GEARHART-PRÜSS THEOREM AND LINEAR STABILITY FOR RIEMANN SOLUTIONS OF CONSERVATION LAWS

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Dedicated to Professor Pavol Brunovsky's 70th birthday

Abstract. We study the spectral and linear stability of Riemann solutions with multiple Lax shocks for systems of conservation laws. Using a self-similar change of variables, Riemann solutions become stationary solutions for the system $u_t+(Df(u)-xI)u_x=0$. In the space of $O((1+|x|)^{-\eta})$ functions, we show that if $\Re\lambda>-\eta$, then λ is either an eigenvalue or a resolvent point. Eigenvalues of the linearized system are zeros of the determinant of a transcendental matrix. On some vertical lines in the complex plane, called *resonance lines*, the determinant can be arbitrarily small but nonzero. A C^0 semigroup is constructed. Using the Gearhart-Prüss Theorem, we show that the solutions are $O(e^{\gamma t})$ if γ is greater than the real parts of the eigenvalues and the coordinates of resonance lines. We study examples where Riemann solutions have two or three Lax-shocks.

1. Introduction

We study the spectral and linear stability of Riemann solutions of a system of n conservation laws in one-dimensional spatial variable,

$$u_\tau+f(u)_\xi=0,\quad u\in\mathbb{R}^n. \tag{1.1}$$

A Riemann problem is an initial/boundary value problem for (1.1) having a piecewise constant initial data with a jump at $\xi=0$. A solution of the Riemann problem has the form $u=\bar u(\xi/\tau)$. Using the change of variables $x=\xi/\tau,\ t=\ln\tau$, system (1.1) becomes

$$u_t+(Df(u)-xI)u_x=0. \tag{1.2}$$

Riemann solutions to (1.1), usually non-stationary in the (ξ,τ) coordinates, are stationary to (1.2) in (x,t) coordinates. This allows us to construct a C^0 semigroup of the linearized system and use the spectral method to study its stability.

The main assumptions of this paper are:

(H1) The Riemann solution has m consecutive Lax i-shock shocks: $\Lambda^i,\ i=\alpha,\alpha+1,\dots,\beta,$, with speeds $\bar s^i$. Here $1\le\alpha\le\beta\le n$ and $m=\beta-\alpha+1\le n$. Let $\bar s^{\alpha-1}=-\infty$ and $\bar s^{\beta+1}=\infty$, then $\bar u(\xi,\tau)=\bar u^i\quad$ if $\bar s^i<\xi/\tau<\bar s^{i+1}$.

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(H2) The system is assumed to be strictly hyperbolic at each $\bar{u}^i$.

(H3) The Rankine-Hugoniot jump condition is satisfied at each shock Λ^i.

(H4) Majda's stability condition is satisfied at each Λ^i.

The main results of this paper are stated in the abstract, and listed in this section when we overview the rest of the paper.

In [10], we obtained the growth rate of solutions in L^2 norm. In this paper we will obtain the growth rate in sup norm. To be precise, let E be a Banach space and $f : \mathbb{R}^+ \to E$ be locally L^2. We say $f(t)$ is $O(e^{\gamma t})$ in L^2 norm if $\int_0^\infty |e^{-\gamma t} f(t)|_E^2 dt < \infty$. If $|f(t)|_E = O(e^{\gamma t})$, then we say $f(t)$ is $O(e^{\gamma t})$ in sup norm.

We allow the number of shocks to be any positive integer $m \leq n$ while in [8, 10] $m = n$ is assumed. The spectrum for a system with $n = m = 2$ has been studied in [11]. In that case, except for $\lambda = -1$, all the other eigenvalues are equally spaced on a vertical line in the complex plane.

Using the characteristic method, sufficient conditions for linear stability have been obtained by many authors [6, 7, 8, 19] in BV and L^1 spaces. Lewicka and Zumbrun showed that if a certain scattering matrix is positive, then the conditions for BV and L^1 stability correspond to the condition that the real parts of eigenvalues of the linearized system are less than 0 and -1 respectively [8].

If A is a bounded operator then the growth rate of $e^{At}, t \geq 0$, is determined by the largest real parts of the eigenvalues of A. This result was known to Lyapunov (1892) for systems of ODEs, and extends to parabolic equations [4] and initial/boundary value problems for hyperbolic systems in one spatial dimension [12]. Lyapunov's theorem does not apply to hyperbolic systems in spaces of dimension greater than one [2], [14]. In [8] Lewicka and Zumbrun constructed an example where the eigenvalues of a Riemann solution with shocks have negative real parts but the system is unstable.

For Riemann solutions consisting of Lax shocks, we define the so called *resonance values* and *resonance lines* where certain determinant can be arbitrarily small, see Definition 7.1 of this paper, and [10]. We will show that if γ is greater than the real parts of any eigenvalues and the coordinates of any resonance lines, then the growth or decay rate for solutions of linearized equation around the Riemann solution is $O(e^{\gamma t})$ in sup norm.

Let A be the infinitesimal generator of a strongly continuous semigroup e^{At} on a Banach space. We define the exponential growth bound (or type) of e^{At}, $\omega(A)$ and the spectral bound $s(A)$ as

$$\omega_0(e^{At}) := \inf\{w \in \mathbb{R} | \text{ there exists } M_w \geq 1 \text{ such that } \|e^{At}\| \leq M_w e^{wt} \text{ for all } t \geq 0\},$$

$$\omega(A) = t^{-1} \log \sup\{|z| : z \in \sigma(e^{tA})\}, t \neq 0, \qquad s(A) = \sup\{\Re z : z \in \sigma(A)\}.$$

It is known that $s(A) \leq \omega(A) = \omega_0(e^{At})$ [21]. The following theorem can be found in [2, 3]:

Theorem 1.1. *[Gearhart-Prüss theorem](Gearhart 1978, Prüss 1984, Greiner 1985, Huang 1985) For a strongly continuous semigroup on a Hilbert space, $\omega(A) < \gamma$ if and only if $s(A) < \gamma$ and*

$$\sup\{(\|z - A)^{-1}\| : Rez > \gamma\} < \infty. \tag{1.3}$$

Definition 1.1. A complex number λ is called an ϵ-pseudo-eigenvalue if $\lambda \in \rho(A)$ and $\|(\lambda I - A)^{-1}\| \geq \epsilon^{-1}$. A vertical line $\Re s = \sigma_0$ in the complex plane that contains ϵ-pseudo-eigenvalues for all $\epsilon > 0$ is called a *vertical line of pseudo-eigenvalues* and σ_0 is said to be the coordinate of the vertical line of pseudo-eigenvalues.

Pseudo-eigenvalues are wellknown in literature [20] but are not related to the growth rate of C^0 semigroups in any publication. The definition of the vertical line of pseudo-eigenvalues seems to be new. For Riemann solutions with shocks, we will prove that the vertical lines of pseudo-eigenvalues defined by the abstract operator A are precisely the resonance lines defined by certain determinant, see Theorem 7.3. We present some simple results relating the vertical line of pseudo-eigenvalues to the growth rate of solutions. The proof will be omitted.

Theorem 1.2. *(1) In the complex plane $\mathbb{C}$, to the right of $\Re\lambda = s(A)$, the ϵ-pseudo eigenvalues approach the set of vertical lines of pseudo-eigenvalues as $\epsilon \to 0$.*

(2) If $s(A) < \omega(A)$, then there exists at least one vertical line of pseudo-eigenvalues whose coordinate is greater than $s(A)$. The converse is also true.

(3) If $\|e^{At}\| = O(e^{\gamma t})$, then to the right of the line $\Re s > \gamma$ there cannot be any eigenvalue or vertical line of pseudo-eigenvalues. If γ is greater than $s(A)$ and the coordinate of any vertical line of pseudo-eigenvalues, then $\|e^{At}\| = O(e^{\gamma t})$.

In §2, we define the weighted function space L^2_η. The linear variational system around the Riemann solution can be written as $V_t = \mathcal{A}(V)$ where $D(\mathcal{A}) \subset L^2_\eta$, see (2.8).

In §3, the Laplace transform is applied to the linear variational system. The solution in the dual variable satisfies a system of integral equations in each R^i – the ith region between two consecutive shocks, with undetermined boundary values that correspond to the characteristic waves entering R^i.

In §4, we derive priori estimates for solutions of the integral equations. The integral term is similar to the convolution, and can also be interpreted as a Fourier transform. In Lemma 4.2, we prove an inequality similar to Young's inequality, and an equality parallel to the Plancherel's equality.

In §5, we solve the Laplace transformed linear non-homogeneous system in L^2_η. We define a constant σ_M which is the least upper bound of the coordinates of all the resonance lines and the real parts of all the eigenvalues. If a constant γ satisfies $\gamma > \max\{-\eta, \sigma_M\}$, then for the dual variable s of t, with $\Re s \geq \gamma$, the inverse Laplace transform exists and is the weak solution of (2.2) (Theorem 5.3). Moreover, the solution is of $O(e^{\gamma t})$ in sup norm (Theorem 5.4).

In §6, Theorem 6.1, we show that if the initial data is in $D(\mathcal{A})$, then the solution is a classical solution of (2.1) . Thus, there exists a semigroup $e^{t\mathcal{A}}$ with $\mathcal{A}$ as the infinitesimal generator. The growth rate of $e^{t\mathcal{A}}$ is obtained by the Gearhart-Prüss theorem (Theorem 6.2).

In §7, we study the eigenvalue problem related to the linearized system around the Riemann solutions. We show that the region $\Re\lambda > -\eta,\ \eta > 0$ consists of either resolvent points or eigenvalues (Theorem 7.2). The eigenvalues are zeros of the determinant of a transcendental matrix. Examples with $n = 3, m = 2$ and $m = n = 3$ are studied in details. In general, eigenvalues that are not equal to -1 lie on vertical lines in $\mathbb{C}$. There can also be vertical *resonance lines* containing δ*-resonance values* with arbitrarily small $\delta > 0$ (Definition 7.1). In Theorem 7.3 we show that the resonance values and resonance lines are exactly the pseudo-eigenvalues and vertical lines of pseudo-eigenvalues defined in Definition 1.1.

During the preparation of this paper, I have benefited from discussions with Y. Latushkin on semigroups and S. Schecter and M. Shearer on hyperbolic conservation laws.

2. Basic settings

Consider the Riemann solutions of (1.2) with m Lax shocks. In the (x, t) coordinates, the location of the ith shock Λ^i is at $x = x^i = \bar{s}^i, \alpha \le i \le \beta$. Regions between shocks will be called the regular layers, denoted by

$$R^i = (x^i, x^{i+1}),\ \alpha - 1 \le i \le \beta;\ \text{where } x^{\alpha-1} = -\infty,\ x^{\beta+1} = \infty.$$

The Riemann solution becomes

$$u(x,t) = \bar{u}^i, \quad \text{if } x \in R^i = (x^i, x^{i+1}), \quad \alpha - 1 \le i \le \beta.$$

Assuming that the position of the ith shock is $\xi = \xi^i(\tau)$ and using $x^i(t) = \xi^i(\tau)/\tau$, then the jump conditions can be derived from the Rankine-Hugoniot conditions of (1.1) as follows.

$$\begin{aligned}
&f(u(\xi^i+,\tau)) - f(u(\xi^i-,\tau)) = \frac{d}{d\tau}\xi^i(\tau)(u(\xi^i+,\tau) - u(\xi^i-,\tau)),\\
&\frac{d\xi^i}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau}\tau + x^i(t) = \dot{x}^i(t) + x^i(t),\\
&f(u(x^i+,t)) - f(u(x^i-,t)) = (\dot{x}^i(t) + x^i(t))(u(x^i+,t) - u(x^i-,t)).
\end{aligned}$$

Let $\Delta^i = \bar{u}^i - \bar{u}^{i-1}$, U be the variation of u and X^i be the variation of the shock position x^i. For a stationary solution $x^i(t) = x^i$ is constant and $\dot{x}^i(t) = 0$. Linearizing around the jump condition, we have

$$Df(\bar{u}^i)U(x^i+,t) - Df(\bar{u}^{i-1})U(x^i-,t) = (\dot{X}^i(t) + X^i(t))\Delta u^i + x^i \cdot (U(x^i+,t) - U(x^i-,t)).$$

Thus the linear variational system of (1.2) with jump conditions at each x^i is

$$
\begin{aligned}
&U_t + (Df - xI)U_x = 0, \quad U(x,0) = U_0(x), \\
&[(Df(u(x)) - xI)U]_{x^i} = [\dot{X}^i(t) + X^i(t)]\Delta^i.
\end{aligned} \tag{2.1}
$$

Here $[F(x)]_{x^i} := F(x^i+) - F(x^i-)$ denotes the jump discontinuity of a function $F(x)$ at x^i.

For brevity, we use Df for $Df(u(x))$ or $Df(\bar{u}^i)$ if no confusion should arise. By a nonsingular change of variables

$$V = e^t(Df - xI)U,\ Y^i(t) = e^t X^i(t),\ h(x) = (Df - xI)U_0(x),$$

system (2.1) is equivalent to

$$
\begin{aligned}
&V_t + (Df - xI)V_x = 0, \quad V(x,0) = h(x), \\
&[V(x,t)]_{x^i} = \dot{Y}^i(t)\Delta^i.
\end{aligned} \tag{2.2}
$$

The change of variables $U \to V$ brings a change of the growth rates of solutions. If $(V, \{Y^i\}_\alpha^\beta) = O(e^{\gamma t})$ then $(U, \{X^i\}_\alpha^\beta) = O(e^{(\gamma-1)t})$. In §7, we show $\lambda = 0$ is an eigenvalue for (2.2) corresponding to the dynamics of the layer positions $\dot{Y}^i = \lambda Y^i$. This implies that $\lambda = -1$ is always an eigenvalue for (2.1).

