Arch. Rational Mech. Anal. 114 (1991) 267–295.

Existence Theory for Damped Gravity Waves in a Closed Rectangular Basin

XIAO-BIAO LIN

Communicated by C. DAFERMOS

Abstract

We study existence and uniqueness of solutions of the equations for the free surface motion of an incompressible, irrotational fluid in a rectangular basin subject to vertical oscillation. After adding artificial damping, which leaves the flow irrotational but correctly represents the physical rate of energy loss at high wave numbers, we prove global existence and uniqueness results in the appropriate Sobolev spaces, provided that the initial data and forcing amplitudes have sufficiently small norms. Convergence of spatially discretized (finite-dimensional) projections is also discussed.

§ 1. Introduction

Free surface waves of inviscid, incompressible, irrotational fluids have attracted great attention among applied scientists, engineers and mathematicians in recent years. By introducing a potential function for the velocity field, one can rewrite the Euler equation with fewer unknown functions. Moreover, the resulting equations possess a (canonical) Hamiltonian structure and also admit both Lagrangian and Eulerian formulations. These "gravity wave" equations will be given below: see WHITHAM [1974, § 13] for a derivation.

Existence theorems for solutions in special classes of analytic functions have been proved for both Lagrangian and Eulerian formulations: See NALIMOV [1969], OVSJANNIKOV [1971, 1974], SHINBROT [1976], KANO & NISHIDA [1979]. Possibly due to technical difficulties with the Eulerian formulation, existence theorems in Sobolev spaces have only been obtained for the Lagrangian formulation: See NALIMOV [1974] and YOSHIHARA [1982]. Moreover, all the existence results up to now are for short time; no global results are available in the literature.

Nevertheless, the Eulerian formulation has the advantage of simplicity and clarity, and the majority of work by physicists and engineers is based on this formulation. In particular, the finite-dimensional Hamiltonian systems obtained after (Galerkin) projection have been studied extensively, *cf.* MILES [1984–1985], and much information about gravity waves has been obtained in this way. When such truncations are studied, they are generally justified by noting that, in the presence of viscosity, modes with high spatial frequency decay rapidly and so do not play a very important role in the long term behavior. In fact, authors such as MILES [1967, 1976] have added "phenomenological" damping terms after truncation of the Hamiltonian system at some (possibly large) number of modes. Discussion of the nature of damping and the effect of damping on different wave numbers can be found in LIGHTHILL [1978], MILES [1967] and the references therein.

Interesting results concerning the stability and bifurcations of steady and time-periodic branches of solutions have been obtained for these finite-dimensional problems. See the series of papers by MILES for examples. Recently, HOLMES [1986] has shown that finite-dimensional truncations of arbitrarily large order, after the addition of suitable (weak) damping, exhibit chaotic motions. His argument uses the method of Melnikov, in which smooth homoclinic manifolds of an unperturbed (averaged) system are shown to split and to intersect transversely due to the perturbation of periodic forcing terms. Similar phenomena have also been studied by GU & SETHNA [1986], GU, SETHNA & NARAIN [1986] and VIRNING, BERMAN & SETHNA [1986]. In this analysis the Hamiltonian structure and the existence of (approximate) integrals of motion is crucial. It is therefore of considerable interest to establish rigorous existence results for the Hamiltonian system perturbed by the addition of weak damping.

The purpose of this paper is twofold: to prove a global existence theorem for gravity waves with weak damping in the Eulerian formulation, and to justify the truncation methods used earlier in studies of such waves. Ultimately, I hope that rigorous results on the undamped system may be obtained by taking the limit of zero damping.

The addition of some form of dissipation is essential for the results of this paper, but rather than include the true dissipation due to kinematic viscosity, I have chosen to add an artificial damping term to the Eulerian equations which exhibits the correct damping rate of the total energy. According to STOKES [1851] or LIGHTHILL [1978], the modes with high wave number, which are the major source of trouble, decay at a rate proportional to $8\nu\lambda^2 \times$ (total kinetic energy), where ν is the kinematic viscosity and λ is the wave number. The energy for the linearized gravity wave is $\frac{1}{2} g \, |v|_0^2 + \frac{1}{2} \gamma \, |\nabla v|_0^2 + (F(0) \, u, u)$, where v describes the surface of the fluid and u is the potential of the velocity field at that surface. The first two terms are the potential energy due to gravity and surface tension, and $(F(0) \, u, u)$ is the kinetic energy. Let μ be a damping coefficient. By adding the term $-\mu \, \nabla^2 u$ to the usual system for gravity waves, we obtain a linearly correct damping rate. For a justification, see the energy estimate in § 6.

The complete system is presented in (2.1)–(2.5) below. Apart from the damping term, this system has been derived by many authors from the Euler equations (BENJAMIN & URSELL [1954]) and also from Hamilton's Principle (MILES [1977]).

In much of the work up-to now, the free surface contact with the container wall is assumed to be orthogonal, based on the assumption that the frictional resistance and the capillary forces between the solid wall and the fluid are negligible. For a more accurate discussion of the contact angle of the equilibrium surface, based on the concept of wetting energy and the principle of virtual work, see FINN [1986]. Unfortunately, at present there is controversy concerning the validity of dynamic contact angle measurements. Despite its doubtful physical basis, the assumption that the fluid surface is orthogonal to the wall has been studied mathematically. BENJAMIN & URSELL [1954] proved that for the linear equation, if the fluid surface is initially orthogonal to the wall, it stays that way. Whether this is the case also for the nonlinear equations is still an open problem. Our existence theorems provide a partial answer to this question.

The method we use is classical. First the linearized system (5.1)–(5.3) is studied. The estimate of the solution (u, v) in terms of the initial data (u_0, v_0) is obtained in two steps, as the sum of an instantaneous solution $(\bar{u}, \bar{v})$ that satisfies the initial data at $t = 0$ and a long-time solution $(\bar{\bar{u}}, \bar{\bar{v}})$ that satisfies (5.1)–(5.3) with zero initial data. To derive estimates for $(\bar{\bar{u}}, \bar{\bar{v}})$, the Laplace transform with respect to the time variable is employed. If the total forcing energy is finite and small (Theorem 2.1), the nonlinear problem is solved by a simple argument involving the Implicit Function Theorem. If the density of the forcing energy is finite and small (Theorem 2.2), an energy inequality is derived to ensure the global existence of solutions. The estimates for the linear problem are also used to prove the convergence of the discretization method.

In § 2, we give a precise description of the problem and state our main results. In § 4, we study the elliptic free boundary value problem which links the velocity potential ϕ to the canonical variables (u, v). In § 5, we establish the basic isomorphism concerning the inhomogeneous linear system. The final results for the nonlinear system are proved in § 6. The convergence of the semi-discretization method is also proved in § 6.

The method we employ is parallel to the method of BEALE [1984], who established the long-time existence and regularity of solutions of the initial value problem for the Navier-Stokes equation with a free surface. For other works on this topic, see ALLAIN [1985] and FUJITA [1985]. The estimates we obtain for the linear problem are close to those for the Navier-Stokes equations, which provides further evidence that the artificial damping we introduce gives the correct rate of energy attenuation.

We conclude this introduction by pointing out that the linear system (5.1)–(5.3) generates an analytic semigroup in the appropriate function space. However, the smoothing effect is not sufficiently strong to make the classical methods work (*cf.* FRIEDMAN [1976], HENRY [1981] and PAZY [1983]).

§ 2. Statement of the problem and the main results

The fluid we consider is contained in a 3-dimensional rectangular basin in (x_1, x_2, y)-space, with a cross section $D = \{0 < x_1 < l_1, 0 < x_2 < l_2\}$. The fluid is bounded below by a flat bottom $S_B = \{y = -d\}$ and above by a free surface $S_F = \{y = v(x_1, x_2, t)\}$. Let

$$\Omega_v = \{(x_1, x_2, y) \mid (x_1, x_2) \in D \text{ and } -d < y < v(x_1, x_2, t)\}.$$

If the motion of the fluid is irrotational, the potential of the velocity field, denoted by $\phi(x_1, x_2, y, t)$, satisfies the following elliptic boundary value problem:

$$
\begin{aligned}
\nabla^2\phi &= 0 \text{ in } \Omega_v, \\
\phi|_{S_F} &= u(x_1, x_2, t), \\
\left.\frac{\partial\phi}{\partial y}\right|_{S_B} &= 0, \\
\left.\frac{\partial\phi}{\partial n}\right|_{\partial D\times(-d,v)} &= 0.
\end{aligned}
\tag{2.1}
$$

It is clear that ϕ is completely determined by (u, v). After solving (2.1), we let $w \stackrel{\text{def}}{=} \left.\frac{\partial\phi}{\partial y}\right|_{S_F}$. Thus w is a function of (u, v), and is denoted by $F(v)\,u$. Suppose the rectangular basin is subject to a vertical oscillation $y''(t) = \alpha(t)$. The motion of the fluid can be determined from the solution of the following Cauchy problem in terms of (u, v):

$$u_t - \mu\,\nabla_x^2 u - \gamma\,\nabla_x^2 v + gv = P\{\alpha(t)\,v - \tfrac{1}{2}\,|\nabla_x u|^2 + \tfrac{1}{2}\,w^2(1 + |\nabla_x v|^2)\}, \tag{2.2}$$

$$v_t = w + w\,|\nabla_x v|^2 - \nabla_x u\cdot\nabla_x v, \tag{2.3}$$

$$u(0) = u_0, \qquad v(0) = v_0, \tag{2.4}$$

where

$$w = F(v)\,u, \tag{2.5}$$

g is the acceleration of gravity, γ is the surface tension, and μ is the damping coefficient. P is the projection operator defined by $Pf = f - (l_1 l_2)^{-1} \int_D f\,dx$. Because of the term $w = F(v)\,u$, (2.2)–(2.4) is not a standard system of partial differential equations. If μ, γ, and the right-hand side of (2.2) were zero, equations (2.2) and (2.3) would be Bernoulli's equation and the kinematic boundary condition, respectively, rewritten so that everything is in terms of u, v, w and their x-derivatives. See MILES [1977].

The second term in (2.2) is the artificial damping. The third term in (2.2) is the linearized surface tension [BENJAMIN & URSELL, 1954]. We can, in fact, handle the more precise, nonlinear, form of surface tension by adding higher order terms to the right-hand side of (2.2), without changing much of the analysis. We shall only look for solutions in the range of the projection P, *i.e.*, solutions satisfying $\int_D u\,dx = 0$, which may be obtained by normalization, and $\int_D v\,dx = 0$, which expresses the incompressibility of the fluid. It can be proved that the right-hand side of (2.3) is in the range of P (see § 6). Therefore, the presence of P in equation (2.2) makes the range of P invariant under the nonlinear system.

The appropriate boundary conditions for (u, v) at ∂D are not clear physically. Also the corners of ∂D will cause technical complications. To simplify matters, we shall restrict our study to a special class of solutions. Assume that the traces of odd order of u and v vanish at ∂D to the highest order. We extend u and v

as even functions in the domain $\{-l_1 < x_1 < l_1, -l_2 < x_2 < l_2\}$. We then further extend u and v as periodic functions of period $2l_1$ in x_1 and $2l_2$ in x_2. The extended (u, v) still satisfies (2.2)–(2.4). Accordingly, ϕ is also extended to $\{(x_1, x_2) \in \mathbb{R}^2, -d < y < v(x_1, x_2, t), t \in \mathbb{R}^+\}$ as an even function in (x_1, x_2) and with period $(2l_1, 2l_2)$ in (x_1, x_2). A classical theorem asserts that ϕ satisfies (2.1) after extension; see COURANT & HILBERT [1962]. Hereafter, we assume that u_0 and v_0 are extendable to even periodic functions with period $(2l_1, 2l_2)$ and we study (2.1)–(2.5) in the above class of periodic functions. If (u, v) is such a solution in a certain function space, then a uniqueness theorem implies that u and v are even functions in (x_1, x_2). Hence, the restriction of (u, v) to $D = \{0 < x_1 < l_1, 0 < x_2 < l_2\}$ will be a desired solution of problem (2.1)–(2.5), which is unique in a certain class of functions.

It is well known that periodic distributions are temperate and may be studied by the Fourier transform. Since ϕ is periodic only in (x_1, x_2), it is convenient to avoid Fourier series expansions and use only Fourier transformations. Let $\tilde{\mathbb{R}}^2$ be the torus obtained by identifying the opposite edges of $(-l_1, l_1)\times(-l_2, l_2)$. Let $H^r(\tilde{\mathbb{R}}^2)$, $r \in \mathbb{R}^+$, be the Sobolev space of L^2-functions with L^2-derivatives of order r in $\tilde{\mathbb{R}}^2$. We assume that the functions in $H^r(\tilde{\mathbb{R}}^2)$ have been lifted to $(2l_1, 2l_2)$-periodic functions in $\tilde{\mathbb{R}}^2$. Define $\tilde{\mathbb{R}}^3 = \tilde{\mathbb{R}}^2 \times \mathbb{R}$. For $r > 1$, $H^r(\tilde{\mathbb{R}}^2)$ is continuously embedded in the Banach space $\tilde{B}(\mathbb{R}^2)$ of bounded continuous functions defined on $\tilde{\mathbb{R}}$. Let $d > 0$ be given. Then for each $v \in H^r(\tilde{\mathbb{R}}^2)$ with $|v|_r$ small, we have $|v|_{B(\tilde{\mathbb{R}}^2)} < d$. Let $\tilde{\Omega}_v = \{(x_1, x_2, y) \mid (x_1, x_2) \in \tilde{\mathbb{R}}^2, -d < y < v(x_1, x_2)\}$. $\tilde{\Omega}_v$ is thus an open subset of $\tilde{\mathbb{R}}^3$. Let $H^r(\tilde{\mathbb{R}}^3)$ and $H^r(\tilde{\Omega}_v)$ be the Sobolev spaces in $\tilde{\mathbb{R}}^3$ and $\tilde{\Omega}_v$, and assume that each function in $H^r(\tilde{\mathbb{R}}^3)$ and $H^r(\tilde{\Omega}_v)$ has been lifted to a $(2l_1, 2l_2)$-periodic function in (x_1, x_2). Of course the lifted function is not L^2-integrable. This difficulty can be circumvented by introducing a special measure in $z = (x_1, x_2, y)$ (or $x = (x_1, x_2)$). Define a measure $d\mu(z) = d\mu_1(x_1, x_2) \otimes d\mu_2(y)$ (or $d\mu(x) = d\mu_1(x_1, x_2)$) on $\tilde{\mathbb{R}}^3$ (or $\tilde{\mathbb{R}}^2$), where $d\mu_1$ is the measure with compact support in $(-l_1, l_1)\times(-l_2, l_2)$ and uniform density $\pi^2/(l_1, l_2)$, and $d\mu_2$ is the usual Lebesgue measure on $\mathbb{R}$. For the dual variable $\zeta = (\xi_1, \xi_2, \eta)$, we define the measure $d\nu(\zeta) = d\nu_1(\xi_1, \xi_2) \otimes d\nu_2(\eta)$, (or $\zeta = (\xi_1, \xi_2)$, $d\nu(\xi) = d\nu_1(\xi_1, \xi_2)$), where $d\nu_1$ is the sum of the Dirac measures with unit mass at each lattice point $\boldsymbol{k} \in ((\pi/l_1)\,\mathbb{Z}, (\pi/l_2)\,\mathbb{Z})$, and $d\nu_2$ is the usual Lebesgue measure in $\mathbb{R}$. For $f \in L^2(\tilde{\mathbb{R}}^3)$, the Fourier transform $g = \hat{f} = \mathscr{F} f$ and the inverse transform $f = \check{g} = \bar{\mathscr{F}} g$ are defined as

$$g(\zeta) = \int_{\tilde{\mathbb{R}}^3} e^{-iz\zeta} f(z)\, d\mu(z),$$

$$f(z) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{iz\zeta} g(\zeta)\, d\nu(\zeta).$$

Clearly, $\mathscr{F} : L^2(\tilde{\mathbb{R}}^3, d\mu) \to L^2(\mathbb{R}^3, d\nu)$ is an isomorphism and f is in $H^s(\tilde{\mathbb{R}}^3)$ if and only if

$$|f|_s = \left\{ \int_{\mathbb{R}^3} (1 + |\zeta|^2)^s |\hat{f}(\zeta)|^2\, d\nu \right\}^{1/2} < \infty.$$

In the same manner we can define $H^s(\tilde{\mathbb{R}}^2)$ and its norm in terms of a Fourier transformation.

