{L(D_A(\theta+1),D_A(\theta))}$, we can prove that $\|J_3\|_{Y(s+\Delta,s+h)}\to 0$ as $\Delta\to 0$. We have

$$\begin{aligned}
&\|J_4\|_{Y(s+\Delta,s+h)}\\
&\quad=\left\|e^{A(s)(t-s-\Delta)}\cdot\int_s^{s+\Delta}e^{A(s)(s+\Delta-\tau)}B(\tau)\,U(\tau,s)\,x\,d\tau\right\|_{Y(s+\Delta,s+h)}\\
&\quad\leqslant C\left\|\int_s^{s+\Delta}e^{A(s)(s+\Delta-\tau)}B(\tau)\,U(\tau,s)\,x\,d\tau\right\|_\theta\\
&\quad\leqslant C\int_s^{s+\Delta}\|e^{A(s)(s+\Delta-\tau)}\|_{L(X,D_A(\theta))}\,\|B(\tau)\|_{L(D_A,X)}\,\|U(\tau,s)\|_{D_A}\,d\tau\\
&\quad\leqslant C\sup_{s\leqslant\tau\leqslant s+\Delta}\|B(\tau)\|_{L(D_A,X)}\int_s^{s+\Delta}(s+\Delta-\tau)^{-\theta}\tau^{\theta-1}\,\|U(\tau,s)\|_{Y(s,s+\Delta)}\,d\tau\\
&\quad\leqslant C\sup_{s\leqslant\tau\leqslant s+\Delta}\|B(\tau)\|_{L(D_A,X)}\int_0^1(1-r)^{-\theta}r^{\theta-1}\,d\tau\\
&\quad\to 0\qquad\text{as }\Delta\to 0.
\end{aligned}$$

Thus $\|\eta(\cdot,s,\Delta)x-U(\cdot,s)x\|_{Y(s+\Delta,s+h)}\to 0$ as $\Delta\to 0$, implying that $\eta(t,s,\Delta)\to -T(t,s)\,A(s)x$ in $D_A(\theta)$ as $\Delta\to 0$.

The case $\Delta<0$ can be treated similarly.

We now prove (2.12). Let $x\in D_A(\theta+1)$, $f\in C([s,t];D_A(\theta))$, and $u(t)$ be the solution of (2.6). Then $\tau\to T(t,\tau)\,u(\tau)$, $s\leqslant\tau\leqslant t$, is differentiable with values in $D_A(\theta)$, and $(\partial/\partial\tau)[T(t,\tau)\,u(\tau)]=T(t,\tau)\,f(\tau)$. Thus, (2.12) follows, and (2.13) follows from part (i) of this theorem.

Finally (2.11) follows from (2.7). ∎

3. Exponential Dichotomies for the Linear Systems

Exponential dichotomies for ODE systems were discussed in detail by Coppel (5). Henry (8) gave a definition for systems generated by semilinear parabolic equations. Lunardi (11, 12) used the idea of exponential dichotomies to study periodic parabolic systems with maximal regularity. The definition given below only concerns the space regularity to be used in this paper. Since the phase space is $D_A(\theta+1)$, it is natural to define the exponential dichotomy in $D_A(\theta+1)$. For completeness, decay properties in $D_A(\theta)$ will be proved from those in $D_A(\theta+1)$, but will not be used in this paper.

Let X be the Banach space and $A(t):D\to X$ as mentioned in Section 2. Let $T(t,s)$ be the evolution operator as in Theorem 2.3.

DEFINITION 3.1. The evolution operator $T(t, s) \in L(D_A(\theta+1))$ is said to have a pseudo exponential dichotomy on $I=[a, b]$ (or $(-\infty, b]$, or $(a, +\infty]$, or $(-\infty, \infty)$) if there exist projections $P_u(t)+P_s(t)=I$ in $D_A(\theta+1)$ for each $t \in I$, and constants $K_1, K_2>0$ and $\beta<\alpha$ with the following properties for s, t in I:

(i) $$P_v(t)\, T(t, s)x = T(t, s)\, P_v(s)x, \qquad v=u, s,$$

for each $x \in D_A(\theta+1)$ and $t \geqslant s$.

(ii) $$\|T(t, s)\, P_s(s)\|_{L(D_A(\theta+1))} \leqslant K_1\, e^{\beta(t-s)}, \qquad t \geqslant s.$$

(iii) $T(t, s): P_u(s)\, D_A(\theta+1) \rightarrow P_u(t)\, D_A(\theta+1)$, $t \geqslant s$ is a homeomorphism with the inverse denoted by $T(s, t): P_u(t)\, D_A(\theta+1) \rightarrow P_u(s)\, D_A(\theta+1)$.

(iv) $$\|T(s, t)\, P_u(t)\|_{L(D_A(\theta+1))} \leqslant K_2\, e^{-\alpha(t-s)}, \qquad t \geqslant s.$$

(v) There exist $h>0$ and $C_0>0$ such that

$$\left\|\int_\tau^t T(t, s) f(s)\, ds\right\|_{\theta+1} \leqslant C_0 \sup_{\tau \leqslant s \leqslant t} \|f(s)\|_\theta, \qquad \text{for all } t-\tau \leqslant h \text{ and } [\tau, t] \subset I.$$

The evolution operator is said to have an exponential dichotomy if, additionally, $\alpha>0$ and $\beta<0$.

For $x \in D_A(\theta)$ and $s<b$, define $P_u(s)x = T(s, b_1)\, P_u(b_1)\, T(b_1, s)x$, where $s<b_1 \leqslant b$. Since $T(b_1, s)x \in D_A(\theta+1)$, the definition makes sense. Also $P_u(s)x$ does not depend on b_1.

Let $0<\theta<1$, $\gamma \in \mathbb{R}$ and $I=[a, b]$, possibly $a=-\infty$ and/or $b=+\infty$. Define $C_{\gamma,\theta}(I)$ to be the Banach space of continuous functions in $D_A(\theta)$ with the norm

$$\|f\|_{\gamma,\theta} = \sup_{t \in I} \|e^{-\gamma t} f(t)\|_\theta$$

being finite.

THEOREM 3.2. *Assume that $T(t, s)$ $(t, s \in I)$ has a pseudo exponential dichotomy with constants $K_1, K_2, C_0>0$ and exponents $\alpha>\beta$. Assume that $f \in C_{\gamma,\theta}(I)$.*

(i) *If* $\beta < \gamma < \alpha$, *then*

$$F(t) = \int_a^t P_s(t)\, T(t, s)\, f(s)\, ds + \int_b^t T(t, s)\, P_u(s) f(s)\, ds, \qquad t \in I$$

is a continuous function with values in $D_A(\theta + 1)$. *Moreover*,

$$\sup_{t \in I} \|F(t)\|_{\theta+1} \leqslant C_1\, e^{\gamma t}\, \|f(t)\|_{\gamma,\theta},$$

where $C_1(\gamma) = C_0 K_1\, e^{|\gamma| h}(1 - e^{(\beta-\gamma)h})^{-1} + C_0 K_2\, e^{(|\alpha| + |\gamma|)h}(1 - e^{(\gamma-\alpha)h})^{-1}$.

(ii) *If* $\gamma < \beta$ *and* $-\infty < a$, *then* $F(t)$ *is continuous in* $D_A(\theta+1)$. *Moreover*

$$\sup_{t \in I} \|F(t)\|_{\theta+1} \leqslant C\, e^{\beta(t-a)}\, e^{\gamma a}\, \|f\|_{\gamma,\theta},$$

where $C = C_0 K_1\, e^{|\gamma| h}(1 - e^{(\gamma-\beta)h})^{-1} + C_0 K_2\, e^{(|\alpha| + |\gamma|)h}(1 - e^{(\gamma-\alpha)h})^{-1}$.

(iii) *If* $\gamma > \alpha$ *and* $b < +\infty$, *then* $F(t)$ *is continuous in* $D_A(\theta+1)$. *Moreover*

$$\sup_{t \in I} \|F(t)\|_{\theta+1} \leqslant C\, e^{\alpha(t-b)} e^{\gamma b}\, \|f\|_{\gamma,\theta},$$

where $C = C_0 K_1\, e^{|\gamma| h}(1 - e^{(\beta-\gamma)h})^{-1} + C_0 K_2\, e^{(|\alpha| + |\gamma|)h}(1 - e^{(\alpha-\gamma)h})^{-1}$.