From the hypothesis (H2), in each R^i, $Df(\bar{u}^i)$ has n distinct eigenvalues,

$$\lambda_1(\bar{u}^i) < \lambda_2(\bar{u}^2) < \cdots < \lambda_n(\bar{u}^i).$$

We use λ_j^i to denote the eigenvalue $\lambda_j(\bar{u}^i)$. Since Λ^i is a Lax i-shock and in (x,t) coordinates x^i is the shock speed, we have for each $\alpha \le i \le \beta$:

$$
\begin{aligned}
&\lambda_{i-1}^{i-1} < x^i < \lambda_{i+1}^i, \quad \text{characteristics leaving } \Lambda^i, \\
&\lambda_i^{i-1} > x^i > \lambda_i^i, \quad\quad \text{characteristics hitting } \Lambda^i.
\end{aligned} \tag{2.3}
$$

In the bounded regions $R^i = (x^i, x^{i+1}), \alpha \le i \le \beta - 1$, from (2.3),

$$\lambda_i^i < x^i < x^{i+1} < \lambda_{i+1}^i.$$

It follows that $x \ne \lambda_j^i$ for any $1 \le j \le n$. The relation between the characteristic and shock waves in bounded regions is illustrated in Figure 2.1. Although the characteristic lines are curved in (x,t) coordinates, see (2.7), we draw straight lines for convenience.

In an unbounded region $R^{\alpha-1}$ or R^β, it is possible to find x such that $x = \lambda_j^{\alpha-1}$ or λ_j^β. See Figure 2.2. More specifically,

(1) in $R^{\alpha-1}$ for each $\lambda_j^{\alpha-1}, 1 \le j \le \alpha - 1$, there exists x such that

$$\lambda_j^{\alpha-1} = x < x^\alpha, \quad j = 1, \ldots, \alpha - 1. \tag{2.4}$$

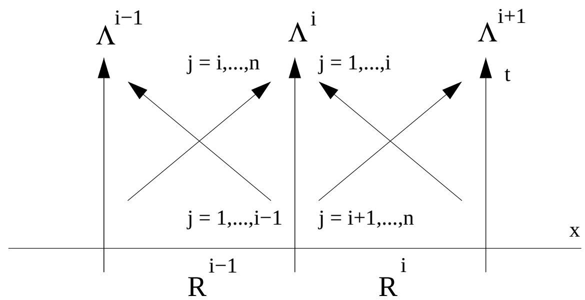

FIGURE 2.1. The left and right going characteristics in R^{i-1} and R^i.

(2) In R^β for each $\lambda_j^\beta, \beta+1 \le j \le n$, there exists x such that

$$\lambda_j^\beta = x > x^\beta, \quad j = \beta+1, \ldots, n. \tag{2.5}$$

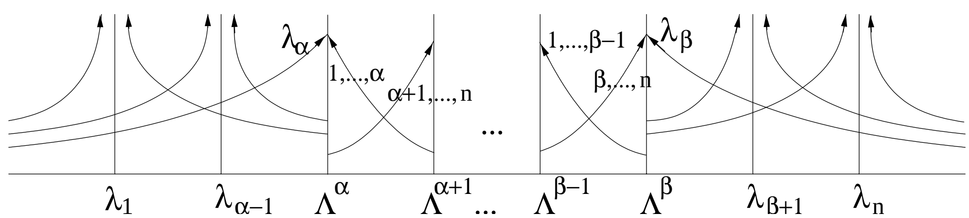

FIGURE 2.2. An example with $1 < \alpha$ and $\beta < n$. In $R^{\alpha-1}$ and R^β, the characteristics have vertical asymptotes at $x = \lambda_1, \ldots, \lambda_{\alpha-1}$ and $\lambda_{\beta+1}, \ldots, \lambda_n$ respectively

Assume that in R^i, the left and right eigenvectors associated to λ_j^i are $\mathbf{l}_j^i$ and $\mathbf{r}_j^i$, and

$$V = \sum_1^n v_j^i(x,t)\mathbf{r}_j^i, \quad h = \sum_1^n h_j^i(x)\mathbf{r}_j^i.$$

From (2.2), then the jth wave satisfies

$$v_{jt} + (\lambda_j - x)v_{jx} = 0, \quad v_j(x,0) = h_j(x). \tag{2.6}$$

The characteristics of (2.6) in R^i is

$$\frac{dx}{dt} = \lambda_j^i - x, \quad x(t) = \lambda_j^i + (x(0) - \lambda_j^i)e^{-t}. \tag{2.7}$$

As $t \to \infty$, the characteristics line has a vertical asymptote: $x(t) \to \lambda_j^i$, which is not in R^i if $\alpha \le i \le \beta - 1$, but can be in $R^{\alpha-1}$ or R^β if $1 < \alpha$ or $\beta < n$ respectively.

Definition 2.1. In R^i, the jth wave $v_j^i(x,t)$ is called a left wave if $\lambda_j^i < x$, or a right wave if $\lambda_j^i > x$ (the characteristic line moves to the left or right as t increases, see Figure 2.2).

In a bounded region $R^i, \alpha \le i \le \beta - 1$, the vertical asymptotes $x = \lambda_j^i \notin R^i$ so that $v_j^i(x,t)$ is either a left (if $j \le i$) or a right wave (if $j \ge i+1$) for all $x \in R^i$. Since $x = \lambda_j^i$ can occur in unbounded regions $R^{\alpha-1}$ and R^{β} if $\alpha>1$ or $\beta<n$, there may exit some j for which $v_j^i(x,t)$ can be a left or right wave, depending $x<\lambda_j^i$ or $x>\lambda_j^i$. This motivates the following definition:

Definition 2.2. (1) In R^i, the jth characteristic mode is called a left mode if $1\le j\le i$ and $i\neq\alpha-1$, a right mode if $i+1\le j\le n$ and $i\neq\beta$. If the jth mode is a left (or right) mode, the corresponding wave $v_j^i(x), x\in R^i$ is a left (or right) wave.

(2) In R^i, the jth characteristic mode is called a mixed mode if $i=\alpha-1,\ 1\le j\le\alpha-1$ or $i=\beta,\ \beta+1\le j\le n$. If jth mode is a mixed mode than the wave $v_j^i(x)$ is a left wave if $x>\lambda_j^i$, or a right wave if $x<\lambda_j^i$.

(3) Let $\Omega:=\cup\{R^i:\alpha\le i\le\beta-1\}\cup\{R^{\alpha-1}\backslash\{\lambda_j^{\alpha-1}:j\le\alpha-1\}\}\cup\{R^{\beta}\backslash\{\lambda_j^{\beta},j\ge\beta+1\}\}$.

Remark 2.1. System (2.6) has singularity at shocks $x=x^i$ and vertical asymptotes in $R^{\alpha-1}$ or R^{β}, but is completely regular in Ω.

Definition 2.3. Let L^2_η be the Hilbert space of locally L^2 functions with the following weighted norm being finite: If the restriction of U to R^i is U^i and if $U^i=\sum_1^n u_j^i(x)\mathbf{r}_j^i$, then

$$\|U\|=\|U\|_{L^2_\eta}:=\left(\sum_{i=\alpha-1}^{\beta}\sum_{j=1}^{n}\|u_j^i\|^2\right)^{1/2},$$

$$\|u_j^i\|:=\left(\int_{R^i}\left|\left(\lambda_j^i-x\right)^\eta u_j^i(x)\right|^2\frac{dx}{|x-\lambda_j^i|}\right)^{1/2}.$$

If the weighted norm for the restriction of U to R^i is finite then we say that U^i and the scalar function u_j^i are in $L^2_\eta(R^i)$.

Most of the estimates in §4 work only in the right half plane $\{\lambda\in\mathbb{C}:\Re\lambda>-\eta\}$ where η is the constant in Definition 2.3. We assume that the constant $\eta>0$. In the unbounded regions $R^{\alpha-1},R^{\beta}$, $\left|\lambda_j^i-x\right|^\eta\to\infty$. Thus as $x\to\pm\infty$, $u_j^i(x)\to0$ algebraically of order $|x-\lambda_j|^{-\eta}$ in the L^2 norm with respect to the measure $dx/|x-\lambda_j|$. Choosing larger η can increase the region in the complex plane $\Re\lambda>-\eta$, but the function space will be smaller.

In the bounded regions $R^i,\alpha\le i\le\beta-1$, the weight is added for convenience only.

System (2.2) can be written as an abstract equation in the Hilbert space L^2_η:

$$\begin{aligned}
&V_t=\mathcal{A}(V),\ V(0)=h,\ \text{with } \mathcal{A}(V)=-(Df-xI)V_x,\quad \text{on each } R^i,\\
&D(\mathcal{A}):=\{V:V,\ (Df-xI)V_x\in L^2_\eta,\\
&\text{with jump conditions at shocks: } [V(x)]_{x^i}\in\operatorname{span}(\Delta^i)\ \text{for } \alpha\le i\le\beta,\ \text{and}\\
&\text{in } R^{\alpha-1},\ R^{\beta}:\ [\mathbf{l}_k\cdot V(x)]_{\lambda_j^i}=0,\ k\neq j,\ \text{for } i=\alpha-1, j\le\alpha-1 \text{ or } i=\beta, j\ge\beta+1\}.
\end{aligned}\tag{2.8}$$

The jump conditions across the vertical asymptotes $x = \lambda_j^i$ in $R^{\alpha-1}$ and R^β mean that when crossing the asymptotes λ_j^i, all the other waves are continuous except for the one that turns vertical at both side of λ_j^i. Even the existence of one sided limits is not assumed for that mode.

3. Laplace transform and a system of integral equations

A function $y(s)$ is in the Hardy-Lebesgue class $\mathcal{H}(\gamma), \gamma \in \mathbb{R}$, if

(i) $y(s)$ is analytic in $\Re(s) > \gamma$;

(ii) $\{\sup_{\sigma>\gamma}(\int_{-\infty}^{\infty} |y(\sigma + i\omega)|^2 d\omega)^{1/2}\} < \infty$.

$\mathcal{H}(\gamma)$ is a Banach space with the norm defined by the left side of (ii). For $\gamma = 0$, this definition and the following lemma can be found in [21].

Lemma 3.1. *The function $z(t) \in e^{\gamma t}L^2(\mathbb{R}^+)$ iff its Laplace transform $y(s) = \mathcal{L}z(s) \in \mathcal{H}(\gamma)$.*

Moreover, if $y(s) \in \mathcal{H}(\gamma)$, then the inverse transform

$$z(t) = (2\pi)^{-1/2} \lim_{N\to\infty} \int_{-N}^{N} y(\gamma + i\omega)e^{it\omega} d\omega$$

vanishes for $t < 0$ and $y(s)$ may be obtained as the one-sided Laplace transform of $z(t)$. Further more,

$$\int_{t=0}^{\infty} e^{-2\gamma t}|z(t)|^2 dt = \int_{\omega=-\infty}^{\infty} |y(\gamma + i\omega)|^2 d\omega.$$

Applying the Laplace transform to (2.2), we have

$$\begin{aligned} & s\hat{V} + (Df - xI)\hat{V}_x = h(x), \quad h \in L^2_\eta \\ & [\hat{V}(x,s)]_{x^i} = [s\hat{Y}^i(s) - Y^i(0)]\Delta^i, \ \alpha \le i \le \beta, \\ & [\mathbf{l}_k \cdot \hat{V}(x)]_{\lambda_j^i} = 0, \ k \neq j, \ \text{for } i = \alpha - 1, j \le \alpha - 1, \ \text{or } i = \beta, j \ge \beta + 1. \end{aligned} \tag{3.1}$$

If for some $\gamma \in \mathbb{R}$, we can find a solution $\hat{V} \in \mathcal{H}(\gamma)$ for (3.1), then the inverse transform shows that $V(x,t)$ is a weak solution of (2.2) with $e^{-\gamma t}V(\cdot,t)$ being an L^2 function in $\mathbb{R}^+$.

To simplify the notation, we will drop the hat on $\hat{V}(x,s)$ if no confusion should arise. The use of the dual variable s already indicates that this is the Laplace transform of $V(x,t)$. The convention also applies to other time dependent functions and their L-transforms.

We now drop the hat in (3.1). If $V(x,s) = \sum_1^n v_j(x,s)\mathbf{r}_j^i$ satisfies (3.1), the jth wave satisfies

$$\begin{aligned} & sv_j + (\lambda_j - x)v_{jx} = h_j(x), \\ & v_{jx} + s(\lambda_j - x)^{-1}v_j = (\lambda_j - x)^{-1}h_j(x), \quad \text{if } x \neq \lambda_j, \\ & v_j(x,s) = h_j(x)/s, \ \text{if } x = \lambda_j. \end{aligned}$$

Observe that if $\Re s > 0$, the system has an algebraic dichotomy in each R^i. See [1, 13, 15, 16, 10] for discussions of exponential and non-exponential dichotomies. As $\Re s \to \infty$, the growth/decay rate of the dichotomy gets larger. To have a solution $V(x,s)$ that is uniformly bounded in s, we must solve each wave along the decay direction of the dichotomy, i.e., solve the right going waves ($x < \lambda_j^i$) from x^i to x^{i+1} and the left going waves ($\lambda_j^i < x$) from x^{i+1} to x^i. This approach is consistent with the characteristic method which requires that each wave is prescribed at the point where the characteristics enter R^i.