Let $\tilde{H}^s(\tilde{\mathbb{R}}^2) = \left\{ f \mid f \in H^s(\tilde{\mathbb{R}}^2), \int_{\mathbb{R}^2} f \, d\mu = 0 \right\}$, $s \geqq 0$. Clearly, the mapping $Pf = f - (2\pi)^{-2} \int_{\mathbb{R}^3} f \, d\mu$ is continuous and surjective from $H^s(\tilde{\mathbb{R}}^2)$ to $\tilde{H}^s(\tilde{\mathbb{R}}^2)$ $s \geqq 0$.

Let $K(\tilde{\mathbb{R}}^3 \times \mathbb{R}; r, s) = H^0(\mathbb{R}, H^r(\tilde{\mathbb{R}}^3)) \cap H^{(r-s)/2}(\mathbb{R}, H^s(\tilde{\mathbb{R}}^3))$, $r \geqq s \geqq 0$. We shall use the abbreviated notation $K(r, s)$ if no confusion can arise. The norm of $f \in K(r, s)$ is equivalent to the L^2-norm of

$$|\hat{f}(\tau)|_r + |\tau|^{(r-s)/2} \, |\hat{f}(\tau)|_s,$$

where $\hat{f}(\tau)$ is the Fourier transform of $f(t)$ and τ is the dual variable of t. Other spaces similar to $K(\tilde{\mathbb{R}} \times \mathbb{R}; r, s)$ can be defined with $\tilde{\mathbb{R}}^3$ replaced by $\tilde{\mathbb{R}}^2$ or $\tilde{\Omega}_v$, with the time $t \in \mathbb{R}$ replaced by $t \in \mathbb{R}^+$ or $t \in (t_1, t_2)$. We next define the space $X^r(t_1, t_2)$ to which the solution of the system (2.2)–(2.5) will belong:

$$\begin{aligned} X^r(t_1, t_2) = \{(u, v) \mid (u, v) &\in K(\tilde{\mathbb{R}}^2 \times (t_1, t_2); r, 0) \times K(\tilde{\mathbb{R}}^2 \times (t_1, t_2); r, 0), \\ &v_t \in K(r-1, 1), \textstyle\int u \, d\mu = \int v \, d\mu = 0 \text{ for all } t \in (t_1, t_2)\}, \end{aligned}$$

$$\begin{aligned} Y^r(t_1, t_2) = \{(f_1, f_2) \mid f_1 &\in K(\tilde{\mathbb{R}}^2 \times (t_1, t_2); r-2, 0), f_2 \in K(\tilde{\mathbb{R}}^2 \times (t_1, t_2); r-1, 1), \\ &\textstyle\int f_1 d\mu = \int f_2 \, d\mu = 0 \text{ for all } t \in (t_1, t_2)\}, \end{aligned}$$

$$r \geqq 2, t_1 > t_2.$$

We define the norms as

$$\|(u, v)\|_{X^r} = |u|_{K(r,0)} + |v|_{K(r,0)} + |v_t|_{K(r-1,1)},$$

$$\|(f_1, f_2)\|_{Y^r} = |f_1|_{K(r-2,0)} + |f_2|_{K(r-1,1)}.$$

Our main results are the following:

Theorem 2.1. *Suppose $r > 3$ is given. There is a $\delta_1 > 0$ such that if $0 < \delta < \delta_1$, $u_0 \in \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2)$, $v_0 \in \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2)$, and $\alpha \in H^{(r-2)/2}(\mathbb{R}^+; \mathbb{R})$ with*

$$|u_0|_{r-1} + |v_0|_{r-1/2} + |\alpha|_{(r-2)/2} < \delta,$$

then there exists a unique $(u, v) \in X^r(\mathbb{R}^+)$, satisfying (2.2)–(2.5) with both sides of (2.2) in $K(r-2, 0)$ and both sides of (2.3) in $K(r-1, 1)$. Moreover, $u \in \mathscr{B}(\bar{\mathbb{R}}^+; \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2))$, $v \in \mathscr{B}(\bar{\mathbb{R}}^+; \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2))$, and (2.4) is satisfied. The mapping $(u_0, v_0, \alpha) \to (u, v)$ is in C^∞ with respect to the indicated norms. The solution satisfies the estimate

$$\|(u, v)\|_{X^r(\mathbb{R}^+)} \leqq C\{|u_0|_{r-1} + |v_0|_{r-1/2} + C_1(\delta) \, |\alpha|_{(r-2)/2}\},$$

where C does not depend on δ and $C_1(\delta) \to 0$ as $\delta \to 0$. Moreover (u, v) is also a C^∞-function of $(\gamma, \mu, g) \in (\mathbb{R}^+)^3$.

The hypotheses on $\alpha(t)$ in Theorem 2.1 are not valid if $\alpha(t)$ is a periodic function. This important case will be covered by Theorem 2.2, in which it is assumed that the "energy density" rather than the total energy of $\alpha(t)$ is small.

Theorem 2.2. *Suppose $r > 3$ and $T > 0$ are given. There is a $\delta_1 > 0$ with the following property: If $0 < \delta < \delta_1$, $u_0 \in \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2)$, $v_0 \in \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2)$, and $\alpha \in H^{(r-2)/2}((\bar{t}, \bar{t}+T); \mathbb{R})$ for each $\bar{t} \geqq 0$, with*

$$|u_0|_{r-1} + |v_0|_{r-1/2} + \|\alpha\| < \delta$$

where $\|\alpha\| = \sup_{\bar{t} \in \bar{\mathbb{R}}^+} |\alpha|_{H^{(r-2)/2}((\bar{t}, \bar{t}+T); \mathbb{R})}$, then there exists a unique global solution (u, v) of (2.2)–(2.5). The solution (u, v) belongs to $X^r(\bar{t}, \bar{t}+T)$, and for each $\bar{t} \geqq 0$, it satisfies (2.2) in $K(\tilde{\mathbb{R}}^2 \times (\bar{t}, \bar{t}+T); r-2, 0)$ and (2.3) in $K(\tilde{\mathbb{R}}^2 \times (\bar{t}, \bar{t}+T); r-1, 1)$. Moreover, $(u, v) \in \mathscr{B}(\bar{\mathbb{R}}^+; \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2)) \times \mathscr{B}(\bar{\mathbb{R}}^+; \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2))$, (2.4) is satisfied, and

$$\sup_{\bar{t} \in \bar{\mathbb{R}}^+} \|(u, v)\|_{X^r(\bar{t}, \bar{t}+T)} \leqq C\delta.$$

Furthermore, $(u, v)|_{(\bar{t}, \bar{t}+T)} \in X^r(\bar{t}, \bar{t}+T)$ has a C^∞-dependence on $(u_0, v_0) \in \tilde{H}^{r-1} \times \tilde{H}^{r-1/2}$ and $\alpha|_{(0, \bar{t}+T)} \in H^{(r-2)/2}((0, \bar{t}+T); \mathbb{R})$ with respect to the topology induced by the indicated norms.

Let $\{\varphi_i\}_{i=0}^\infty$ be the orthonormal basis in $L^2(\tilde{\mathbb{R}}^2)$ induced by the Fourier harmonic modes. Let $P_N : L^2(\tilde{\mathbb{R}}^2) \to L^2(\tilde{\mathbb{R}}^2)$ be the projection to the subspace spanned by $\{\varphi_i\}_{i=1}^N$. (We set $\varphi_0 \equiv$ constant in $\tilde{\mathbb{R}}^2$). Obviously P_N is a continuous map from $H^s(\tilde{\mathbb{R}}^2)$ to $H^s(\tilde{\mathbb{R}}^2)$, $s \geqq 0$. We use the following notation:

$$f_1(\alpha, u, v) = P\{\alpha v - \tfrac{1}{2} |\nabla_x u|^2 + \tfrac{1}{2} w^2(1 + |\nabla_x v|^2)\}, \tag{2.6}$$

$$f_2(u, v) = w - F(0)\, u + w\, |\nabla_x v|^2 - \nabla_x u \cdot \nabla_x v, \tag{2.7}$$

where w is given by (2.5). We state our semi-discretization problem using the projection P_N. Find $(u^N, v^N) : \mathbb{R}^+ \to P_N L^2(\tilde{\mathbb{R}}^2)$ satisfying

$$u_t^N - \mu \nabla_x^2 u^N - \gamma \nabla_x^2 v^N + g v^N = P_N f_1(\alpha, u^N, v^N), \tag{2.8}$$

$$v_t^N - F(0)\, u^N = P_N f_2(u^N, v^N), \tag{2.9}$$

$$u^N(0) = P_N u_0, \qquad v^N(0) = P_N v_0. \tag{2.10}$$

We can state existence theorems for system (2.8)–(2.10), which are completely analogous to Theorems 2.1 and 2.2 and which have the same proofs. Moreover, we shall prove the following approximation theorem.

Theorem 2.3. (i) *Suppose that all the hypotheses of Theorem 2.1 are valid. There is a $\delta_1 > 0$ such that if $|u_0|_{r-1} + |v_0|_{r-1/2} + |\alpha|_{(r-2)/2} < \delta_1$, then the global solution $(u, v) \in X^r(\mathbb{R}^+)$ for (2.2)–(2.4) exists, and the global solution $(u^N, v^N) \in$*

$X^r(\mathbb{R}^+)$ *for* (2.8)–(2.10) *exists for all* $N > 0$. *Moreover* $(u^N, v^N) \to (u, v)$ *in* $X^r(\mathbb{R}^+)$ *as* $N \to +\infty$.

(ii) *Suppose that all the hypotheses of Theorem 2.2 are valid. There is a* $\delta_2 > 0$ *such that if* $|u_0|_{r-1} + |v_0|_{r-1/2} + \|\alpha\| < \delta_2$, *then the global solutions* (u, v) *for* (2.2)–(2.4) *and* (u^N, v^N) *for* (2.8)–(2.10) *exist,* $(u, v)|_{(\bar{t}, \bar{t}+T)} \in X^r(\bar{t}, \bar{t} + T)$, $(u^N, v^N)|_{(\bar{t}, \bar{t}+T)} \in X^r(\bar{t}\ \bar{t} + T)$, *and* $\sup\limits_{\bar{t} \in \bar{\mathbb{R}}^+} \|(u^N - u, v^N - v)\|_{X^r(\bar{t}, \bar{t}+T)} \to 0$ *as* $N \to +\infty$.

§ 3. Basic Lemmas

In this section we collect several basic results concerning nonlinear operations on functions in Sobolev spaces, namely, multiplication, composition and change of coordinates. They will be useful in estimating the right hand sides of (2.2) and (2.3) especially in estimating the nonlinear functional $w = F(v)\, u$. For general references see the paper of BOURGUIGNON & BREZIS [1974] and the book of MIZOHATA [1973].

It is well known that $W_p^s(\mathbb{R}^n)$ is an algebra for $s > n/p$. A more general result has been proved by ZOLESIO [1977], which in particular yields

Lemma 3.1. *Let* $r > \dfrac{n}{2}$ $n = 2, 3$. *Let* r_1, r_2 *be such that* $0 \leqq r_1 \leqq r, 0 \leqq r_2 \leqq r$ *and* $r_1 + r_2 \geqq r$. *If* $h_i \in H^{r_i}(\tilde{\mathbb{R}}^n)$, $i = 1, 2$, *then* $h_1 \cdot h_2 \in H^{r_1+r_2-r}(\tilde{\mathbb{R}}^n)$ *with*

$$|h_1 \cdot h_2|_{r_1+r_2-r} \leqq C\, |h_1|_{r_1} \cdot |h_2|_{r_2}.$$

We shall need similar estimates for the products of functions in $K(\tilde{\mathbb{R}}^n \times \mathbb{R}; r, s)$. For a special case of the following lemma, see BEALE [1984].

Lemma 3.2. (i) *Let* $r > \dfrac{n+2}{2}$, $n = 2, 3$, $s \geqq 0$ *and* $r - s > 1$. *Let* r_1, r_2 *be such that* $s \leqq r_1 \leqq r, s \leqq r_2 \leqq r$ *and* $r_1 + r_2 \geqq r + s$. *If* $g \in K(\tilde{\mathbb{R}}^n \times \mathbb{R}; r_1, s)$ *and* $h \in K(\tilde{\mathbb{R}}^n \times \mathbb{R}; r_2, s)$, *then* $gh \in K(r_1 + r_2 - r, s)$ *with*

$$|gh|_{K(r_1+r_2-r,s)} \leqq C\, |g|_{K(r_1,s)}\, |h|_{K(r_2,s)}.$$

(ii) *With the same* r, s *as in* (i), *let* $0 \leqq \delta \leqq s$. *If* $g \in K(r, s)$ *and* $h \in K(r - \delta, s - \delta)$, *then* $gh \in K(r - \delta\ s - \delta)$ *with*

$$|gh|_{K(r-\delta,s-\delta)} \leqq C\, |g|_{K(r,s)}\, |h|_{K(r-\delta,s-\delta)}.$$

Proof. Let X be a Hilbert space. For a measurable function $f: \mathbb{R} \to X$ the Fourier transformation $\hat{f}(\tau) = \int\limits_{-\infty}^{\infty} e^{-i\tau t} f(t)\, dt$ can be defined if $f \in L^2(\mathbb{R}, X)$. We say that $f \in H^s(\mathbb{R}, X)$ if $(1 + |\tau|^s)\, \hat{f}(\tau) \in L^2(\mathbb{R}, X)$. We have the following results, which are similar to those for real-valued functions.

(a) If $0 < s < \frac{1}{2}$, then $H^s(\mathbb{R}, X) \subset L^p(\mathbb{R}, X)$ with continuous embedding for $\frac{1}{p} = \frac{1}{2} - s$. If $s > \frac{1}{2}$, then $H^s(\mathbb{R}, X) \subset \mathscr{B}(\mathbb{R}, X)$ with continuous embedding.

(b) Let Y be a dense subspace in X with continuous injection. If $f \in L^2(\mathbb{R}, Y) \cap H^s(\mathbb{R}, X)$, $s > \frac{1}{2}$, then $f \in \mathscr{B}(\mathbb{R}, [Y, X]_{1/2s})$, where $[Y, X]_\theta$, $0 \leqq \theta \leqq 1$ is the interpolation space between the spaces Y and X (see LIONS & MAGENES [1972]).