Proof. Let $t - (n+1)h \leqslant a < t - nh$. Let $n = \infty$ if $a = -\infty$. Then

$$\begin{aligned}
&\left\| \int_a^t P_s(t)\, T(t, s)\, f(s)\, ds \right\|_{\theta+1} \\
&\quad \leqslant \sum_{i=0}^{n} \|T(t, t - ih)\, P_s(t - ih)\|_{L(D_A(\theta+1))} \\
&\qquad \times \left\| \int_{\max\{(t-ih-h,a)\}}^{t-ih} e^{\gamma s}\, T(t - ih, s)(e^{-\gamma s} f(s))\, ds \right\|_{\theta+1} \\
&\quad \leqslant \sum_{i=0}^{n} K_1\, e^{\beta i h}\, e^{\gamma(t-ih) + |\gamma| h} C_0\, \|f\|_{\gamma,\theta}.
\end{aligned}$$

Thus, if $\beta < \gamma$, then

$$\begin{aligned}
\left\| \int_a^t P_s(t)\, T(t, s)\, f(s)\, ds \right\|_{\theta+1} &\leqslant C_0 K_1\, e^{|\gamma| h}\, e^{\gamma t} \sum_{i=0}^{\infty} e^{(\beta-\gamma)ih}\, \|f\|_{\gamma,\theta} \\
&\leqslant C_0 K_1\, e^{|\gamma| h}(1 - e^{(\beta-\gamma)h})^{-1}\, e^{\gamma t}\, \|f\|_{\gamma,\theta}. \qquad (3.1)
\end{aligned}$$

If $\gamma < \beta$ and $-\infty < a$, then

$$\left\| \int_a^t P_s(t)\, T(t,s) f(s)\, ds \right\|_{\theta+1}$$

$$\leqslant C_0 K_1 e^{|\gamma| h} e^{\beta t} \sum_{i=0}^{n} e^{(\gamma-\beta)(t-ih)} \|f\|_{\gamma,\theta}$$

$$\leqslant C_0 K_1 e^{|\gamma| h} e^{(\gamma-\beta)a} (1-e^{(\gamma-\beta)h})^{-1} e^{\beta t} \|f\|_{\gamma,\theta}. \tag{3.2}$$

On the other hand, let $t + mh < b \leqslant t + (m+1)h$. Let $m = \infty$ if $b = +\infty$. Then

$$\left\| \int_t^b T(t,s)\, P_u(s) f(s)\, ds \right\|_{\theta+1}$$

$$\leqslant \sum_{i=0}^{m} \Big\| T(t, \min\{t+ih+h, b\})\, P_u(\min\{t+ih+h, b\})$$

$$\times \int_{t+ih}^{\min\{t+ih+h,b\}} e^{\gamma s}\, T(\min\{t+ih+h, b\}, s)(e^{-\gamma s} f(s))\, ds \Big\|_{\theta+1}$$

$$\leqslant \sum_{i=0}^{m} K_2 e^{-\alpha ih} e^{\gamma(t+ih)} \cdot e^{(|\alpha|+|\gamma|)h} \cdot C_0 \|f\|_{\gamma,\theta}.$$

Thus, if $\gamma < \alpha$, then

$$\left\| \int_t^b T(t,s)\, P_u(s) f(s)\, ds \right\|_{\theta+1} \leqslant C_0 K_2 e^{(|\gamma|+|\alpha|)h} (1-e^{(\gamma-\alpha)h})^{-1} e^{\gamma t} \|f\|_{\gamma,\theta}. \tag{3.3}$$

If $\gamma > \alpha$ and $b < +\infty$, then

$$\left\| \int_t^b T(t,s)\, P_u(s) f(s)\, ds \right\|_{\theta+1}$$

$$\leqslant C_0 K_2 e^{(|\gamma|+|\alpha|)h} e^{\alpha t} \sum_{i=0}^{m} e^{(\gamma-\alpha)(t+ih)}$$

$$\leqslant C_0 K_2 e^{(|\gamma|+|\alpha|)h} e^{(\gamma-\alpha)b} (1-e^{(\alpha-\gamma)h})^{-1} e^{\alpha t} \|f\|_{\gamma,\theta}. \tag{3.4}$$

Assertion (i) then follows from (3.1) and (3.3). Assertion (ii) follows from (3.2) and (3.3). Assertion (iii) follows from (3.1) and (3.4).

Assume that $T(t,s)$ has a pseudo exponential dichotomy on I with constants K_1, K_2, $C_0 > 0$ and exponents $\alpha > \beta$. For $\beta < \gamma < \alpha$ and $f \in C(I; D_A(\theta))$, define

$$\|f\|_{\gamma,\theta,\tau_0} = \sup_{t \in I} \|e^{-\gamma(t-\tau_0)} f(t)\|_\theta, \qquad \tau_0 \in \mathbb{R}.$$

The functions f with $\|f\|_{\gamma,\theta,\tau_0} < +\infty$ form a Banach space, denoted by $C_{\gamma,\theta,\tau_0}(I)$.

THEOREM 3.3. (a) *Let* $I=[a,b]$ *be a finite interval. For* $\phi_a \in P_s(a) D_A(\theta+1)$ *and* $\phi_b \in P_u(b) D_A(\theta+1)$ *and* $f \in C(I; D_A(\theta))$, *there exists a unique* $u \in C(I; D_A(\theta+1)) \cap C^1(I; D_A(\theta))$ *such that*

$$u'(t) = A(t)\,u(t) + f(t),$$

$$P_s(a)\,u(a) = \phi_a,$$

$$P_u(b)\,u(b) = \phi_b.$$

We can write

$$u(t) = T(t,a)\,\phi_a + T(t,b)\,\phi_b + \int_a^t P_s(t)\,T(t,s)\,f(s)\,ds + \int_b^t T(t,s)\,P_u(s)\,f(s)\,ds. \tag{3.5}$$

Moreover

$$\|u(t)\|_{\theta+1} + \|u'(t)\|_\theta \leqslant K_1 e^{\beta(t-a)} \|\phi_a\|_{\theta+1} + K_2 e^{\alpha(t-b)} \|\phi_b\|_{\theta+1} + C_1 e^{\gamma(t-\tau_0)} \|f\|_{\gamma,\theta,\tau_0}, \tag{3.6}$$

where $C_1(\gamma)$ *is given in Theorem* 3.2 (i).

(b) *Let* $I=[a,+\infty)$. *For* $\phi_a \in P_s(a) D_A(\theta+1)$ *and* $f \in C_{\gamma,\theta;a}(I)$, *there exists a unique* $u \in C(I; D_A(\theta+1)) \cap C^1(I; D_A(\theta)) \cap C_{\gamma,\theta,a}(I)$ *such that*

$$u'(t) = A(t)\,u(t) + f(t),$$

$$P_s(a)\,u(a) = \phi_a.$$

We can write

$$u(t) = T(t,a)\phi_a + \int_a^t P_s(t)\,T(t,s)\,f(s)\,ds + \int_\infty^t T(t,s)\,P_u(s)\,f(s)\,ds.$$

Moreover

$$\|u(t)\|_{\theta+1} + \|u'(t)\|_\theta \leqslant Ce^{\gamma(t-a)}\{\|\phi_a\|_{\theta+1} + \|f\|_{\gamma,\theta,a}\}.$$

(c) *Let* $I=[-\infty, b)$. *For* $\phi_b \in P_u(b) D_A(\theta+1)$ *and* $f \in C_{\gamma,\theta,b}(I)$, *there exists a unique* $u \in C(I; D_A(\theta+1)) \cap C^1(I; D_A(\theta)) \cap C_{\gamma,\theta,b}(I)$ *such that*

$$u'(t)=A(t)\,u(t)+f(t),$$
$$P_u(b)\,u(b)=\phi_b.$$

We can write

$$u(t)=T(t,b)\phi_a+\int_{-\infty}^{t} P_s(t)\,T(t,s)\,f(s)\,ds+\int_{b}^{t} T(t,s)\,P_u(s)\,f(s)\,ds.$$

Moreover

$$\|u(t)\|_{\theta+1}+\|u'(t)\|_{\theta}\leqslant C_1\,e^{\gamma(t-b)}\{\|\phi_b\|_{\theta+1}+\|f\|_{\gamma,\theta,b}\}.$$

(d) *Let* $I=\mathbb{R}$. *For any* $f \in C_{\gamma,\theta,\tau_0}(I)$, *there exists a unique* $u \in C(I; D_A(\theta+1)) \cap C^1(I; D_A(\theta)) \cap C_{\gamma,\theta,\tau_0}(I)$ *such that*

$$u'(t)=A(t)\,u(t)+f(t).$$

We can write

$$u(t)=\int_{-\infty}^{t} P_s(t)\,T(t,s)\,f(s)\,ds+\int_{\infty}^{t} T(t,s)\,P_u(s)\,f(s)\,ds.$$

Moreover

$$\|u(t)\|_{\theta+1}+\|u'(t)\|_{\theta}\leqslant C_1\,e^{\gamma(t-\tau_0)}\,\|f\|_{\gamma,\theta,\tau_0}.$$