We shall use v_ℓ (v_r) to denote the left (right) wave respectively. Using integration factors, we find that for the right wave, $\lambda_r > x$, or the left wave $x > \lambda_\ell$,

$$((\lambda_r - x)^{-s} v_r)_x = (\lambda_r - x)^{-s-1} h_r, \quad ((x - \lambda_\ell)^{-s} v_\ell)_x = -(x - \lambda_\ell)^{-s-1} h_\ell.$$

The solution in $R^i = (x^i, x^{i+1})$ satisfies the integral equations:

$$v_r(x,s) =(\frac{\lambda_r - x}{\lambda_r - x^i})^s v_r(x^i, s) + \int_{x^i}^{x} (\frac{\lambda_r - x}{\lambda_r - y})^s h_r(y) \frac{dy}{\lambda_r - y}, \tag{3.2}$$
$$x^i < x < \min\{x^{i+1}, \lambda_r\}, \quad \text{if } \lambda_r > x^i,$$

$$v_\ell(x,s) =(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}})^s v_\ell(x^{i+1}, s) + \int_{x^{i+1}}^{x} (\frac{\lambda_\ell - x}{\lambda_\ell - y})^s h_\ell(y) \frac{dy}{\lambda_\ell - y}, \tag{3.3}$$
$$\max\{x^i, \lambda_\ell\} < x < x^{i+1}, \quad \text{if } \lambda_\ell < x^{i+1}.$$

In the right hand sides of (3.2) and (3.3), $v_r(x^i, s)$ and $v_\ell(x^{i+1}, s)$ are unknown variables. As a convention, $x^{\alpha-1} = -\infty, x^{\beta+1} = \infty$, and the terms involving $v_r(x^{\alpha-1}, s)$ and $v_\ell(x^{\beta+1}, s)$ are ignored. Note that $0 < \frac{\lambda_r - x}{\lambda_r - x^i} < \frac{\lambda_r - x}{\lambda_r - y} < 1, \quad 0 < \frac{\lambda_\ell - x}{\lambda_\ell - y} < \frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}} < 1.$

Remark 3.1. (i) In bounded regions $R^i, \alpha \le i \le \beta - 1$, (3.2) works for $r = i+1, \ldots, n$ and (3.3) works for $\ell = 1, \ldots, i$.

(ii) In $R^{\alpha-1}$, (3.2) works for $r = 1, \ldots, n$ and (3.3) works for $\ell = 1, \ldots, \alpha - 1$. For $1 \le j \le \alpha - 1$ the jth mode is a mixed mode.

(iii) In R^β, (3.3) works for $\ell = 1, \ldots, n$ and (3.2) works for $r = \beta + 1, \ldots, n$. For $\beta + 1 \le j \le n$ the jth mode is a mixed mode.

(iv) If the jth mode is a mixed mode, then to compute $v_j(x,s)$, we use (3.2) for $x \in (x^i, \lambda_j)$ and (3.3) for $x \in (\lambda_j, x^{i+1})$. Notice in this case either $x^i = -\infty$, or $x^{i+1} = \infty$.

(v) In $R^{\alpha-1}$ or R^β, if the jth mode is a mixed mode, then from (2.6), we must set $v_j(x,t) = h_j(x)$ at the singularity $x = \lambda_j^i$ (or $v_j(x,s) = h_j(x)/s$). This is meaningful if $h_j(x)$ is continuous in x. If $h_j \in L^2_\eta$, the value at the singularity is undefined.

4. Estimates of the integral terms

In this section, we derive some estimates for the integral terms of (3.2) and (3.3). To simplify the notation, we make use of the information carried in the names of variables, e.g., x is the spatial variable, t is the time variable and $s = \sigma + i\omega$ is the dual to t after the Laplace transform.

Definition 4.1. We say $V(x,t)$ is in $L^2_\eta(x)$ if $V(\cdot,t)\in L^2_\eta$ for each fixed t. We say $V(x,t)$ is in $L^2(t)$ if it is in $L^2(\mathbb{R}^+)$ for each fixed x. We say $V(x,t)$ is in $L^2_\eta(x,t)$ if it is locally an L^2 function of (x,t), and for almost every t, $V(\cdot,t)\in L^2_\eta$ with

$$\int_0^\infty \|V(\cdot,t)\|^2_\eta dt<\infty.$$

After the Laplace transform, $V(x,t)$ becomes $V(x,s)$ with $s=\sigma+i\omega$. We say $V(x,s)$ is in $L^2_\eta(x)$ if $V(\cdot,s)\in L^2_\eta$ for each fixed s. We say $V(x,s)$ is in $L^2(\omega)$ if $V(x,s)$ is in $L^2(\mathbb{R})$ for each fixed x and σ. We say $V(x,s)$ is in $L^2_\eta(x,\omega)$ if for each fixed σ, it is locally a L^2 function of (x,ω), and for almost every ω, $V(\cdot,s)\in L^2_\eta$ with

$$\int_{-\infty}^\infty \|V(\cdot,s)\|^2_\eta d\omega<\infty.$$

These definitions also extend to functions defined only in one regular layer $R^i, i=\alpha-1,\dots,\beta$.

Let $s=\sigma+i\omega$ with $\sigma>-\eta$. In each R^i, define

$$F_r(x,s):=\int_{x^i}^x\left(\frac{\lambda_r-x}{\lambda_r-y}\right)^s h_r(y)\frac{dy}{\lambda_r-y},\quad \text{right mode: } x^i<x<\min\{\lambda_r,x^{i+1}\}$$
$$F_\ell(x,s):=\int_x^{x^{i+1}}\left(\frac{x-\lambda_\ell}{y-\lambda_\ell}\right)^s h_\ell(y)\frac{dy}{y-\lambda_\ell},\quad \text{left mode: } \max\{\lambda_\ell,x^i\}<x<x^{i+1}.$$

We present a generalized Minkowski inequality in L^2_η but omit the proof:

Lemma 4.1. *If $f(\cdot,\xi)\in L^2_\eta$ for almost every $\xi\in\mathbb{R}$, and the mapping $\xi\to L^2_\eta$ is integrable with $\int\|f(\cdot,\xi)\|_{L^2_\eta}d\xi<\infty$, and if $g\in L^\infty(\mathbb{R})$ then $F(\cdot):=\int g(\xi)f(\cdot,\xi)d\xi\in L^2_\eta$, and*

$$\|F\|_{L^2_\eta}\le\int|g(\xi)|\|f(\cdot,\xi)\|_{L^2_\eta}d\xi.$$

Lemma 4.2. *Assume that $h\in L^2_\eta$, i.e. in $R^i, i=\alpha-1,\dots,\beta$, the weighted norms $\|h_j\|$ of $h_j, j=1,\dots,n$, as in Definition 2.3, are finite. Then for any $\sigma>-\eta$, we have:*

(1) The function $(x,s)\to F_j(x,s)\in\mathbb{C}^n$ is continuous for $\Re s=\sigma>-\eta$, $x\in\Omega$ (See Definition 2.2). Furthermore,

$$|F_j(x,s)|\le\frac{\|h_j\|}{\sqrt{2(\sigma+\eta)}\cdot|\lambda_j-x|^\eta}.$$

(2) $F_j(\cdot,\sigma+i\omega)\in L^2_\eta(R^i)$ with

$$\|F_j(\cdot,\sigma+i\omega)\|_{L^2_\eta(R^i)}\le\frac{1}{\sigma+\eta}\|h_j\|_{L^2_\eta(R^i)}$$

uniformly with respect to $\omega\in\mathbb{R}$.

(3) As a function of x, for almost every ω, $F_j(\cdot,s)\in L^2_\eta(R^i)$. Moreover, in $R^i, \alpha\le i\le\beta-1$,

$$\int_{\omega=-\infty}^\infty\|F_j(\cdot,s)\|^2d\omega\le C(\eta)\|h_j\|^2.$$

In $R^{\alpha-1}$ and R^β,

$$\int_{\omega=-\infty}^\infty\|F_j(\cdot,s)\|^2d\omega\le\frac{1}{\sigma+\eta}\|h_j\|^2.$$

(4) The function $F_j(x,s)$ is in $L^2(\omega)$ for each $\sigma > -\eta$, $x \in \Omega$. Moreover, $x \to F_j(x,s)$ is a continuous function for such x to $L^2(\omega)$ with one-sided limits at $x = x^i, \alpha \le i \le \beta$.

The following estimates holds where $C(\eta)$ depends only on η:

$$\int_{\omega=-\infty}^{\infty} |F_j(x,s)|^2 d\omega \le C(\eta)\|h_j\|^2,$$

Proof. We prove the lemma on the unbounded interval $R^\beta = (x^\beta, \infty)$ only, since the other cases can be treated similarly.

To prove (1), we consider $x^\beta < x < \lambda_j^\beta, j \ge \beta + 1$. By the definition we have

$$\begin{aligned}
|F_j(x,s)|^2 &\le (\lambda_j - x)^{2\sigma} \left| \int_{x^\beta}^{x} (\lambda_j - y)^{-(\eta+\sigma)} \cdot |(\lambda_j - y)^\eta h_j(y)| \frac{dy}{\lambda_j - y} \right|^2 \\
&\le (\lambda_j - x)^{2\sigma} \int_{x^\beta}^{x} (\lambda_j - y)^{-2(\eta+\sigma)-1} dy \cdot \int_{x^\beta}^{x} (\lambda_j - y)^{2\eta} |h_j(y)|^2 \frac{dy}{\lambda_j - y} \\
&\le \frac{1}{2(\eta+\sigma)(\lambda_j - x)^{2\eta}} \|h_j\|^2.
\end{aligned}$$

The case $\lambda_j^\beta < x < \infty$ can be treated similarly.

To prove (2), first consider the left mode $1 \le j \le \beta$, where $\lambda_j < x^\beta$. Let $e^\xi = (y - \lambda_j)/(x - \lambda_j) \ge 1$. Then

$$\begin{aligned}
F_j(x,s) &= \int_x^\infty \left(\frac{x - \lambda_j}{y - \lambda_j} \right)^s h_j(y) \frac{dy}{y - \lambda_j} \\
&= \int_0^\infty e^{-s\xi} h_j(\lambda_j + (x - \lambda_j)e^\xi) d\xi, \quad x^\beta < x < \infty.
\end{aligned}$$

Using Lemma 4.1, we have

$$\|F_j(\cdot, s)\| \le \int_0^\infty e^{-\sigma\xi} \|h_j(\lambda_j + (\cdot - \lambda_j)e^\xi)\| d\xi.$$

If we set $z = \lambda_j + (x - \lambda_j)e^\xi, z^\beta = \lambda_j + (x^\beta - \lambda_j)e^\xi \ge x^\beta$, then

$$\begin{aligned}
&\|h_j(\lambda_j + (\cdot - \lambda_j)e^\xi)\|^2 = \int_{x^\beta}^\infty (x - \lambda_j)^{2\eta} |h_j(\lambda_j + (x - \lambda_j)e^\xi)|^2 \frac{dx}{x - \lambda_j} \\
&= e^{-2\eta\xi} \int_{z^\beta}^\infty (z - \lambda_j)^{2\eta} |h_j(z)|^2 \frac{dz}{z - \lambda_j} \le e^{-2\eta\xi} \|h_j\|^2.
\end{aligned}$$

$$\|h_j(\lambda_j + (\cdot - \lambda_j)e^\xi)\| \le e^{-\eta\xi} \|h_j\|. \tag{4.1}$$

Thus

$$\|F_j(\cdot, s)\| \le \int_{\xi=0}^\infty e^{-(\sigma+\eta)\xi} d\xi \|h_j\| = \frac{1}{\eta + \sigma} \|h_j\|.$$

If $\beta = n$ then all the modes are the left modes. But if $\beta < n$, there are mixed modes $\beta + 1 \le j \le n$. Let j be one of the mixed mode and let

$$R^\beta - \{\lambda_j\} = \Omega_1 \cup \Omega_2, \text{ where } \Omega_1 = (x^\beta, \lambda_j),\ \Omega_2 = (\lambda_j, \infty).$$

For $x \in \Omega_2 = (\lambda_j, \infty)$ where the characteristics is left moving, we have:

$$F_j(x,s) = \int_x^\infty \left(\frac{x-\lambda_j}{y-\lambda_j}\right)^s h_j(y)\frac{dy}{y-\lambda_j}.$$

Similar to the left modes $j \le \beta$, we can show that

$$\|F_j(\cdot,s)\|_{L^2_\eta(\Omega_2)} \le \frac{1}{\eta+\sigma}\|h_j\|_{L^2_\eta(\Omega_2)}. \tag{4.2}$$

For $x \in \Omega_1 = (x^\beta, \lambda_j)$, the characteristics is right moving. We have:

$$F_j(x,s) := \int_{x^\beta}^x \left(\frac{\lambda_j - x}{\lambda_j - y}\right)^s h_j(y)\frac{dy}{\lambda_j - y}.$$

Let $e^\xi = (\lambda_j - y)/(\lambda_j - x) \ge 1$ and assume $h_j(x) = 0$ for $x \notin R^\beta$. Then

$$F_j(x,s) = \int_0^\infty e^{-s\xi} h_j(\lambda_j - (\lambda_j - x)e^\xi)d\xi.$$

Using Lemma 4.1, we have

$$\|F_j(\cdot,s)\| \le \int_0^\infty e^{-\sigma\xi}\|h_j(\lambda_j - (\lambda_j - \cdot)e^\xi)\|d\xi.$$

If we set $z = \lambda_j - (\lambda_j - x)e^\xi, z^\beta = \lambda_j - (\lambda_j - x^\beta)e^\xi \le x^\beta$, then

$$\begin{aligned}
\|h_j(\lambda_j - (\lambda_j - \cdot)e^\xi)\|^2 &= \int_{x^\beta}^{\lambda_j} |x-\lambda_j|^{2\eta}|h_j(\lambda_j - (\lambda_j - x)e^\xi)|^2 \frac{dx}{|x-\lambda_j|} \\
&= e^{-2\eta\xi}\int_{z^\beta}^{\lambda_j} |z-\lambda_j|^{2\eta}|h_j(z)|^2\frac{dz}{|z-\lambda_j|} \le e^{-2\eta\xi}\|h_j\|^2.
\end{aligned}$$

$$\|h_j(\lambda_j - (\lambda_j - \cdot)e^\xi)\| \le e^{-\eta\xi}\|h_j\|. \tag{4.3}$$

Thus

$$\|F_j(\cdot,s)\|_{L^2_\eta(\Omega_1)} \le \int_{\xi=0}^\infty e^{-(\sigma+\eta)\xi}d\xi\|h_j\| = \frac{1}{\eta+\sigma}\|h_j\|_{L^2_\eta(\Omega_1)}. \tag{4.4}$$

Observe that $\|h_j\|^2_{L^2_\eta(R^\beta)} = \|h_j\|^2_{L^2_\eta(\Omega_1)} + \|h_j\|^2_{L^2_\eta(\Omega_2)}$. Similar formula holds for $\|F_j\|_{L^2_\eta(R^\beta)}$. Combining (4.2) and (4.4), we have

$$\|F_j(\cdot,s)\|_{L^2_\eta(R^\beta)} \le \frac{1}{\eta+\sigma}\|h_j\|_{L^2_\eta(R^\beta)}.$$

To prove (3), we will interpret $F_j(x,s)$ as a Fourier transform of h_j. First consider the left mode $1 \le j \le \beta$, where $\lambda_j < x^\beta$. Let $e^\xi = (y-\lambda_j)/(x-\lambda_j) \ge 1$. Then

$$\begin{aligned}
&F_j(x,\sigma+i\omega) = \int_{\xi=0}^\infty e^{-(\sigma+i\omega)\xi}h_j(\lambda_j + (x-\lambda_j)e^\xi)d\xi = \mathcal{F}(f(\xi,x)) \\
&\text{where } f(\xi,x) = e^{-\sigma\xi}H(\xi)h_j(\lambda_j + (x-\lambda_j)e^\xi), \quad H(\xi) \text{ is the Heaviside function.}
\end{aligned} \tag{4.5}$$

For each ξ, from (4.1) we have that $f(\xi,\cdot) \in L^2_\eta$ and

$$\|f(\xi,\cdot)\| = e^{-\sigma\xi}e^{-\eta\xi}\|h_j\|.$$

It is now clear that $\xi \to f(\xi,\cdot)$ is in $L^2(\mathbb{R}, L^2_\eta)$. Using Plancherel's Theorem, we have

$$\int_{-\infty}^{\infty} \|F_j(\cdot,\sigma+i\omega)\|^2 \le \frac{1}{2\pi}\frac{1}{2(\sigma+\eta)}\|h_j\|^2.$$

We have proved (3) for the case that the jth mode is a left mode in R^β. If the jth mode is a mixed mode, then the inequalities must be proved in $\Omega_1 = (x^\beta, \lambda_j)$ and $\Omega_2 = (\lambda_j, \infty)$ separately and combine the results. Details are left to the readers.