Proof of (i): Let $r_1 + r_2 - r = \varrho$. We have $\varrho \geqq s$, and

$$\begin{aligned}
1 + |\zeta|^\varrho + |\tau|^{(\varrho-s)/2}(1 + |\zeta|^s) \\
&\leqq C\{1 + |\zeta'|^\varrho + |\tau'|^{(\varrho-s)/2}(1 + |\zeta'|^s)\} \\
&\quad + C\{1 + |\zeta - \zeta'|^\varrho + |\tau - \tau'|^{(\varrho-s)/2}(1 + |\zeta - \zeta'|^s)\} \\
&\quad + C\{|\tau'|^{(\varrho-s)/2}|\zeta - \zeta'|^s\} + C\{|\tau - \tau'|^{(\varrho-s)/2}|\zeta'|^s\}.
\end{aligned}$$

It follows from the identity $(g \cdot h)\hat{} = \hat{g} * \hat{h}$ that

$$\begin{aligned}
|gh|_{K(\varrho,s)} &\leqq C\,|w(\zeta, \tau, \varrho, s)\,\hat{g} * \hat{h}|_{L^2} + C|\hat{g} * w(\zeta, \tau, \varrho, s)\,\hat{h}|_{L^2} \\
&\quad + C\,||\tau|^{(\varrho-2s)/2}\,\hat{g} * (1 + |\zeta^s)\,\hat{h}|_{L^2} \\
&\quad + C\,|(1 + |\zeta|^s)\hat{g} * |\tau|^{(\varrho-s)/2}\,\hat{h}|_{L^2}
\end{aligned}$$

where $w(\zeta, \tau, \varrho, s) = 1 + |\zeta|^\varrho + |\tau|^{(\varrho-s)/2}(1 + |\zeta|^s)$. Let $g_1 = (w(\zeta, \tau, \varrho, s)\,\hat{g})\check{}$, then $g_1 \in K(r_1 - \varrho, 0)$, $|w(\zeta, \tau, \varrho, s)\,\hat{g} * \hat{h}|_{L^2} \leqq C\,|g_1 h|_{L^2}$.

Case I: $r_2 - s > 1$. We have $h \in \mathscr{B}(\mathbb{R}, H^{r_2-1})$ and $g_1 \in L^2(\mathbb{R}, H^{r_1-\varrho})$. Since $r - 1 > \frac{n}{2}$, it follows from Lemma 3.1 that $|g_1(t)\,h(t)|_{L^2(\tilde{\mathbb{R}}^n)} \leqq C\,|g_1(t)|_{r_1-\varrho}|h(t)|_{r_2-1}$ for $(r_1 - \varrho) + (r_2 - 1) - (r - 1) = 0$ and $H^0 = L^2$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} |g_1(t)\,h(t)|^2_{L^2(\tilde{\mathbb{R}}^n)}\,dt &\leqq C\int_{-\infty}^{\infty} |g_1(t)|^2_{r_1-\varrho}\,|h(t)|^2_{r_2-1}\,dt \\
&\leqq C\,|h|^2_{\mathscr{B}(\mathbb{R},r_2-1)}\int_{-\infty}^{\infty} |g_1(t)|^2_{r_1-\varrho}\,dt \\
&\leqq C\,|h|^2_{K(r_2,s)}\,|g_1|^2_{K(r_1-\varrho,0)} \\
&\leqq C\,|g|^2_{K(r_1,s)}\,|h|^2_{K(r_2,s)}.
\end{aligned}$$

Case II: $r_2 - s < 1$. Then $h \in L^p(\mathbb{R}; H^s)$ with $\frac{1}{p} = \frac{1}{2} - \frac{r_2 - s}{2}$. Also $g_1 \in H^{(1-r_2+s)/2}(\mathbb{R}, H^{r-s-1}) \subset L^q(\mathbb{R}, H^{r-s-1})$ with $\frac{1}{q} = \frac{1}{2} - \frac{1 - r_2 + s}{2}$.

Note that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. Hence

$$\int_{-\infty}^{\infty} |g_1(t)\,h(t)|^2_{L^2(\tilde{\mathbb{R}}^n)}\,dt$$

$$\leqq C\int_{-\infty}^{\infty} |g_1(t)|^2_{r-s-1}\,|h(t)|^2_s\,dt \qquad (\text{for } r-s-1+s=r-1)$$

$$\leqq C\left\{\int_{-\infty}^{\infty} |g_1(t)|^q_{r-s-1}\,dt\right\}^{\frac{2}{q}}\left\{\int_{-\infty}^{\infty} |h(t)|^p_s\,dt\right\}^{\frac{2}{p}}$$

$$\leqq C\,|g|^2_{K(r,s)}\,|h|^2_{K(r_2,s)}.$$

Case III: $r_2-s=1$. By reducing r_2 and r slightly, Case III is reduced to Case II.

Similarly, we obtain the estimate for $|\hat{g}*w(\zeta,\tau,\varrho,s)\,\hat{h}|_{L^2}$. To estimate $||\tau|^{(\varrho-s)/2}\,\hat{g}*(1+|\zeta|^s)\,\hat{h}|_{L^2}$, we observe that $g_2=(|\tau|^{(\varrho-s)/2}\hat{g})^{\check{}}\in K(r_1-\varrho+s,s)$ and $h_2=((1+|\zeta|^s)h)^{\check{}}\in K(r_2-s,0)$. There are two cases to be considered: first, $r_2-s>1$, for which $h_2\in\mathscr{B}(\mathbb{R};H^{r_2-s-1})$ and $g_2\in L^2(\mathbb{R},H^{r_1-\varrho+s})$, and second, $r_2-s\leqq 1$. The latter case can be reduced to $r_2-s<1$ to obtain $h_2\in L^p(\mathbb{R};H^0)$ and $g_2\in L^q(\mathbb{R};H^\beta)$ with $\frac{1}{p}=\frac{1}{2}-\frac{r_2-s}{2}$, $\frac{1}{q}=\frac{r_2-s}{2}$ and $\beta=r-1$. The estimate for $|g_2h_2|_{L^2}$ can then be obtained by the same method as for $|g_1h|_{L^2}$.

Proof of (ii): The proof is similar to that of (i) and is omitted.

Lemma 3.3. (i) *Let* $g_1,\dots,g_m\in H^r(\tilde{\mathbb{R}}^n)$, $r>\frac{n}{2}$, $n=2,3$. *Then* $\prod_{i=1}^m g_i\in H^r(\tilde{\mathbb{R}}^n)$. *If* α *is a multi-index with* $|\alpha|\leqq r$, *then*

$$D^\alpha\prod_{i=1}^m g_i=\sum_{\sum_{j=1}^m\alpha_j=\alpha} C^\alpha_{\alpha_1,\dots,\alpha_m}\prod_{j=1}^m D^{\alpha_j}g_j\in H^{r-|\alpha|}(\tilde{\mathbb{R}}^n),\tag{3.1}$$

which is Leibniz's rule.

(ii) *Let* $g_1,\dots,g_m\in K(\tilde{\mathbb{R}}\times\mathbb{R}^n;r,s)$, $r>\frac{n+2}{2}$, $n=2,3$, $s\geqq 0$, $r-s>1$. *Then* $\prod_{i=1}^m g_i\in K(r,s)$. *If* α *is a multi-index and if* β *is a non-negative integer with* $|\alpha|+2\beta\leqq r-s$, *then*

$$D_t^\beta D_x^\alpha\prod_{i=1}^m g_i=\sum_{\substack{\sum_{j=1}^m\alpha_j=\alpha\\ \sum_{j=1}^m\beta_j=\beta}} C^{(\alpha,\beta)}_{(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m)}\prod_{j=1}^m D_t^{\beta_j}D_x^{\alpha_j}g_j\in K(r-|\alpha|-2\beta,s),\tag{3.2}$$

where $C^\alpha_{\alpha_1,\dots,\alpha_m}$ *and* $C^{(\alpha,\beta)}_{(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m)}$ *are non-negative integers depending on the multi-indices only.*

Proof. (3.1) and (3.2) are valid for smooth functions $g_1, \ldots, g_m$. In the general case, we use mollifiers and pass to the limit. Note that to estimate the right-hand sides of (3.1) and (3.2), we need Lemmas 3.1 and 3.2.

We denote by $\mathrm{Cl}(U)$ or $\bar{U}$ the closure of a subset U in a topological space. We denote by $F \circ g$ the composition of a function F with g.

Lemma 3.4. *Let $F \in C^\infty(U)$, where U is an open subset of $\mathbb{R}$ containing 0, and $F(0) = 0$. Then:*

(i) *$g \to F \circ g$ is a C^∞-mapping from V to $H^r(\tilde{\mathbb{R}}^n)$ for $r > \frac{n}{2}$, $n = 2, 3$, where $V = \{g \mid g \in H^r(\tilde{\mathbb{R}}), \mathrm{Cl}(\mathrm{Range}\, g) \subset U\}$ is an open set in $H^r(\tilde{\mathbb{R}}^n)$. Moreover, the usual chain rule is valid when computing $D^\alpha(F \circ g) \in H^{r-|\alpha|}(\tilde{\mathbb{R}}^n)$, $|\alpha| \leqq r$.*

(ii) *$g \to F \circ g$ is a C^∞-mapping from V_1 to $K(\tilde{\mathbb{R}}^n \times \mathbb{R}; r, s)$ for $r > \frac{n+2}{2}$, $s \geqq 0$, $r - s > 1$, where $V_1 = \{g \mid g \in K(r, s), \mathrm{Cl}(\mathrm{Range}\, g) \subset U\}$ is an open set in $K(r, s)$. Moreover, the usual chain rule is valid when computing $D_t^\beta D_x^\alpha(F \circ g) \in K(r - |\alpha| - 2\beta, s)$, $|\alpha| + 2\beta \leqq r - s$.*

Proof. We only give the proof of (i), since the proof of (ii) is similar. If r is an integer, the proof that $F \circ g \in H^r(\tilde{\mathbb{R}}^n)$ can be found in MIZOHATA [1973]. Let us use a mollifier $g_\delta = g * \varphi_\delta$, $g_\delta \to g$ in $H^r(\tilde{\mathbb{R}}^n)$ as $\delta \to 0$. Since $r > \frac{n}{2}$, $H^r(\tilde{\mathbb{R}}^n) \subset \mathscr{B}(\mathbb{R}^n)$. Without loss of generality, $\mathrm{Cl}(\mathrm{Range}\, g_\delta) \subset U$ for all $\delta \leqq \delta_0$. It is easy to show that $F \circ g_\delta \to F \circ g$ in $L^2(\tilde{\mathbb{R}}^n)$ *as* $\delta \to 0$. Also, $D^\alpha(F \circ g_\delta) \to D^\alpha(F \circ g)$ in $\mathscr{D}'(\tilde{\mathbb{R}}^n)$, where $|\alpha| \leqq r$. Let $r = [r] + r_1$, and write

$$D^\alpha(F \circ g_\delta) = \sum_{\substack{l \leqq |\alpha| \\ \sum_1^l |\alpha_j| \leqq |\alpha|}} C_{\alpha_1, \ldots, \alpha_l}(g_\delta(x)) \prod_{j=1}^{l} D^{\alpha_j} g_\delta(x), \quad |\alpha| \leqq [r]. \tag{3.1}$$

By Lemma 3.1, the right-hand side of (3.1) approaches

$$\sum_{\substack{l \leqq |\alpha| \\ \sum_1^l |\alpha_j| \leqq |\alpha|}} C_{\alpha_1, \ldots, \alpha_l}(g(x)) \prod_{j=1}^{l} D^{\alpha_j} g(x)$$

in $H^{r_1}(\tilde{\mathbb{R}}^n)$. Therefore, $D^\alpha(F \circ g_\delta) \to D^\alpha(F \circ g)$ in $H^{r_1}(\tilde{\mathbb{R}}^n)$, where $|\alpha| \leqq r$. We conclude that $F \circ g_\delta \to F \circ g$ in $H^r(\tilde{\mathbb{R}}^n)$. Letting $\delta \to 0$ in (3.1), we find that g_δ can be replaced by g. Thus the chain rule has been verified.

To show that $g \to F \circ g$ is a C^∞-mapping we observe that $(D_g^l(F \circ g))(h_1, \ldots, h_l) = ((D^l F) \circ g)(h_1, \ldots, h_l)$, which approaches $((D_g^l F) \circ \tilde{g})(h_1, \ldots, h_l)$ as $g \to \tilde{g}$ in $H^r(\tilde{\mathbb{R}}^n)$ uniformly with respect to $h_i \in H^r(\tilde{\mathbb{R}}^n)$, $|h_i|_r \leqq 1$, $i = 1, \ldots, l$.

Corollary 3.5. (i) *Let $V_1 = \{g \mid g \in H^r(\tilde{\mathbb{R}}^n), \mathrm{Cl}(\mathrm{Range}\, g) > -1\}$, $r > \frac{n}{2}$, $n = 2,3$. Then $g \to (g + 1)^{-1} - 1$ is a C^∞-mapping from V_1 to $H^r(\tilde{\mathbb{R}}^n)$.*

(ii) *Let* $V_2 = \{g \mid g \in K(\tilde{\mathbb{R}}^n \times \mathbb{R}; r, s), \mathrm{Cl}(\mathrm{Range}\, g) > -1\}$, $r > \frac{n+2}{2}$, $s \geqq 0$, $r - s > 1$. *Then* $g \to (g+1)^{-1} - 1$ *is a* C^∞*-mapping from* V_2 *to* $K(r, s)$.
(iii) *Similar results are true for the mapping* $g \to (g+1)^{1/2} - 1$.

Lemma 3.6. *Let* θ *be a* C^1*-diffeomorphism from* $\tilde{\mathbb{R}}^3$ *to* $\tilde{\mathbb{R}}^3$. *Suppose that the Jacobian matrix* $d\theta$ *satisfies* $d\theta - I \in (H^{r-1}(\tilde{\mathbb{R}}^3))^9$, $r > \frac{5}{2}$, *with* $\inf\{\det(d\theta)\} \geqq \varepsilon > 0$. *For each* $g \in H^s(\tilde{\mathbb{R}}^3)$, $s \leqq r$, *the functions* $g \circ \theta$ *and* $g \circ \theta^{-1}$ *are in* $H^s(\tilde{\mathbb{R}}^3)$ *with* $|g \circ \theta|_s \leqq C|g|_s$ *and* $|g \circ \theta^{-1}|_s \leqq C|g|_s$. *Moreover,* $d(\theta^{-1})$ *and* $(d\theta)^{-1}$ *satisfy* $(d\theta)^{-1} - I \in (H^{r-1}(\tilde{\mathbb{R}}^3))^9$ *and* $d(\theta^{-1}) - I \in (H^{r-1}(\tilde{\mathbb{R}}^3))^9$.

Proof. That $g \circ \theta$ and $g \circ \theta^{-1}$ are in $H^s(\tilde{\mathbb{R}}^3)$ has been proved by many authors (see BEALE [1984]). Observe that each entry $((d\theta)^{-1})_{i,j}$, $1 \leqq i, j \leqq 3$, can be obtained by adding the products of the entries of $d\theta$ and then dividing by $\det(d\theta)$. From Lemma 3.1 and Corollary 3.5, $(d\theta)^{-1} - I \in (H^{r-1}(\tilde{\mathbb{R}}^3))^9$. Hence $(\theta^{-1}) - I = (d\theta)^{-1} \circ \theta^{-1} - I \in (H^{r-1}(\tilde{\mathbb{R}}^3))^9$.

