COLLAPSING AND EXPLOSION WAVES IN PHASE TRANSITIONS WITH METASTABILITY, EXISTENCE, STABILITY AND RELATED RIEMANN PROBLEMS

HAITAO FAN AND XIAO-BIAO LIN

Dedicated to Professor Jack Hale's 80th birthday

ABSTRACT. Collapsing waves were observed numerically before and were used to explain the ring formations in dynamic flows involving phase transitions with metastability. In this paper, necessary and sufficient conditions for collapsing type of waves to exist are given. The conditions are that the wave speed of the collapsing wave is not less than a number and is supersonic on both sides of the wave. Existence and non-existence conditions for the explosion waves are also found. The stability of these waves are studied numerically. Although there are infinitely many collapsing (or explosion) waves for a fixed downstream state, the collapsing (or explosion) wave appeared in the solution of Riemann problem is numerically verified to be the one with the slowest speed. Although a Riemann problem in the zero viscosity limit may have two solutions, one with, the other without, a collapsing (or explosion) wave, from the vanishing viscosity point of view, the one with a collapsing (or explosion) wave is numerically verified to be admissible.

1. INTRODUCTION

Dynamic flows involving liquid/vapor phase transition is an important phenomenon occurring in many engineering processes. For retrograde fluids, i.e. fluids with high specific heat capacities, such flows can be approximated by assuming the temperature is a constant. The one-dimensional case of the system describing such flows in Lagrangian

Date: August 11, 2009.
Research of Lin was supported in part by the National Science Foundation under grant DMS-0708386.

coordinates is

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_t + p(\lambda, v)_x = \epsilon u_{xx},\\
&\lambda_t = \frac{1}{\gamma} w(\lambda, v) + \beta \lambda_{xx},
\end{aligned}
\tag{1.1}
$$

where v is the specific volume, u the velocity of the fluid, λ the weight portion of vapor in the liquid/vapor mixture, ϵ the viscosity, β the diffusion coefficient and $\gamma > 0$ the typical reaction time. The pressure function in (1.1), $p(\lambda, v)$, satisfies

$$p_v < 0 < p_\lambda. \tag{1.2a}$$

In this paper, we further assume

$$p_{vv} > 0. \tag{1.2b}$$

Figure 1.1 shows the graph of a typical pressure function, where p_e is the equilibrium pressure at which liquid and vapor can coexist and m, M are the Maxwell points. In this paper, we use the following scaling

$$\gamma = \epsilon / a, \ \ \beta = b\epsilon. \tag{1.2c}$$

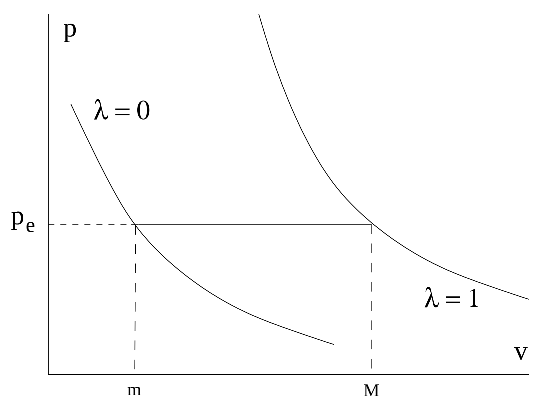

FIGURE 1.1. The pressure function $p = p(\lambda, v)$ for some fixed λ.

The function $w(\lambda, v)$ represents the rate of vapor initiation and growth. To study the traveling waves of (1.1) and the related issues, we take

$$w(v, \lambda) = (p(\lambda, v) - p_e)\lambda(\lambda - 1), \tag{1.3}$$

where p_e is the equilibrium pressure. System (1.1-3) not only exhibits all major one-dimensional wave patterns observed in actual experiments on retrograde fluids [4], but also explains the puzzling ring formations observed experiments [7].

The comparison of experimental observations and the behavior of (1.1-3) done in [4, 7] are through the study of the system's traveling waves and Riemann problems. A traveling wave of (1.1) is a solution of (1.1-3) of the form $(u, v, \lambda)(\frac{x-ct}{\epsilon})$, where c is the speed of the wave. Plugging the form of the solution into (1.1), we see that it is a solution of

$$\begin{aligned} & -cv' - u' = 0, \\ & -cu' + p' = u'', \\ & -c\lambda' = aw(\lambda, v) + b\lambda'', \\ & (u, v, \lambda)(\pm\infty) = (u_\pm, v_\pm, \lambda_\pm), \end{aligned} \tag{1.4}$$

where $a = \epsilon/\gamma$, $b = \beta/\epsilon$. Because λ is the weight portion of the vapor in the liquid/vapor mixture, we only admit solutions with $0 \leq \lambda \leq 1$. The Rankine-Hugoniot condition

$$\begin{aligned} & -c(v_+ - v_-) - (u_+ - u_-) = 0, \\ & -c(u_+ - u_-) + p(v_+, \lambda_+) - p(\lambda_-, v_-) = 0 \end{aligned} \tag{1.5}$$

is necessary for (1.4) to have a solution. For collapsing and explosion waves it will be shown in Lemma 2.1 that the wave speed c is positive. Therefore, from (1.5)

$$c = \sqrt{-\frac{p_(\lambda_+, v_+) - p(\lambda_-, v_-)}{v_+ - v_-}} \tag{1.6}$$

holds. Besides (1.6), equilibrium points $(v_\pm, \lambda_\pm)$ must also satisfy one of these three equations $\lambda = 0$ or $\lambda = 1$ or $p(\lambda, v) = p_e$.

Definition 1.1. A liquefaction wave is a solution of (1.4) with

$$\begin{aligned} & \lambda_- = 0,\ 0 < \lambda_+ \leq 1, \\ & p(\lambda_\pm, v_\pm) \geq p_e,\ c^2 + p_v(\lambda_\pm, v_\pm) < 0, \end{aligned}$$

while an evaporation wave is that with

$$\begin{aligned} & \lambda_- = 1,\ 0 \leq \lambda_+ < 1, \\ & p(\lambda_\pm, v_\pm) \leq p_e,\ c^2 + p_v(\lambda_\pm, v_\pm) < 0. \end{aligned}$$

In [4, 6], Fan proved that liquefaction and evaporation waves exist if the wave speed $c > 0$ satisfies $c \geq 2\sqrt{ab|p(\lambda_-, v_-) - p_e|}$. On the other hand, if the speeds satisfy $c \leq 2\sqrt{ab|p(\lambda_+, v_+) - p_e|}$, then there is no liquefaction and evaporation waves. Since the system (1.1-3) consists of conservation laws and a reaction-diffusion equation, Fan [5], investigated the the stability of traveling waves of (1.1-3) by studying a simplified prototype system consists of one conservation law and a

KPP equation. The results on the simplified system suggest that the stability of its traveling waves is decided by the KPP equation in the system. Another simpler prototype system in [5] modeling the effect of nucleation rate term on the speed of liquefaction traveling waves showed that the nucleation rate term will speed up the liquefaction wave. Fan and Corli [3] showed the existence and uniqueness of the solution of Riemann problem for (1.1) with $\epsilon = \gamma = \beta = 0$. Amadori and Corli established the existence of global solutions inviscid case of (1.1-3) with $a = b = 0$ for a class of initial data of large total variations, [1]. Trivisa proved the existence of variational solutions of the non-isothermal multi-dimensional case of (1.1) under various assumptions [9].