To prove (4), as in the proof of (3), we use (4.5). We claim that for each fixed x, $f(x,\xi)$ is in $L^2(\xi)$. Consider the left mode, $x^\beta < x < \infty$. If we set $z = \lambda_j + (x-\lambda_j)e^\xi, z^\beta = \lambda_j + (x^\beta - \lambda_j)e^\xi \ge x^\beta$, then

$$\begin{aligned}
&\int_{-\infty}^{\infty} |f(\xi,x)|^2 d\xi = \int_0^\infty e^{-2\sigma\xi}|h_j(\lambda_j + (x-\lambda_j)e^\xi)|^2 d\xi\\
&\le \left(\frac{1}{x-\lambda_j}\right)^{2\eta}\int_{z^\beta}^{\infty} e^{-2(\sigma+\eta)\xi}\,(z-\lambda_j)^{2\eta}\,|h_j(z)|^2\frac{dz}{z-\lambda_j}\\
&\le \left(\frac{1}{x^\beta-\lambda_j}\right)^{2\eta}\|h_j\|^2, \quad \text{since } \sigma+\eta>0 \text{ and } x^\beta-\lambda_j < x-\lambda_j.
\end{aligned}$$

Applying Plancherel's formula to (4.5), we have that $\int_{-\infty}^{\infty}\|F_j(x,\sigma+i\omega)\|^2 d\omega \le (2\pi)^{-1}|x^\beta-\lambda_j|^{-2\eta}\|h_j\|^2$. Moreover, the mapping $x \to f(\xi,x)$ is continuous from R^β to $L^2(\xi)$. Therefore, $F_j(x,\sigma+i\omega)$ depends continuously on x as a function in $L^2(\omega)$.

The proof is similar for the mixed modes. Details shall be omitted. □

5. L^2 solutions via the Laplace transform

The definition of χ and the scattering matrix M are motivated by Lewicka's earlier works and some of the calculation in this section is similar to that in [8, 10].

We will derive an explicit formula for the solution $V(x,s)$ of equations (3.2), (3.3). We show that these solutions are in $L^2_\eta(x,\omega)$ for $\sigma \ge \gamma$ where $s = \sigma + i\omega$ and the real constant $\gamma > \max\{-\eta, \sigma_M\}$. The constant σ_M is defined in Definition 5.1. Moreover, for $\sigma \ge \gamma$, the solution $V(x,s)$ is a continuous function of $x \in R^i$ in $L^2(\omega)$ with one-sided limits at $x = x^i$. In each R^i, $V(x,s)$ is also a point-wise continuous function for x near x^i with one-sided limit in the sup norm. The jump condition as in (3.1) at the ith shock is satisfied in the sense that both sides are functions in $L^2(\omega)$ and in $C(\omega)$.

Based on this, the inverse Laplace transform $V(x,t)$ of $V(x,s)$ is a weak solution in the sense of distribution. The function $e^{-\gamma t}V(x,t)$ is in $L^2_\eta(x,t)$ as in Definition 4.1. Moreover, $e^{-\gamma t}V(x,t)$ is in $L^2(t)$ and is continuous with respect to x in each region R^i with one-sided limits at $x = x^i$. The value of the jump at x^i is understood as a L^2 function in time.

If $V(x^i+,s)$ and $V(x^i-,s)$ along the shock Λ^i are specified then the above can be used to determine $s\hat{Y}^i(s) - Y^i(0)$. If the initial condition $Y^i(0)$ is given, we can compute $\hat{Y}^i(s)$ and $Y^i(t)$. From now on, we require that the jumps of $V(x,t)$ and hence $\hat{V}(x,s)$ are along the direction of Δ^i but ignore the values of the jumps. The jump conditions thus simplify to

$$[V(x,s)]_{x^i} = 0, \text{ mod } \Delta^i. \tag{5.1}$$

The rest of the section is devoted to solving the system (3.2), (3.3) and (5.1).

5.1. **An algebraic system with jump conditions.** We consider an algebraic system for $V(x,s) = \sum_1^n v_j(x,s)\mathbf{r}_j^i$, $R^i = (x^i, x^{i+1}), i = \alpha-1, \ldots, \beta$:

(5.2)
$$\begin{aligned}
&v_r(x,s) = \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s v_r(x^i,s) + H_r(x,s), && r = i+1, \ldots, n, \text{ if } \alpha \le i \le \beta - 1,\\
&v_\ell(x,s) = \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s v_\ell(x^{i+1},s) + H_\ell(x,s), && \ell = 1, \ldots, i, \text{ if } \alpha \le i \le \beta - 1,\\
&v_r(x,s) = \left(\frac{\lambda_r - x}{\lambda_r - x^\beta}\right)^s v_r(x^\beta,s) + H_r(x,s), && x^\beta \le x < \lambda_r^\beta, \text{ if } r = \beta+1, \ldots, n,\\
&v_\ell(x,s) = \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^\alpha}\right)^s v_\ell(x^\alpha,s) + H_\ell(x,s), && \lambda_\ell^{\alpha-1} < x \le x^\alpha, \text{ if } \ell = 1, \ldots, \alpha - 1,\\
&v_j(x,s) = H_j(x,s), \qquad -\infty < x \le \min\{x^\alpha, \lambda_j^{\alpha-1}\}, && \text{or } \max\{x^\beta, \lambda_j^\beta\} \le x < \infty, j = 1, \ldots, n,\\
&[V(x,s)]_{x^i} = 0, && \text{mod } \Delta^i, \quad i = \alpha, \ldots, \beta,\\
&[\mathbf{l}_k \cdot V(x)]_{\lambda_j^i} = 0, && k \ne j, \text{ for } i \le \alpha - 1, \text{ or } i \ge \beta + 1.
\end{aligned}$$

The terms H_j satisfy

H 5.1. The function $(x,s) \to H_j(x,s) \in \mathbb{C}^n$ is continuous for $\Re s = \sigma \ge \gamma$, $x \in \Omega$. Moreover, H_j is bounded for all $\sigma \ge \gamma$ by

$$|H_j(x, \sigma + i\omega)| \le C(\gamma)|\lambda_j - x|^{-\eta}.$$

H 5.2. For $\sigma \ge \gamma$, $H_j(x,s) \in L_w^2(x,\omega)$. That is, with the fixed σ, for almost every ω, $H_j(\cdot,s) \in L_\eta^2$ with

$$\int_{\omega=-\infty}^{\infty} \|H_j(\cdot,s)\|_\eta^2 d\omega < \infty.$$

The function $x \to H_j(x, \sigma + i\omega)$ is continuous from x to $L^2(\omega)$ for $x \in \Omega$, with one-sided limits at $x = x^i, \alpha \le i \le \beta$.

H 5.3. $\sup\{\|H_j(\cdot,s)\|_\eta : \Re s = \sigma \ge \gamma\} < \infty$.

We first solve $v_j(x,s)$ in $R^i, \alpha \le i \le \beta - 1$, then use the jump condition and Majda's stability condition to find enough boundary conditions on $x^\alpha -$ and $x^\beta +$ so that $v_j(x,s)$ can be determined in $R^{\alpha-1}$ and R^β.

From the first two equations of (5.2), in each region $R^i, \alpha \le i \le \beta - 1$, we have

$$v_r(x^{i+1}-, s) = ((\lambda_r - x^{i+1})/(\lambda_r - x^i))^s v_r(x^i+, s) + H_r(x^{i+1}, s), \quad r = i+1, \ldots, n, \tag{5.3}$$

$$v_\ell(x^i+, s) = ((x^i - \lambda_\ell)/(x^{i+1} - \lambda_\ell))^s v_\ell(x^{i+1}-, s) + H_\ell(x^i, s), \quad \ell = 1, \ldots, i. \tag{5.4}$$

We now find $(v_r(x^i+, s), v_\ell(x^{i+1}-, s))$ from (5.3), (5.4) and (5.1). These are the values of left/right going waves leaving the shocks. For $\alpha + 1 \le i \le \beta - 1$, we place the left going waves $v_\ell(x^i-)$ and then the right going waves $v_r(x^i+)$ leaving Λ^i in an $(n-1)$ dimensional vector χ_i. See Figure 5.1. However, only the right (or left) going waves leaving Λ^α (or Λ^β) are placed in χ_α (or in χ_β). Next, we define χ as a block structured vector. To simplify the notations, we drop s in $v_j(x^i, s)$:

$$\chi = \begin{pmatrix} \chi_\alpha \\ \vdots \\ \chi_\beta \end{pmatrix}, \ \chi_\alpha = \begin{pmatrix} v_{\alpha+1}(x^\alpha+), \\ \vdots \\ v_n(x^\alpha+) \end{pmatrix}, \ \chi_\beta = \begin{pmatrix} v_1(x^\beta-) \\ \vdots \\ \chi_{\beta-1}(x^\beta-) \end{pmatrix}, \ \chi_i = \begin{pmatrix} v_1(x^i-) \\ \vdots \\ v_{i-1}(x^i-) \\ v_{i+1}(x^i+) \\ \vdots \\ v_n(x^i+) \end{pmatrix}, \quad \alpha+1 \le i \le \beta-1.$$

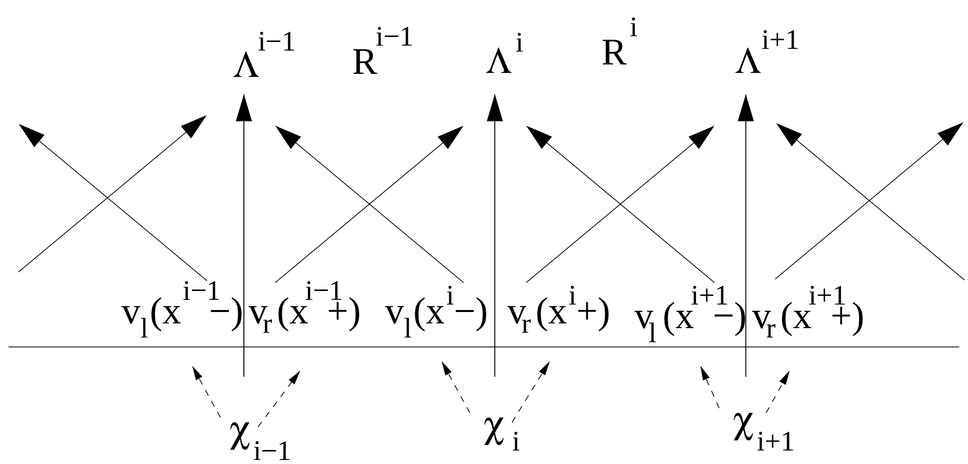

FIGURE 5.1. χ_i consists of the left and right going waves leaving Λ^i: $v_\ell(x^i-), \ell = 1, \ldots, i-1$ and $v_r(x^i+), r = i+1, \ldots, n$. The characteristics should be curved.

Define the following matrices $D = \text{diag}(D_1 \dots D_n)$ where

$$D_i = \text{diag}\left(\left(\frac{\lambda_\ell^{i-1} - x^{i-1}}{\lambda_\ell^{i-1} - x^i} \right)_{\ell=1}^{i-1}, \left(\frac{\lambda_r^i - x^{i+1}}{\lambda_r^i - x^i} \right)_{r=i+1}^{n} \right), \quad \alpha+1 \le i \le \beta-1,$$

$$D_\alpha = \text{diag}\left(\left(\frac{\lambda_r^\alpha - x^{\alpha+1}}{\lambda_r^\alpha - x^\alpha} \right)_{r=\alpha+1}^{n} \right),$$

$$D_\beta = \text{diag}\left(\left(\frac{\lambda_\ell^{\beta-1} - x^{\beta-1}}{\lambda_\ell^{\beta-1} - x^\beta} \right)_{\ell=1}^{\beta-1}, \right).$$

Then from (5.3) and (5.4), for all $\alpha+1 \le i \le \beta-1$,

$$\begin{pmatrix} v_1(x^{i-1}+) \\ \vdots \\ v_{i-1}(x^{i-1}+) \\ v_{i+1}(x^{i+1}-) \\ \vdots \\ v_n(x^{i+1}-) \end{pmatrix} = D_i^s \chi_i + \begin{pmatrix} H_1(x^{i-1}+, s) \\ \vdots \\ H_{i-1}(x^{i-1}+, s) \\ H_{i+1}(x^{i+1}-, s) \\ \vdots \\ H_n(x^{i+1}-, s) \end{pmatrix}. \tag{5.5}$$

For $i = \alpha$ or β, (5.5) still holds with some modification: Delete the first $\alpha - 1$ rows if $i = \alpha$ and delete the last $n - \beta$ rows if $i = \beta$.