In proving Lemmas 3.1–3.6, we have used the Fourier transform for functions defined on $\tilde{\mathbb{R}}^n$ (or $\tilde{\mathbb{R}}^n \times \mathbb{R}$). It is known that each $g \in H^r(\tilde{\Omega}_0)$ admits a continuous extension $\tilde{g} \in H^r(\tilde{\mathbb{R}}^3)$ and that each $g \in K(\tilde{\mathbb{R}}^n \times (t_1, t_2); r, s)$ admits a continuous extension $\tilde{g} \in K(\tilde{\mathbb{R}}^n \times \mathbb{R}; r, s)$. Therefore Lemmas 3.1–3.6 are valid with $\mathbb{R}$ replaced by $(t_1, t_2) \subset \mathbb{R}$ or $\tilde{\mathbb{R}}^3$ replaced by $\tilde{\Omega}_0$. Later we shall show that under certain conditions on v, $H^s(\tilde{\Omega}_v)$ admits a continuous extension to $H^s(\tilde{\mathbb{R}}^3)$. Thus, we can replace $\tilde{\mathbb{R}}^3$ by $\tilde{\Omega}_v$ in Lemmas 3.1–3.6. Our final remark is that in Lemma 3.6 we did not discuss the case in which $\theta, g \in K(r, s)$. We shall see in § 4 that only a change of variable for functions in $H^r(\tilde{\mathbb{R}}^3)$, not in $K(r, s)$, is needed.

§ 4. An Elliptic Problem with a Free Boundary

The potential ϕ of the velocity field is determined by (u, v) via the elliptic boundary value problem

$$\begin{aligned} \nabla^2\phi &= 0 \quad \text{in } \tilde{\Omega}_v, \\ \phi|_{S_F} &= u(x_1, x_2, t), \\ \left.\frac{\partial\phi}{\partial y}\right|_{S_B} &= 0. \end{aligned} \tag{4.1}$$

If $v = 0$ and $u \in H^s(\tilde{\mathbb{R}}^2)$, $s \geqq \frac{3}{2}$, then $\phi \in H^{s+1/2}(\bar{\Omega}_0)$. Thus, $w = \left.\frac{\partial\phi}{\partial y}\right|_{S_F} \in H^{s-1}(\tilde{\mathbb{R}}^2)$. Explicit formulas for ϕ and ω can be found when $v = 0$. Let $u = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\zeta x}\, \hat{u}\, dv$, where dv is atomic as defined in § 2. Then

$$\phi = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi x} \{\cosh \xi(y + d)/\cosh \xi\, d\}\, \hat{u}(\xi)\, dv,$$

$$w = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi x} [\xi \tanh \xi\, d]\, \hat{u}(\xi)\, dv.$$

It follows that $\hat{w}(\xi) = (\xi \tanh \xi\, d)\, \hat{u}(\xi)$ and that the mapping $w = F(0)\, u$ extends to all $u \in H^s(\tilde{\mathbb{R}}^2)$, $s \geqq 0$, with $|w|_{s-1} \leqq C\, |u|_s$. Suppose $\int_{\mathbb{R}^2} u(x)\, d\mu = 0$. Then $\int_{\tilde{\Omega}_0} \phi\, d\mu = 0$ and $\int_{\mathbb{R}^2} w\, d\mu = 0$. Therefore $\hat{u}(\xi) = \hat{w}(\xi) = 0$ almost everywhere for $\{|\xi_1| < \pi/l_1\} \cap \{|\xi_2| < \pi/l_2\}$ with respect to the measure $d\nu$. Thus, we have a bounded inverse $u = (F(0))^{-1}\, w$, $|u|_s < C\, |w|_{s-1}$. Also

$$\langle F(0)\, u, u\rangle_{d\mu} = \int (\xi \tanh \xi\, d)\, |\hat{u}(\xi)|^2\, d\nu \geqq C\, |u|_{1/2}^2$$

for all $u \in \tilde{H}^{3/2}(\tilde{\mathbb{R}}^2)$ with C being independent of u. Therefore, $u \to F(0)\, u$ is a self-adjoint positive-definite operator in $\tilde{H}^{1/2}(\tilde{\mathbb{R}}^2)$.

The purpose of the rest of this section is to determine the dependence of w on (u, v) in suitable Sobolev spaces. Assume that $v \in H^r(\tilde{\mathbb{R}}^2) \subset C^1(\mathbb{R}^2)$, $r > 2$, $|v|_r < \delta$. When δ is small, sup $|v(x)| < d$ and the domain $\tilde{\Omega}_v$ is well defined. However, the upper surface of $\tilde{\Omega}_v$ is not smooth enough to apply the classical theory of elliptic boundary value problems. We shall use a change of variable $\theta : \tilde{\Omega}_0 \to \tilde{\Omega}_v$, where θ is a diffeomorphism and $\phi \to \phi_0 = \phi \circ \theta$ maps $H^s(\tilde{\Omega}_v)$ to $H^s(\tilde{\Omega}_0)$. The function ϕ_0 will satisfy an elliptic boundary problem with its coefficients depending on θ, and hence on v. Since ϕ_0 is defined on a smooth domain $\tilde{\Omega}_0$, the classical theory applies. Hereafter we use x to denote (x_1, x_2).

Consider an auxiliary Dirichlet problem:

$$\begin{aligned} \nabla^2 \bar{v}(x, y, t) &= 0 \quad \text{in } \tilde{\Omega}_0, \\ \bar{v}(x, 0, t) &= v(x, t), \\ \bar{v}(x, -d, t) &= 0. \end{aligned} \tag{4.2}$$

Here we introduce t as an independent variable for future use. For the time being et t be fixed. For $v \in H^r(\tilde{\mathbb{R}}^2)$, (4.2) has a unique solution $\bar{v}(\cdot, t) \in H^{r+1/2}(\tilde{\Omega}_0)$, and $|\bar{v}|_{r+1/2} \leqq C\, |v|_r \leqq C\delta$. We can extend $\bar{v}$ to $\tilde{\mathbb{R}}^3$ so that $|\bar{v}|_{H^{r+1/2}(\tilde{\mathbb{R}}^3)} \leqq C_1\, \delta$. Let $\theta : \tilde{\mathbb{R}}^3 \to \tilde{\mathbb{R}}^3$ be defined as

$$\begin{aligned} x &= x_0, \\ y &= y_0 + \bar{v}(x_0, y_0, t) \stackrel{\text{def}}{=} V(x_0, y_0, t). \end{aligned} \tag{4.3}$$

When δ is small, one can prove that θ is a C^1-diffeomorphism on $\mathbb{R}^3$ and maps $\tilde{\Omega}_0$ onto $\tilde{\Omega}_v$. Note that $|d\theta - I|_{r-1/2} \leqq C\, |D\bar{v}|_{r-1/2} \leqq C\, |v|_{r+1/2} \leqq C\delta$. Thus, when δ is small, all the hypotheses in Lemma 3.6 are satisfied and the map $g \to g \circ \theta$ defines a change of variable in $H^s(\tilde{\mathbb{R}}^3)$, for $0 \leqq s \leqq r + \frac{1}{2}$ and $r > 2$. After a close look at the proof of Lemma 3.6, *cf.* Beale [1984], one finds that the same proof yields that the mappings $g \to g \circ \theta$ from $H^s(\tilde{\Omega}_v)$ to $H^s(\tilde{\Omega}_0)$ and $g \to g \circ \theta^{-1}$ from $H^s(\tilde{\Omega}_0)$ to $H^s(\tilde{\Omega}_v)$ are both linear and bounded, for $0 \leqq s \leqq r + \frac{1}{2}$ and $r > 2$. The proof does require a continuation of functions in $H^s(\tilde{\Omega}_0)$ to $H^s(\tilde{\mathbb{R}}^3)$, but does not need a continuation of functions in $H^s(\tilde{\Omega}_v)$ to $H^s(\tilde{\mathbb{R}}^3)$.

Lemma 4.1. *If* $v \in H^r(\tilde{\mathbb{R}}^2)$, $r > 2$, $|v|_r < \delta$, δ *is sufficiently small, and if* $0 \leqq s \leqq r + \frac{1}{2}$, *then there exists a linear bounded operator* $E: H^s(\tilde{\Omega}_v) \to H^s(\tilde{\mathbb{R}}^3)$ *such that* $REg = g$ *for all* $g \in H^s(\tilde{\Omega}_v)$, *where* R *is the restriction of functions in* $H^s(\tilde{\mathbb{R}}^3)$ *to* $\tilde{\Omega}_v$.

Proof. Using θ defined by (4.3), we set $\tilde{g} = E(g \circ \theta) \circ \theta^{-1}$. Then $g \to \tilde{g}$ is the desired mapping. By Lemma 3.6, $|\tilde{g}|_{H^s(\tilde{\mathbb{R}}^3)} \leqq C\,|E(g \circ \theta)|_{H^s(\tilde{\mathbb{R}}^3)} \leqq C\,|g \circ \theta|_{H^s(\tilde{\Omega}_0)} \leqq C|g|_{H^s(\tilde{\Omega}_v)}$. Notice that we have employed the extension $E: H^s(\tilde{\Omega}_0) \to H^s(\tilde{\mathbb{R}}^3)$.

More general results concerning the extension operator $E: H^s(\Omega) \to H^s(\mathbb{R}^n)$, $s > 0$, can be found in GRISVARD [1985, Theorem 1.4.3.1], where Ω is bounded and open with a Lipschitz boundary. As a consequence of Lemma 4.1, all the basic lemmas in § 3, which have been proved for functions defined in $\tilde{\mathbb{R}}^3$, are valid with $\tilde{\mathbb{R}}^3$ replaced by $\tilde{\Omega}_v$.

If $v \in C^\infty(\tilde{\mathbb{R}}^2)$ and $\phi \in C^\infty(\tilde{\Omega}_v)$, then the traces $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{r^*}\}$ are defined for ϕ in the sense that $\gamma_j\phi = \partial^j\phi/\partial n^j|_{\partial\tilde{\Omega}_v}$, where n is the outward unit normal vector, $0 \leqq j \leqq r^*$, and r^* is the largest integer such that $r^* < r - \frac{1}{2}$ and $r > \frac{1}{2}$. We have

$$\sum_{j=0}^{r^*} |\gamma_j\phi|_{H^{r-j-1/2}(\tilde{\mathbb{R}}^2)} \leqq C\,|\phi|_{H^r(\tilde{\Omega}_v)}.$$

Thus, Γ extends to $H^r(\tilde{\Omega}_v) \to \Pi_{j=0}^{r^*} H^{r-j-1/2}(\tilde{\mathbb{R}}^2)$ as a bounded operator. The operator Γ is surjective and has a bounded right inverse G. For $v \in H^r(\tilde{\mathbb{R}}^2)$, $r > 2$, $|v|_r < \delta$ small, we can prove a similar result for $H^s(\tilde{\Omega}_v)$.

Lemma 4.2. *Suppose* v *and* r *are as above,* $\frac{1}{2} < s \leqq r + \frac{1}{2}$, *and* s^* *is the largest integer such that* $s^* < s - \frac{1}{2}$. *Then* $\Gamma = \{\gamma_0, \ldots, \gamma_{s^*}\}$ *is well defined from* $H^s(\tilde{\Omega}_v)$ *to* $\Pi_{j=0}^{s^*} H^{s-j-1/2}(\tilde{\mathbb{R}}^2)$ *and is linear, bounded, and surjective. This operator has a bounded right inverse* G.

Proof. Let $\phi \in C^\infty(\Omega_v)$. Let θ be defined by (4.3). Obviously, $\gamma_0\phi = (\gamma_0(\phi \circ \theta)) \circ \theta^{-1}$. Moreover $|\gamma_0\phi|_{s-1/2} \leqq C\,|\gamma_0(\phi \circ \theta)|_{s-1/2} \leqq C\,|\phi \circ \theta|_s \leqq C\,|\phi|_s$, by virtue of Lemma 3.6 and the fact that the upper surface of $\tilde{\Omega}_v$ is modelled by $\tilde{\mathbb{R}}^2$, *i.e.*, $\gamma_0\phi(x_1, x_2) = \phi(x_1, x_2, v(x_1, x_2))$. Similarly, we have $|\gamma_0(D^j\phi)|_{s-j-1/2} \leqq C\,|\phi|_s$, $j \leqq s^*$. Now the normal of S_F is $\boldsymbol{n} = (1 + v_{x_1}^2 + v_{x_2}^2)^{-1/2}$ $(-v_{x_1}, -v_{x_2}, 1) \in H^{r-1}(\tilde{\mathbb{R}}^2)$, by virtue of Lemma 3.1 and Corollary 3.5. Therefore $\gamma_j\phi = \{\gamma_0(D^j\phi)\} \cdot \underbrace{(\boldsymbol{n}, \ldots, \boldsymbol{n})}_{j\text{-fold}} \in H^{s-j-1/2}(\tilde{\mathbb{R}}^2)$ for $\phi \in H^s(\tilde{\Omega}_v)$ and $|\gamma_j\phi|_{s-j-1/2} \leqq C\,|\phi|_s$, $0 \leqq j \leqq s^*$.