Proof. We first prove case (a). Solutions of Eq. (2.4) can be written as

$$u(t)=T(t,a)\,u(a)+\int_{a}^{t} T(t,s)\,f(s)\,ds.$$

Thus

$$P_s(t)\,u(t)=P_s(t)\,T(t,a)\,u(a)+\int_{a}^{t} P_s(t)\,T(t,s)\,f(s)\,ds$$
$$=T(t,a)\,\phi_a+\int_{a}^{t} P_s(t)\,T(t,s)\,f(s)\,ds.$$

Since

$$u(b)=T(b,t)\,u(t)+\int_{t}^{b} T(b,s)\,f(s)\,ds,$$

$$\phi_b=P_u(b)\,u(b)=T(b,t)\,P_u(t)\,u(t)+\int_{t}^{b} P_u(b)\,T(b,s)\,f(s)\,ds,$$

we have

$$P_u(t)\,u(t) = T(t, b)\,\phi_b - \int_t^b T(t, b)\,P_u(b)\,T(b, s)\,f(s)\,ds$$

$$= T(t, b)\,\phi_b + \int_b^t T(t, s)\,P_u(s)\,f(s)\,ds.$$

Since $u(t) = (P_u(t) + P_s(t))\,u(t)$, we have (3.5). Estimate (3.6) follows from Theorem 3.2.

Cases (b), (c), and (d) can be proved similarly by letting $a \to -\infty$ and/or $b \to +\infty$. ∎

We now derive decay estimates of $T(t, s)$ on $P_u(s)\,D_A(\theta)$ and $P_s(s)\,D_A(\theta)$. We make the follwing hypothesis:

(H) There exist positive constants h_1, C_1 such that

$$\|T(t, s)x\|_\theta \leqslant C_1\,\|x\|_\theta \qquad \text{for all} \quad t - s \leqslant h_1 \text{ and } [s, t] \subset I.$$

Remark. Hypothesis (H) and the condition (v) in Definition 3.1 are satisfied if $I = [a, b]$ is a compact interval. But they are needed if I is not compact. According to Buttu (2), the constant C in (2.5) depends on θ, h_0, h_1, $\sup_{t \geqslant 0}\{t^k\,\|A^k e^{tA}\|_{L(X)}\}$ and $\sup_{t \geqslant 0}\{t^{k-\theta}\,\|A^k e^{tA}\|_{L(D_A(\theta), X)}\}$ with $k = 1, 2$. Therefore, it can be shown that if $A(t)$ is periodic in t or if $A(t) \to A_\infty$ as $t \to \pm\infty$ in appropriate function spaces, then (H) and (v) in Definition 3.1 are satisfied.

Recall that for $x \in D_A(\theta)$ and $s < b$, ($I = [a, b]$, b can be $+\infty$), $P_u(s)x = T(s, b_1)\,P_u(b_1)\,T(b_1, s)x$, for $s < b_1 \leqslant b$. Let $h_0 = \min\{h, h_1\}$, where h is the constant in Definition 3.1 (v). For $b - s \geqslant h_0$, we have

$$P_u(s)\,x = T(s, s + h_0)\,P_u(s + h_0)\,T(s + h_0, s)x.$$

By Definition 3.1 (v),

$$\begin{aligned}\|T(s + h_0, s)x\|_{\theta+1} &\leqslant h_0^{-1} \left\| \int_0^{h_0} T(s + h_0, s + \tau)\,T(s + \tau, s)x\,d\tau \right\|_{\theta+1} \\ &\leqslant C_0 h_0^{-1} \sup_{0 \leqslant \tau \leqslant h_0} \|T(s + \tau, s)x\|_\theta \\ &\leqslant C_0 C_1 h_0^{-1}\,\|x\|_\theta.\end{aligned}$$

This inequality should be compared with (2.11), where the compactness of $[a, b]$ is assumed. Using Definition 3.1 (iv), we have $\|P_u(s)x\|_{\theta+1} \leqslant C\,\|x\|_\theta$ if $b - s \geqslant h_0$.

For $b-s<h_0$, we have

$$P_u(s)x = T(s, b)\, P_u(b)\, T(b, s)x.$$

Similarly we can prove that $\|P_u(s)x\|_{\theta+1} \leqslant C(b-s)^{-1}\, \|x\|_\theta$ if $b-s<h_0$. In general, we have

$$\|P_u(s)x\|_{\theta+1} \leqslant C(1+(b-s)^{-1})\, \|x\|_\theta. \tag{3.7}$$

Let $P_s(s)x = x - P_u(s)\,x$ for $x \in D_A(\theta)$. It is easy to verify that

$$P_u^2(s) = P_u(s) \qquad \text{and} \qquad P_s^2(s) = P_s(s) \qquad \text{in } D_A(\theta).$$

Moreover, we have the following

THEOREM 3.4. *Assume that $T(t, s)$ has an exponential dichotomy on $[a, b]$, and* (H) *is satisfied. Then $P_u(t)$ and $P_s(t)$ can be defined for $t<b$, $x \in D_A(\theta)$, by continuation. Moreover, we have*

$$P_v(t)\, T(t, s)x = T(t, s)\, P_v(s)x, \qquad x \in D_A(\theta),\ v = u, s. \tag{3.8}$$

$$\|T(t, s)\, P_s(s)x\|_\theta \leqslant \bar{K}_1\, e^{\beta(t-s)}\, \|x\|_\theta, \qquad s<t \leqslant b, \tag{3.9}$$

$$\|T(s, t)\, P_u(t)\|_{\theta+1} \leqslant \bar{K}_2\, e^{-\alpha(t-s)}\, \|x\|_\theta, \qquad s \leqslant t<b, \tag{3.10}$$

where $\bar{K}_1 = C(1+(b-s)^{-1})$ and $\bar{K}_2 = C(1+(b-t)^{-1})$.

Proof. For any $x \in D_A(\theta)$, (3.8) can be proved by letting $y \in D_A(\theta+1)$ and $y \to x$ in $D_A(\theta)$. We can also obtain (3.10) from (3.7) and Definition 3.1.

To prove (3.9), observe that from (3.7), $\|P_s(s)x\|_\theta \leqslant C(1+(b-s)^{-1})\, \|x\|_\theta$. Thus, if $t-s \leqslant h_0$, (3.9) follows from (H). Let $t-s>h_0$; then

$$\begin{aligned}\|T(t, s)\, P_s(s)x\|_{\theta+1} &= \|T(t, s+h_0)\, P_s(s+h_0)\, T(s+h_0, s)x\|_{\theta+1}\\ &\leqslant K\, e^{\beta(t-s-h_0)}\, \|T(s+h_0, s)x\|_{\theta+1}\\ &\leqslant CKh_0^{-1}\, e^{\beta(t-s)}\, \|x\|_\theta.\end{aligned}$$

This proves (3.9). ∎

Let $t \to B(t) \in C(I; L(D_A, X)) \cap C(I; L(D_A(\theta+1), D_A(\theta))$. Assume that

$$\sup_{t \in I} \{\|B(t)\|_{L(D_A, X)} + \|B(t)\|_{L(D_A(\theta+1), D_A(\theta))}\} \leqslant \delta.$$

When δ is sufficiently small, we can show that

$$u'(t) = (A(t)+B(t))\, u(t) \tag{3.11}$$

also has an evolution operator $T_B(t, s)$ defined on $D_A(\theta)$ and $D_A(\theta+1)$. Let $T(t, s)$ have a pseudo exponential dichotomy on I with projections $P_s(t)$, $P_u(t)$, constants K_1, K_2, C_0, and exponents $\beta < \alpha$.