The last column vector in (5.5) shall be denoted by $\mathcal{H}_i, i = \alpha, \dots, \beta$. We also have

$$\begin{pmatrix} v_\alpha(x^\alpha-) \\ \vdots \\ v_n(x^\alpha-) \end{pmatrix} = \mathcal{H}_{\alpha-1} := \begin{pmatrix} H_\alpha(x^\alpha-, s) \\ \vdots \\ H_n(x^\alpha-, s) \end{pmatrix}, \tag{$(5.5)_{\alpha-1}$}$$

$$\begin{pmatrix} v_1(x^\beta+) \\ \vdots \\ v_\beta(x^\beta+) \end{pmatrix} = \mathcal{H}_{\beta+1} := \begin{pmatrix} H_1(x^\beta+, s) \\ \vdots \\ H_\beta(x^\beta+, s) \end{pmatrix}. \tag{$(5.5)_{\beta+1}$}$$

From (H4), at the shock Λ^i, the following vectors

$$\mathbf{r}_1^{i-1}, \dots, \mathbf{r}_{i-1}^{i-1}, \Delta^i, \mathbf{r}_{i+1}^i, \dots, \mathbf{r}_n^i, \tag{5.6}$$

are linearly independent and form a basis in $\mathbb{R}^n$. Any $\mathbf{u} \in \mathbb{R}^n$ can be expressed uniquely as:

$$\mathbf{u} = \sum_{j=1}^{i-1} \alpha_j \mathbf{r}_j^{i-1} + \alpha_i \Delta^i + \sum_{j=i+1}^{n} \alpha_j \mathbf{r}_j^i.$$

Let B_i be the matrix of which the columns are vectors (5.6). Then

$$(\alpha_1, \dots, \alpha_n)^\tau = B_i^{-1} \mathbf{u}.$$

Let

$$\bar{E}_i = \begin{pmatrix} I_{(i-1)\times(i-1)} & 0 & 0 \\ 0 & 0 & -I_{(n-i)\times(n-i)}. \end{pmatrix}$$

$$\bar{E}_i^{(1)} = \begin{pmatrix} I_{(i-1)\times(i-1)} & 0 & 0. \end{pmatrix}, \quad \bar{E}_i^{(2)} = \begin{pmatrix} 0 & 0 & -I_{(n-i)\times(n-i)}. \end{pmatrix}$$

Then

$$(\alpha_1, \dots, \alpha_{i-1}, -\alpha_{i+1}, \dots, -\alpha_n)^\tau = \tilde{E}_i B_i^{-1}\mathbf{u},$$

$$(\alpha_1, \dots, \alpha_{i-1})^\tau = \tilde{E}_i^{(1)} B_i^{-1}\mathbf{u}, \quad (-\alpha_{i+1}, \dots, -\alpha_n)^\tau = \tilde{E}_i^{(2)} B_i^{-1}\mathbf{u}.$$

Let the left and right going waves in R^i and R^{i-1} be

$$V_\ell^i(x) = \sum_1^i v_\ell(x)\mathbf{r}_\ell^i, \qquad V_r^i(x) = \sum_{i+1}^n v_r(x)\mathbf{r}_r^i, \quad x \in R^i,$$

$$V_\ell^{i-1}(x) = \sum_1^{i-1} v_h(x)\mathbf{r}_h^{i-1}, \qquad V_r^{i-1}(x) = \sum_i^n v_k(x)\mathbf{r}_k^{i-1}, \quad x \in R^{i-1}.$$

From the jump condition at x^i, we have

$$V_r^i(x^i+) + V_\ell^i(x^i+) = V_r^{i-1}(x^i-) + V_\ell^{i-1}(x^i-) + S^i\Delta^i.$$

Written in the coordinates,

$$\sum_1^{i-1} v_h(x^i-)\mathbf{r}_h^{i-1} - \sum_{i+1}^n v_r(x^i+)\mathbf{r}_r^i + S^i\Delta^i = \sum_1^i v_\ell(x^i+)\mathbf{r}_\ell^i - \sum_i^n v_k(x^i-)\mathbf{r}_k^{i-1}.$$

Applying $\tilde{P}_i := \tilde{E}_i B_i^{-1}$, $\tilde{P}_i^{(1)} := \tilde{E}_i^{(1)} B_i^{-1}$ and $\tilde{P}_i^{(2)} := \tilde{E}_i^{(2)} B_i^{-1}$ to the equation above, we have

$$\begin{aligned} &(v_1(x^i-), \dots, v_{i-1}(x^i-), v_{i+1}(x^i+), \dots, v_n(x^i+))^\tau \\ =&\tilde{P}_i(\sum_1^i v_\ell(x^i+)\mathbf{r}_\ell^i - \sum_i^n v_k(x^i-)\mathbf{r}_k^{i-1}), \end{aligned}$$

$$(v_1(x^i-), \dots, v_{i-1}(x^i-))^\tau = \tilde{P}_i^{(1)}(\sum_1^i v_\ell(x^i+)\mathbf{r}_\ell^i - \sum_i^n v_k(x^i-)\mathbf{r}_k^{i-1}),$$

$$(v_{i+1}(x^i+), \dots, v_n(x^i+))^\tau = \tilde{P}_i^{(2)}(\sum_1^i v_\ell(x^i+)\mathbf{r}_\ell^i - \sum_i^n v_k(x^i-)\mathbf{r}_k^{i-1}).$$

For $\alpha \le i \le \beta$, define the $(n-1)\times(n-i+1)$ and $(n-1)\times i$ matrices

$$M_{i-1}^i = -\tilde{P}_i(\mathbf{r}_i^{i-1}, \dots, \mathbf{r}_n^{i-1}), \qquad M_{i+1}^i = \tilde{P}_i(\mathbf{r}_1^i, \dots, \mathbf{r}_i^i),$$

$$M_{i-1}^{i,(1)} = -\tilde{P}_i^{(1)}(\mathbf{r}_i^{i-1}, \dots, \mathbf{r}_n^{i-1}), \qquad M_{i+1}^{i,(1)} = \tilde{P}_i^{(1)}(\mathbf{r}_1^i, \dots, \mathbf{r}_i^i),$$

$$M_{i-1}^{i,(2)} = -\tilde{P}_i^{(2)}(\mathbf{r}_i^{i-1}, \dots, \mathbf{r}_n^{i-1}), \qquad M_{i+1}^{i,(2)} = \tilde{P}_i^{(2)}(\mathbf{r}_1^i, \dots, \mathbf{r}_i^i).$$

Here M^i_{i-1} or M^i_{i+1} represents the contribution to the waves leaving Λ^i from the waves leaving Λ^{i-1} or Λ^{i+1} respectively.

The waves leaving Λ^i can be expressed by the waves hitting Λ^i from the left and right as

$$\begin{pmatrix} v_1(x^i-) \\ \vdots \\ v_{i-1}(x^i-) \\ v_{i+1}(x^i+) \\ \vdots \\ v_n(x^i+) \end{pmatrix} = M^i_{i-1} \begin{pmatrix} v_i(x^i-) \\ \vdots \\ v_n(x^i-) \end{pmatrix} + M^i_{i+1} \begin{pmatrix} v_1(x^i+) \\ \vdots \\ v_i(x^i+) \end{pmatrix}, \tag{5.7}$$

$$\begin{pmatrix} v_1(x^i-) \\ \vdots \\ v_{i-1}(x^i-) \end{pmatrix} = M^{i,(1)}_{i-1} \begin{pmatrix} v_i(x^i-) \\ \vdots \\ v_n(x^i-) \end{pmatrix} + M^{i,(1)}_{i+1} \begin{pmatrix} v_1(x^i+) \\ \vdots \\ v_i(x^i+) \end{pmatrix}, \tag{5.8}$$

$$\begin{pmatrix} v_{i+1}(x^i+) \\ \vdots \\ v_n(x^i+) \end{pmatrix} = M^{i,(2)}_{i-1} \begin{pmatrix} v_i(x^i-) \\ \vdots \\ v_n(x^i-) \end{pmatrix} + M^{i,(2)}_{i+1} \begin{pmatrix} v_1(x^i+) \\ \vdots \\ v_i(x^i+) \end{pmatrix}. \tag{5.9}$$

Note that in the right hand sides of (5.7), (5.8), and (5.9), $(v_i(x^i-), \dots, v_n(x^i-))^\tau$ comes from the lower half of (5.5) with i replaced by $i-1$, and $(v_1(x^i+, \dots, v_i(x^i+))^\tau$ comes from the upper half of (5.5) with i replaced by $i+1$. In other words, the waves hitting Λ^i come from Λ^{i-1} and Λ^{i+1}. This motivates the definition of matrix $\tilde{M}$ with the following block structure:

$$\tilde{M} = \begin{pmatrix} M^{\alpha,(2)}_{\alpha-1} & [\Theta] & [M^{\alpha,(2)}_{\alpha+1}\Theta] & & & & & \\ & [\Theta M^{\alpha+1}_{\alpha}] & [\Theta] & [M^{\alpha+1}_{\alpha+2}\Theta] & & & & \\ & & [\Theta M^{\alpha+2}_{\alpha+1}] & [\Theta] & [M^{\alpha+2}_{\alpha+3}\Theta] & & & \\ & & & & \vdots & \vdots & & \\ & & & & & [\Theta M^{\beta,(1)}_{\beta-1}] & [\Theta] & M^{\beta,(1)}_{\beta+1} \end{pmatrix}.$$

In the above, the three matrices $[\Theta M^i_{i-1}], [\Theta], [M^i_{i+1}\Theta]$ on the same row are used to express the total scattered waves from Λ^i due to the impinging waves hitting it from Λ^{i-1} or Λ^{i+1}. The three matrices $[M^{i-1}_i\Theta], [\Theta], [\Theta M^{i+1}_i]$ on the same column are used to express the contribution of outgoing waves leaving Λ^i eventually hitting Λ^{i-1} from the right or Λ^{i+1} from the left. Exceptions are the matrices $M^{\alpha,(2)}_{\alpha-1}$ and $M^{\beta,(1)}_{\beta+1}$, which are of $(n-\alpha) \times (n-\alpha+1)$ and $(\beta-1) \times \beta$ and are used to express the contribution of $\mathcal{H}_{\alpha-1}$ and $\mathcal{H}_{\beta+1}$ to χ_α and χ_β respectively. The zero matrix $[\Theta]$ on the main diagonal is of size $(n-1) \times (n-1)$. Notice that the column size of the matrix M^i_{i-1} (or M^i_{i+1}) decreases (or increases) as i runs from α to β. The matrix Θ to the left of M^i_{i-1} has $(i-2)$ columns and Θ to the right of M^i_{i+1} has $(n-i-1)$ columns so that the combined size of matrices in each [ ] is of $(n-1)\times(n-1)$.

Let M be the $(n-1)m \times (n-1)m$ matrix resulting from removing the first block of $n-\alpha+1$ columns and the last block of β columns from $\tilde{M}$. It means that the columns containing $M^{\alpha,(2)}_{\alpha-1}$ and $M^{\beta,(1)}_{\beta+1}$ will be dropped. Using the matrix M, from (5.5) and (5.7), we have the following equation for $\{\chi_i\}_{i=\alpha}^{\beta}$:

$$\begin{aligned} &\chi = MD^s\chi + \tilde{M}\mathcal{H}, \\ &\text{or } (I - MD^s)\chi = \tilde{M}\mathcal{H}, \end{aligned} \tag{5.10}$$

where D^s is the power of the diagonal matrix D, $\mathcal{H}$ is from the right hand side of (5.5), $(5.5)_{\alpha-1}$, $(5.5)_{\beta+1}$.

$$\mathcal{H} := (\mathcal{H}^\tau_{\alpha-1}, \ldots, \mathcal{H}^\tau_{\beta+1})^\tau.$$

Definition 5.1. Let $\Xi(s) := \det(I - MD^s)$ and

$$\sigma_M = \sup\{\sigma : \inf_\omega |\Xi(\sigma + i\omega)| = 0\}.$$

In the next section, we show that the roots of $\Xi(s)$ correspond to eigenvalues of the linearized system.

Lemma 5.1. *Let γ be any constant that satisfies $\gamma > \max\{-\eta, \sigma_M\}$. Then the inverse matrix $(I - MD^s)^{-1}$ exists and*

$$\|(I - MD^s)^{-1}\| \leq C(\gamma)$$

uniformly in the region $\sigma \geq \gamma$.

Proof. From the definition of the diagonal matrix D^s, there exists $N > 0$ such that if $\sigma > N$ then $|MD^s| < 1/2$, and thus $\|(I - MD^s)^{-1}\| < 2$. So what left is the region $\Sigma := \{\gamma \leq \sigma \leq N\}$.

Observe that each entry of MD^s is uniformly bounded with respect to $s = \sigma + i\omega \in \Sigma$. Using minors to express the inverse matrix $(I - MD^s)^{-1}$, the numerators are bounded with respect to s. The denominator is $\Xi(s) := \det(I - MD^s)$. Since $\gamma > \sigma_M$, we have that $\Xi(s) \neq 0$ for $s \in \Sigma$.

Assume that there is a sequence $\{s_n\}_1^\infty = \{\sigma_n + i\omega_n\}_1^\infty \subset \Sigma$ such that $\Xi(s_n) \to 0$ as $n \to \infty$. Since the real parts of s_n are bounded, without loss of generality, assume that $\sigma_n \to \sigma_0, \gamma \leq \sigma_0 \leq N$. Let $\tau_n = \sigma_0 + i\omega_n$. Since $\Xi(s)$ is uniformly continuous with respect to σ, we find that $\Xi(\tau_n) \to 0$ as $n \to \infty$. This is a contradiction to $\Re\tau_n = \sigma_0 > \sigma_M$. Therefore, there exists $C_1(\gamma) > 0$ such that $|\Xi(s)| > C_1(\gamma)$ for $s \in \Sigma$.