We now derive the boundary value problem satisfied by $\phi_0 \overset{\text{def}}{=} \phi \circ \theta$. Assume that the constant δ is small. From $V = y_0 + \bar{v}(x_0, y_0, t)$, $|\bar{v}|_{r+1/2} \leqq C\,\delta$, $r > 2$, we see that $V \in C^1$ and $(\partial v/\partial y_0) - 1$ is small. The Implicit Function Theorem implies that there exists a C^1-function $V^{-1}(x_0, y, t)$ such that

$$V(x_0, V^{-1}(x_0, y, t), t) = y. \tag{4.4}$$

Moreover θ^{-1} has the form

$$x_0 = x,$$
$$y_0 = V^{-1}(x, y, t).$$

For the time being, assume that V and V^{-1} are C^∞-functions, and that $V(x_0, y_0, t) - y_0$ and $V^{-1}(x, y, t) - y$ both have compact support in the y-direction. It is easy to obtain from (4.4) that:

$$\frac{\partial V^{-1}}{\partial y} - 1 = \left(\frac{\partial V}{\partial y_0}\right)^{-1} \circ \theta^{-1} - 1, \tag{4.5}$$

$$\frac{\partial V^{-1}}{\partial x} = -\left(\frac{\partial V}{\partial x_0}\right)\left(\frac{\partial V}{\partial y_0}\right)^{-1} \circ \theta^{-1}, \tag{4.6}$$

$$\frac{\partial^2 V^{-1}}{\partial y^2} = -\left(\frac{\partial^2 V}{\partial y_0^2}\right)\left(\frac{\partial V}{\partial y_0}\right)^{-3} \circ \theta^{-1}, \tag{4.7}$$

$$\begin{aligned}\frac{\partial^2 V^{-1}}{\partial x^2} = \Bigg\{ &-\left(\frac{\partial^2 V}{\partial x}\right)\left(\frac{\partial V}{\partial y_0}\right)^{-1} + 2\frac{\partial^2 V}{\partial x_0\, \partial y_0}\left(\frac{\partial V}{\partial x_0}\right)\left(\frac{\partial V}{\partial y_0}\right)^{-2} \\ &- \left(\frac{\partial V}{\partial x_0}\right)^2\left(\frac{\partial^2 V}{\partial y_0^2}\right)\left(\frac{\partial V}{\partial y_0}\right)^{-3}\Bigg\} \circ \theta^{-1}.\end{aligned} \tag{4.8}$$

Since $\phi_0 = \phi \circ \theta$, we have $\phi = \phi_0 \circ \theta^{-1}$, *i.e.*, $\phi(x, y, t) = \phi_0(x, V^{-1}(x, y, t), t)$. Assuming that $\phi_0 \in C^\infty(\tilde{\Omega}_0)$, we easily deduce

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi_0}{\partial x_0} \circ \theta^{-1} + \left(\frac{\partial \phi_0}{\partial y_0} \circ \theta^{-1}\right) \cdot \frac{\partial V^{-1}}{\partial x}, \tag{4.9}$$

$$\frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi_0}{\partial y_0} \circ \theta^{-1}\right) \cdot \frac{\partial V^{-1}}{\partial y}, \tag{4.10}$$

$$\begin{aligned}\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi_0}{\partial x_0^2} \circ \theta^{-1} &+ 2\left(\frac{\partial^2 \phi_0}{\partial x_0\, \partial y_0} \circ \theta^{-1}\right) \cdot \frac{\partial V^{-1}}{\partial x} \\ &+ \left(\frac{\partial^2 \phi_0}{\partial y_0^2} \circ \theta^{-1}\right)\left(\frac{\partial V^{-1}}{\partial x}\right)^2 + \left(\frac{\partial \phi_0}{\partial y_0} \circ \theta^{-1}\right) \cdot \frac{\partial^2 V^{-1}}{\partial x^2},\end{aligned} \tag{4.11}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\partial^2 \phi_0}{\partial y_0^2} \circ \theta^{-1}\right) \cdot \left(\frac{\partial V^{-1}}{\partial y}\right)^2 + \left(\frac{\partial \phi_0}{\partial y_0} \circ \theta^{-1}\right)\left(\frac{\partial^2 V^{-1}}{\partial y^2}\right). \tag{4.12}$$

Since $\nabla^2 \phi = 0$, it follows that ϕ_0 satisfies the following equation

$$\begin{aligned}(\nabla^2 \phi_0) \circ \theta^{-1} &+ \left[\left(\frac{\partial V^{-1}}{\partial x}\right)^2 + \left(\frac{\partial v^{-1}}{\partial y}\right)^2 - 1\right]\left(\frac{\partial^2 \phi_0}{\partial y_0^2} \circ \theta^{-1}\right) \\ &+ 2\left(\frac{\partial V^{-1}}{\partial x}\right)\left(\frac{\partial^2 \phi_0}{\partial x_0\, \partial y_0} \circ \theta^{-1}\right) \\ &+ \left(\frac{\partial^2 V^{-1}}{\partial x^2} + \frac{\partial^2 V^{-1}}{\partial y^2}\right)\left(\frac{\partial \phi_0}{\partial y_0} \circ \theta^{-1}\right) = 0.\end{aligned}$$

Applying $\circ\,\theta$ to both sides, we see that ϕ_0 satisfies the following boundary value problem:

$$-\nabla^2\phi_0 = A(v)\frac{\partial^2\phi_0}{\partial y_0^2} + B(v)\frac{\partial^2\phi_0}{\partial x_0\,\partial y_0} + C(v)\frac{\partial\phi_0}{\partial y_0} \quad \text{in } \tilde{\Omega}_0, \tag{4.13}$$

$$\phi_0(x_0, 0, t) = u(x_0, t), \tag{4.14}$$

$$\frac{\partial\phi_0}{\partial y_0}(x_0, -d, t) = 0, \tag{4.15}$$

where

$$A(v) = \left[\left(\frac{\partial V^{-1}}{\partial x}\right)^2 + \left(\frac{\partial V^{-1}}{\partial y}\right)^2 - 1\right]\circ\theta, \qquad B(v) = 2\cdot\frac{\partial V^{-1}}{\partial x}\circ\theta,$$
$$C(v) = \left(\frac{\partial^2 V^{-1}}{\partial x^2} + \frac{\partial^2 V^{-1}}{\partial y^2}\right)\circ\theta. \tag{4.16}$$

We have formally obtained the boundary value problem satisfied by ϕ_0. Our next goal is to put all these formal computations in suitable function spaces and solve (4.13)–(4.15) by the Implicit Function Theorem. The following is the main result of this section:

Theorem 4.3. *Let* $u, v \in K(\tilde{\mathbb{R}}^2\times\mathbb{R}^+;\, r, 0)$, $r > 3$, $|v|_{K(r,0)} < \delta$, *where* δ *is small. Then* $\phi_0 \in K(\tilde{\Omega}_0\times\mathbb{R}^+;\, r+\frac{1}{2}, 2)$ *and* $w = F(v)\,u \in K(\tilde{\mathbb{R}}^2\times\mathbb{R}^+;\, r-1, \frac{1}{2})$. *Moreover,* ϕ_0 *and* w *are* C^∞*-functions of* (u, v) *in the indicated spaces.*

Proof. Assume that $|v|_{r-1}$ is small for any fixed $t \in \mathbb{R}^+$. Since $r-1 > 2$ we can define $\bar{v}$, V, V^{-1} and θ as we did for the smooth functions. Using mollifiers, we can prove that (4.5) and (4.6) are satisfied in $H^{r-3/2}(\tilde{\Omega}_v)$ and that (4.7) and (4.8) are satisfied in $H^{r-5/2}(\tilde{\Omega}_v)$. Also by the use of mollifiers, we can prove that for $\phi \in H^{r-1/2}(\tilde{\Omega}_v)$ and for each t, we have that $\phi_0 \in H^{r-1/2}(\tilde{\Omega}_v)$, that (4.9) and (4.10) are satisfied in $H^{r-3/2}(\tilde{\Omega}_v)$, and that (4.11) and (4.12) are satisfied in $H^{r-5/2}(\tilde{\Omega}_v)$. Finally, looking for a solution $\phi \in H^{r-1/2}(\tilde{\Omega}_v)$ of (4.1) is equivalent to looking for a solution $\phi_0 \in H^{r-1/2}(\tilde{\Omega}_0)$ of (4.13)–(4.15), for any fixed t. Observe that $A(v) \in H^{r-3/2}(\tilde{\Omega}_0)$, $B(v) \in H^{r-3/2}(\tilde{\Omega}_0)$ and $C(v) \in H^{r-5/2}(\tilde{\Omega}_0)$, and that they are small in the indicated spaces if $|v|_{r-1}$ is small. Also they are C^∞-functions of $v \in H^{r-1}(\tilde{\mathbb{R}}^2)$, for any fixed t. Since $-\nabla^2 : H^{r-1/2}(\tilde{\Omega}_0) \to H^{r-5/2}(\tilde{\Omega}_0)$ is an isomorphism with the boundary conditions (4.14) and (4.15), we use the Implicit Function Theorem to conclude that the problem (4.13)–(4.15) has a unique solution $\phi_0 \in H^{r-1/2}(\tilde{\Omega}_0)$ and that ϕ_0 is a C^∞-function of (u, v) if $|v|_{r-1}$ is small, for any fixed $t \in \mathbb{R}^+$.

Now according to the hypotheses of the lemma, $|v|_{K(r,0)} < \delta$ is small, $r > 3$. We have $|v|_{\mathscr{B}(\bar{\mathbb{R}}^+, H^{r-1}(\tilde{\mathbb{R}}^2))} < C\,\delta$, $r-1 > 2$, by virtue of the trace theorem, *cf.* LIONS & MAGENES [1972]. Therefore, if δ is small, then for each $t \in \bar{\mathbb{R}}^+$, (4.13)–(4.15) is solvable and has a unique solution $\phi_0 \in \mathscr{B}(\bar{\mathbb{R}}^+, H^{r-1/2}(\tilde{\Omega}_0))$ with C^∞-dependence on $u, v \in K(r, 0)$. (In fact $\phi \in \mathscr{B}(\bar{\mathbb{R}}^+, H^{r-1/2}(\tilde{\Omega}_v))$ and has C^∞-dependence on $u, v \in K(r, 0)$; however, we shall not need this result.)

More regularity can be obtained for the solution ϕ_0 of (4.13)–(4.15). Observe that the inequalities $|v|_{K(r,0)} < \delta$ and $r > 3$ imply that $|\bar{v}|_{K(r+\frac{1}{2},\frac{1}{2})} < C\,\delta$ by (4.2). From our basic lemmas in § 3, $A(v)$ and $B(v) \in K(r-\frac{1}{2}, 0)$, $C(v) \in K(r-\frac{3}{2}, 0)$, and they are C^∞-functions of $v \in K(r, 0)$ and are small in the indicated spaces if δ is small. Notice that the $\circ\,\theta$ operator in (4.16) cancels $\circ\,\theta^{-1}$ in (4.5)–(4.8). We now look for a solution $\phi_0 \in K(r+\frac{1}{2}, 2)$ for (4.13)–(4.15). Here $A(v)$ and $B(v)$ are good multipliers by Lemma 3.1(i), and by the fact that $r-\frac{1}{2} > (3+2)/2$. Since $C(v) \in K(r-\frac{3}{2}, 0)$ and $\partial\phi_0/\partial y_0 \in K(r-\frac{1}{2}, 1)$, by Lemma 3.1(ii), we have that $C(v)\,(\partial\phi_0/\partial y_0) \in K(r-\frac{3}{2}, 0)$. As both sides of (4.13) are functions in $K(r-\frac{3}{2}, 0)$ by virtue of the fact $-\nabla^2 : K(r+\frac{1}{2}, 2) \to K(r-\frac{3}{2}, 0)$ is an isomorphism with the boundary conditions (4.14) and (4.15), we can use the Implicit Function Theorem to conclude that (4.13)–(4.15) possesses a unique solution $\phi_0 \in K(r+\frac{1}{2}, 2)$, which has C^∞-dependence on $(u, v) \in K(r, 0) \times K(r, 0)$. Since $K(r+\frac{1}{2}, 2) \subset \mathscr{B}(\overline{\mathbb{R}}^+, H^{r-1/2}(\tilde{\Omega}_0))$ with continuous embedding, by the uniqueness of the solution in $\mathscr{B}(\overline{\mathbb{R}}^+, H^{r-1/2}(\tilde{\Omega}_0))$, our previous solution ϕ_0 defined by $\phi_0 = \phi \circ \theta$ is in $K(r+\frac{1}{2}, 2)$.

From the continuity of the trace γ_0 and from (4.10), we obtain

$$w = \gamma_0 \left(\frac{\partial\phi}{\partial y}\right) = \gamma_0 \left[\frac{\partial\phi_0}{\partial y_0} \circ \frac{\partial V}{\partial y}^{-1} \circ \theta\right] \in \gamma_0 K(r-\tfrac{1}{2}, 1) \subset K(r-1, \tfrac{1}{2});$$

w is a C^∞-function of $(u, v) \in K(r, 0) \times K(r, 0)$. Here we have employed the fact that $\partial\phi_0/\partial y_0 \in K(\tilde{\mathbb{R}}^2 \times \mathbb{R}^+, r-\frac{1}{2}, 1)$ is a good multiplier since $r-\frac{1}{2} > \frac{3+2}{2}$ and $r-1-\frac{1}{2} > 1$. Also $\dfrac{\partial V}{\partial y}^{-1} \circ \theta$ has the form $1 + K(r-\frac{1}{2}, 0) \subset 1 + K(r-\frac{1}{2}, 1)$. That γ_0 commutes with $\circ\,\theta$ (or $\circ\,\theta^{-1}$) is easy to verify. The proof of Theorem 4.3 has been completed.

§ 5. Estimates for the Linear Equations

The linear inhomogeneous problem corresponding to (2.2)–(2.4) is

$$u_t - \mu\,\nabla^2 u - \gamma\,\nabla^2 v + gv = f_1, \tag{5.1}$$

$$v_t - F(0)\,u = f_2, \tag{5.2}$$

$$u(0) = u_0, \quad v(0) = v_0. \tag{5.3}$$

Let $r \geqq 2$. For each $(u, v) \in X^r(\mathbb{R}^+)$, define the linear operator $\mathscr{A}(u, v) = (\mu\,\nabla^2 u + \gamma\,\nabla^2 v - gv, F(0)\,u)$ and the linear operator $L(u, v) = (u_t, v_t) - \mathscr{A}(u, v)$. Obviously, $L : X\ |(\mathbb{R}^+) \to Y^r(\mathbb{R}^+)$ is continuous. Also, from the trace theorem, for such (u, v) we have that $(u_0, v_0) = (\gamma_0 u, \gamma_0 v) \in \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2) \times \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2)$ with $|u_0|_{r-1} + |v_0|_{r-1/2} \leqq C\,\|(u, v)\|_{X^r}$. Thus, (5.1)–(5.3) define a linear continuous mapping $\mathscr{L} : (u, v) \to (u_0, v_0, f_1, f_2)$, in the indicated product spaces. The objective of this section is to show that $\mathscr{L}$ admits a bounded inverse $\mathscr{L}^{-1} : \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2) \times \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2) \times Y^r(\mathbb{R}^+) \to X^r(\mathbb{R}^+)$, so that (5.1)–(5.3) has a unique solution.

First we use the Laplace transform to estimate (5.1)–(5.3) with "zero initial data to the highest order".

Definition 5.1. Let $X_{00}^r(\mathbb{R}^+)$, $Y_{00}^r(\mathbb{R}^+)$, $K_{00}(\tilde{\mathbb{R}}^2\times\mathbb{R}^+; r, s)$ be the subspace of functions in $X^r(\mathbb{R}^+)$, $Y^r(\mathbb{R}^+)$, $K(\tilde{\mathbb{R}}^2\times\mathbb{R}^+; r, s)$ that extend to functions in $X^r(\mathbb{R})$, $Y^r(\mathbb{R})$, $K(\tilde{\mathbb{R}}^2\times\mathbb{R}; r, s)$, respectively, and that are 0 for $t < 0$.

Assume that $(u, v)\in X_{00}^r(\mathbb{R}^+)$, $(f_1, f_2)\in Y_{00}^r(\mathbb{R}^+)$, and $(u_0, v_0) = 0$. In view of Definition 5.1, we are assuming that (5.1) and (5.2) hold for all $t\in\mathbb{R}$. Applying the Laplace transform to (5.1) and (5.2), we have

$$\lambda\hat{u} - \mu\nabla^2\hat{u} + (g - \gamma\nabla^2)\hat{v} = \hat{f}_1, \tag{5.4}$$

$$\lambda\hat{v} - F(0)\hat{u} = \hat{f}_2, \tag{5.5}$$

where $\lambda = \sigma + i\tau$. Note that $\hat{f}_1$ and $\hat{f}_2$ are defined for $\sigma \geqq 0$ and are analytic for $\sigma > 0$. The norm of $(f_1, f_2)\in Y_{00}^r(\mathbb{R})$ is equivalent to the L^2-norm of

$$|\hat{f}_1(i\tau)|_{r-2} + |\tau|^{(r-2)/2}|\hat{f}_1(i\tau)|_0 + |\hat{f}_2(i\tau)|_{r-1} + |\tau|^{(r-2)/2}|\hat{f}_2(i\tau)|_1.$$

Lemma 5.2. *Let* $r\geqq 2$. *Suppose that* $(\hat{f}_1(\lambda), \hat{f}_2(\lambda))\in\tilde{H}^{r-2}(\tilde{\mathbb{R}}^2)\times\tilde{H}^{r-1}(\tilde{\mathbb{R}}^2)$ *for each* λ *with* $\operatorname{Re}\lambda = \sigma > 0$. *Then* (5.4) *and* (5.5) *have a unique solution* $(\hat{u}(\lambda), \hat{v}(\lambda))\in\tilde{H}^r(\tilde{\mathbb{R}}^2)\times\tilde{H}^r(\tilde{\mathbb{R}}^2)$ *satisfying*

$$\begin{aligned}&|\hat{u}| + |\lambda|^{r/2}|\hat{u}|_0 + |\hat{v}|_r + |\lambda|^{r/2}|\hat{v}|_0\\ &\qquad\leqq C(|\hat{f}_1|_{r-2} + |\lambda|^{(r-2)/2}|\hat{f}_1|_0 + |\hat{f}_2|_{r-1} + |\lambda|^{(r-2)/2}|\hat{f}_2|_1).\end{aligned} \tag{5.6}$$

The proof of Lemma 5.2 will be postponed to the end of this section. As a consequence of Lemma 5.2, we can prove the following.