Definition 1.2. Collapsing waves are traveling waves of (1.1) with

$$\begin{aligned} &\lambda_+ = 1, \quad p(\lambda_+, v_+) > p_e, \\ &0 \le \lambda_- \le 1, \quad p(\lambda_-, v_-) = p_e, \quad \text{or} \quad \lambda_- = 0, \; p(\lambda_-, v_-) > p_e. \end{aligned} \tag{1.7}$$

Explosion waves of (1.1) are traveling waves with

$$\begin{aligned} &\lambda_+ = 0, \quad p(\lambda_+, v_+) < p_e, \\ &0 \le \lambda_- \le 1, \quad p(\lambda_-, v_-) = p_e, \quad \text{or} \quad \lambda_- = 1, \; p(\lambda_-, v_-) < p_e. \end{aligned} \tag{1.8}$$

We require $0 \le \lambda \le 1$ for both waves.

The locations of $(v_\pm, \lambda_\pm)$ for both waves are depicted in Figure 1.1.

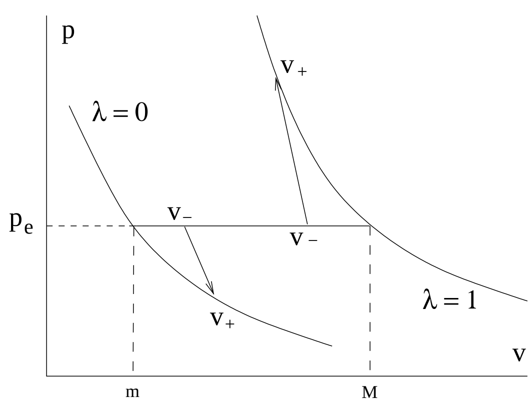

FIGURE 1.2. The points $(v_\pm, \lambda_\pm)$ of collapsing wave $(p_+ > p_-)$ or explosion wave $(p_+ < p_-)$ for the case $p_- = p_e$. The arrows point to the fronts of the waves.

For the existence of a collapsing wave, the supersonic condition

$$c^2 \ge -p_v(\lambda_+, v_+),$$

must hold (cf. Theorem 2.8). There are up to two v_+ that satisfy (1.6) and (1.7) for a given pair of v_- and c. Only the unique v_+ that satisfies the supersonic condition is shown in the Figure 1.2.

For explosion waves, the inequality

$$c^2 \geq -p_v(\lambda_+, v_+)$$

is a consequence of the assumption (1.2).

The collapsing wave is used to explain the formation of cloudy rings in closed end shock tube experiments [7]. In fact, the outer front of the cloudy ring is a collapsing wave. However, in [7], the existence of collapsing and explosion waves was verified only numerically.

One of the goals of this paper is to find the conditions for the existence and non-existence of collapsing and explosion waves solutions satisfying $0 \leq \lambda(\xi) \leq 1$ for all ξ.

In Section 2, we shall prove that the necessary and sufficient conditions for the existence of collapsing waves are

$$\text{(H1)} \qquad c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_\pm, v_\pm) \geq 0.$$

For the existence of explosion waves, we show that the following conditions are sufficient:

$$\text{(H2)} \qquad c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_-, v_-) \geq 0.$$

We also find a set of necessary conditions for the existence of explosion waves:

$$\text{(H3)} \qquad c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_+, v_+) > 0.$$

These collapsing and explosion waves shown to exist are monotone.

A real liquid/vapor phase transition model may involve some very small or large parameters. In section 3, we study the dynamics of the collapsing/explosion waves as $b \to 0$ (or ∞). This corresponds to systems with the ratio of diffusion coefficient to viscosity being very small (or large). We find that if b is small, the traveling waves must fast jump from $\lambda = \lambda_e$ to near $\lambda = \lambda_+$ while v remains almost unchanged; then $v(\xi)$ moves slowly from v_- to v_+ along $\lambda = \lambda_+$. When b is large, the solution of the traveling wave cannot have any fast jump. Starting from the equi-presure line, the traveling wave solution stays on a two dimensional slow manifold that is normally stable. The traveling wave from (λ_-, p_-, v_-) to (λ_+, p_+, v_+) restricted to the slow manifold is of fisher type – a saddle to attractor connection.

Given (v_+, λ_+) satisfying (1.7) or (1.8), there are infinity many (v_-, λ_-) satisfying (1.7) and (H1) (or (1.8) and (H2)), and hence infinity many traveling waves connecting to the given (v_+, λ_+) with different speeds. A natural question is which one of them will actually appear in the solution of (1.1) for large t. We consider the perturbed Riemann initial value

$$(u, v, \lambda)(x, 0) = (U_0, V_0, \Lambda_0)(x) + \begin{cases} (u_+, v_+, \lambda_+), & x > 0 \\ (u_-, v_-, \lambda_-), & x < 0 \end{cases} \tag{1.9}$$

where $0 \leq \lambda(x) \leq 1$. The last equation in (1.1) is similar to the KPP equation

$$\begin{aligned} &\lambda_t = \lambda(\lambda - 1) + \lambda_{xx}, \\ &\lambda(x, 0) = \Lambda_0(x) + \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \end{aligned} \tag{1.10}$$

The traveling wave occurring in the solution of initial value problem of (1.10) for large t is the one with decay rate similar to that of $\Lambda_0(x)$ as $x \to \infty$. See [2]. When $\Lambda_0(x) \equiv 0$ for $x > 0$, the traveling wave with the slowest speed appears in the solution. We expect that similar phenomenon occurs for collapsing and explosion waves of (1.1). Our numerical computation, shown in section §4, supports this conjecture.

To study the behavior of (1.1-3) when $\epsilon > 0$ is small, under the scaling $\gamma = \epsilon/a$ and $\beta = b\epsilon$, we assume the strong limit $\lim_{\epsilon \to 0+}$ of the solution exist, and study the behavior of the limit. Such limit will satisfy the limiting equation of (1.1-3),

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p_x = 0, \\ &(p - p_e)\lambda(\lambda - 1) = 0. \end{aligned} \tag{1.11}$$

The Riemann problem of (1.11) is (1.11) with the initial data

$$(u, v, \lambda)(x, 0) = \begin{cases} (u_+, v_+, \lambda_+), & x > 0 \\ (u_-, v_-, \lambda_-), & x < 0 \end{cases}$$

which is (1.9) with $(U_0, V_0, \Lambda_0)(x) \equiv 0$. Then the behavior of traveling waves of (1.10) cited in the last paragraph suggested that when there are multiple liquefaction waves available for constructing solutions of (1.11), resulting in nonuniqueness, the one with the slowest speed should be chosen, at least for the time period not too far from $t = 0$ so that the nucleation effect is negligible. The nucleation process will initiate droplets of the other phase, making $\Lambda_0 \neq 0$ at some point (x, t), and hence speeding up the motion of phase boundaries. The same restriction should be imposed on evaporation, collapsing, and explosion waves.

In [3], by excluding collapsing and explosion waves, existence and uniqueness of the solution of the Riemann problem for (1.11) is established under kinetic relations mimicking the behavior of the slowest liquefaction and evaporation waves and excluding collapsing and explosion waves. The exclusion of collapsing and explosion waves is artificial and should be removed. However, admitting liquefaction (evaporation) waves and collapsing (explosion) waves simultaneously results in two solutions for the same Riemann initial data. To resolve this nonuniqueness, we have to find a criterion on when to use liquefaction (evaporation) waves and when to use collapsing (explosion) waves. In §5, we show that when a pure phase is in contact with mixture or the other pure phase, then collapsing and explosion waves of the slowest speed are preferred if they exist. Whether this criterion will settle the uniqueness of the Riemann problem is left for future research.