This proves that $|(I - MD^s)^{-1}|$ is uniformly bounded by a constant $C(\gamma)$. ☐

From Lemma 5.1, for $\sigma \geq \gamma$, system (5.10) has a unique solution

$$\chi(s) = (I - MD^s)^{-1}\tilde{M}\mathcal{H}.$$

The algebraic system (5.2) can be solved for $(v_r(x,s), v_\ell(x,s))$ in R^i if we extract the boundary values $(v_r(x^i+,s), v_\ell(x^{i+1}-,s))$ from χ and substitute them into (5.2). Since χ satisfies (5.10), the jump conditions are satisfied at each x^i.

We have the following lemma about the solution of (5.2).

Lemma 5.2. *Assume that $\gamma > \max\{-\eta, \sigma_M\}$ where η is the constant in the definition of L^2_η and σ_M is as in the Definition 5.1. For the forcing terms H_j of (5.2), assume that the Hypotheses H 5.1 H 5.2 and H 5.3 are satisfied. Assume that $\Re s = \sigma \geq \gamma$.*

(i) Then there exists a unique solution $V(x,s)$ to the system (5.2) that is in $L^2_w(x,\omega)$. Moreover,

$$\int_{\omega=-\infty}^{\infty} \|V(\cdot,s)\|^2_\eta d\omega < C(\gamma)[\int_{\omega=-\infty}^{\infty}(\|H(\cdot,s)\|^2_\eta + \sum_{i=\alpha}^{\beta}\sum_{\ell=1}^{i}|H_\ell(x^i+,s)|^2 + \sum_{i=\alpha}^{\beta}\sum_{r=i}^{n}|H_r(x^i-,s)|^2)d\omega].$$

(ii) The mapping $x \to V(x,s)$ is continuous with values in $L^2(\omega)$ for $x \in \Omega$. It has one-sided limit at each x^i. Moreover, in the same domain of x and $\sigma \geq \gamma$, $(x,s) \to V(x,s)$ is point-wise continuous and $x \to V(x,s)$ is continuous in $C^0(\omega)$ with one-sided limits as $x \to x^i\pm$.

(iii)

$$\sup\{\|V(\cdot,s)\|_\eta : \sigma \geq \gamma\} \leq C \sup\{\|H(\cdot,s)\|_\eta : \sigma \geq \gamma\}.$$

Proof. From the assumption, the vectors $\mathcal{H}_i, i = \alpha-1,\dots,\beta$, are in $L^2(\omega)$ and bounded by

$$\int_{\omega=-\infty}^{\infty}\sum_{i=\alpha-1}^{\beta}|\mathcal{H}_i(s)|^2 d\omega = [\int_{\omega=-\infty}^{\infty}(\sum_{i=\alpha}^{\beta}\sum_{\ell=1}^{i}|H_\ell(x^i+,s)|^2 + \sum_{i=\alpha}^{\beta}\sum_{r=i}^{n}|H_r(x^i-,s)|^2)d\omega].$$

From Lemma 5.1, the matrix $(I - MD^s)^{-1}\tilde{M}$ is uniformly bounded with respect to s, we found that $\chi(s)$ is in $L^2(\omega) \cap C^0(\omega)$, and is uniformly bounded with respect to $\sigma \geq \gamma$, so are the vectors $(v_r(x^i+,s), v_\ell(x^{i+1}-,s)), \alpha \leq i \leq \beta-1$.

STEP I, Solving (5.2) in bounded regions $R^i, \alpha \leq i \leq \beta-1$: Using $(v_r(x^i+,s), v_\ell(x^{i+1}-,s))$, we can compute $v_j(x,s)$ in each R^i. Observe that $v_j(x.s) = \psi_j(x,s) + H_j(x,s)$ in R^i where

$$\psi_r(x,s) = \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s v_r(x^i,s),\ i+1 \leq r \leq n, \quad \psi_\ell(x,s) = \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s v_\ell(x^{i+1},s),\ 1 \leq \ell \leq i.$$

Since the regions are finite and s is bounded to the left in $\mathbb{C}$, there exists $C(\gamma) > 0$ such that

$$\| \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s \| \leq C(\gamma), \quad \| \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s \| \leq C(\gamma) \text{ if } \sigma \geq \gamma. \tag{5.11}$$

From this it follows that in a bounded R^i,

$$\int_{\omega=-\infty}^{\infty}\sum_{j=1}^{n}\|\psi_j(\cdot,s)\|^2 d\omega \leq C(\gamma)\int_{\omega=-\infty}^{\infty}(\sum_{1}^{i}|v_\ell(x^i+,s)|^2 + \sum_{i+1}^{n}|v_r(x^{i+1}-,x)|^2 d\omega.$$

Based on this, we have that in each bounded R^i, $V(x,s)$ is in $L^2_\eta(x,\omega)$. This proves (i).

The proof of (ii) and (iii) are also easy based on the boundedness of R^i and (5.11).

STEP II, Solving (5.2) in unbounded regions $R^{\alpha-1}$ and R^{β}: Using

$$
\begin{aligned}
\begin{pmatrix} v_1(x^\alpha-,s) \\ \vdots \\ v_{\alpha-1}(x^\alpha-,s) \end{pmatrix} &= M^{\alpha,(1)}_{\alpha-1}\mathcal{H}_{\alpha-1} + M^{\alpha,(1)}_{\alpha+1}\begin{pmatrix} v_1(x^\alpha+,s) \\ \vdots \\ v_{\alpha}(x^\alpha+,s) \end{pmatrix}, \\
\begin{pmatrix} v_{\beta+1}(x^\beta+,s) \\ \vdots \\ v_{n}(x^\beta+,s) \end{pmatrix} &= M^{\beta,(2)}_{\beta-1}\begin{pmatrix} v_\beta(x^\beta-,s) \\ \vdots \\ v_{n}(x^\beta-,s) \end{pmatrix} + M^{\beta,(2)}_{\beta+1}\mathcal{H}_{\beta+1},
\end{aligned}
\tag{5.12}
$$

and by (5.2), we find $v_1(x^\alpha-),\ldots,v_{\alpha-1}(x^\alpha-)$ in $R^{\alpha-1}$ and $v_{\beta+1}(x^\beta+),\ldots,v_n(x^\beta+)$ in R^β, all in $L^2(\omega)\cap C^0(\omega)$, uniformly bounded for $\sigma\ge\gamma$.

We consider $x\in R^\beta$ only since $x\in R^{\alpha-1}$ can be treated similarly.

Observe that $v_r(x,s)=\psi_r(x,s)+H_r(x,s)$ in R^β. Since $H_r(x,s)\in L^2_\eta(x,\omega)$, we only need to show that $\psi_r(x,s)=\left(\frac{\lambda_r-x}{\lambda_r-x^\beta}\right)^s v_r(x^\beta,s)\in L^2_\eta(x,\omega)$. The difficulty lies on the fact $\left(\frac{\lambda_r-x}{\lambda_r-x^\beta}\right)^s\to\infty$ if $x\to\lambda_r$ and $-\eta<\sigma=\Re s<0$. We shall see that this trouble is resolved by the help of he wight function in R^β. For any right going waves in R^β with $x^\beta<x<\lambda_r$, using $|(\lambda_r-x)^s|=(\lambda_r-x)^\sigma$,

$$
\begin{aligned}
\|\psi_r(\cdot,s)\|^2_{L^2_\eta(x)} &= |v_r(x^\beta,s)|^2\int_{x^\beta}^{\lambda_r}\left|\left(\frac{\lambda_r-x}{\lambda_r-x^\beta}\right)^s\right|^2(\lambda_r-x)^{2\eta}\frac{dx}{|\lambda_r-x|}, \\
&\le \frac{|\lambda_r-x^\beta|^{2\eta}}{2(\sigma+\eta)}|v_r(x^\beta,s)|^2
\end{aligned}
\tag{5.13}
$$

To prove (iii), from H 5.1, $H_j(x^i\pm,s)$ is bounded uniformly for $\sigma\ge\gamma$. Therefore, for a right moving wave $v_r, r\ge\beta+1$, $x^\beta<x<\lambda^\beta_r$, (iii) follows from (5.13). For $x\in R^\beta, j=1,\ldots,\beta$, or for $x^\beta_j<x<\infty$, $j=\beta+1,\ldots,n$, we have $v_j(x,s)=H_j(x,s)$. For those modes, $\|v_j(\cdot,s)\|=\|H_j(\cdot,s)\|_{L^2_\eta(x,\omega)}$ and (iii) is satisfied.

From (5.13). we also have,

$$
\|\psi_r(\cdot,s)\|^2_{L^2_\eta(x,\omega)}\le\frac{|\lambda_r-x^\beta|^{2\eta}}{2(\sigma+\eta)}|v_r(x^\beta,s)|^2_{L^2(\omega)},\quad \beta+1\le r\le n.
$$

From the above, (i) follows easily.

The proof of (ii) is straightforward and will be omitted.

□

5.2. **Solving the system of integral equations.** To solve $v_j(x,s)$ from (3.2), (3.3) and (5.1), we only need to write the integral terms as $H_j(x,s)$ and use Lemma 5.2. The main result of this section is the following theorem:

Theorem 5.3. *Assume that $h(x)\in L^2_\eta(x)$. Then*

(i) for any constant $\gamma>\max\{-\eta,\sigma_M\}$, there exists a unique solution $V(x,s)\in L^2_\eta(x,\omega)$ to (3.2),

(3.3) and (5.1) if $\Re s = \sigma \geq \gamma$. The mapping $x \to V(x,s) \in L^2(\omega)$ is continuous for $x \in \Omega$ with one sided limits at $x^i, \alpha \leq i \leq \beta$. Moreover

$$\int_{\omega=-\infty}^{\infty} \|V(\cdot,s)\|_\eta^2 d\omega \leq C(\gamma)\|h\|_\eta^2,$$

$$\sup\{\|V(\cdot,s)\|_{L^2_\eta} : \sigma \geq \gamma\} \leq C(\gamma)\|h\|_{L^2_\eta}.$$

(ii) The inverse transform $V(x,t) = \mathcal{L}^{-1}V(x,s)$ satisfies that $e^{-\gamma t}V(x,t) \in L^2_\eta(x,t)$ with

$$\int_{t=0}^{\infty} e^{-2\gamma t}\|V(\cdot,t)\|_\eta^2 dt \leq C(\gamma)\|h\|_\eta^2.$$

It is a weak solution to (2.2) in the sense of distribution.

(iii) The mapping $x \to e^{-\gamma t}V(x,t)$ is continuous from $x \to L^2(t)$, for $\Re s \geq \gamma$, $x \in \Omega$. In this sense the jump conditions are satisfied in the weighted $L^2(t)$ space.

Proof. For $\sigma \geq \gamma$, $\alpha - 1 \leq i \leq \beta$, $F_j(x,s)$ as in §4, let

$$H_j(x,s) = F_j(x,s), \quad j = 1,\dots,n,$$

From Lemma 4.2, $F_j(x,s)$ satisfy conditions H 5.1, H 5.2 and H 5.3. Therefore, from Lemma 5.2, there exists a unique solution $V(x,s)$ to (5.2) with these $H_j(x,s)$. It is the unique solution to (3.2) (3.3) and (5.1). Clearly, all the conditions as in (i) are satisfied.

Let $V(x,t) = \mathcal{L}^{-1}V(x,s)$. Since $V(x,s) \in L^2_\eta(x,\omega)$, from the Paley-Wiener Theorem, $e^{-\gamma t}V(x,t) \in L^2_\eta(x,t)$ and is a weak solution of (2.2). This proves (ii).

Since $x \to V(x,s)$ is continuous for $\Re s \geq \gamma$, from $x \in \Omega \to L^2(\omega)$, therefore, $x \to e^{-\gamma t}V(x,t)$ is continuous from the same domain of $x \to L^2(t)$. This proves (iii) □

Theorem 5.4. *The solution $V(x,t)$ constructed as $\mathcal{L}^{-1}V(x,s)$ is a continuous function of t in the space $L^2_\eta(x)$ with $V(x,0) = h(x)$. Moreover,*

$$\sup_t\{e^{\gamma t}\|V(\cdot,t)\|_{L^2_\eta}\} < C(\gamma)\|h\|_{L^2_\eta}.$$

Proof. Although our solutions are constructed by the Laplace transform, we can show that the solution expressed by integrals in dual space is the same as constructed by the characteristic method in (x,t) space. In particular, several shift operators in L^2 spaces are involved, which are well-known to be continuous in the space of L^2 functions. Based on this, the continuous dependence on time can be proved. Notice that we cannot derive the integral formulas directly by the characteristic method, since we do not assume that the solutions are differentiable.

Consider the mode $V_r(x,t) = \mathcal{L}^{-1}V_r(x,s), r = i+1,\dots,n$ in R^i, $i = \alpha-1,\dots,\beta-1$ where $x^i \leq x \leq \min\{x^{i+1}, \lambda^i_r\}$, $x^{\alpha-1} = -\infty$ and $x^{\beta+1} = \infty$. The other modes can be treated similarly.

Recall that

$$
\begin{aligned}
&v_r(x,t) = \mathcal{L}^{-1} v_r(x,s) \\
=&\mathcal{L}^{-1}\big(\big(\frac{\lambda_r - x}{\lambda_r - x^i}\big)^s v_r(x^i+,s)\big) + \mathcal{L}^{-1}\big(\int_{x^i}^{x} \big(\frac{\lambda_r - x}{\lambda_r - y}\big)^s h_r(y)\frac{dy}{\lambda_r - y}\big) \\
:=&w_1(x,t) + w_2(x,t).
\end{aligned}
$$

Make the spatial change of variable $e^{\xi} = (\lambda_r - x^i)/(\lambda_r - x) \geq 1$. Then $\xi = 0$ corresponds to $x = x^i$. Denote $\bar{v}_r(\xi, s) = v_r(x, s)$.

$$
\begin{aligned}
w_1(x,t) &= \mathcal{L}^{-1}(e^{-s\xi}\bar{v}_r(0,s)) \\
&= \bar{v}_r(0, t-\xi)H(t-\xi).
\end{aligned}
$$

where $H(t)$ is the Heaviside function. In particular, this means that $w_1(x,t)$ is a locally L^2 function in the new spatial variable ξ since $v_r(x^i,t)$ is locally L^2 in time.