THEOREM 3.5 (Roughness of Exponential Dichotomies). *Let $\tilde{\beta}$ and $\tilde{\alpha}$ be two constants with $\beta < \tilde{\beta} < \tilde{\alpha} < \alpha$ and let $C_1(\tilde{\beta})$ and $C_1(\tilde{\alpha})$ be the constants as in Theorem* 3.2 (i) *with γ replaced by $\tilde{\beta}$ and $\tilde{\alpha}$ respectively. Let $\tilde{C}_1 = \max\{C_1(\tilde{\beta}), C_1(\tilde{\alpha})\}$ and $C_2 = \tilde{C}_1(K_1+K_2)/(1-\tilde{C}_1\delta)$. If $C_0\delta < 1$, $C_1(\tilde{\beta})\cdot\delta < 1$, $C_1(\tilde{\alpha})\cdot\delta < 1$, and $C_2\delta < 1$ then* (3.11) *also has a pseudo exponential dichotomy on I with the projections denoted by $\tilde{P}_s(t)$ and $\tilde{P}_u(t)$. Moreover*

$$\|T_B(t, s)\,\tilde{P}_s(s)\|_{\theta+1} \leqslant \tilde{K}_1\, e^{\tilde{\beta}(t-s)}, \qquad s \leqslant t, \tag{3.12}$$

$$\|T_B(s, t)\,\tilde{P}_u(t)\|_{\theta+1} \leqslant \tilde{K}_2\, e^{-\tilde{\alpha}(t-s)}, \qquad s \leqslant t, \tag{3.13}$$

$$\left\|\int_\tau^t T_B(t, s)\, f(s)\, ds\right\|_{\theta+1} \leqslant \tilde{C}_0\,\|f\|_\theta, \qquad |t-\tau| \leqslant h, \tag{3.14}$$

and

$$\|\tilde{P}_s(t) - P_s(t)\|_{\theta+1} \leqslant \frac{C_2\,\delta}{1-C_2\,\delta}. \tag{3.15}$$

Here $\tilde{K}_1 = K_1(1-\tilde{C}_1\delta)^{-1}(1-C_2\delta)^{-1}$, $\tilde{K}_2 = K_2(1-\tilde{C}_1\delta)^{-1}(1-C_2\delta)^{-1}$ and $\tilde{C}_0 = C_0(1-C_0\delta)^{-1}$. We use $\|\cdot\|_{\theta+1}$ to denote norms of operators in $L(D_A(\theta+1))$.

Proof. We only prove the theorem when $I = [a, b]$ and $-\infty < a < b < +\infty$. The other cases can be proved similarly. Let $\phi \in D_A(\theta+1)$. From Theorem 3.3, the following mapping $\mathscr{T}(s)$ maps $C_{\tilde{\beta}, \theta+1, s}[s, b]$ into itself:

$$(\mathscr{T}(s)\, y)(t) = T(t, s)\, P_s(s)\phi + \int_s^t P_s(t)\, T(t, \xi)\, B(\xi)\, y(\xi)\, d\xi$$

$$- \int_t^b T(t, \xi)\, P_u(\xi)\, B(\xi)\, y(\xi)\, d\xi.$$

$$\|(\mathscr{T}(s)\, y)(t)\|_{\theta+1} \leqslant K_1\, e^{\beta(t-s)}\, \|\phi\|_{\theta+1} + C_1(\tilde{\beta})\, e^{\tilde{\beta}(t-s)}\, \delta\, \|y\|_{\tilde{\beta}, \theta+1, s},$$

where $C_1(\tilde{\beta})$ is given by Theorem 3.2 (i).

Since $C_1(\tilde{\beta})\delta < 1$, then $\mathscr{T}(s)$ is a contraction in $C_{\tilde{\beta}, \theta+1, s}[s, b]$. Let the unique fixed point be $y(t; s, \phi)$, $s \leqslant t \leqslant b$. $y(t; s, \phi)$ is a solution for (3.11) with

$$\|y(t; s, \phi)\|_{\tilde{\beta}, \theta+1, s} \leqslant \frac{K_1}{1-C_1(\tilde{\beta})\delta}\, \|\phi\|_{\theta+1}. \tag{3.16}$$

Conversely, any solution $y(t)$, $s \leqslant t \leqslant b$, with $P_s(s)\, y_s = P_s(s)\phi$ and $P_u(b)\, y(b) = 0$ is a fixed point for $\mathscr{T}(s)$. The assertion is also true if $b = +\infty$ and $y \in C_{\bar{\beta}, \theta+1, s}[s, +\infty)$.

Similarly, any solution $z(t)$ of (3.11) for $a \leqslant t \leqslant s$ with $P_u(s)\, z(s) = P_u(s)\phi$ and $P_s(a)\, z(a) = 0$ is a fixed point for the equation

$$z(t) = T(t, s)\, P_u(s)\phi - \int_t^s T(t, \xi)\, P_u(\xi)\, B(\xi)\, z(\xi)\, d\xi$$

$$+ \int_a^t P_s(t)\, T(t, \xi)\, B(\xi)\, z(\xi)\, d\xi \tag{3.17}$$

Since $C_1(\tilde{\alpha})\delta < 1$, Eq. (3.17) has a unique solution in $C_{\tilde{\alpha}, \theta+1, \delta}[a, s]$, denoted by $z(t; s, \phi)$. We have

$$\|z(t; s, \phi)\|_{C_{\tilde{\alpha}, \theta+1, s}[a, s]} \leqslant \frac{K_2}{1 - C_1(\tilde{\alpha})\, \delta}\, \|\phi\|_{\theta+1}. \tag{3.18}$$

Let $W^s(s) = \{y(s; s, \phi) \mid \phi \in D_A(\theta+1)\}$ and $W^u(s) = \{z(s; s, \phi) \mid \phi \in D_A(\theta+1)\}$. Then one can show that

$$W^s(s) = \{u \in D_A(\theta+1) \mid T_B(b, s)u \in P_s(b)\, D_A(\theta+1)\}, \tag{3.19}$$

$$W^u(s) = \{T_B(s, a)u \mid u \in P_u(a)\, D_A(\theta+1)\}, \tag{3.20}$$

if $a > -\infty$ and $b < +\infty$. (If $a = -\infty$ then $W^u(s) = \{u \in D_A(\theta+1) \mid$ a backward solution $T_B(t, s)u$, $t \leqslant s$ is defined with $\|T_B(t, s)u\|_{C_{\tilde{\alpha}, \theta+1, s}(-\infty, s]} < +\infty\}$. If $b = +\infty$, then $W^s(s) = \{u \in D_A(\theta+1)$ with $\|T_B(t, s)u\|_{C_{\bar{\beta}, \theta+1, s}[s, +\infty)} < +\infty\}$.) It is also obvious that $\phi \to y(s; s, \phi)$ and $\phi \to z(s; s, \phi)$ are homeomorphisms from $P_s(s)\, D_A(\theta+1)$ to $W^s(s)$ and from $P_u(s)\, D_A(\theta+1)$ to $W^u(s)$, respectively. Here $W^s(s)$ and $W^u(s)$ are equipped with the norms induced from $D_A(\theta+1)$.

We now show that

$$W^s(s) + W^u(s) = D_A(\theta+1).$$

For $\phi \in D_A(\theta+1)$, consider

$$\begin{aligned} x &= y(s; s, \phi) + z(s; s, \phi) \\ &= \phi + \int_a^s P_s(s)\, T(s, \xi)\, B(\xi)\, z(\xi; s, \phi)\, d\xi \\ &\quad - \int_s^b T(s, \xi)\, P_u(\xi)\, B(\xi)\, y(\xi; s, \phi)\, d\xi. \end{aligned}$$

We can show that

$$\|x-\phi\|_{D_A(\theta+1)} \leqslant C_2\,\delta\,\|\phi\|_{\theta+1},$$

where C_2 has been defined before. Since $C_2\,\delta<1$, $y(s;s,\phi)+z(s;s,\phi)$: $\phi\to x$ is a homeomorphism in $D_A(\theta+1)$ and we denote the inverse by $\phi=\Phi(s,x)$. We have

$$\|\Phi(s,x)\|_{\theta+1} \leqslant (1-C_2\,\delta)^{-1}\,\|x\|_{\theta+1}. \tag{3.21}$$

We now show that $W^s(s)\cap W^u(s)=\{0\}$. Let $\phi_1, \phi_2 \in D_A(\theta+1)$ with

$$y(s;s,\phi_1)=z(s;s,\phi_2).$$

Let $\phi=P_s(s)\,\phi_1-P_u(s)\,\phi_2$. It can be verified that

$$x=0=y(s;s,\phi)+z(s;s,\phi).$$

We have $\phi=\Phi(s,0)=0$, implying that $P_s(s)\,\phi_1=0$ and $P_u(s)\,\phi_2=0$. Therefore $W^s(s)\cap W^u(s)=\{0\}$.

Define

$$\tilde{P}_s(s)x=y(s;s,\Phi(s,x)),$$
$$\tilde{P}_u(s)x=z(s;s,\Phi(s,x)).$$

We have shown that $\tilde{P}_s(s)$ and $\tilde{P}_u(s)$ are projections associated to the splitting $W^s(s)\oplus W^u(s)=D_A(\theta+1)$. The equation

$$T_B(t,s)\,\tilde{P}_s(s)=\tilde{P}_s(t)\,T_B(t,s)$$

can be proved by the invariance of $W^s(s)$ and $W^u(s)$; see (3.19) and (3.20). Details are omitted. To show (3.15), observe that

$$\begin{aligned}
&\|(\tilde{P}_s(s)-P_s(s))x\|_{\theta+1}\\
&\quad\leqslant \|y(s;s,\Phi(s,x))-P_s(s)\,\Phi(s,x)\|_{\theta+1}\\
&\qquad+\|P_s(s)\,\Phi(s,x)-P_s(s)x\|_{\theta+1}\\
&\quad\leqslant \left\|\int_s^b T(s,\xi)\,P_u(\xi)\,B(\xi)\,y(\xi;s,\Phi(s,x))\,d\xi\right\|_{\theta+1}\\
&\qquad+\left\|\int_a^s P_s(s)\,T(s,\xi)\,B(\xi)\,z(\xi;s,\Phi(s,x))\,d\xi\right\|_{\theta+1}\\
&\quad\leqslant \tilde{C}_1\,\delta(\|y\|_{\tilde{\beta},\theta+1,s}+\|z\|_{\tilde{\alpha},\theta+1,s}).
\end{aligned}$$

From (3.16), (3.18), and (3.21), we have (3.15).