Lemma 5.3. $L: X_{00}^r(\mathbb{R}^+)\to Y_{00}^r(\mathbb{R}^+)$ *is an algebraic and topological isomorphism.*

Proof. The L^2-norm of $|\hat{f}_1(\sigma + i\tau)|_{r-2} + |\lambda|^{(r-2)/2}|\hat{f}_1(\sigma + i\tau)|_0 + |\hat{f}_2(\sigma + i\tau)|_{r-1} + |\lambda|^{(r-2)/2}|\hat{f}_2(\sigma + i\tau)|_1$, for $\sigma\geqq 0$ fixed, $-\infty < \tau < \infty$, is bounded by $C\,\|(f_1, f_2)\|_{Y_{00}^r}$, where C is independent of $\sigma\geqq 0$. From Lemma 5.2, $(\hat{u}(\lambda)), (\hat{v}(\lambda))$ can be constructed for $\sigma > 0$, and is analytic in λ. By the Paley-Wiener Theorem, $u(t) = v(t) = 0$ for $t < 0$. Observe that on the line $\operatorname{Re}\lambda \equiv \sigma$ with $\sigma > 0$, $\hat{u}(\sigma + i\tau)$ and $\hat{v}(\sigma + i\tau)$ are the Fourier transforms of $e^{-\sigma t}u(t)$ and $e^{-\sigma t}v(t)$, as functions of $-\infty < \tau < +\infty$. From Lemma 5.2, $(e^{-\sigma t}u, e^{-\sigma t}v)\in\{K_{00}(r, 0)\}^2$ and $\|(e^{-\sigma t}u, e^{-\sigma t}v)\|_{\{K_{00}(r,0)\}^2}\leqq C\,\|(f_1, f_2)\|_{Y_{00}^r}$. It follows from Fatou's Lemma and the Lebesgue Convergence Theorem that $(u, v)\in\{K_{00}(r, 0)\}^2$ and $(e^{-\sigma t}u, e^{-\sigma t}v)\to(u, v)$ in $\{K_{00}(r, 0)\}^2$, whence $|(u, v)|_{\{K_{00}(r,0)\}^2}\leqq C\,\|(f_1, f_2)\|_{Y_{00}^r}$. The estimate $|v_t|_{K(r-1,1)}\leqq C\,\|(f_1, f_2)\|_{Y_{00}^r}$ follows directly from (5.2).

We now construct $\mathscr{L}^{-1}: (u_0, v_0, f_1, f_2)\to(u, v)$. First consider the case $r = 2$. Let $(u_0, v_0)\in\tilde{H}^1\times\tilde{H}^{3/2}$. Let G be the right inverse of the trace of (u, v) at $t = 0$. Define $(\bar{u}, \bar{v}) = (Gu_0, Gv_0)$. Then $(\bar{u}, \bar{v})\in K^2(\mathbb{R}^+)$ and $\|(\bar{u}, \bar{v})\|_{X^2}\leqq C(|u_0|_1 + |v_0|_{3/2})$. Let $(u, v) = (\bar{u}, \bar{v}) + (\bar{\bar{u}}, \bar{\bar{v}})$. We are to solve

$$L(\bar{\bar{u}}, \bar{\bar{v}}) = (f_1, f_2) - L(\bar{u}, \bar{v}). \tag{5.7}$$

The right-hand side is in $K(0,0)\times K(1,1)=K_{00}(0,0)\times K_{00}(1,1)=Y^2_{00}(\mathbb{R}^+)$. Lemma 5.2 implies that (5.7) has a unique solution $(\bar{\bar{u}},\bar{\bar{v}})\in X^2_{00}$ with

$$\|(\bar{\bar{u}},\bar{\bar{v}})\|_{X^2_{00}}\leqq C\{\|(f_1,f_2)\|_{Y^2_{00}}+\|L(\bar{u},\bar{v})\|_{Y^2_{00}}\}$$
$$\leqq C\{\|(f_1,f_2)\|_{Y^2_{00}}+|u_0|_1+|v_0|_{3/2}\}.$$

Therefore we have proved that $\mathscr{L}^{-1}:\tilde{H}^1\times\tilde{H}^{3/2}\times Y^2\to X^2$ is bounded.

Suppose that $r>2$ and $r\notin\{\text{integers}\}\cup\{\text{integers}+\frac{1}{2}\}$. Obviously $\mathscr{L}^{-1}:\tilde{H}^{r-1}\times\tilde{H}^{r-1/2}\times Y^r\to X^2$ is bounded. We shall show that the image of $\mathscr{L}^{-1}$ is in X^r and that $\mathscr{L}^{-1}:\tilde{H}^{r-1}\times\tilde{H}\times Y^r\to X^r$ is bounded. The method is similar to that for the case $r=2$. We write $(u,v)=(\bar{u},\bar{v})+(\bar{\bar{u}},\bar{\bar{v}})$, where $(\bar{\bar{u}},\bar{\bar{v}})\in X^r_{00}$ so that Lemma 5.3 applies to (5.7). However, $(\bar{u},\bar{v})$ has to satisfy more compatibility conditions at $t=0$. We assert that for each $(u_0,v_0,f_1,f_2)\in\tilde{H}^{r-1}\times\tilde{H}^{r-1/2}\times K(r-2,0)\times K(r-1,1)$, there exists a unique $(\{u_j\}_{j=1}^{r^*},\{v_j\}_{j=1}^{r^*})\in\Pi_{j=1}^{r^*}\tilde{H}^{r-2j-1}(\tilde{\mathbb{R}}^2)\times\Pi_{j=1}^{r^*}\tilde{H}^{r-2j}(\tilde{\mathbb{R}}^2)$ such that if we set $\bar{u}=G(\{u_j\}_{j=0}^{r^*})$, $\bar{v}=G(\{v_j\}_{j=0}^{r^*})$, then

$$\bar{\Gamma}L(\bar{u},\bar{v})=\bar{\Gamma}(f_1,f_2). \tag{5.8}$$

Here r^* is the largest integer less than $(r-1)/2$, $\bar{\Gamma}(f_1,f_2)=(\bar{\Gamma}f_1,\bar{\Gamma}f_2)$ is the trace operator with $\bar{\Gamma}=\{\gamma_0,\gamma_1,\dots,\gamma_{r^*-1}\}$, and G is the right inverse of $\Gamma=\{\gamma_0,\dots,\gamma_{r^*}\}$, $\Gamma(u,v)=(\Gamma u,\Gamma v)$. We need the following basic results.

Lemma 5.4.

$$\bar{\Gamma}\colon Y^r(\mathbb{R}^+)\to\prod_{j=0}^{r^*-1}\tilde{H}^{r-2j-3}\times\tilde{H}^{r-2j-2},$$

$$\Gamma\colon X^r(\mathbb{R}^+)\to\prod_{j=0}^{r^*}\tilde{H}^{r-2j-1}\times\left(\tilde{H}^{r-1/2}\times\prod_{j=1}^{r^*}\tilde{H}^{r-2j}\right)$$

are bounded and surjective and admit bounded right inverses $\bar{G}$ and G respectively.

Proof. For each $\{v_j\}_{j=0}^{r^*}\in\tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2)\times\Pi_{j=1}^{r^*}\tilde{H}^{r-2j}(\tilde{\mathbb{R}}^2)$, we construct v such that $\Gamma v=\{v_j\}_{j=0}^{r^*}$. The rest of the proof is left to the reader. (*Cf.* LIONS & MAGENES [1972]). Let us write $v=v^{(1)}+v^{(2)}$. Let $v^{(1)}\in H^0(\mathbb{R}^+,\tilde{H}^r(\tilde{\mathbb{R}}^2))\cap H^r(\mathbb{R}^+,\tilde{H}^0(\tilde{\mathbb{R}}^2))$ be defined as the q-tuple $v^{(1)}=G_1(v_0,0,\dots,0)$, where q is the largest integer $<r-\frac{1}{2}$ and G_1 is the right inverse of $\Gamma_1:H^0(\mathbb{R}^+,\tilde{H}^r(\tilde{\mathbb{R}}^2))\cap H^r(\mathbb{R}^+,\tilde{H}^0(\tilde{\mathbb{R}}^2))\to\Pi_{j=0}^q\tilde{H}^{r-j-1/2}(\tilde{\mathbb{R}}^2)$. Let $v^{(2)}\in K(\tilde{\mathbb{R}}^2\times\mathbb{R}^+;\,r+1,1)$, $v^{(2)}=G_2(0,v_1,\dots,v_{r^*})$, where G_2 is the right inverse of $\Gamma_2:K(r+1,1)\to\Pi_{j=0}^{r^*}\tilde{H}^{r-2j}(\tilde{\mathbb{R}}^2)$. Obviously $v=v^{(1)}+v^{(2)}\in K(r,0)$ and $v_t\in K(r-1,1)$, with

$$|v|_{K(r,0)}+|v|_{K(r-1,1)}\leqq C\left\{|v_0|_{r-1/2}+\sum_{j=1}^{r^*}|v_j|_{r-2j}\right\}. \quad\square$$

The kernel of Γ in $X^r(\mathbb{R}^+)$ ($\bar{\Gamma}$ in $Y^r(\mathbb{R}^+)$) will be denoted by $X_0^r(\mathbb{R}^+)$ ($Y_0^r(\mathbb{R}^+)$). For $r \notin \{\text{integers}\} \cup \{\text{integers} + \frac{1}{2}\}$, $r > 2$, we have $X_0^r(\mathbb{R})^+) = X_{00}^r(\mathbb{R}^+)$, and $Y_0^r(\mathbb{R}^+) = Y_{00}^r(\mathbb{R}^+)$.

We can easily prove our assertion concerning (5.8) now. Observe that $\bar{\Gamma} D_t = \{\gamma_1, \ldots, \gamma_{r^*}\}$ and $D_{xx}\bar{\Gamma} = \bar{\Gamma} D_{xx}$ on X^r. Thus $\{u_j\}_{j=1}^{r^*}$ and $\{v_j\}_{j=1}^{r^*}$ can easily be computed from (5.8) provided that (u_0, v_0) is given. For the same reason, $\|(\bar{u}, \bar{v})\|_{X^r} \leqq C\{\Sigma_{j=0}^{r^*}\{|u_j|_{r-2j-1} + \Sigma_{j=1}^{r^*}|v_j|_{r-2j} + |v_0|_{r-1/2}\} \leqq C\{|u_0|_{r-1} + |v_0|_{r-1/2} + \|(f_1, f_2)\|_{Y^r}\}$.

Now since $(\bar{\bar{u}}, \bar{\bar{v}}) = (u, v) - (\bar{u}, \bar{v})$, we are led to (5.7) again. From (5.8) and the identity $Y_0^r(\mathbb{R}^+) = Y_{00}^r(\mathbb{R}^+)$ it follows that the right-hand side of (5.8) belongs to $Y_{00}^r(\mathbb{R}^+)$. Applying Lemma 5.3 to (5.7), we have a unique solution $(\bar{\bar{u}}, \bar{\bar{v}}) \in X_{00}^r = X_0^r$ and

$$\|(\bar{\bar{u}}, \bar{\bar{u}}\|_{X_{00}^r} \leqq C\{|u_0|_{r-1} + |v_0|_{r-1/2} + \|(f_1, f_2)\|_{Y^r}\}.$$

Thus, the construction of $\mathscr{L}^{-1}$ for $r \notin \{\text{integers}\} \cup \{\text{integers} + \frac{1}{2}\}$, $r > 2$ has been completed.

Finally, the restriction $r \notin \{\text{integers}\} \cup \{\text{integers} + \frac{1}{2}\}$ can be removed by the interpolation method. The interpolation of the spaces $H^{r,s}(\Omega \times \mathbb{R})$ can be found in LIONS & MAGENES [1972]; see also the notations there. It follows easily that

$$\begin{aligned} [X^{r_1}, X^{r_2}]_\theta &= X^{r_1(1-\theta)+r_2\theta}, \\ [Y^{r_1}, Y^{r_2}]_\theta &= Y^{r_1(1-\theta)+r_2\theta}, \end{aligned} \qquad r_1 > r_2 \geqq 2, \quad 0 < \theta < 1.$$

We summarize our results in the following theorem.

Theorem 5.5. *Let* $r \geqq 2$. *Then* $\mathscr{L} : (u, v) \to (u_0, v_0, f_1, f_2)$ *is an algebraic and topological isomorphism of* $X^r(\mathbb{R}^+)$ *onto* $\tilde{H}^{r-1}(\tilde{\mathbb{R}}^2) \times \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2) \times Y^r(\mathbb{R}^+)$.

We conclude this section with the proof of Lemma 5.2.

Proof of Lemma 5.2 We decompose the right-hand side of (5.4) and (5.5) into $(\hat{f}_1, 0)^\tau + (0, \hat{f}_2)^\tau$, and apply the principle of superposition. First let $\hat{f}_2 = 0$. Substituting $\hat{v} = \lambda^{-1} F(0)\, \hat{u}$ into (5.4), we obtain

$$\lambda \hat{u} + A\hat{u} + \lambda^{-1} B\hat{u} = \hat{f}_1, \tag{5.9}$$

where $A = -\mu \nabla^2$ and $B = (g - \gamma\nabla^2) \cdot F(0)$. There exist positive constants C_1 and C_2 such that

$$(A\hat{u}, \hat{u}) \geqq C_1 |\hat{u}|_1^2,$$

$$(B\hat{u}, \hat{u}) \geqq C_2 |\hat{u}|_{3/2}^2.$$

Here we have used the facts that the damping $\mu \neq 0$ and that $\int_{\mathbb{R}^2} u \, d\mu = 0$. It is clear that (5.9) has a unique solution $\hat{u} \in \tilde{H}^{r+1}(\tilde{\mathbb{R}}^2)$. Assume that $\hat{u}$ is sufficiently smooth with respect to the spatial variables. Equation (5.9) implies that

$$\lambda(\hat{u}, |D|^{2k}\, \hat{u} + (A\hat{u}, |D|^{2k}\, \hat{u}) + \lambda^{-1}(B\hat{u}, |D|^{2k}\, \hat{u}) = (\hat{f}_1, |D|^{2k}\hat{u}), \tag{5.10}$$

where $|D|$ is the singular integral operator with the symbol $|\xi|$. Taking the real part of (5.10), we have

$$|\hat{u}|_{k+1}^2 \leqq C\,|\hat{f}_1|_{k-1}\,|\hat{u}|_{k+1}$$

since $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\lambda^{-1} > 0$. Letting $k = r-1$, we find that

$$|\hat{u}|_r \leqq C|\hat{f}_1|_{r-2}. \tag{5.11}$$

The smoothness assumption on $\hat{u}$ can be removed by applying $(\cdot, |D|^{2k}\,\hat{u}_l)$ to (5.9), where $\hat{u}_l$ is the Fourier truncation of $\hat{u}$, yielding $|\hat{u}_l|_r \leqq c\,|\hat{f}_1|_{r-2}$. Letting $l \to \infty$, we have (5.11). This kind of obvious procedure will be omitted in the sequel.