Using that $e^{-\gamma t} v_r(x^i,t) \in L^2(t)$, we have (assuming bounded R^i for simplicity)

$$
\begin{aligned}
\|w_1(\cdot,t)\|^2_{L^2_\eta} \leq& \int_{x^i}^{x^{i+1}} |\lambda_r - x|^{2\eta} |w_1(x,t)|^2 \frac{dx}{|\lambda_r - x|} \\
&\leq (\lambda_r - x^i)^{2\eta} \int_0^\infty e^{-2\xi\eta} |\bar{v}_r(0,t-\xi)|^2 d\xi \\
&\leq e^{2\gamma t}(\lambda_r - x^i)^{2\eta} \int_0^\infty e^{-2\gamma\tau} |\bar{v}_r(0,\tau)|^2 d\tau < \infty.
\end{aligned}
$$

This proves that $w_1(x,t)$ is a continuous function from $t \to L^2_\eta(x)$ and the growth rate is $e^{\gamma t}$, $\gamma > \max\{-\eta, \sigma_M\}$.

Consider $w_2(x,t) = \mathcal{L}^{-1}F_r(x,s)$, Using the new variable ξ for y: $e^{\xi} = (\lambda_r - y)/(\lambda_r - x)$, we have

$$
\begin{aligned}
F_r(x,s) &= \int_{\xi=0}^{\infty} e^{-s\xi} h_r(\lambda_r - (\lambda_r - x)e^{\xi}) d\xi \\
&= \mathcal{L} h_r(\lambda_r - (\lambda_r - x)e^{\cdot}).
\end{aligned}
$$

Thus,

$$
w_2(x,t) = h_r(\lambda_r - (\lambda_r - x)e^t).
$$

This is fully expected by the characteristic method. It also shows that $w_2(x,t)$ is a continuous function from $t \to L^2_\eta(x)$ since h is composed with a smooth change of variable that depends on t. Indeed,

$$
\begin{aligned}
\|h_r(\lambda_r - (\lambda_r - x)e^t)\|_{L^2_\eta(x)} &\leq e^{-\eta t}\|h_r\|_{L^2_\eta} \\
&\leq e^{\gamma t}\|h_r\|_{L^2_\eta}.
\end{aligned}
$$

as shown in the proof of Lemma 4.2.

Finally, when $t = 0$, $w_1(x,0) = 0$ and $w_2(x,0) = h(x)$. Thus $v_r(x,0) = h_r(x)$. $\square$

6. Differentiability of solutions for initial data in $D(\mathcal{A})$

Recall that the differential operator $\mathcal{A}$ is defined as in (2.8)

H 6.1. Assume that $h(x) = \sum h_j(x)\mathbf{r}_j^i \in D(\mathcal{A})$.

Remark 6.1. Note that the condition $(Df - xI)V_x \in L^2_\eta$ is not equivalent to that $V_x \in L^2_\eta$ since the variable x is unbounded and $Df - xI$ is singular at $x = \lambda_j^i$ in unbounded regions $R^{\alpha-1}, R^\beta$.

Theorem 6.1. *If h satisfies Hypothesis 6.1, then the L^2 solution constructed in §5 is differentiable. That is, as a continuous function of t, $V(\cdot,t) \in D(\mathcal{A})$ and $V_t(\cdot,t) \in L^2_\eta$ with*

$$\int_{t=0}^\infty e^{-2\gamma t}(\|V(\cdot,t)\|^2 + \|V_t(\cdot,t)\|^2 + \|\mathcal{A}V(\cdot,t)\|^2)dt \le C\|h\|^2,$$

$$\sup_{t\ge 0}\{e^{-\gamma t}(\|V(\cdot,t)\| + \|V_t(\cdot,t)\| + \|\mathcal{A}V(\cdot,t)\|)\} \le C\|h\|.$$

Moreover, for $x \in \Omega$, the L^2 functions $e^{-\gamma t}V(x,t)$, $e^{-\gamma t}V_t(x,t)$ and $e^{-\gamma t}\mathcal{A}V$ depend continuously on x with values in $L^2(t)$.

Proof. Let F_j be defined as in §4. For $i+1 \le r \le n$,

$$\begin{aligned}
sF_r(x,s) &= \int_{x^i}^x s\frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}}h_r(y)dy \\
&= \int_{x^i}^x (\lambda_r - x)^s\partial_y((\lambda_r - y)^{-s})h_r(y)dy \\
&= h_r(x) - \frac{(\lambda_r - x)^s}{(\lambda_r - x^i)^s}h_r(x^i) - \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^s}\partial_y h_r(y)dy \\
&= h_r(x) - \frac{(\lambda_r - x)^s}{(\lambda_r - x^i)^s}h_r(x^i) - \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}}(\lambda_r - y)\partial_y h_r(y)dy
\end{aligned}$$

From (3.2), we have

$$sv_r(x,s) = \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s sv_r(x^i,s) + sF_r(x,s).$$

Let $z_j(x,s) = sv_j(x,s) - h_j(x)$, then

$$z_r(x,s) = \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s z_r(x^i,s) + \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}}(\mathcal{A}h)_r(y)dy. \tag{6.1}$$

Here $(\mathcal{A}h)_j$ is the jth component of vector valued function $\mathcal{A}h$.

Similarly we can show that for $1 \le \ell \le i$,

$$z_\ell(x,s) = \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s z_\ell(x^{i+1},s) + \int_{x^{i+1}}^x \frac{(\lambda_\ell - x)^s}{(\lambda_\ell - y)^{s+1}}(\mathcal{A}h)_\ell(y)dy. \tag{6.2}$$

Let

$$\begin{aligned}
H_r(x,s) &:= \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}}(\mathcal{A}h)_r(y)dy, \quad i+1 \le r \le n, \\
H_\ell(x,s) &:= \int_{x^{i+1}}^x \frac{(\lambda_\ell - x)^s}{(\lambda_\ell - y)^{s+1}}(\mathcal{A}h)_\ell(y)dy, \quad 1 \le \ell \le i.
\end{aligned}$$

Based on H 6.1, we can apply lemmas 4.2 to show that $H_j(x,s) \in L^2_\eta(x,\omega)$ that depends continuously on $x \in R^i$ with values in $L^2(\omega)$.

Let $Z(x,s) = \sum_1^n z_j(x,s)\mathbf{r}_j^i$, then the jump conditions on V and h imply that

$$[Z(x,s)]_{x^i} = 0 \bmod \Delta^i. \tag{6.3}$$

Applying Lemma 5.2 to the systems (6.1), (6.2) and (6.3), we find the function $Z(x,s) = sV(x,s) - h(x)$ is in $L^2_\eta(x,\omega)$ and depends continuously on $x \in \Omega$ with values in $L^2(\omega)$. Note that as shown in Theorem 5.4, $V(x,0) = h(x)$.

Observe that

$$sV(x,s) - h(x) = s\mathcal{L}(V(x,t) - H(t)h(x)).$$

In the Hilbert space L^2_η, the inverse Laplace transform of the right hand side is $\partial_t V(\cdot,t) - \delta(0)h(\cdot)$. From the Plancherel's theorem, $e^{-\gamma t}(\partial_t V(\cdot,t) - \delta(0)h(\cdot))$ is an $L^2(t)$ function in L^2_η.

We now consider the spatial regularity. From (3.2) and (3.3), one easily obtain that

$$(\lambda_j - x)\partial_x v_j(x,s) = -sv_j(x,s) + h_j(x), \quad 1 \le j \le n.$$

Therefore,

$$(Df - xI)V_x(x,s) = -(sV(x,s) - h(x)).$$

If $\sigma \ge \gamma$ then $sV(x,s) - h(x) \in L^2_\eta(x,\omega)$. Therefore $\mathcal{A}V(x,s) \in L^2_\eta(x,\omega)$. By inspecting terms in the right hand side we conclude that $(Df - xI)V_x(x,s)$ depends continuously on $x \in \Omega$ with values in $L^2(\omega)$. Using the inverse Laplace transform, we find that $e^{-\gamma t}\mathcal{A}V(x,t)$ is in $L^2_\eta(x,t)$ and depends continuously on $x \in \Omega$ with values in $L^2(t)$.

Using Theorem 5.4 to $Z(x,s)$, we can show that $e^{-\gamma t}V_t(x,t), e^{-\gamma t}\mathcal{A}V(x,t)$ are continuous, uniformly bounded functions of $t \ge 0$ with values in $L^2_\eta(x)$. □

Theorem 6.2. *(1) $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $e^{t\mathcal{A}}$ in L^2_η.*

(2) The exponential growth rate of the semigroup satisfies

$$\|e^{At}\| \le Ce^{\gamma t}.$$

Proof. In Theorem 5.3, we proved that the densely defined linear operator $\mathcal{A}$ has a nonempty resolvent set $\Re s \ge \gamma$ for any $\gamma > \max\{-\eta, \sigma_M\}$. In Theorem 6.1, we showed that the initial value problem

$$\frac{du(t)}{dt} = \mathcal{A}u(t),\ t > 0, \quad u(0) = h$$

has a unique classical solution $u(t) \in L^2_\eta$, which is continuously differentiable on $[0,\infty)$, for every initial value $h \in D(\mathcal{A})$. Thus, based on a theorem in semigroup theory [17], page 102, $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $e^{t\mathcal{A}}$ in L^2_η.

From Theorem 5.3, part (i), we have

$$\sup\{\|(s-\mathcal{A})^{-1}\| : \Re s \geq \gamma\} < \infty.$$

Based on the Gearhart-Prüss Theorem, the exponential growth rate of the semigroup satisfies

$$\|e^{At}\| \leq Ce^{\gamma t}.$$

□

Remark 6.2. The same growth rate of solutions $V(x,t)$ has been obtained independently in Theorem 5.4, using a prior estimates of the solutions.

7. Eigenvalues and resonance values

Let λ be an eigenvalue for system (2.2) where $\Re\lambda > -\eta$. Let the corresponding eigenvector be $(V(x),\ \{\bar{Y}^i\}_{i=1}^n) \in L^2_\eta \times \mathbb{R}^n$. This a case with $h \equiv 0$. Therefore, $V \in D(\mathcal{A})$ and

$$\lambda V + (Df - xI)V_x = 0, \tag{7.1}$$

$$[V]_{x^i} - \lambda \bar{Y}^i \Delta^i = 0.$$

Let $V(x) = \sum v_j^i(x)\mathbf{r}_j^i$ in R^i.

Lemma 7.1. *Assume that $(V(x),\ \{\bar{Y}^i\}_{i=1}^n)$ is an eigenvector associated to an eigenvalue λ with $\Re\lambda > -\eta$ where η is from the definition of L^2_η. Then $V(x)_j \equiv 0$ for $x \in R^{\alpha-1}, \alpha \leq j \leq n$, and for $x \in R^\beta, 1 \leq j \leq \beta$.*

Proof. In $R^{\alpha-1}$ or R^β, for those values of j, we have

$$v_j^{\alpha-1}(x) = \left(\frac{\lambda_j - x}{\lambda_j - x^\alpha}\right)^\lambda v_j^{\alpha-1}(x^\alpha-), \quad -\infty < x < x^\alpha,$$

$$v_j^\beta(x) = \left(\frac{\lambda_j - x}{\lambda_j - x^\beta}\right)^\lambda v_j^\beta(x^\beta+), \quad x^\beta < x < \infty.$$

Since $\Re\lambda > -\eta$, in order to satisfy the conditions

$$\|v_j^{\alpha-1}\|_{L^2_\eta} < \infty, \quad \|v_j^\beta\|_{L^2_\eta} < \infty,$$

we must have $v_j(x^{\alpha-1}-) = v_j(x^\beta+) = 0$. Therefore $V(x) = 0$ if $x \in R^{\alpha-1}, \alpha \leq j \leq n$, or $x \in R^\beta, 1 \leq j \leq \beta$. □

Similarly, one can show that all the right moving waves $v_j(x,t) = 0$ if $-\infty < x < \lambda_j^{\alpha-1}$ and all the left moving waves $v_j(x,t) = 0$ if $\lambda_j^\beta < x < \infty$. But the characteristic waves leaving x^α to the left or leaving x^β to the right may not be zero.

Remark 7.1. $\lambda = 0$ is alway an eigenvalue. The corresponding eigenspace contains the n-dimensional linear subspace of $L^2_\eta \times \mathbb{R}^n$: $V(x) \equiv 0$, $\{\bar{Y}^i\}_1^n \in \mathbb{R}^n$.

In the U variable, $\lambda = -1$ is alway an eigenvalue which reflect the dynamics of the shock position $x^i(t)$ for the original system (2.1):

$$\dot{X}^i(t) + X^i(t) = 0.$$

The eigenvalue $\lambda = -1$ has another simple interpretation. In the vanishing viscosity approach to shock waves, shocks are limits of traveling waves that have 0 as an eigenvalue corresponding to the shift of traveling waves (shock positions), say by $\Delta\xi^i$. In the similarity coordinate $x = \xi/\tau$, the shift of shock position decays algebraically like $\Delta x^i = \Delta\xi^i/\tau$. If we use $t = \ln\tau$ as time, the decay becomes exponentially in time with the rate $\lambda = -1$.