2. Existence of collapsing and explosion waves

In this section we present the proof of the existence of collapsing and explosion waves for the case $p(\lambda_-, v_-) = p_e$. The same proof applies to the case $\lambda_- = 0, 1$, $p(\lambda_-, v_-) \neq p_e$ with some minor change which will not be given in this paper.

The system (1.1) can be written as

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p(\lambda, v)_x = \epsilon u_{xx}, \\ &\lambda_t = \frac{a}{\epsilon} w(\lambda, v) + b\epsilon\lambda_{xx}. \end{aligned}$$

We do not assume ϵ to be small in this section. Recall that the pressure $p(\lambda, v)$ satisfies

$$p_v < 0 < p_\lambda,$$

and the growth rate w is

$$w(\lambda, v) = (p - p_e)\lambda(\lambda - 1).$$

Since $p_\lambda > 0$, the function $p = p(\lambda, v)$ can be solved for $\lambda = \lambda^*(v, p)$. For each $v = v_0$, with $v_m < v_0 < v_M$, $w = w(\lambda, v_0)$ has three zeros: $\lambda = 0,\ \lambda_e = \lambda^*(v_0, p_e),\ \lambda = 1$, depicted in Figure 2.1.

Consider the difference quotients:

$$\begin{aligned} Q_1(\lambda, v_0) &:= (w(\lambda, v_0) - w(0, v_0))/(\lambda - 0) \\ &= (p(\lambda, v_0) - p_e)(\lambda - 1), \quad 0 < \lambda < \lambda_e. \\ Q_2(\lambda, v_0) &:= (w(\lambda, v_0) - w(1, v_0))/(\lambda - 1) \\ &= (p(\lambda, v_0) - p_e)\lambda, \quad \lambda_e < \lambda < 1. \end{aligned}$$

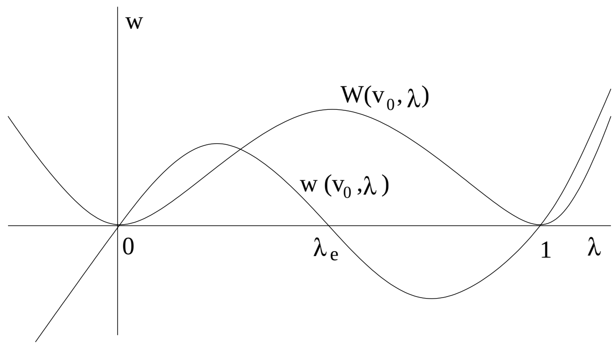

FIGURE 2.1. The graphs of w and $W = \int w d\lambda$.

It is easy to check that

$$\partial_\lambda Q_1(\lambda, v_0) = (\lambda - 1)p_\lambda + p(\lambda, v_0) - p_e < 0, \qquad \text{for } 0 < \lambda < \lambda_e,$$
$$\partial_\lambda Q_2(\lambda, v_0) = \lambda p_\lambda + p(\lambda, v_0) - p_e > 0, \qquad \text{for } \lambda_e < \lambda < 1.$$

Thus, Q_1 (or Q_2) is a decreasing (or increasing) function of λ on its domain. In particular, we have

$$Q_1(\lambda, v_0) < \frac{\partial w(\lambda = 0, v_0)}{\partial \lambda}, \qquad 0 < \lambda < \lambda_e, \tag{2.1}$$

$$Q_2(\lambda, v_0) < \frac{\partial w(\lambda = 1, v_0)}{\partial \lambda}, \qquad \lambda_e < \lambda < 1. \tag{2.2}$$

These properties are reflected in Figure 2.1 as concavity of the function $w(\lambda, v_0)$ for $\lambda < \lambda_e$ and $\lambda > \lambda_e$ respectively.

The system can be recast into

$$v_{tt} + p_{xx} = \epsilon v_{txx},$$
$$\epsilon \lambda_t = a w(\lambda, v) + b\epsilon^2 \lambda_{xx}.$$

We look for explosion waves connecting the equilibrium pressure line $p(\lambda, v) = p_e$ to the pure liquid state $\lambda = 0$ and collapsing waves connecting $p(\lambda, v) = p_e$ to pure vapor state $\lambda = 1$.

Let c be the wave speed of traveling solutions, so that $v = v(x - ct)$, $\lambda = \lambda(x - ct)$. Let $\dot{y}$ denote $\frac{dy}{ds}$ where $s = x - ct$. The traveling wave equation for v is:

$$c^2 \ddot{v} + \ddot{p} = -c\epsilon \frac{d^3 v}{ds^3}.$$

If we integrate the above twice from $-\infty$ to s, we obtain the following system for traveling waves:

$$\begin{aligned}-\epsilon c\dot{v} &= c^2(v-v_-)+(p-p_-),\\ -\epsilon c\dot{\lambda} &= aw+b\epsilon^2\ddot{\lambda}.\end{aligned}$$

If we rescale the time $s=\epsilon b\xi$, and use y' to denote $dy/d\xi$, we have obtained the the traveling waves system that will be considered in this section:

$$\begin{aligned}&\lambda''+c\lambda'+abw(\lambda,v)=0,\\ &cv'+bc^2(v-v_-)+b(p(\lambda,v)-p_-)=0.\end{aligned} \tag{2.3}$$

The above can be rewritten as a first order system of three variables (λ,μ,v):

$$\begin{aligned}\lambda' &= \mu,\\ \mu' &= -c\mu-abw(\lambda,v),\\ v' &= -bc(v-v_-)-\frac{b}{c}(p(\lambda,v)-p_-).\end{aligned} \tag{2.4}$$

We look for a heteroclinic solution of (2.4) connecting the equilibrium points $E_\pm:=\{(\lambda_\pm,\mu_\pm,v_\pm)\}$.

Equilibrium states are the zeros of the right hand side of (2.4).

$$\begin{aligned}&\mu=0,\\ &w(\lambda,v)=0,\\ &c^2(v-v_-)+p(\lambda,v)-p_-=0.\end{aligned} \tag{2.5}$$

The solutions of $w=0$ form three branches: $\lambda=0,\ 1$ and $p(\lambda,v)=p_e$. The graph of (2.5) with a given c is a straight line in Figure 1.2. The equilibrium points in (v,p) coordinates corresponding to the $0<\lambda_-<1$ case of collapsing and explosion waves, $E_\pm:=\{(v_\pm,p_\pm)\}$, are plotted in Figure 1.2. Since we look for a traveling wave solution from the equi-pressure line $p=p_e$ to $\lambda=0$ or 1, we need $p_-=p_e,\lambda_+=0,1$.