We now prove that

$$|\lambda|\,|\hat{u}|_0 \leqq C\,|\hat{f}_1|_0. \tag{5.12}$$

By letting $k = 0$ in (5.10) we obtain

$$|\lambda|^2\,|\hat{u}|_0^2 \leqq C\{|\hat{u}|_{3/2}^2 + |\lambda|\,|\hat{u}|_1^2 + |\lambda|\,|\hat{u}|_0\,|\hat{f}_1|_0\}.$$

From

$$|\hat{u}|_{3/2}^2 \leqq C\,|\hat{f}_1|_0^2,$$

$$|\lambda|\,|\hat{u}|_1^2 \leqq C\,|\lambda|\,|\hat{u}|_0\,|\hat{u}|_2 \leqq C\,|\lambda|\,|\hat{u}|_0\,|\hat{f}_1|_0,$$

$$|\lambda|\,|\hat{u}|_0\,|\hat{f}_1|_0 \leqq \frac{\varepsilon}{2}\,|\lambda|^2\,|\hat{u}|_0^2 + \frac{1}{2\varepsilon}\,|\hat{f}_1|_0^2,$$

we have $|\lambda|^2\,|\hat{u}|_0^2 \leqq C\,|\hat{f}_1|_0^2$, which implies (5.12).

For $\hat{f}_2 = 0$, we have proved that $|\hat{u}|_r + |\lambda|^{r/2}\,|\hat{u}|_0 \leqq C\,\|\hat{f}_1\|_{r-2}$, where we have used the notation $\|\hat{f}_1\|_{r-2} = |\hat{f}_1|_{r-2} + |\lambda|^{(r-2)/2}\,|\hat{f}_1|_0$. We shall derive the estimates for $\hat{v}$ under the same assumption that $\hat{f}_2 = 0$. Letting $k = r - \frac{1}{2}$ in (5.10), we obtain

$$|\lambda|^{-1}\,|\hat{u}|_{k+3/2} \leqq C\{|\lambda|\,|\hat{u}|_k^2 + |\hat{u}|_{k+1}^2 + |\hat{f}_1|_{k-3/2}\,|\hat{u}|_{k+3/2}\}.$$

From

$$|\hat{u}|_{k+1}^2 \leqq C\,|\hat{u}|_{k+1/2}\,|\hat{u}|_{k+3/2} \leqq C\,|\hat{u}|_{k+3/2}\,|\hat{f}_1|_{k-3/2},$$

$$|\lambda|\,|\hat{u}|_k^2 \leqq C\,|\lambda|\,|\hat{f}_1|_{k-3/2}^2,$$

$$|\hat{f}_1|_{k-3/2}|\hat{u}|_{k+3/2} \leqq \frac{\varepsilon}{2}\,|\lambda|^{-1}\,|\hat{u}|_{k+3/2}^2 + \frac{1}{2\varepsilon}\,|\lambda|\,|\hat{f}_1|_{k-3/2}^2,$$

we have $|\lambda|^{-1}\,|\hat{u}|_{k+3/2}^2 \leqq C\,|\lambda|\,|\hat{f}_1|_{k-3/2}^2$, whence

$$|\lambda|^{-1}\,|\hat{u}|_{r+1} \leqq C\,|\hat{f}_1|_{r-2}. \tag{5.13}$$

Since $\hat{f}_2 = 0$, we obtain from (5.5) that

$$|\hat{v}|_r \leqq C\,|\lambda|^{-1}\,|\hat{u}|_{r+1} \leqq C\,|\hat{f}_1|_{r-2}. \tag{5.14}$$

We now derive the estimate for $|\hat{v}|_0$ under the same hypothesis that $\hat{f}_2 = 0$. By the convexity of the norm,

$$|\hat{u}|_1 \leqq C\,|\hat{u}|_0^{(r-1)/r}\,|\hat{u}|_r^{1/r}. \tag{5.15}$$

Since $|\hat{u}|_r| \leqq C\,\|\hat{f}_1\|_{r-2}$ and $|\hat{u}|_0 \leqq C\,|\lambda|^{-1}\,|\hat{f}_1|_0 \leqq C\,|\lambda|^{-r/2}\,\|\hat{f}_1\|_{r-2}$ by (5.12), we obtain from (5.15) that

$$|\hat{u}|_1 \leqq C\,|\lambda|^{-(r-1)/2}\,\|\hat{f}_1\|_{r-2}.$$

Therefore

$$|\lambda|^{r/2}\,|\hat{v}|_0 \leqq C\,|\lambda|^{-1/2}\,\|\hat{f}_1\|_{r-2}. \tag{5.16}$$

From (5.14) we infer that $|v|_0 \leqq C\,\|\hat{f}_1\|_{r-2}$; thus

$$|\lambda|^{r/2}\,|\hat{v}|_0 \leqq C\,|\lambda|^{r/2}\,\|\hat{f}_1\|_{r-2}. \tag{5.17}$$

Combining (5.16) and (5.17), we have $|\lambda|^{r/2}\,|\hat{v}|_0 \leqq C\,\|\hat{f}_1\|_{r-2}$. Adding this to (5.14), we obtain

$$|\hat{v}|_r + |\lambda|^{r/2}\,|\hat{v}|_0 \leqq C\,\|\hat{f}_1\|_{r-2}.$$

Lemma 5.2 has been proved for the case $\hat{f}_2 = 0$.

Next, we set $\hat{f}_1 = 0$ and derive the estimates for $(\hat{u}, \hat{v})$ in terms of $\hat{f}_2$. From (5.4) and (5.5), we obtain

$$\hat{u} = -(\lambda + A)^{-1}\,(g - \gamma\,\nabla^2)\,\hat{v}, \tag{5.18}$$

$$\hat{v} + \lambda^{-1}\,B(\lambda + A)^{-1}\,\hat{v} = \lambda^{-1}\,\hat{f}_2. \tag{5.19}$$

Letting $\hat{e} = (\lambda + A)^{-1}\,\hat{v}$, we have

$$\lambda\hat{e} + A\hat{e} + \lambda^{-1}\,B\hat{e} = \lambda^{-1}\,\hat{f}_2. \tag{5.20}$$

This is completely analogous to (5.9), with $\lambda^{-1}\hat{f}_2$ in place of $\hat{f}_1$. Therefore, recalling that $\hat{f}_2 \in \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2)$ by our assumption, from (5.11) we have $|\hat{e}|_{r+1} \leqq C\,|\lambda|^{-1}\,|\hat{f}_2|_{r-1}$. From (5.13), $|\hat{e}|_{r+2} \leqq C\,|\hat{f}_2|_{r-1}$; thus

$$|\hat{v}|_r \leqq |(\lambda + A)\,\hat{e}|_r \leqq C\{|\lambda|\,|\hat{e}|_r + |\hat{e}|_{r+2}\} \leqq C\,|\hat{f}_2|_{r-1}. \tag{5.21}$$

From (5.12) it follows that $|\lambda|^2\,|\hat{e}|_0 \leqq C\,|\hat{f}_2|_0$. Inequality (5.11) yields $|\lambda|\,|\hat{e}|_2 \leqq C\,|\hat{f}_2|_0$. Thus

$$|\lambda|\,|\hat{v}|_0 \leqq |\lambda|\cdot|(\lambda + A)\,\hat{e}|_0 \leqq |\lambda|^2\,|\hat{e}|_0 + C\,|\lambda|\,|\hat{e}|_2 \leqq C|\hat{f}_2|_0. \tag{5.22}$$

Combining (5.21) and (5.22), we obtain

$$|v|_r + |\lambda|^{r/2} |\hat{v}|_0 \leqq C\{|\hat{f}_2|_{r-1} + |\lambda|^{(r-2)/2} |\hat{f}_2|_0\}. \tag{5.23}$$

From (5.18), and the fact that

$$\|(\lambda + A)^{-1}\|_{\mathscr{L}(H^{k-2}, H^k)} \leqq C,$$

where C is independent of $\sigma \geqq 0$, we have $|\hat{u}|_r \leqq C|\hat{v}|_r$, and $|\hat{u}|_0 \leqq C|\hat{v}|_0$. Therefore,

$$|\hat{u}|_r + |\lambda|^{r/2} |\hat{u}|_0 \leqq C\{|\hat{f}_2|_{r-1} + |\lambda|^{(r-2)/2} |\hat{f}_2|_0\}. \tag{5.24}$$

Lemma 5.2 has been proved also for $\hat{f}_1 = 0$. Therefore the proof of Lemma 5.2 has been completed.

§ 6. Nonlinear Problems and Semi-discretizations

The nonlinear problem (2.2)–(2.4) can be written as

$$\mathscr{L}(u, v) - (u_0, v_0, f_1(\alpha, u, v), f_2(u, v)) = 0. \tag{6.1}$$

The operator $\mathscr{L}$ is defined in § 5 and $(f_1(\alpha, u, v), f_2(u, v))$ is defined in (2.6) and (2.7). We shall prove Theorem 2.1 by the Implicit Function Theorem. From Theorem 5.5, $\mathscr{L}: X^r(\mathbb{R}^+) \to \tilde{H}^{r-1}(\tilde{\mathbb{R}}^2) \times \tilde{H}^{r-1/2}(\tilde{\mathbb{R}}^2) \times Y^r(\mathbb{R}^+)$ is an isomorphism for $r \geqq 2$. To estimate the nonlinear functions we assume that $r > 3$. Let $\|(u, v)\|_{X^r} < \varepsilon_1$. If $\varepsilon_1 > 0$ is sufficiently small, then $w = F(v)\, u \in K(\tilde{\mathbb{R}}^2 \times \mathbb{R}^+; r-1, \frac{1}{2})$ by Theorem 4.3, and w is a C^∞-function of $(u, v) \in X^r(\mathbb{R}^+)$. Clearly $\nabla_x u \in K(r-1, 0)$, $\nabla_x v \in K(r-1, 0)$, and $K(r-1, 0) \subset K(r-1, \frac{1}{2})$. From Lemma 3.2(i), $K(r-1, \frac{1}{2})$ is an algebra since the space dimension is $n = 2$, $r - 1 > \dfrac{2+2}{2}$, and $r - 1 - \frac{1}{2} > 1$. Let us view $\alpha \in H^{(r-2)/2}(\mathbb{R}^+, \mathbb{R})$ as an element in $K(\tilde{\mathbb{R}}^+ \times \mathbb{R}; r, 2)$, constant with respect to the spatial variables. Then Lemma 3.2(i) also implies that $\alpha \cdot v \in K(r-1; 1)$. We have just proved for each $(u, v) \in X^r(\mathbb{R}^+)$ and each $\alpha \in H^{(r-2)/2}(\mathbb{R}^+, \mathbb{R})$, small in their norms, that $f_1(\alpha, u, v) \in K(r-1, 1) \subset K(r-2, 0)$ and $f_2(u, v) \in K(r-1, 1)$, with $\|(f_1, f_2)\| = O\{|u|^2 + |v|^2 + |\alpha|\,|v|\}$ with respect to the indicated norms. We know that $\int_{\mathbb{R}^2} f_1(\alpha, u, v)(t)\, d\mu = 0$, for $t \geqq 0$, by virtue of the projection P. To show that $\int_{\mathbb{R}^2} f_2(u, v)(t)\, d\mu = 0$ for $t \geqq 0$, one has to evaluate $\int_{\mathbb{R}^2} \{w(1 + |\nabla_x v|^2) - \nabla_x u \cdot \nabla_x v\}\, d\mu$. But this comes from $\displaystyle\int_{S_F} \frac{\partial \phi}{\partial n}(t)\, dS = \int_{\tilde{\Omega}_v} \nabla^2 \phi(t)\, d\mu = 0$, where ϕ is the solution of (2.1). Therefore, $(f_1(\alpha, u, v), f_2(u, v)) \in Y^r(\mathbb{R}^+)$. Denoting the left-hand side of (6.1) by $Q(u, v, u_0, v_0, \alpha)$, which is, in each variable, a C^∞-function into $\tilde{H}^{r-1}\tilde{H}^{r-1/2} \times Y^r$, we have that $Q(0, 0, 0, 0, 0) = 0$ and $D_{u,v}Q(0, 0, 0, 0, 0) = \mathscr{L}$, which is an isomorphism as mentioned before. Therefore, the Implicit Function Theorem asserts that all the solutions of (6.1), in a neighborhood of zero, have the form

$$(u, v) = \mathscr{Q}(u_0, v_0, \alpha)$$

which is defined for $|u_0|_{r-1} + |v_0|_{r-1/2} + |\alpha|_{(r-2)/2} < \delta$. $\mathscr{Q}(u_0, v_0, \alpha)$ is a C^∞-function in the indicated spaces. $\mathscr{Q}(0, 0, 0) = 0$ and $D_\alpha \mathscr{Q}(0, 0, 0) = 0$. Hence, the estimate for (u, v) in Theorem 2.1 follows.

To study the dependence of the solutions on $(\gamma, \mu, g) \in (\mathbb{R}^+)^3$, we write $\gamma = \gamma_0 + \gamma_1$, $\mu = \mu_0 + \mu_1$ and $g = g_0 + g_1$, where $(\gamma_0, \mu_0, g_0) \in (\mathbb{R}^+)^3$ is fixed and (γ_1, μ_1, g_1) is a small perturbation in $\mathbb{R}^3$. We leave all terms involving (γ_0, μ_0, g_0) on the left-hand side of (2.2) and move all terms involving (γ_1, μ_1, g_1) to the right-hand side of (2.2) and proceed in the obvious way. Details are omitted. This completes the proof of Theorem 2.1.

Remark 6.1. In his work, BEALE [1984] proved that solutions of the free-surface Navier-Stokes equations become arbitrarily smooth after any specified time interval. A similar result for damped gravity waves is also valid provided that α is smoother than assumed in Theorem 2.1. The solution (u, v) will reach the regularity determined by α after any specified time interval. Moreover, even without further regularity assumptions on α, the solution still becomes arbitrarily smooth with respect to the spatial variables after any specified time interval. To show this, we observe that $\alpha \in H^{(r-2)/2}(\mathbb{R}^+, \mathbb{R}) \subset K(r, 2)$, and the term involving α is $\alpha v \in K(r, 2)$ since $v \in K(r, 0)$. Given any $t_1 > 0$, arguing as in BEALE [1984], we can find $(u(t_0), v(t_0)) \in \tilde{H}^{r-1/2} \times \tilde{H}^r$, $t_0 \in (0, t_1)$. We can prove that $(u, v)|_{(t_0, +\infty)} \in K(r + \frac{1}{2}, \frac{1}{2}) \times K(r + \frac{1}{2}, \frac{1}{2})$, with $v_t|_{(t_0, +\infty)} \in K(r - \frac{1}{2}, \frac{3}{2})$, by rewriting all the linear estimates in § 5 and the elliptic free boundary problem in § 4. The regularity can be increased by repeating the whole procedure in $(t_1, 2t_1)$, *etc.* Details are omitted. That (u, v) is smoother than asserted in Theorem 2.1 is useful in proving Theorem 2.3(ii).