From (3.16) and (3.21), we have for $t \geqslant s$,

$$\|(T_B(t,s)\,\tilde{P}_s(s)x\|_{\tilde{\beta},\theta+1,s} = \|y(t;s,\Phi(s,x))\|_{\tilde{\beta},\theta+1,s} \leqslant K_1(1-\tilde{C}_1\delta)^{-1}(1-C_2\delta)^{-1}\|x\|_{\theta+1}.$$

And from (3.18) and (3.21) we have, for $t \leqslant s$,

$$\|(T_B(t,s)\,\tilde{P}_u(s)x\|_{\tilde{\alpha},\theta+1,s} = \|z(t;s,\Phi(s,x))\|_{\tilde{\alpha},\theta+1,s} \leqslant K_2(1-\tilde{C}_1\delta)^{-1}(1-C_2\delta)^{-1}\|x\|_{\theta+1}.$$

Thus (3.12) and (3.13) are proved. The proof of (3.14) is easy and is not given here.

4. Stability of Periodic Solutions in Conway and Smoller's Example

The Global behavior of the following predator–prey equations was analyzed by Conway and Smoller.

$$\begin{aligned} u'(t) &= u(f(u)-v), \\ v'(t) &= v(u-\gamma). \end{aligned} \tag{4.1}$$

Assume that

(A) $f(b)=f(k)=0$, where $0<b<k$; $f(u)>0$ for $b<u<k$, and $f(u)<0$ otherwise;

(B) $f'>0$ on $[b,m)$, $f'<0$ on $(m,k]$, $f''<0$, and $f'''(m)$ exists;

(C) $b<\gamma<k$.

Under the above assumptions, the positive quadrant ($u>0$, $v>0$) is invariant for (4.1). There are four equilibria: $O=(0,0)$, $B=(b,0)$, $K=(k,0)$, and $P=(\gamma, f(\gamma))$. It is easy to check that B and K are hyperbolic saddle points. Conway and Smoller showed that there is a unique γ_0, $b<\gamma_0<k$, such that when $\gamma=\gamma_0$ the unstable manifold $W^u(K)$ of K meets the stable manifold $W^s(B)$ of B to form a heteroclinic solution $\vec{q}^1(t)$ in the first quadrant.There is also a heteroclinic solution $\vec{q}_2(t)$ from B to K which lies on the u-axis for all $\gamma \in \mathbb{R}$. Let the two components of $\vec{q}_i(t)$ be $q_{iu}(t)$ and $q_{iv}(t)$, $i=1,2$. It was shown that $q'_{1u}(t)<0$ and $q'_{2u}(t)>0$, $t \in \mathbb{R}$. Conway and Smoller also proved that $W^u(K)$ and $W^s(B)$ must meet $\Sigma=\{(u,v), u=\gamma\}$, $b<\gamma<k$. Let $Y_1=(\gamma, y_1(\gamma))$ (or $Y_2=(\gamma, y_2(\gamma))$) be the first point where $W^u(K)$ (or $W^s(B)$) meet Σ. Then $G(\gamma)=y_1(\gamma)-y_2(\gamma)$ measures the distance between $W^u(K)$ and $W^s(B)$. Clearly $G(\gamma_0)=0$.

LEMMA 4.1. $\partial G(\gamma_0)/\partial\gamma < 0$.

Proof. Let $U = (u, v)^\tau$ and rewrite (4.1) as

$$U'(t) = F(U, \gamma). \tag{4.2}$$

The only bounded solution for the linear equation

$$U'(t) = D_u F(\vec{q}_1(t), \gamma_0)\, U(t),$$

is $\vec{q}_1'(t)$ up to constant multiples. The adjoint equation

$$\Psi'(t) = (D_u F(\vec{q}_1(t), \gamma_0))^* \Psi(t)$$

has a unique bounded solution $\Psi^*(t) = (\psi_1(t), \psi_2(t))$, $t \in \mathbb{R}$, up to constant multiples. Moreover, $\Psi^*(t) \to 0$ exponentially as $|t| \to \infty$. See Palmer (13). Let $\vec{q}_1(0) \in \Sigma$. Since $\Psi^*(0) \perp \vec{q}_1'(0)$, without loss of generality we can assume that $\psi_2(0) = 1$. Therefore $G(\gamma) = \langle \Psi^*(0), (Y_1 - Y_2)\rangle$. One can prove (see Lin (9), for example) that

$$\begin{aligned}\frac{\partial G(\gamma_0)}{\partial\gamma} &= \int_{-\infty}^{\infty} \langle \Psi^*(t), D_\gamma F(\vec{q}_1(t), \gamma_0)\rangle\, dt \\ &= -\int_{-\infty}^{\infty} \psi_2(t)\, q_{1v}(t))\, dt.\end{aligned}$$

Observe that $\psi_1(t)\, q_{1u}'(t) + \psi_2(t)\, q_{1v}'(t) = 0$, $t \in \mathbb{R}$. Thus $\psi_2(0) > 0$ and $q_{1u}'(t) < 0$, $t \in \mathbb{R}$, imply that $\psi_2(t) > 0$ for all $t \in \mathbb{R}$. The desired conclusion of Lemma 4.1 follows from the fact that $q_{1v}(t) > 0$, $t \in \mathbb{R}$.

Let the eigenvalues of the linearized ODE be $-\lambda_B^- < 0 < \lambda_B^+$ and $-\lambda_K^- < 0 < \lambda_K^+$ at the equilibria B and K. Though it is well known that the stability of the heteroclinic cycle $\vec{q}_1(t)$ and $\vec{q}_2(t)$ from inside is determined by the quantity

$$\lambda_B^+ \lambda_K^+ - \lambda_B^- \lambda_K^-,$$

we still give some detail here since it is used in the sequel.

Let $\gamma = \gamma_0$. Let Σ_0, Σ_1, Σ_2 and Σ_3 be cross sections that are transverse to $W^s_{\text{loc}}(B)$, $W^u_{\text{loc}}(B)$, $W^s_{\text{loc}}(K)$, and $W^u_{\text{loc}}(K)$, respectively. Assume that each Σ_i is in a small neighborhood of the corresponding equilibrium. Let $Y(t) = (u(t), v(t))$ be a solution of (4.1) that is inside the heteroclinic cycle. See Figure 4.1. Let $Y(0) \in \Sigma_0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, $\Sigma_0 = \Sigma_4$, and each τ_i be

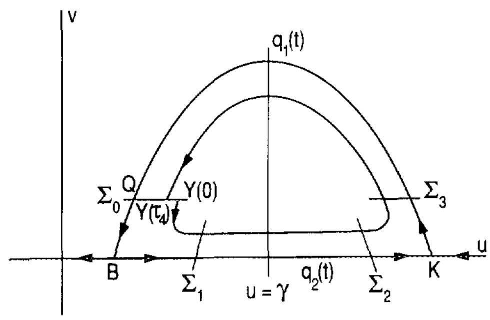

FIGURE 4.1

the first time that $Y(\tau_i) \in \Sigma_i$. Let $\omega_B = \tau_1$ and $\omega_K = \tau_3 - \tau_2$. We can show that there exists a constant $c > 0$ such that

$$c^{-1} e^{-\lambda_B^- \omega_B} \leqslant e^{-\lambda_K^+ \omega_K} \leqslant c e^{-\lambda_B^- \omega_B}.$$

Or equivalently, there exists $C > 0$, independent of ω_B, such that

$$\frac{\lambda_B^-}{\lambda_K^+} - \frac{c}{\omega_B} \leqslant \frac{\omega_K}{\omega_B} \leqslant \frac{\lambda_B^-}{\lambda_K^+} + \frac{c}{\omega_B}. \tag{4.3}$$

The proof is trivial if one observes that the ODE near B and K can be C^1 linearized. Using C^1 linearization again we can show that

$$\begin{aligned} c_1^{-1} e^{-\lambda_B^+ \omega_B} &\leqslant \operatorname{dist}(Y(0),\ W^s_{\text{loc}}(B)) \leqslant c_1 e^{-\lambda_B^+ \omega_B}, \\ c_2^{-1} e^{-\lambda_K^- \omega_K} &\leqslant \operatorname{dist}(Y(\tau_3),\ W^u_{\text{loc}}(K)) \leqslant c_2 e^{-\lambda_K^- \omega_K}. \end{aligned} \tag{4.4}$$

Here $\operatorname{dist}(\cdot, \cdot)$ is the distance along Σ_0 or Σ_3. The constants C_1 and C_2 depend on Σ_i, $0 \leqslant i \leqslant 3$. We then have

$$\begin{aligned} \frac{\operatorname{dist}(Y(\tau_3),\ W^u_{\text{loc}}(K))}{\operatorname{dist}(Y(0),\ W^s_{\text{loc}}(B))} &\leqslant C e^{(\lambda_B^+ \omega_B - \lambda_K^- \omega_K)} \\ &\leqslant C e^{(\lambda_K^+ \lambda_B^+ - \lambda_K^- \lambda_B^-) \omega_B / \lambda_K^+}, \end{aligned} \tag{4.5}$$

due to (4.3) and (4.4).