For any v_- that satisfies $v_m<v_-<v_M$, equation $p(\lambda_-,v_-)=p_e$ has a unique solution

$$\lambda_-=\lambda_e=\lambda^*(v_-,p_e).$$

The equilibrium E_- is parameterized by v_-:

$$E_-:=\{(\lambda_-,\mu_-,v_-)=(\lambda^*(v_-,p_e),0,v_-)\}. \tag{2.6}$$

The equilibrium E_+ is on the line $\lambda = 0$ (explosion wave) or 1 (collapsing wave) and is parameterized by v_+:

$$E_+ := \{(\lambda_+, \mu_+, v_+) | \lambda_+ = 0 \text{ or } 1, \mu_+ = 0, \\ v_+ > v_- \text{ for explosion waves }, v_+ < v_- \text{ for collapsing waves}\}. \tag{2.7}$$

Let $p_+ = p(\lambda_+, v_+)$ where $\lambda_+ = 0$ or 1. The wave speed c and $v_\pm$ are now related by:

$$c^2(v_+ - v_-) + (p_+ - p_-) = 0. \tag{2.8}$$

This also confirms that for each pair of (E_-, E_+), the wave speed c satisfies (1.6).

It will be proved in Lemma 2.1 that if $c < 0$, then E_+ is an unstable equilibrium to (2.4). A traveling wave from E_- to E_+ cannot exist if $c < 0$. In this paper we always assume that $c > 0$.

For each triple (E_-, E_+, c) we will assume conditions (H1) (or (H2)) to construct collapsing waves (or explosion waves).

From (2.8), for the collapsing wave $p_+ > p_-$, we must have $v_+ < v_-$; while for the explosion wave $p_+ < p_-$, we must have $v_+ > v_-$.

2.1. **Eigenvalues and eigenvectors at equilibrium points.** In this subsection, we first show that if $c > 0$, then the equilibrium E_- is a saddle with exactly one positive real eigenvalue; while E_+ is an attractor with all eigenvalues real and negative. The traveling wave solution we look for is a heteroclinic solution connecting the saddle to the attractor. Moreover, as $\xi \to \pm\infty$, the orbit of the traveling wave is tangent to the linear space spanned by the eigenvectors corresponding to real eigenvalues, thus non-oscillatory. In the next subsection, we will prove the existence of traveling wave solutions and obtain more precise information about the traveling wave solutions, e.g., the (λ, v) components of the traveling wave solutions are monotone.

The linear variational system for the traveling wave equations (2.4) is

$$\begin{pmatrix} \Lambda \\ M \\ V \end{pmatrix}' = A(b) \begin{pmatrix} \Lambda \\ M \\ V \end{pmatrix}, \quad \text{where } A(b) = \begin{pmatrix} 0 & 1 & 0 \\ -abw_\lambda & -c & -abw_v \\ -\frac{b}{c}p_\lambda & 0 & -bc - \frac{b}{c}p_v \end{pmatrix}.$$

Eigenvalues r are determined by

$$\det(rI - A(b)) = \begin{vmatrix} r & -1 & 0 \\ abw_\lambda & r + c & abw_v \\ \frac{b}{c}p_\lambda & 0 & r + \frac{b}{c}(p_v + c^2) \end{vmatrix} = 0.$$

We first study eigenvalues at the equilibrium E_+. Observe at $\lambda_+ = 0$, or 1,

$$w_\lambda(\lambda_+, v_+) = |p - p_e| > 0, \quad w_v(\lambda_+, v_+) = 0,$$
$$\det(rI - A(b)) = (r^2 + cr + w_\lambda)(r + \frac{b}{c}(p_v + c^2)).$$

Eigenvalues at E_+ are

$$r_1 = -c/2 - \sqrt{(c/2)^2 - ab|p(\lambda_+, v_+) - p_e|},$$
$$r_2 = -c/2 + \sqrt{(c/2)^2 - ab|p(\lambda_+, v_+) - p_e|},$$
$$r_3 = -\frac{b}{c}(p_v(\lambda_+, v_+) + c^2).$$

Lemma 2.1. *(1) If $c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|$ and $c^2 + p_v(\lambda_+, v_+) \geq 0$ are satisfied and $c > 0$, then the equilibrium E_+ has three real negative eigenvalues.*

(2) If $c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|$ and $c^2 + p_v(\lambda_+, v_+) \geq 0$ are satisfied and $c < 0$, then E_+ has three real positive eigenvalues.

The proof of the lemma is straightforward and shall be omitted.

We next study the eigenvalues at the equilibrium E_- where $p = p_e$. Observe that

$$\det(-A(b)) = \frac{ab^2}{c}\begin{vmatrix} w_\lambda & w_v \\ p_\lambda & p_v + c^2 \end{vmatrix},$$
$$\mathrm{tr}(-A(b)) = c + \frac{b}{c}(p_v(\lambda_-, v_-) + c^2).$$

Lemma 2.2. *If $c > 0$ and $c^2 + p_v(\lambda_-, v_-) \geq 0$ are satisfied, then E_- has two stable eigenvalues $r_j, j = 1, 2$, real or complex, with $Re r_j < 0$, and one real, unstable eigenvalue $r_3 > 0$. The eigenvector corresponding to r_3 is*

$$(\Lambda, M, V) = (1, r_3, -\frac{bp_\lambda/c}{r_3 + b(p_v + c^2)/c}),$$

with $\Lambda > 0, M > 0$ and $V < 0$. The system has an one-dimensional unstable manifold passing through E_-.

Proof. First, we prove the lemma for the case of $p_- \neq p_e$ in the Definition 1.2. In this case, we have

$$\det(rI - A(b)) = \begin{vmatrix} r & -1 & 0 \\ -ab|p_- - p_e| & r + c & 0 \\ \frac{b}{c}p_\lambda & 0 & r + \frac{b}{c}(p_v + c^2) \end{vmatrix} = 0.$$

One of the eigenvalue of $A(b)$,

$$r = \frac{1}{2}\left(-c + \sqrt{c^2 + 4ab|p_- - p_e|}\right),$$

is positive with an eigenvector

$$(\Lambda, M, V) = (1, r_3, -\frac{bp_\lambda/c}{r_3 + b(p_v + c^2)/c}).$$

Now, we prove the lemma for the case of Definition 1.2 where $p_- = p_e$. From the conditions of this lemma,

$$r_1 + r_2 + r_3 = -\mathrm{tr}(-A(b)) < 0. \tag{2.9}$$

Observe that at E_-, $p = p_e$,

$$\begin{aligned} w_\lambda &= p_\lambda \lambda(\lambda - 1) < 0, \\ w_v &= p_v \lambda(\lambda - 1) > 0. \end{aligned}$$

Then we have at E_-,

$$r_1 r_2 r_3 = -\det(-A(b)) = -\frac{ab^2}{c} c^2 p_\lambda \lambda_-(\lambda_- - 1) > 0. \tag{2.10}$$

Now, we can have two cases:
(1) The system has a pair of complex-conjugate eigenvalues, denoted $r_1, r_2 = \bar{r}_1$, and one real eigenvalue r_3. Since $r_1 \cdot r_2 > 0$, from (2.10), $r_3 > 0$. Moreover, from (2.9), the real parts of r_1, r_2 are negative. The system has one unstable eigenvalue that corresponds to an one-dimensional unstable manifold at E_-.
(2) The system has three real eigenvalues. From (2.9), at least one of which is negative. From (2.10), exactly two of which are negative, denoted by $r_1,\ r_2$. The third eigenvalue r_3 must be positive. Again, the system has exactly one unstable real eigenvalue.

The eigenvector (Λ, M, V) for r_3 can be solved directly from the matrix $rI - A(b)$. Finally, from $V = -\frac{bp_\lambda/c}{r_3 + b(p_v + c^2)/c}$, $r_3 > 0,\ c > 0$ and $c^2 + p_v(\lambda_-, v_-) \geq 0$, we have $V < 0$. □

Information about the eigenvector corresponding to the eigenvalue r_3 will be useful in the future.