Theorem 7.2. *(1) In the region $\Re\lambda > -\eta$, $\lambda \neq 0$ is an eigenvalue iff $\Xi(\lambda) := \det(I - MD^\lambda) = 0$. (2) The region $\Re\lambda > -\eta$ contains only normal points of the resolvent equation.*

Proof. (1) If $\det(I - MD^\lambda) = 0$, then system $(I - MD^\lambda)\chi = 0$ has a non-trivial solution χ. This means that with $H_j \equiv 0$ and $s = \lambda$, in $R^i, \alpha \leq i \leq \beta - 1$, system (5.3), (5.4) has a nontrivial solution $v_r(x^i, \lambda), r = i+1, \dots, n$ and $v_\ell(x^{i+1}, \lambda), \ell = 1, \dots, i$. Then an eigenfunction $V(x, \lambda)$ corresponding to λ can be constructed using (3.2), (3.3) and $h \equiv 0$. In $R^{\alpha-1}$ and R^β, $v_j(x, \lambda)$ can be constructed for $\lambda_j^{\alpha-1} < x < x^\alpha, j \leq \alpha - 1$ and $x^\beta < x < \lambda_j^\beta, j \geq \beta + 1$. The rest of the waves in $R^{\alpha-1}$ and R^β are zeros as by Lemma 7.1.

On the other hand if $V(x, \lambda)$ is an eigenfunction corresponding to a non-zero eigenvalue λ, then the system $(I - MD^\lambda)\chi = 0$ has a non-trivial solution χ. Therefore $\det(I - MD^\lambda) = 0$.

(2) If $\det(I - MD^\lambda) \neq 0$, then $(I - MD^\lambda)^{-1}$ exits for such λ. From the previous section, if $h \in L^2_\eta$, then we have a unique solution $V \in L^2_\eta$ for the resolvent equation

$$V_x + \lambda(Df - xI)^{-1}V = (Df - xI)^{-1}h, \quad [V]_{x^i} = 0 \bmod \Delta^i.$$

Certainly $\|V\| \leq C(\lambda)\|h\|$ for some constant C (uniform boundedness theorem in Banach spaces). This shows if λ is not an eigenvalue then it is a resolvent point in $\Re\lambda > -\eta$.

□

Let σ_m be the largest real parts of the zeros of $\Xi(s) = \det(I - MD^s)$, i.e:

$$\sigma_m = \sup\{\sigma : \text{there exists } \omega \text{ such that } \Xi(\sigma + i\omega) = 0\}.$$

Then $\sigma_m \leq \sigma_M$ and the two can be different.

For any $\sigma_0 > -\eta$, as a function of ω, $\Xi(\sigma_0 + i\omega)$ is quasi-periodic, with the frequencies defined by finite linear combinations of $\ln|x^{i+1} - \lambda_j^i| - \ln|x^i - \lambda_j^i|$, $i = \alpha, \dots, \beta - 1$, $j = 1, \dots, n$.

If the frequencies are rationally related, then $\Xi(\sigma_0 + i\omega)$ is periodic in ω. In this case $\inf_\omega |\Xi(\sigma_0 + i\omega)| = 0$ generally implies that there exists ω_0 such that $\Xi(\sigma_0 + i\omega_0) = 0$. There are countably many eigenvalues lying on the vertical line $\{\sigma = \sigma_0\}$ with equal vertical spacings. This has been verified in examples consisting of two shocks [11].

If the frequencies are not rationally related, then it is possible to find $\sigma_0 > \sigma_m$ such that $\inf_\omega |\Xi(\sigma_0 + i\omega)| = 0$.

Definition 7.1. If $s \in \mathbb{C}$ with $0 < |\Xi(s)| \leq \delta$, then s is called a *resonance value* of order δ, or a δ resonance value. A vertical line $\Re s = \sigma_0$ in the complex plane that contains resonance values of arbitrarily small order is called a *resonance line* and σ_0 is said to be the *coordinate of the resonance line.*

A number σ_0 is the coordinate of a resonance line, if and only if

$$\inf_\omega |\Xi(\sigma_0 + i\omega)| = 0.$$

It is not hard to show that at a *resonance line* $\{\sigma + i\omega | \sigma = \sigma_0\}$, by choosing ω, the system response to forcing terms with frequency ω can be arbitrarily large.

Theorem 7.3. *There exists a constant $C > 0$ such that if $s \in \mathbb{C}$ is a resonance value of order δ, then*

$$C/\delta < \|R(A, s)\|.$$

The resonance lines are exactly the vertical lines of pseudo-eigenvalues

If a resonance line with coordinate greater than σ_m exists, then $\sigma_m < \sigma_M$. The existence of resonance lines has not been verified by numerical computations.

Example 7.1. (1) For a system of n equations with m Lax shocks, $m \leq n$, the determinant has the form

$$\Xi(s) = 1 - \sum_{j=1}^{q} a_j e^{b_j s}, \quad b_j < 0$$

with a possibly large q. Since $b_j < 0$, there exists a sufficiently large $\gamma > 0$ such that if $\Re s \geq \gamma$ then $\Xi(s) \neq 0$ and

$$\inf\{|\Xi(s)| : \Re s \geq \gamma\} \geq C(\gamma) > 0.$$

(2) For a system of two equations with two Lax shocks, under general conditions,

$$\Xi(\sigma + i\omega) = 1 - a e^{b(\sigma + i\omega)}$$

is periodic in ω. Eigenvalues not equal to -1 are are equally spaced on a unique vertical line [11].

(3) For a system of three equations with three shocks, in the this section we will show that

$$\Xi(s) = 1 - \sum_{j=1}^{8} a_j e^{b_j s}, \quad b_j < 0. \tag{7.2}$$

There may be resonance values on a vertical line if the frequencies $b_j, j = 1, \ldots, 8$, are not rationally related.

(4) As an simplified artificial example, consider the function

$$\Xi(s) = 1 - a_1 e^{-(\sigma+i\omega)} - a_2 e^{-\alpha(\sigma+i\omega)}.$$

Assume that there exits $\sigma_0 \in \mathbb{R}$ such that $a_1 e^{-\sigma_0} + a_2 e^{-\alpha\sigma_0} = 1$. If p/q is rational, then $s = \sigma_0 + 2k\pi iq$, with $k \in \mathbb{Z}$ is an eigenvalue. If p/q is irrational, then $\Re s = \sigma_0$ is the coordinate of a resonance line where $|\Xi(\sigma_0 + i\omega)|$ can be arbitrarily small but nonzero.

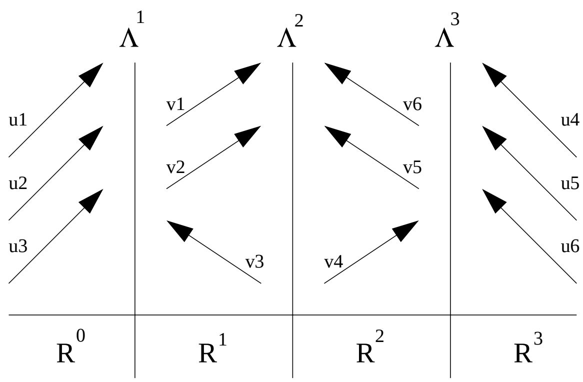

FIGURE 7.1. An example of three Lax shocks and four regions.

7.1. **A system of three equations with three Lax shocks.** We look for an eigenfunction that is defined in 4 regions and has three component in each region [Figure 7.1]. In the unbounded regions, $u_j = 0,\ 1 \le j \le 6$. We look for characteristic modes $v_j,\ 1 \le j \le 6$ in regions R^1 and R^2.

The scattering matrix M is of 6×6.

$$M = \begin{pmatrix} 0 & 0 & n_1 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 \\ a & b & 0 & 0 & e & f \\ c & d & 0 & 0 & g & h \\ 0 & 0 & 0 & n_3 & 0 & 0 \\ 0 & 0 & 0 & n_4 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

Each entry m_{ij} in M is the rate of the scattering wave v_i from the shocks produced by the impinging wave v_j hitting the shocks. For example, n_1 and n_2 represent the rate the out going modes v_1 and v_2 produced by v_3, after scattered by the shock Λ^1. The entries (a, b, c, d) represent the conversion of (v_1, v_2) to (v_3, v_4) after hitting Λ_2, etc.

Let ϕ_j be the growth rate the wave v_j moving from shock to shock:

$$\phi_1 := \left(\frac{\lambda_2^1 - x^2}{\lambda_2^1 - x^1}\right)^s, \quad \phi_2 := \left(\frac{\lambda_3^1 - x^2}{\lambda_3^1 - x^1}\right)^s, \quad \phi_3 := \left(\frac{\lambda_1^1 - x^1}{\lambda_1^1 - x^2}\right)^s,$$
$$\phi_4 := \left(\frac{\lambda_3^2 - x^3}{\lambda_3^2 - x^2}\right)^s, \quad \phi_5 := \left(\frac{\lambda_1^2 - x^2}{\lambda_1^2 - x^3}\right)^s, \quad \phi_6 := \left(\frac{\lambda_2^2 - x^2}{\lambda_2^2 - x^3}\right)^s.$$

For s to be an eigenvalue, the following matrix must be singular:

$$(-1 + MD^s) = \begin{pmatrix} -1 & 0 & n_1\phi_3 & 0 & 0 & 0 \\ 0 & -1 & n_2\phi_3 & 0 & 0 & 0 \\ a\phi_1 & b\phi_2 & -1 & 0 & e\phi_5 & f\phi_6 \\ c\phi_1 & d\phi_2 & 0 & -1 & g\phi_5 & h\phi_6 \\ 0 & 0 & 0 & n_3\phi_4 & -1 & 0 \\ 0 & 0 & 0 & n_4\phi_4 & 0 & -1 \end{pmatrix}.$$

Using the column operations to eliminate $(n_1\phi_3, n_2\phi_3, n_3\phi_4, n_4\phi_4)$ and by setting $\phi_{ij} := \phi_i\phi_j$, $\phi_{ijk\ell} = \phi_i\phi_j\phi_k\phi_\ell$, we find that

$$\begin{aligned}\Xi(s) &= \begin{pmatrix} -1 + an_1\phi_{13} + bn_2\phi_{23} & en_3\phi_{45} + fn_4\phi_{46} \\ cn_1\phi_{13} + dn_2\phi_{23} & -1 + gn_3\phi_{45} + hn_4\phi_{46} \end{pmatrix} \\ &= 1 - an_1\phi_{13} - bn_2\phi_{23} - gn_3\phi_{45} - hn_4\phi_{46} \\ &+ (ag - ce)n_1n_3\phi_{1345} + (ah - cf)n_1n_4\phi_{1346} \\ &+ (bg - de)n_2n_3\phi_{2345} + (bh - df)n_2n_4\phi_{2346},\end{aligned}$$

Using a change of variable, it is easy to express each of the term $\phi_j = e^{b_j s}$ and $\Xi(s)$ has the desired form of (7.2). Note that Since $\phi_{1345} = \phi_{13}\phi_{45}$, etc, the exponents of (7.2) satisfy

$$b_5 = b_1b_3, \quad b_6 = b_1b_4, \quad b_7 = b_2b_3, \quad b_8 = b_2b_4.$$

There are only 4 basic frequencies determined by $b_1, \dots, b_4$. The other 4 are linear combinations of the first 4. Therefore, if $b_1 \dots, b_4$ are rationally related, then $\Xi(s)$ is periodic in ω for the fixed σ. Otherwise, $\Xi(s)$ is quasi-periodic in ω.

7.2. **A system of three equations with two Lax shocks.** Let the location of the two shocks $\Lambda^i, i = 1, 2$ be $x^i, i = 1, 2$. The x axis is divided by the shocks into three regions R^i. For the solution $\bar{u}^i$ is R^i assume that $Df(\bar{u}^i)$ has i eigenvalues that is less than x^i and $n - i$ eigenvalues that is greater than x^{i+1}. The numbers of left-right waves are depicted in Fig. 7.2

It has been proved in Lemma 7.1, if the eigenfunctions is in L^2_η for $\Re\lambda > -\eta$, then $u_j = 0, j = 1, \dots, 5$. We look for the modes $v_j, j = 1, \dots, 3$. Then v_4 can be calculated later.

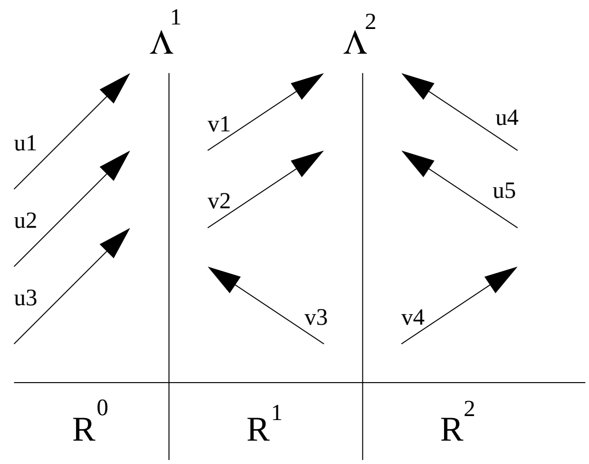

FIGURE 7.2. There are three regions divided by two Lax shocks

We set $\chi_1 = (v_1, v_2)^\tau$, $\quad \chi_2 = (v_3)$. The scattering matrix can be obtained by dropping the last three rows and three columns of the matrix M in (7.3).

$$M = \begin{pmatrix} 0 & 0 & n_1 \\ 0 & 0 & n_2 \\ a & b & 0 \end{pmatrix}.$$

Let ϕ_1, ϕ_2, ϕ_3 be the growth rate of the waves v_1, v_2, v_3 from one shock to another. In order for s to be an eigenvalue, the following matrix must be singular:

$$(-1 + MD^s) = \begin{pmatrix} -1 & 0 & n_1\phi_3 \\ 0 & -1 & n_2\phi_3 \\ a\phi_1 & b\phi_2 & -1 \end{pmatrix}.$$

Thus,

$$\Xi(s) := 1 - an_1\phi_{13} - bn_2\phi_{23} = 0.$$

There are two frequencies involved, $\phi_{13} = \phi_1\phi_3$ and $\phi_{23} = \phi_2\phi_3$. They are determined by the total times that the wave v_3 traveling from Λ_2 to Λ^1 and then reflected from Λ_1 and following the directions of v_1 and v_2 to Λ^2 again. Resonance values may exist if the time of traverse of v_3 and v_1 is not rationally related to the time of traverse of v_3 and v_2.

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