In the sequel we assume that

(D) $\lambda_K^+ \lambda_B^+ - \lambda_K^- \lambda_B^- < 0.$

Condition (D) is not an empty assumption. It is satisfied by at least an example studied in Conway and Smoller (4).

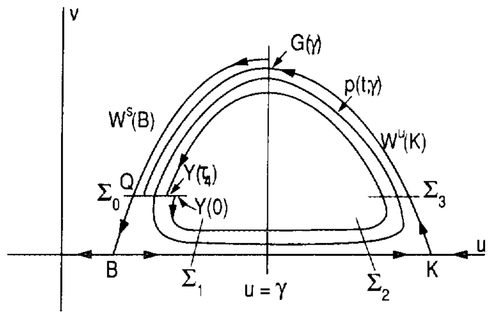

FIGURE 4.2

From (4.5), if (D) is satisfied, then

$$\frac{\operatorname{dist}(Y(\tau_4), W^s_{\mathrm{loc}}(B))}{\operatorname{dist}(Y(0), W^s_{\mathrm{loc}}(B))} \to 0$$

as $\omega_B \to \infty$. From (4.4) again, if $\operatorname{dist}(Y(0), W^s_{\mathrm{loc}}(B))$ is sufficiently small, then ω_B is sufficiently large and $Y(\tau_4)$ is closer to $W^s_{\mathrm{loc}}(B)$ than $Y(0)$. Let $Q = \Sigma_0 \cap W^s_{\mathrm{loc}}(B)$. A segment $\widehat{QY}(0)$ on Σ_0 is mapped to itself by the Poincaré map $\Pi(\gamma_0): \Sigma_0 \to \Sigma_0$ induced by the flow. Using the continuous dependence of the flow on γ and Lemma 4.1, we see that $\Pi(\gamma): \Sigma_0 \to \Sigma_0$ maps $\widehat{QY}(0)$ to itself for $\gamma_0 \leqslant \gamma \leqslant \gamma_0 + \varepsilon$, where $\varepsilon > 0$ is small. See Fig. 4.2.

We can use the C^1 linearization of the flow near K and B again to prove that $\Pi(\gamma)$, $\gamma_0 \leqslant \gamma \leqslant \gamma_0 + \varepsilon$, is a contradiction on $\widehat{QY}(0)$. Thus, there must exist a unique periodic solution $p(t, \gamma)$, $\gamma_0 < \gamma \leqslant \gamma_0 + \varepsilon$, $p(0, \gamma) \in \Sigma_0$, which is near the heteroclinic cycle and is asymptotically stable. We also observe that as $\gamma \to \gamma_0$, $p(0, \gamma) \to W^s_{\mathrm{loc}}(B)$ and $\omega_B \to \infty$. Therefore the rate of contraction of $\Pi(\gamma): \widehat{QY}(0) \to \widehat{QY}(0)$ approaches zero as $\varepsilon \to 0$.

Consider now the diffusively perturbed system

$$\begin{aligned} u_t(t, x) &= d_1 u_{xx}(t, x) + u(f(u) - v) + \Delta d_1 u_{xx}, \\ v_t(t, x) &= d_2 v_{xx}(t, x) + v(u - \gamma) + \Delta d_2 v_{xx}, \qquad 0 < x < 1 \end{aligned} \tag{4.6}$$

with the boundary condition

$$u_x(t, 0) = u_x(t, 1) = v_x(t, 0) = v_x(t, 1) = 0 \tag{4.7'}$$

or

$$\begin{aligned} u(t, 0) &= u(t, 1), \qquad u_x(t, 0) = u_x(t, 1), \\ v(t, 0) &= v(t, 1), \qquad v_x(t, 0) = v_x(t, 1). \end{aligned} \tag{4.7''}$$

Here assume f is a C^∞ function, $d_1, d_2 > 0$, and Δd_1 and Δd_2 are perturbations of d_1 and d_2. Let $U = (u, v)^\tau$, $A = \text{diag}\{d_1 \partial_{xx}, d_2 \partial_{xx}\}$, and $F(U, \gamma, \Delta d_1, \Delta d_2) = (\tilde{f}(u, v, \gamma) + \Delta d_1 u_{xx},\ \tilde{g}(u, v, \gamma) + \Delta d_2 v_{xx})^\tau$, where $\tilde{f}(u, v, \gamma) = u(f(u) - v)$ and $\tilde{g}(u, v, \gamma) = v(u - \gamma)$. We now rewrite (4.6) as an abstract equation

$$U'(t) = AU(t) + F(U, \gamma, \Delta d_1, \Delta d_2). \tag{4.8}$$

Let $X = [L^2(0, 1)]^2$. $D_A = \{U \in [H^2(0, 1)]^2 \mid \text{Boundary conditions in (4.7) are satisfied}\}$. $A : D_A \to X$. The closed linear operator A clearly generates an analytic semigroup e^{At} on X. Let $D_A(\theta)$ and $D_A(\theta + 1)$, $0 < \theta < 1$ be defined as in Section 2. We seek the strict solution $U(t)$ for (4.8), i.e., $U \in C^1([t_0, t_0 + \tau); D_A(\theta)) \cap C([t_0, t_0 + \tau); D_A(\theta + 1))$.

Since the spatial domain is one dimensional, $\tilde{f}$ and $\tilde{g} : H^m(0, 1) \times \mathbb{R}^3 \to H^m(0, 1)$ is a C^∞ mapping for $m \geqslant 1$. When Neumann boundary conditions are imposed, we observe that

$$D_{A^m} = \{U \in [H^{2m}(0, 1)]^2 \mid \partial_x^{2k-1} U = 0,\ 1 \leqslant k \leqslant m, \text{ at } x = 0, 1\}.$$

Thus it can be directly verified that $\tilde{f}(U, \gamma)$ and $\tilde{g}(U, \gamma) : D_{A^m} \times \mathbb{R} \to D_{A^m}$ are C^∞. Therefore from the nonlinear interpolation property

$$\tilde{f}(u, v, \gamma) \text{ and } \tilde{g}(u, v, \gamma) : D_A(\theta + m) \times \mathbb{R} \to D_A(\theta + m), \qquad m \geqslant 1,$$

are C^∞. It is now easy to see that

$$F(U, \gamma, \Delta d_1, \Delta d_2) : D_A(\theta + m) \times \mathbb{R}^3 \to D_A(\theta + m - 1) \tag{4.9}$$

is C^∞. (4.9) is also valid if periodic boundary conditions are imposed.

By using the results in Section 2, we find that (4.8) generates a dynamical system on $D_A(\theta + m)$, $m \geqslant 1$, that is C^∞ is the parameters Δd_1, Δd_2, and γ. More precisely, we have a local solution in time for any initial condition $x \in D_A(\theta + m)$.

THEOREM 4.2. *Let $\tilde{f}(u, v, \gamma)$ and $\tilde{g}(u, v, \gamma)$ be C^∞ functions in $\mathbb{R}^3$ and let Δd_1 and Δd_2 be sufficiently small. Then for all $x \in D_A(\theta + m)$, $m \geqslant 1$, Eq. (4.8) with $U(t_0) = x$ admits a unique strict solution on $[t_0, t_0 + \varepsilon]$. Here $\varepsilon > 0$ may depend on x. Moreover, $U(t)$ is a C^∞ function of $(x, \gamma, \Delta d_1, \Delta d_2)$. Here $x \in D_A(\theta + m)$, $b < \gamma < k$, and $|\Delta d_1| + |\Delta d_2| < \eta$, $U \in C^1([t_0, t_0 + \varepsilon], D_A(\theta + m - 1)) \cap C([t_0, t_0 + \varepsilon], D_A(\theta + m))$, $m \geqslant 1$.*

The following result shows that the time interval does not need to be small if the solution is near a given solution $\tilde{U}(t)$.