Lemma 2.3. *(i) Assume that $\lambda'(\xi) > 0$ for $\xi \in [0, T]$ and $v'(0) \leq 0$. Then $v'(\xi) < 0$ for $\xi \in (0, T]$.*

(ii) Assume that $\lambda'(\xi) < 0$ for $\xi \in [0, T]$ and $v'(0) \geq 0$. Then $v'(\xi) > 0$ for $\xi \in [0, T]$.

Proof. (i) The proof follows from the linear variational equation for v'. Let $a(\xi) = -\frac{b}{c}(p_v + c^2)$, $g(\xi) = \frac{b}{c}p_\lambda(\lambda, v)$ and $V(\xi) = v'(\xi)$. Then

$$V' = a(\xi)V - g(\xi)\lambda'(\xi),$$
$$V(\xi) = e^{\int_0^\xi a(t)dt}V(0) - \int_0^\xi e^{\int_\tau^\xi a(t)dt}g(\tau)\lambda'(\tau)\,d\tau < 0.$$

(ii) The proof of part is similar to that of part (i) and will be skipped. □

2.2. **Existence of collapsing waves for any $b > 0$.** We consider the collapsing wave connecting $p_- = p_e$ to $\lambda_+ = 1$. The explosion wave that connects $p_- = p_e$ to $\lambda_+ = 0$ will be constructed in the next subsection. As from Lemma 2.1, we assume that $c > 0$.

Consider the first two equations of the traveling wave system (2.4)

$$\lambda' = \mu, \quad \mu' = -c\mu - abw(\lambda, v),$$

with v as a parameter satisfying $v_+ \le v \le v_+$. From (H1), at $\lambda_+ = 1$ the system has two real eigenvalues $r_1(v) < r_2(v) < 0$.

$$r_1(v) = -c/2 - \sqrt{(c/2)^2 - ab(p(1, v) - p_e)},$$
$$r_2(v) = -c/2 + \sqrt{(c/2)^2 - ab(p(1, v) - p_e)}.$$

Let the smaller of the eigenvalue in norm, $r_2(v)$, be denoted $k(v)$ for simplicity. The eigenvector associated to $r_2(v)$ is $(\lambda, \mu) = (1, k(v))$. From $p_v < 0$, it can be verified that $dk(v)/dv > 0$. Note that $k(v) < 0$, this means that as v decreases, $|k(v)|$ increases and the eigenvector becomes more vertical.

Consider a prism shaped solid W in (λ, μ, v) space bounded by the surfaces (cf. Figure 2.2):

Left side $\mathcal{F}_\ell := \{\lambda = \lambda_e,\ \mu \ge 0\}$;

Bottom side $\mathcal{F}_b := \{\mu = 0,\ \lambda_e \le \lambda \le 1,\ v_+ \le v \le v_-\}$;

Back side $\mathcal{F}_k := \{v = v_-,\ \lambda_e \le \lambda \le 1,\ \mu \ge 0\}$;

Front side $\mathcal{F}_f := \{v = v^+,\ \lambda_e \le \lambda \le 1,\ \mu \ge 0\}$;

Slant side $\mathcal{F}_s := \{k(v)(\lambda - 1) - \mu = 0,\ \lambda_e \le \lambda \le 1, v_+ \le v \le v_-\}$.

Let $\mathbf{t} := (\Lambda, M, V)$ be the eigenvector associated with r_3 as in Lemma 2.2. Then $\Lambda > 0, M > 0$ and $V < 0$. A theorem on invariant manifolds asserts that $\{\alpha\mathbf{t} : \alpha \in \mathbb{R}\}$ is the one-dimensional tangent space of the local one-dimensional unstable manifold $W^u_{loc}(E_-)$ at $E_- = (\lambda_-, 0, v_-)$. For a sufficiently small $\bar{\alpha} > 0$, there is a near identity diffeomorphism $\Pi : \{\alpha\mathbf{t} : |\alpha| < \bar{\alpha}\} \to W^u_{loc}(E_-)$ such that the distance from $\alpha\mathbf{t}$ to its image is of $O(\alpha^2)$. For a sufficiently small $\alpha > 0$, $\alpha\mathbf{t}$ is in W, and

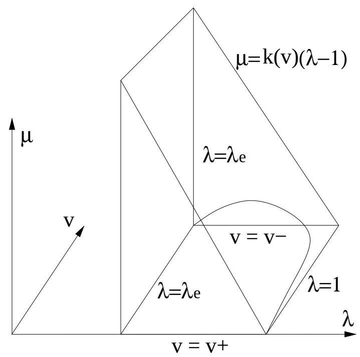

FIGURE 2.2. The prism shaped solid W

its distance to the boundary of W is $O(\alpha)$. Therefore a branch of the one-dimensional local unstable manifold $W^u_{loc}(E_-)$ must enter W from E_-. Let P be a point that is on $W^u_{loc}(E_-) \cap W$ and let the solution passing through P be $\phi(\xi, P)$, $\xi \geq 0$.

Next we want to show that the orbit of $\phi(\xi, P)$, $\xi \geq 0$ cannot leave W through all its surfaces. That is $\phi(\xi, P) \in W$ for all $\xi \geq 0$ and thus $\phi(\xi, P) \to E_+$ as $\xi \to \infty$. Therefore the entire branch of the unstable manifold is in W and is connected to $E_+ = (1, 0, v_+)$. Cf. Figure 2.3.

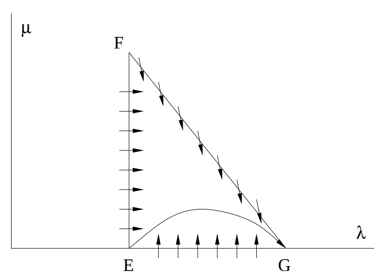

FIGURE 2.3. Front view of the solid W. The flow enters W from all its surfaces. The curve EG is the front view the heteroclinic solution.

Since $d\lambda/d\xi = \mu > 0$, the orbit of $\phi(\xi, P)$ can not hit $\mathcal{F}_\ell$ from inside of W.

On the interior of $\mathcal{F}_b$, we have $d\mu/d\xi = -abw(\lambda, v) > 0$ since $w < 0$ for $\lambda_e < \lambda < 1$. The orbit of $\phi(\xi, P)$ cannot hit the interior of $\mathcal{F}_b$ from inside of W also.

On $\mathcal{F}_k$, we have $v' = -\frac{b}{c}(p(v_-, \lambda) - p_-)$. Since $p_\lambda > 0$ and $\lambda \geq \lambda_e$, we have $p(v_-, \lambda) \geq p(v_-, \lambda_e) = p_-$. Thus, $v' \leq 0$ and the equal sign can only be achieved at $\lambda = \lambda_e$. The orbit of $\phi(\xi.P)$ cannot hit $\mathcal{F}_k$ from inside of W.

On $\mathcal{F}_f$, we have $v' = -\frac{b}{c}(p(v_+, \lambda) - p_+)$. Since $p_\lambda > 0$ and $\lambda \leq 1$, we have $p(v_+, \lambda) \leq p(v_+, 1) = p_+$. Thus, $v' \geq 0$ and the equal sign can only be achieved at $\lambda = 1$ which is where the equilibrium E_+ is. The orbit cannot hit $\mathcal{F}_f$ except by approaching E_+ as $\xi \to \infty$.