Before proving Theorem 2.2, we observe that under the hypotheses of Theorem 2.2, for any $T_1 > 0$, the solution $(u, v) \in X^r(0, T_1)$ exists provided $|u_0|_{r-1} + |v_0|_{r-1/2} + \|\alpha\| < \delta(T_1)$, where $\delta(T_1)$ is a constant depending on T_1. To see this, let $\psi(t)$ be a C^∞-function such that $\psi(t) = 1$ for $t \leqq T_1$ and $\psi(t) = 0$ for $t \geqq T_1 + 1$. Consider $\tilde{\alpha}(t) = \alpha(t)\,\psi(t)$. Clearly, $|\tilde{\alpha}|_{(r-2)/2} \leqq C\,\|\alpha\|$, where the constant C depends on T_1. We can apply Theorem 2.1 to the system (2.2)–(2.4) with $\tilde{\alpha}(t)$ in place of $\alpha(t)$, and the solvability of the system for $t \in (0, T_1)$ follows. If we can show that at $t = T_1$, $|u(T_1)|_{r-1} + |v(T_1)|_{r-1/2}$ is still small, then we can repeat the argument and thus prove Theorem 2.2 by induction. To this end, we need an energy estimate for the linear homogeneous system (5.1)–(5.3).

System (5.1)–(5.3) with $f_1 = f_2 = 0$ has a natural total energy $E = \frac{1}{2} g\,|v|_0^2 + \frac{1}{2}\gamma\,|\nabla v|_0^2 + \frac{1}{2}\,(F(0)\,u, u)$, which is conserved when $\mu = 0$. We define a modified energy

$$E_\varepsilon(t) = \tfrac{1}{2} g\,\big||D|^k v\big|_0^2 + \tfrac{1}{2}\gamma\,\big||D|^k \nabla v\big|_0^2 + \tfrac{1}{2}\,(F(0)\,|D|^k u, |D|^k u) + \varepsilon(|D|^{k+1/2} u, |D|^{k+1/2} v),$$

where $k = r - \frac{3}{2}$ and $\varepsilon > 0$ is a small constant to be determined. We easily find that there is $\varepsilon_1 > 0$ such that if $0 < \varepsilon < \varepsilon_1$, then

$$C_0(|u|_{r-1}^2 + |v|_{r-1/2}^2) \leqq E_\varepsilon(t) \leqq C_1(|u|_{r-1}^2 + |v|_{r-1/2}^2). \tag{6.2}$$

Assuming that u_0, v_0, f_1 and f_2 are sufficiently smooth, and hence that (u, v) is sufficiently smooth (the smoothness assumptions can be removed easily), we have

$$\begin{aligned}\frac{dE_\varepsilon(t)}{dt} &= (v_t, g\,|D|^{2k} v) - (v_t, \gamma\,|D|^{2k} \nabla^2 v) + (u_t, F(0)\,|D|^{2k} u)\\ &\quad + \varepsilon(u_t, |D|^{2k+1} v) + \varepsilon(v_t, |D|^{2k+1} u)\\ &= -\mu(F(0)\,|D|^k \nabla u, |D|^k \nabla u) - \varepsilon\gamma\,\big||D|^{k+1/2} \nabla v\big|_0^2\\ &\quad - \varepsilon g\,\big||D|^{k+1/2} v\big|_0^2 - \varepsilon\mu(|D|^{k+1/2} \nabla u, |D|^{k+1/2} \nabla v)\\ &\quad + \varepsilon(F(0)\,|D|^{k+1/2} u, |D|^{k+1/2} u)\\ &\quad + (f_1, F(0)\,|D|^{2k} u + \varepsilon\,|D|^{2k+1} v)\\ &\quad + (f_2, g\,|D|^{2k} v - \gamma\,|D|^{2k} \nabla^2 v + \varepsilon\,|D|^{2k+1} u).\end{aligned}$$

Notice that if $f_1 = f_2 = 0$ and $k = \varepsilon = 0$, then $\dfrac{dE}{dt} = -\mu(F(0)\nabla u, \nabla u)$. If $\bar{u}$ is an eigenfunction such that $\nabla^2 \bar{u} + \lambda^2 \bar{u} = 0$, then $\dfrac{dE}{dt} = \mu\lambda^2(F(0)\,\bar{u}, \bar{u})$. We have proved that for a normal mode with wave number λ, the energy is lost at a rate $\mu\lambda^2 \times$(total kinetic energy); see § 1. Thus, the artificial damping term represents the physical energy attenuation at high wave numbers. In the general situation, in which $f_1, f_2, \varepsilon, k \neq 0$, we employ the following inequalities:

$$-\mu(F(0)\,|D|^k \nabla u, |D|^k \nabla u) \leqq -c\mu\,|u|_{k+3/2}^2,$$

$$-\varepsilon\gamma \cdot \big||D|^{k+1/2} \nabla v\big|_0^2 \leqq -\varepsilon C\,|v|_{k+3/2}^2,$$

$$-\varepsilon\mu(|D^{k+1/2}|\,\nabla u, |D|^{k+1/2} \nabla v) \leqq \frac{C\sqrt{\varepsilon}\mu}{2}\,|u|_{k+3/2}^2 + \frac{C\varepsilon^{3/2}\mu}{2}\,|v|_{k+3/2}^2,$$

$$\varepsilon(F(0)\,|D|^{k+1/2} u, |D|^{k+1/2} u) \leqq C\varepsilon\,|u|_{k+1}^2,$$

$$\begin{aligned}(f_1, F(0)\,|D|^{2k} u + \varepsilon\,|D|^{2k+1} v) &\leqq C\{|f_1|_{k-1/2}\,|u|_{k+3/2} + \varepsilon\,|f_1|_{k-1/2}\,|v|_{k+3/2}\}\\ &\leqq \varepsilon\,|u|_{k+3/2}^2 + \varepsilon^2 |v|_{k+3/2}^2 + C_1(\varepsilon)\,|f_1|_{k-1/2}^2,\end{aligned}$$

$$\begin{aligned}&(f_2, g\,|D|^{2k} v - \gamma\,|D|^{2k} \nabla^2 v + \varepsilon\,|D|^{2k+1} u)\\ &\leqq C\{|f_2|_{k+1/2}\,|v|_{k+3/2} + \varepsilon\,|f_2|_{k+1/2}\,|u|_{k+1/2}\}\\ &\leqq \varepsilon^2(|v|_{k+3/2}^2 + |u|_{k+3/2}^2) + C_2(\varepsilon)\,|f_2|_{k+1/2}^2,\end{aligned}$$

where $C_1(\varepsilon)$, $C_2(\varepsilon)$ depend on ε. Hence, there is an ε_2, $0 < \varepsilon_2 \leqq \varepsilon_1$ such that if $0 < \varepsilon < \varepsilon_2$, then

$$\frac{dE_\varepsilon(t)}{dt} \leqq -\beta(\varepsilon)\,E_\varepsilon(t) + C(\varepsilon)\,\{|f_1|_{k-1/2}^2 + |f_2|_{k+1/2}^2\},$$

where $\beta(\varepsilon)$, $C(\varepsilon)$ are positive constants depending on ε. By the Gronwall inequality,

$$E_\varepsilon(t) \leqq E_\varepsilon(0)\,e^{-\beta(\varepsilon)t} + C(\varepsilon)\int_0^t \{|f_1|_{r-2}^2 + |f_2|_{r-1}^2\}\,ds,\ t > 0. \tag{6.3}$$

We are now ready to prove Theorem 3.2. Fix $0<\varepsilon<\varepsilon_2$ and then choose $\tilde{t}>0$ such that $e^{-\beta(\varepsilon)\tilde{t}}\leqq\frac{1}{2}$. From Theorem 2.1, we know that there is a $\delta_1>0$ such that if $E_\varepsilon(0)<\delta_1^2$ and $\|\alpha\|<\delta_1$, then the solution (u, v) of (2.2)–(2.4) exists in $(0, \tilde{t})$, by virtue of (6.2). Moreover,

$$\|(u, v)\|_{X^r(0,\tilde{t})}\leqq C\,\delta_1. \tag{6.4}$$

From (6.4), we have $\|(f_1(\alpha, u, v), f_2(u, v))\|_{Y^r(0,\tilde{t})}=o(\delta_1)$. We can choose δ so small that $C(\varepsilon)\int_0^{\tilde{t}}\{|f_1|_{r-2}^2+|f_2|_{r-1}^2\}\,ds<\frac{\delta_1^2}{2}$. Inequality (6.3) implies that $E_\varepsilon(\tilde{t})<\delta_1^2$. It is now clear that the proof of Theorem 2.2 can be completed by induction.

The existence of the solutions (u^N, v^N) for (2.8)–(2.10) under the hypotheses of Theorem 2.3, parts (i) and (ii), is obvious from the proof of Theorem 2.1 and 2.2. We shall now show that the convergence results also follow from the estimates employed in proving Theorems 2.1 and 2.2.

Let us write (2.2)–(2.4) in an abstract form as

$$\begin{aligned} U_t-\mathscr{A}U&=F(\alpha, U),\\ U(0)&=U_0,\end{aligned} \tag{6.5}$$

where $U=(u, v)$, $F(\alpha, U)=(f_1(\alpha, u, v), f_2(u, v))$, and $\mathscr{A}$ is the operator defined in § 5. $F(\alpha, U)$ is of second order with respect to α, U:

$$|F(\alpha, U)|_{Y^r}\leqq C\{|\alpha|_{(r-2)/2}\,|U|_{X^r}+|U|_{X^r}^2\},$$

$$|F(\alpha, U_1)-F(\alpha. U_2)|_{Y^r}\leqq C\{(|\alpha|_{(r-2)/2}+|U_1|_{X^r}+|U_2|_{X^r})\,|U_1-U_2|_{X^r}\}.$$

The discretized system can be written as

$$\begin{aligned} U_t^N-\mathscr{A}U^N&=P_NF(\alpha, U^N),\\ U^N(0)&=U_0^N.\end{aligned} \tag{6.6}$$

We assume that $U_0^N\to U_0$ as $N\to\infty$; for example, we can take $U_0^N=P_NU_0$. Subtracting (6.5) from (6.6), we obtain

$$\begin{aligned} (U^N-U)\,t-\mathscr{A}(U^N-U)&=P_NF(\alpha, U^N)-F(\alpha, U),\\ (U^N-U)(0)&=U_0^N-U_0.\end{aligned} \tag{6.7}$$

First, let us prove Theorem 2.3(i). If $|U_0|_{X^r}+|\alpha|_{(r-2)/2}<\delta$, with δ sufficiently small, then $|U|_{X^r}<\varepsilon(\delta)$ and $|U^N|_{X^r}<\varepsilon(\delta)$ are small, uniformly with respect to N (at least for sufficiently large N's). Let $|U_0^N-U_0|_{H^{r-1}\times H^{r-1/2}}=\bar{\varepsilon}_N$, $\bar{\varepsilon}_N\to 0$ as $N\to\infty$. We also have

$$\begin{aligned} &|P_NF(\alpha\ \ U^N)-F(\alpha\ \ U|_{Y^r})\\ &\quad\leqq|P_NF(\alpha, U^N)-P_NF(\alpha, U)|_{Y^r}+|P_NF(\alpha, U)-F(\alpha, U)|_{Y^r}\\ &\quad\leqq C(\delta+2\varepsilon(\delta))\,|U^N-U|_{X^r}+\varepsilon_N,\end{aligned} \tag{6.8}$$

where $\varepsilon_N \to 0$ as $N \to \infty$. From our linear estimates in § 5 and from (6.7) we obtain

$$|U^N - U|_{X^r} \leqq C\{|U_0^N - U_0|_{H^{r-1}\times H^{r-1/2}} + |P_N F(\alpha, U^N) - F(\alpha, U)|_{Y^r}\}$$
$$\leqq C\bar{\varepsilon}_N + C\varepsilon_N + C(\delta + 2\varepsilon(\delta))\,|U^N - U|_{X^r}.$$

If δ is sufficiently small, then $C(\delta + 2\varepsilon(\delta)) < \frac{1}{2}$, and we have

$$|U^N - U|_{X^r} \leqq C\bar{\varepsilon}_N + C\varepsilon_N \to 0 \quad \text{as } N \to \infty. \tag{6.9}$$

This completes the proof of Theorem 2.3(i).

The proof of Theorem 2.3(ii) uses the energy estimate (6.3), which can be written as

$$|(U^N - U)(t)|^2_{H^{r-1}\times H^{r-1/2}} \leqq C\,|U_0^N - U_0|^2_{H^{r-1}\times H^{r-1/2}}\, e^{-\beta(\varepsilon)t}$$
$$+ C(\varepsilon)\int_0^t |P_N F(\alpha, U^N) - F(\alpha, U)|^2_{H^{r-2}\times H^{r-1}}\, dt.$$

Choosing $\tilde{t} > 0$ sufficiently large so that $Ce^{-\beta(\varepsilon)\tilde{t}} < \frac{1}{3}$ and using (6.8) we have

$$|(U^N - U)(\tilde{t})|^2_{H^{r-1}\times H^{r-1/2}} \leqq \tfrac{1}{3}\,|U_0^N - U_0|^2_{H^{r-1}\times H^{r-1/2}}$$
$$+ C(\varepsilon)\,(\delta + 2\varepsilon(\delta))\,|U^N - U|_{X^r(0,\tilde{t})} + C(\varepsilon)\,\varepsilon_N.$$

From (6.9), we have

$$|(U^N - U)(\tilde{t})|^2_{H^{r-1}\times H^{r-1/2}} \leqq \tfrac{2}{3}\,|U_0^N - U_0|^2_{H^{r-1}\times H^{r-1/2}} + C_1\varepsilon_N,$$

provided that δ is small and $C(\varepsilon)\cdot C\cdot(\delta + 2\varepsilon(\delta)) < \frac{1}{3}$.

Given any $\eta > 0$, we can choose N_0 such that $|U_0^N - U_0|^2_{H^{r-1}\times H^{r-1/2}} < \eta$ and $\cdot\, C_1\varepsilon_N < \dfrac{\eta}{3}$ for $N > N_0$. Then $|(U^N - U)(\tilde{t})|^2_{H^{r-1}\times H^{r-1/2}} < \eta$. The convergence on $(j\tilde{t}, j\tilde{t} + \tilde{t})$, $j = 1, 2, \ldots,$ can be considered similarly. We observe here that $\varepsilon_N^j = \|P_N^j F(\alpha, U) - F(\alpha, U)\|_{Y^r(j\tilde{t}, j\tilde{t}+\tilde{t})} \to 0$ as $N \to \infty$, uniformly with respect to j, by virtue of the additional regularity of $F(\alpha, U)$ for $t > \tilde{t}$ that was mentioned in Remark 6.1. The proof of Theorem 2.3 is now complete.

Acknowledgment. I am grateful for many valuable discussions with P. Holmes, who suggested the problem to me. J. Guckenheimer and G. Raugel informed me about the work of J. T. Beale. T. Healey suggested the possibility of including a damping term in the system. C. Dafermos, K. Mischaikow and S. Schecter corrected many grammatical errors for me. This research was begun while I was visiting Cornell University and completed at Michigan State University. Support from both institutions is gratefully acknowledged.

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Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27695

(Received June 21, 1990)