THEOREM 4.3. *Let $\tilde{f}$ and $\tilde{g}$ be C^∞ functions in $\mathbb{R}^3$ and let Δd_1 and Δd_2 be sufficiently small. If $\tilde{U}(t)$ is a solution for (4.8) that is defined on $[t_0, t_0 + \tau]$, $\tau > 0$. Then there exists $\zeta > 0$, such that if $|x - \tilde{U}(t_0)|_{\theta + m} < \zeta$,* *then* (4.8) *has a unique solution* $U(t) \in C^1([t_0, t_0+\tau], D_A(\theta+m-1)) \cap C([t_0, t_0+\tau], D_A(\theta+m))$, $m \geqslant 1$. *Moreover* U *is* C^∞ *in* $(x, \gamma, \Delta d_1, \Delta d_2)$ *in the indicated domains and norms.*

In the rest of this section we assume that $\Delta d_1 = \Delta d_2 = 0$, $d_1 > 0$ and $d_2 > 0$. The boundary conditions in (4.7) imply that each solution to (4.1) can induce a spatially homogeneous solution to (4.6). Thus, $(u(t, x), v(t, x)) = (k, 0)$ or $(b, 0)$, denoted by K or B again, is an equilibrium for (4.6). Also $(u(t, x), v(t, x)) = p(t, x)$, $\gamma_0 < \gamma < \gamma_0 + \varepsilon$, is a periodic solution for (4.6).

Let $(\tilde{u}(t), \tilde{v}(t))$ be a solution to the ODE (4.1). Linearizing (4.6) around $(\tilde{u}(t), \tilde{v}(t))$ we have

$$\begin{aligned} u_t &= d_1 u_{xx} + [\tilde{u} f'(\tilde{u}) + f(\tilde{u}) - \tilde{v}] u - \tilde{u} v, \\ v_t &= d_2 v_{xx} + \tilde{v} u + (\tilde{u} - \gamma) v. \end{aligned} \tag{4.10}$$

We can rewrite (4.10) as an abstract parabolic equation in $D_A(\theta+1)$,

$$U'(t) = \mathscr{A}(t)\, U(t). \tag{4.11}$$

Let $\mathscr{A}_B = \mathscr{A}(t)$ (or $\mathscr{A}_K = \mathscr{A}(t)$) at the equilibria B (or K). By (4.10), it is easy to show the following

PROPOSITION 4.4. (i) *Suppose that the Neumann boundary conditions in* (4.7) *are imposed. Then the eigenvalues of* $\mathscr{A}_B$ *are* $\{\lambda_{1n}(B), \lambda_{2n}(B)\}$, $n \geqslant 0$, *with*

$$\lambda_{1n}(B) = bf'(b) - n^2\pi^2 d_1$$
$$\lambda_{2n}(B) = b - \gamma - n^2\pi^2 d_2.$$

The eigenvalues of $\mathscr{A}_K$ *are* $\{\lambda_{1n}(K), \lambda_{2n}(K)\}$, $n \geqslant 0$ *with*

$$\lambda_{1n}(K) = k - \lambda - n^2\pi^2 d_2$$
$$\lambda_{2n}(K) = kf'(k) - n^2\pi^2 d_1.$$

The eigenvalue $\lambda_{in}(B)$ (*or* $\lambda_{in}(K)$), $i = 1, 2$, *is a pole of order one for* $(\lambda - \mathscr{A}_B)^{-1}$ (*or* $(\lambda - \mathscr{A}_K)^{-1}$). *The eigenvectors for* $\lambda_{1n}(B)$ *and* $\lambda_{2n}(B)$ (*or* $\lambda_{1n} K$ *and* $\lambda_{2n}(K)$) *span the 2-dimensional subspace* $\{(u_n \cos n\pi x, v_n \cos n\pi x) \mid (u_n, v_n) \in \mathbb{R}^2\}$ *in* $D_A(\theta+1)$.

(ii) *Suppose that the periodic boundary conditions in* (4.7″) *are imposed. Then the eigenvalues of* $\mathscr{A}_B$ *are* $\{\lambda_{1n}(B), \lambda_{2n}(B)\}$, $n \geqslant 0$, *with*

$$\lambda_{1n}(B) = bf'(b) - 4n^2\pi^2 d_1$$
$$\lambda_{2n}(B) = b - \gamma - 4n^2\pi^2 d_2.$$

The eigenvalues of $\mathscr{A}_K$ *are* $\{\lambda_{1n}(K), \lambda_{2n}(K)\}$, $n \geq 0$, *with*

$$\lambda_{1n}(K) = k - \gamma - 4n^2\pi^2 d_2$$

$$\lambda_{2n}(K) = kf'(k) - 4n^2\pi^2 d_1.$$

The eigenvalue $\lambda_{in}(B)$ *(or* $\lambda_{in}(K)$*),* $i = 1, 2$, *is a pole of order one for* $(\lambda - \mathscr{A}_B)^{-1}$ *(or* $(\lambda - \mathscr{A}_K)^{-1}$*). The eigenvectors for* $\lambda_{1n}(B)$ *and* $\lambda_{2n}(B)$ *(or* $\lambda_{1n}(K)$ *and* $\lambda_{2n}(K)$*),* $n \geq 1$, *span the 4-dimensional subspace* $\{(u_n \cos 2n\pi x, v_n \cos 2n\pi x) \oplus (w_n \sin 2n\pi x, z_n \sin 2n\pi x) \mid (u_n, v_n, w_n, z_n) \in \mathbb{R}^4\}$ *in* $D_A(\theta + 1)$.

THEOREM 4.5. (i) *Assume that the Neumann boundary conditions are imposed. Then for each* (d_1, d_2), *satisfying* $d_1 > 0$, $d_2 > 0$ *and*

$$(k - \gamma)(bf'(b) - \pi^2 d_1) + (k - \gamma - \pi^2 d_2)(\gamma - b) < 0, \tag{4.12}$$

there exists an $\varepsilon_0 > 0$ *such that the periodic solution* $p(t, \gamma)$, $\gamma_0 < \gamma < \gamma_0 + \varepsilon_0$, *is asymptotically stable in* $D_A(\theta + 1)$. *Here* ε_0 *depends on* d_1 *and* d_2.

(ii) *Assume that the periodic boundary conditions are imposed. Then the same result as in* (i) *is valid if* (4.12) *is replaced by*

$$(k - \gamma)(bf'(b) - 4\pi^2 d_1) + (k - \gamma - 4\pi^2 d_2)(\gamma - b) < 0,$$

(iii) *The maximal norm of stable characteristic values approach zero as* $\gamma \to \gamma_0^+$ *both in cases* (i) *and* (ii).

Proof. We only need to prove part (i) since the proof of part (ii) is similar. The leading two eigenvalues for $\mathscr{A}_B$ are $\mu_1(B) = \lambda_{10}(B)$ and $\mu_2(B) = \max\{\lambda_{11}(B), \lambda_{20}(B)\}$. Here $\mu_{11}(B)$ is simple and $\mu_2(B)$ is semi-simple. Thus, $U' = \mathscr{A}_B U$ has a pseudo exponential dichotomy with the exponents being $\mu_2(B) < \mu_1(B)$. Similarly, $U' = \mathscr{A}_K U$ has a pseudo exponential dichotomy with the exponents being $\mu_2(K) < \mu_1(K)$, where $\mu_1(K) = \lambda_{10}(K)$ and $\mu_2(K) = \max\{\lambda_{11}(K), \lambda_{20}(K)\}$.

Let $(\tilde{u}(r), \tilde{v}(t)) = p(t, \gamma)$ and $T(t, s)$, $t \geq s$, be the evolution operator for (4.11) in $D_A(\theta + 1)$. Assume that $\gamma > \gamma_0$, $\gamma - \gamma_0$ is small, and Σ_0, Σ_1 (and Σ_2, Σ_3) is near B (and K), we have that $\mathscr{A}(t) - \mathscr{A}_B$, $0 \leq t \leq \tau_1$ and $\mathscr{A}(t) - \mathscr{A}_K$, $\tau_2 \leq t \leq \tau_3$ are uniformly small in the space $L(D_A(\theta + 1), D_A(\theta))$. From Theorem 3.5, $U'(t) = \mathscr{A}(t)\, U(t)$, $0 \leq t \leq \tau_1$ has a pseudo exponential dichotomy with exponents $\beta_B < \alpha_B$, where $\alpha_B \cong \mu_1(B)$ and $\beta_B \cong \mu_2(B)$. Similarly, $U'(t) = \mathscr{A}(t)\, U(t)$, $\tau_2 \leq t \leq \tau_3$ has a pseudo exponential dichotomy with exponents $\beta_K \cong \mu_2(B)$ and $\alpha_K \cong \mu_1(K)$. Let the stable and unstable projections be $P_s(t)$ and $P_u(t)$ for $0 \leq t \leq \tau_1$ and $\tau_2 \leq t \leq \tau_3$.