Before proving the following lemma, we comment on how we choose the the surface $\mathcal{F}_s$:

(i) By Lemma 2.3, $v' < 0$ for the solution $\phi(\xi, P)$. If we can show $\mathbf{n}_v < 0$ then $\mathbf{n}_v \cdot v' > 0$.

(ii) Consider the (λ, μ) components of the flow on $\mathcal{F}_s$. For each fixed v, if we choose the slope of $\mathcal{F}_s$ to be an eigenvector with negative slope, then the 2-d restriction of the flow, $d\mu/d\lambda$, is equal to the slope at $\lambda = 1$. Our formula (2.12) and (2.2) indicate that $d\mu/d\lambda$ is a monotone increasing function for $\lambda_e < \lambda < 1$.

From (i) and (ii), we can conclude that $\phi(\xi, P)$ cannot leave W on the side $\mathcal{F}_s$, where $\lambda < 1$. See below for details.

Lemma 2.4. *The solution $\phi(\xi, P)$ cannot leave W from the slant side $\mathcal{F}_s$.*

Proof. The normal of the slant side $\mathcal{F}_s := \{k(v)(\lambda - 1) - \mu = 0\}$, pointing downward and to the interior of W, is

$$\mathbf{n} = (\mathbf{n}_\lambda, \mathbf{n}_\mu, \mathbf{n}_v) = (k(v), -1, k'(v)(\lambda - 1))$$

The vector fields are

$$\mathbf{f} = (\mathbf{f}_\lambda, \mathbf{f}_\mu, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on $\mathcal{F}_s$,

$$\mathbf{n} \cdot \mathbf{f} = k(v)\mu + c\mu + abw(\lambda, v) + k'(v)(\lambda - 1)v' > 0.$$

For $\mu > 0$, using $\frac{\mu}{\lambda - 1} = k(v)$, we have

$$\frac{\mathbf{n} \cdot \mathbf{f}}{\mu} = k(v) + c + \frac{abw(\lambda, v)}{\mu} + \frac{k'(v)}{k(v)} v'. \tag{2.11}$$

We evaluate the slope field $\frac{d\mu}{d\lambda}$ on $\mathcal{F}_s$,

$$\begin{aligned} \frac{d\mu}{d\lambda} &= -c - abw(\lambda, v)/\mu \\ &= -c - \frac{ab}{k(v)} \frac{w(\lambda, v) - w(1, v)}{\lambda - 1}. \end{aligned} \tag{2.12}$$

This shows that $d\mu/d\lambda$ increases as $\lambda \to 1_-$, cf Figure 2.1 and (2.2).

Letting $\lambda \to 1_-$, we have $\frac{d\mu}{d\lambda} \to k(v)$ and $\frac{w(\lambda,v) - w(1,v)}{\lambda - 1} \to \frac{\partial w(1,v)}{\partial \lambda}$. Thus at $\lambda = 1$,

$$k(v) = -c - \frac{ab}{k(v)} \frac{\partial w(1, v)}{\partial \lambda}.$$

Substituting into (2.11), we have

$$-\frac{k(v)}{\mu}\mathbf{n}\cdot\mathbf{f} = ab\left(\frac{\partial w(1,v)}{\partial\lambda} - \frac{w(\lambda,v)}{\lambda-1}\right) - k'(v)v'. \tag{2.13}$$

From (2.2),

$$\frac{\partial w(1,v)}{\partial\lambda} - \frac{w(\lambda,v)}{\lambda-1} > 0.$$

From $k(v) < 0,\ k'(v) > 0$ and $v' < 0$,

$$-k'(v)v' > 0.$$

The desired result follows by substituting the above into (2.13). □

Lemma 2.5. *Let $(\lambda,\mu,v)(\xi)$ be a solution of the initial value problem with*

$$(\lambda(0),\mu(0),v(0)) \in W.$$

Assume that $v'(0) < 0$, then $v'(\xi) < 0$ for all $\xi > 0$ and the orbit stays in W for all $\xi > 0$.

Proof. Let $t_1 > 0$ be the first time that the orbit hits the boundary of W. Then it must hit $\mathcal{F}_s$ at $\xi = t_1$.

For $0 \le \xi < t_1$ we have $\lambda'(\xi) = \mu(\xi) > 0$. From Lemma 2.3, $v'(\xi) < 0$ at $\xi = t_1$. Either $(\lambda,v)(t_1) \in \mathcal{F}_s, \lambda < 1$ or it is on the line $\lambda = 1, \mu = 0, v_+ \le v \le v_-$.

In the first case, this would imply that the vector field enters W from outside at the point $(\lambda,\mu,v)(t_1)$. There exists a time $0 < t_2 < t_1$ such that the orbit is outside W, contradicting to the assumption that $t_1 > 0$ is the first time that the orbit hits the boundary of W.

In the second case, the orbit must meet the line

$$L := \{(\lambda,\mu,v) | \lambda = 1,\ \mu = 0,\ v_+ \le v < v_-\},$$

at $\xi = t_1$. However, there is another solution of (2.4) passing through the same point. This solution is $(\hat{v}(\xi), \lambda(\xi) \equiv 1, \mu \equiv 0)$, where $\hat{v}$ is a solution of

$$\hat{v}' = -\frac{b}{c}(p(1,\hat{v}) - p_- + c^2(\hat{v} - v_-)),$$
$$\hat{v}(t_1) = v(t_1)$$

This is impossible by the uniqueness of the initial value problems of ODEs. □

Lemma 2.6. *Let*

$$H(\lambda,v) := c^2(v - v_-) + p(\lambda,v) - p_e. \tag{2.14}$$

Then we have $H(\lambda_\pm, v_\pm) = 0$ *and*

$$H(\lambda, v) := c^2(v - v_+) + p(\lambda, v) - p_+.$$

(i) If $c^2 + p_v(\lambda = 1, v_+) \geq 0$, *then* $H(\lambda = 1, v) > 0$ *for* $v > v_+$.
(ii) If

$$c^2 + p_v(\lambda = 1, v_+) < 0, \tag{2.15}$$

then there exists $v_0 \in (v_+, v_-)$ *such that*

$$\begin{aligned} &H(v_0, \lambda = 1) = 0, \\ &H(v, \lambda) < H(\lambda = 1, v) < 0, \text{ for } v_+ < v < v_0, \ \lambda < 1. \end{aligned} \tag{2.16}$$

Proof. $H(\lambda_-, v_-) = 0$ comes from the definition (2.14). Subtracting $c^2(v_+ - v_-) + p(\lambda_+, v_+) = 0$, we have $H(\lambda_+, v_+) = 0$.

(i) Our condition implies that $H'(\lambda = 1, v_+) \geq 0$. Due to the hypothesis $p_{vv} > 0$, we have $H''(\lambda = 1, v) > 0$. Therefore, $H'(\lambda = 1, v) > 0$ for $v > v_+$. Using $H(\lambda = 1, v_+) = 0$, we conclude that $H(\lambda = 1, v) > 0$ for $v > v_+$.

(ii) Using $p_\lambda > 0$, we see that

$$H(v_-, \lambda_+ = 1) > H(v_-, \lambda_-) = 0.$$

We also see that (2.15) implies that $H(v, \lambda_+ = 1) < 0$ for $v > v_+$ and close to v_+. By the Intermediate Value Theorem, there exits a point $v = v_0 \in (v_+, v_-)$ where $H(v_0, \lambda = 1) = 0$. Assuming that v_0 is the smallest of such points, then (2.16) is satisfied for $\lambda < 1, v_+ < v < v_0$. □

Theorem 2.7. *Let* E_- *and* E_+ *be a pair of equilibrium points as in* (2.6) *and* (2.7) *and let* c *be calculated from (1.6). Assume that conditions of (H1) are satisfied. Then there exists a collapsing wave connecting* E_- *on the equilibrium pressure line* $p = p_e$ *to* E_+ *on* $\lambda_+ = 1$. *Moreover, the* (λ, u, v) *components of the traveling wave are monotone.*

Proof. Due to Lemma 2.5, using the fact $\phi(\xi, P)$ satisfies the condition $v'(0) < 0$, we conclude that the orbit of which cannot hit $\mathcal{F}_s$ in finite time and the orbit must stay in W for all $\xi > 0$. Moreover, since $v'(\xi) < 0$ and $\lambda'(\xi) > 0$, the solution must approach a limit with $\lambda = 1$ and $\mu = 0$. The orbit of $\phi(\xi, P)$ cannot hit L in finite time, so it must approach an equilibrium point on L. It remains to show that there is only one equilibrium point on L.

The flow on L is governed by $v' = -(b/c)H(\lambda = 1, v)$. Condition (H1) and part (i) of Lemma 2.6 implies that $H(\lambda = 1, v) > 0$ for $v > v_+$. Hence E_+ is the only equilibrium point on L.

In the proof, we see that (λ, v) of the collapsing wave are monotone. Since v' does not change sign, then the equation $u' = -cv'$ implies $u(\xi)$ is also monotone. □

Theorem 2.8. *For a collapsing wave to exist, it is necessary and sufficient that conditions of (H1) hold:*

$$c \geq 2\sqrt{ab|p_+ - p_e|}, \tag{2.17}$$

and

$$c^2 \geq -p_v(\lambda_\pm, v_\pm) \tag{2.18}$$

Proof. The sufficiency part is already proved in Theorem 2.7.

Now, we prove that (2.17) is necessary. Indeed, if $c < 2\sqrt{ab|p_+ - p_e|}$, then two of the eigenvalues r_2 and r_3 are complex and hence the heteroclinic solution of (2.4) would be oscillating around $\lambda_+ = 1$ for large ξ, forcing the trajectory out of the $0 \leq \lambda \leq 1$ range. More precisely, consider a small δ neighborhood O_δ of E_+ with $\lambda = 1$. Let $\ell = \lambda - 1$. The heteroclinic solution Γ satisfies:

$$\ell' = \mu, \quad \mu' = -c\mu - ab(p - p_e)(\ell + \ell^2).$$

Moreover, $r^2 := \ell^2 + \mu^2 \neq 0$, since $(\ell, \mu, v) = (0, 0, v)$ with $v > v_+$ is a point on L which is invariant under the flow and hence has no intersection with the heteroclinc orbit Γ. Let

$$k := \inf\{ab(p(\lambda, v) - p_e)(\ell + 1) : (\lambda, v) \in O_\delta\}.$$

If $c < 2\sqrt{ab|p_+ - p_e|}$, then we can choose δ sufficiently small such that $c^2 < 4k$. Using the polar coordinates

$$\ell = r\sin(\theta), \quad \mu = r\cos(\theta),$$

we have

$$\begin{aligned}\frac{d\theta}{d\xi} &= \frac{\mu^2 + c\mu\ell + ab(p - p_e)(\ell + 1)\ell^2}{\mu^2 + \ell^2} \\ &\geq \frac{\mu^2 + c\mu\ell + k\ell^2}{\mu^2 + \ell^2} \\ &= \cos^2(\theta) + c\cos(\theta)\sin(\theta) + k\sin^2(\theta).\end{aligned}$$

The condition $c^2 < 4k$ implies that $\frac{d\theta}{d\xi}$ is positive for $0 \leq \theta \leq 2\pi$ with a positive minimum. It is also a 2π periodic function. Thus, $\theta(\xi) \to \infty$ as $\xi \to \infty$. Then $\lambda(\xi) = r(\xi)\sin(\theta) + 1$ will be oscillatory around $\lambda = 1$. We recall that only those traveling waves in the range $0 \leq \lambda \leq 1$ are admitted. Thus, collapsing (or explosion) wave does not exist if $c < 2\sqrt{ab|p_+ - p_e|}$.

To show that $c^2 \geq -p_v(\lambda_+, v_+)$ is a necessary condition by an indirect proof, we assume its contrary (2.15), $c^2 < -p_v(\lambda_+, v_+)$. From part (ii) of Lemma 2.6, there exist $v_0 \in (v_+, v_-)$ where $H(v_0, \lambda = 1) = 0$ and an open set $Q(E_+) := \{(\lambda, v) : v_+ < v < v_0, \lambda < 1\}$ where $H(\lambda, v) < 0$, by using $p_\lambda > 0$. Any heteroclinic orbit connecting E_- to E_+ with $\lambda < 1$ must enter $Q(E_+)$ at some point $\bar{\xi}$ where $v_+ < v(\bar{\xi}) = v_0 < v_-$. However, $v'(\xi) = -(b/c)H(\lambda, v) \geq 0$, preventing the orbit from entering in $Q(E_+)$. This contradicts to $v(\xi) \to v_+$ as $\xi \to \infty$.

We note that in the case of collapsing wave, (1.2b) implies $c^2 \geq -p_v(\lambda_-, v_-)$. This can be proved by contradiction. Assume the contrary: $c^2 < -p_v(\lambda_-, v_-)$. Then on Figure 1.1, the curve $p = p(\lambda_-, v)$, $v_+ < v < v_-$ will be above the straight line segment $\overline{v_- v_+}$, defined by $p - p_e = -c^2(v - v_-)$. Hence $p(\lambda = 1, v_+) < p(\lambda_-, v_+)$, contradicting to $p_\lambda > 0$.

More precisely, if $p_v(\lambda_-, v_-) < -c^2$ then using $p_{vv} > 0$, we have

$$p_v(\lambda_-, v) = p_v(\lambda_-, v_-) - \int_v^{v_-} p_{vv} dv < -c^2, \quad v < v_-,$$
$$p(\lambda_-, v_+) - p_e = -\int_{v_+}^{v_-} p_v dv > c^2(v_- - v_+).$$

Using $p(\lambda = 1, v_+) - p_e = c^2(v_- - v_+)$, we have

$$p(\lambda_-, v_+) > p(\lambda = 1, v_+).$$

This is a contradiction to $p_\lambda > 0$.

Thus (2.18) is a necessary condition for the existence of a collapsing wave.

□

2.3. **Existence of explosion waves.** From Figure 1.2, we see that for explosion waves, $c^2 + p_v(\lambda_+, v_+) \neq 0$ holds because $p_{vv} > 0$.

Theorem 2.9. *(i) For an explosion wave to exist, it is necessary that the following conditions hold:*

$$c \geq 2\sqrt{ab|p_+ - p_e|}, \tag{2.19}$$
$$c^2 + p_v(\lambda = 0, v_+) > 0. \tag{2.20}$$

(ii) For the existence of a monotone explosion wave to exist, it is sufficient that (2.19) *and*

$$c^2 + p_\lambda(\lambda_-, v_-) > 0. \tag{2.21}$$

hold.