Let $\tilde{X} = \{(u, v) \in D_A(\theta+1) \mid \int_0^1 u(t, x)\, dx = \int_0^1 v(t, x)\, dx = 0\}$ be a closed subspace of $D_A(\theta+1)$. Since the coefficients in system (4.10) do not depend on x, it is not hard to show that $T(t, 0)\tilde{X} \subset \tilde{X}$ for all $t \geqslant 0$. From Propsoition 4.4, $\tilde{X}$ is in the stable subspace of $e^{\mathscr{A}_B t}$. From the proof of Theorem 3.5, (3.19), $T(\tau_1, 0)\tilde{X} \subset \tilde{X}$ implies that $\tilde{X} \subset \mathscr{R}P_s(0)$. Similarly, $\tilde{X} \subset \mathscr{R}P_s(\tau_2)$. Let $U \in \tilde{X}$ and $|U|_{\theta+1} = 1$. Then

$$|T(\tau_1, 0)\, U|_{\theta+1} \leqslant Ce^{\beta_B \omega_B},$$

$$|T(\tau_2, 0)\, U|_{\theta+1} \leqslant Ce^{\beta_B \omega_B},$$

$$|T(\tau_3, \tau_2)\, T(\tau_2, 0)\, U|_{\theta+1} \leqslant Ce^{\beta_B \omega_B + \beta_K \omega_K}.$$

Therefore

$$\begin{aligned} |T(\tau_4, 0)\, U|_{\theta+1} &\leqslant Ce^{(\beta_B + (\omega_K/\omega_B)\beta_K)\omega_B} \\ &\leqslant Ce^{[\beta_B \lambda_K^+ + (\lambda_B^- + C_1 \lambda_K^+/\omega_B)\beta_K]\omega_B/\lambda_K^+}, \\ &\leqslant Ce^{J\omega_B/\lambda_K^+}. \end{aligned}$$

Here we have used (4.3). The quantity in the [ ] is denoted by J.

We claim that $J \leqslant \bar{c} < 0$, where $\bar{c}$ does not depend on ω_B. The assertion is easy to prove if $\omega_B = +\infty$, $\beta_B = \mu_2(B)$, and $\beta_K = \mu_2(K)$. In fact, we then have $J = \lambda_K^+ \mu_2(B) + \lambda_B^- \mu_2(K)$. If $\mu_2(B) = \lambda_{11}(B)$ and $\mu_2(K) = \lambda_{11}(K)$, we can use (4.12) to conclude that $J < 0$. If $\mu_2(B) = \lambda_{11}(B)$ and $\mu_2(K) = \lambda_{20}(K)$, we have $J < \lambda_K^+ \lambda_B^+ - \lambda_B^- \lambda_K^- < 0$ from condition (D). If $\mu_2(B) = \lambda_{20}(B) = -\lambda_B^-$, then since $\mu_2(K) < \lambda_K^+$, we still have $J < 0$. It is now clear that if ω_B is sufficiently large and if β_B and β_K are sufficiently close to $\mu_2(B)$ and $\mu_2(K)$, then $J \leqslant \bar{c} < 0$. These conditions are satisfied if we first choose Σ_0 and Σ_1 (and Σ_2 and Σ_3) to be close to B (and K), and then ε to be small. We then have, for $U \in \tilde{X}$, $|U|_{\theta+1} = 1$,

$$|T(\tau_4, 0)\, U|_{\theta+1} = c_1(\omega_B) \to 0 \qquad \text{as} \quad \omega_B \to \infty. \tag{4.13}$$

Based on condition (D), we have seen that $p(t, \gamma)$ is an asymptotically stable periodic solution for (4.1) with the stable characteristic value approaching zero as $\gamma \to \gamma_0^+$. Therefore, there exists $\eta \in \mathbb{R}^2$, $|\eta| = 1$, such that if $\bar{\eta}(x) = \eta$ for $0 < x < 1$, then $\bar{\eta} \in D_A(\theta+1)$ and

$$|T(\tau_4, 0)\bar{\eta}| = c_2(\omega_B) \to 0 \qquad \text{as} \quad \omega_B \to \infty. \tag{4.14}$$

Let $X^\perp = \tilde{X} \oplus \text{span}[\bar{\eta}] \subset D_A(\theta+1)$. Then $T(\tau_4, 0)X^\perp \subset X^\perp$. For $U \in X^\perp$, based on (4.13) and (4.14), we have $|T(\tau_4, 0)U| \to 0$ as $\omega_B \to \infty$. Observe that $X^\perp \oplus \text{span}[\dot{p}(0, \gamma)] = D_A(\theta+1)$, and $|T(\tau_4, 0)\, \dot{p}(0, \gamma)| = |\dot{p}(0, \gamma)|$. It follows from Theorem 2.4 in Lunardi (12) that the periodic solution $p(t, \gamma)$ is asymptotically stable.

APPENDIX

Conway and Smoller (4) studied a special case where $f(u) = a(k-u)(u-b)$. Using a proper rescaling we can set $a=1$. The stability of the heteroclinic cycle from inside is determined by $\lambda_B^+\lambda_K^+ + \lambda_B^-\lambda_K^- = b(k-b)(k-\gamma_0) - k(k-b)(\gamma_0 - b)$, or by the sign of $b(k-\gamma_0) - k(\gamma_0 - b)$. Conway and Smoller showed that periodic solutions bifurcating from the heteroclinic cycle must be stable, which implies that $b(k-\gamma_0) - k(\gamma_0 - b) \leqslant 0$. But this does not rule out the possibility of having an equals sign. To show that $\lambda_B^+\lambda_K^+ + \lambda_B^-\lambda_K^- < 0$, we want to verify that

$$\frac{2}{1/b + 1/k} < \gamma_0. \tag{A1}$$

It is known that $\gamma_0 < (b+k)/2$ and $G((b+k)/2) > 0$; see Conway and Smoller (4). Define $A_{vr}(b, k, \alpha) = [(b^\alpha + k^\alpha)/2]^{1/\alpha}$. It is a classical result that $A_{vr}(b, k, -1) \leqslant A_{vr}(b, k, 1)$, and the difference is small if $k-b$ is small. Proving (A1) is equivalent to finding a fine estimate for γ_0. Here I have used a numerical method to show that $G(A_{vr}(b, k, -1)) < 0$. Thus (A1) is proved numerically. Table A.I lists some values of $v_1 - v_2$, where v_1 and v_2 are the v-coordinates of $W^u(K)$ and $W^s(B)$ evaluated at $u = [b + A_{vr}(b, k, -1)]/2$. The integration was done with the Runge–Kutta–Fehlberg method and a step size $h = 0.01$. Repeating the same computation with $h = 0.05$ shows no significant difference in the results. Similar computation has also been done for $10 \leqslant k \leqslant 20$, showing that $v_1 - v_2$ is strictly positive on all the sample points computed.

TABLE A.I

Values of $v_1 - v_2$

k	$b=0.1k$	$b=0.2k$	$b=0.3k$	$b=0.4k$	$b=0.5k$	$b=0.6k$	$b=0.7k$	$b=0.8k$	$b=0.9k$
1	0.7768	0.4570	0.2672	0.1509	0.0799	0.0382	0.0154	0.0044	0.0006
2	2.0660	1.2485	0.7474	0.4318	0.2347	0.1155	0.0482	0.0147	0.0020
3	3.8142	2.3397	1.4190	0.8309	0.4583	0.2297	0.0982	0.0310	0.0046
4	5.9965	3.7150	2.2727	1.3428	0.7483	0.3798	0.1651	0.0536	0.0084
5	8.5982	5.3647	3.3031	1.9646	1.1031	0.5653	0.2490	0.0825	0.0135
6	11.6100	7.2829	4.5062	2.6942	1.5218	0.7858	0.3497	0.1178	0.0199
7	15.0208	9.4675	5.8793	3.5294	2.0033	1.0408	0.4671	0.1594	0.0276
8	18.8323	11.9103	7.4203	4.4697	2.5474	1.3303	0.6012	0.2074	0.0367
9	23.0331	14.6045	9.1259	5.5154	3.1535	1.6540	0.7519	0.2618	0.0471
10	27.6183	17.5570	10.9966	6.6635	3.8208	2.0115	0.9192	0.3225	0.0590

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