Proof. (i) The necessity of condition (2.19) can be proved just like that of (2.17) in Theorem 2.8.

Now suppose that (2.20) is not satisfied. Then $c^2+p_v(\lambda=0,v_+)\le 0$. Combine this with $p_{vv}>0$, we have

$$H(\lambda=0,v_+)=0,\ \partial_v H(\lambda=0,v_+)\le 0, \partial_v^2 H(\lambda=0,v)>0.$$

Therefore $H(\lambda=0,v)>0$, hence $v'=-\frac{b}{c}H(\lambda,v)<0$ for $v<v_+$. This is an contradiction to $v_-<v_+$.

(ii) The proof of the existence of an explosion wave is similar to that of Theorem 2.7. The two eigenvalues for a fixed v are

$$r_1(v)=-c/2-\sqrt{(c/2)^2+ab(p(0,v)-p_e)},$$
$$r_2(v)=-c/2+\sqrt{(c/2)^2+ab(p(0,v)-p_e)}.$$

Let $k(v)=r_2(v)$, then we have $k'(v)<0$.

Define the prism shaped region W as in Figure 2.4, bounded by $v_-\le v\le v_+$, $0\le\lambda\le\lambda_e$, $\mu\ge 0$ and a slant surface $\mathcal{F}_s:=\{\mu-k(v)\lambda=0\}$.

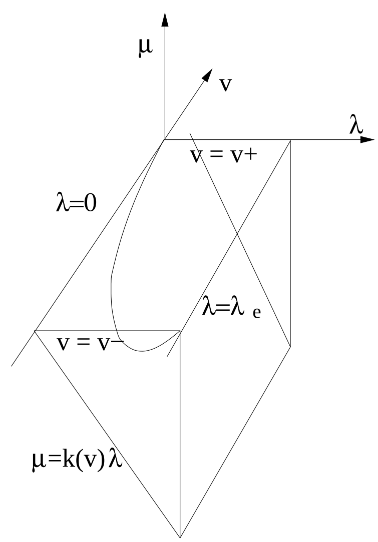

FIGURE 2.4. The prism shaped solid W.

The E_- has a one-dimensional unstable manifold that enters W from E_-. Let $P\in W^u(E_-)\cap W$ and $\phi(\xi,P)$ be the solution with the initial data P. Similar to the proof of Theorem 2.7, it is easy to show that on the four straight sides of W, $\phi(\xi,P)$ cannot leave W. By Lemma 2.3, for the solution $\phi(\xi,P)$, $v'(\xi)>0$ as long as inside W.

The normal of the slant side $\mathcal{F}_s$, pointing upward and to the interior of W, is

$$\mathbf{n}=(-k(v),1,-k'(v)\lambda)$$

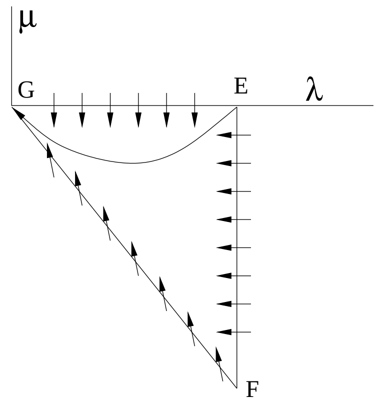

FIGURE 2.5. Front view of the solid W. The flow enters W from all its surfaces. The curve EG is the front view of the heteroclinic solution.

The vector fields are

$$\mathbf{f} = (\lambda', \mu', v').$$

Since $-k'(v)v'(\xi) > 0$, to prove $\mathbf{n} \cdot \mathbf{f} > 0$, it suffices to show $-k(v)\lambda' + \mu' > 0$. Due to $\lambda' < 0$, this is equivalent to $d\mu/d\lambda - k(v) < 0$. On the part of $\mathcal{F}_s$ where $0 < \lambda < \lambda_e$, we have

$$\begin{aligned} \frac{d\mu}{d\lambda} &= -c - \frac{abw(\lambda, v)}{\mu} \\ &= -c - \frac{abw(\lambda, v) - w(0, v)}{k(v)\lambda}. \end{aligned}$$

Let $\lambda \to 0+$, we have

$$k(v) = -c - \frac{ab}{k(v)} \frac{\partial w(\lambda, v)}{\partial \lambda}.$$

From (2.1), $d\mu/d\lambda < k(v)$. Therefore, the solution$\phi(\xi, P)$ cannot leave W on from $\mathcal{F}_s$. Cf. Figure 2.5. The solution must approach a limit where $\lambda = 0, \mu = 0, v \leq v_+$.

It remains to show that there is no equilibrium point satisfies $v_- \leq v < v_+$, $\lambda = 1, \mu = 0$. From $H(\lambda_-, v_-) = 0$ we have $H(\lambda = 0, , v_-) < 0$ due to $p_\lambda > 0$. If there exists a point $v_0 \in (v_-, v_+)$ such $H(\lambda = 0, v_0) = 0$, then using $H(\lambda = 0, v_+) = 0$ and $H(\lambda = 0, v_-) < 0$, a contradiction to $p_{vv} > 0$ can be reached. $\square$

3. Singular limit of the traveling wave solutions as $b \to 0$ or $b \to \infty$

Recall that $b = \beta/\epsilon$ and $a = \epsilon/\gamma$. We consider singular limit of collapsing or explosion waves with either $b \to 0$, or ∞, meanwhile $(ab) = \beta/\gamma$ remain constant. In the (λ, v) plane, we show that as $b \to 0$ the singular limit of the solution is the union of two sides $\overline{\lambda_e v_-}$ and $\overline{v_- v_+}$ in Figure 3.1. While as $b \to \infty$ the singular limit is a smooth decreasing curve defined later in (3.12).

3.1. **Singular limit as $b \to 0$.** In this subsection we assume that b is a small parameter and we are interested in the singular limit of the collapsing and explosion waves as $b \to 0$.

In singular perturbation literatures, (2.4) is the so called "fast system" where the time scale is ξ and $y' = dy/d\xi$:

$$\lambda' = \mu, \tag{3.1}$$

$$\mu' = -c\mu - (ab)w(\lambda, v), \tag{3.2}$$

$$v' = -\frac{b}{c}(c^2(v - v_-) + p(\lambda, v) - p_e) = 0. \tag{3.3}$$

Introducing the slow time $\tau = b\xi$, and using $\dot{y}$ to denote $dy/d\tau$, (2.4) becomes the so called "slow system":

$$b\dot{\lambda} = \mu, \tag{3.4}$$

$$b\dot{\mu} = -c\mu - (ab)w(\lambda, v), \tag{3.5}$$

$$\dot{v} = -c(v - v_-) - \frac{1}{c}(p(\lambda, v) - p_e), \tag{3.6}$$

We want to show that as $b \to 0$, the collapsing and explosion waves approach the so called singular traveling wave solutions that consist of two layers: a fast initial layer where the solution jumps from $\lambda = \lambda_e$ to $\lambda = 0$ or $\lambda = 1$ in a time scale of $O(b)$, while v remains constant; a slow regular layer where the solution stays on the slow manifold defined by $\lambda = 0$ or 1. See the diagram in Figure 3.1.

The slow manifolds and the flow on the slow manifolds:

In the slow system (3.4) and (3.5), as $b \to 0$, the region where $(\dot{\lambda}, \dot{\mu})$ do not blow up and where (λ, μ, v) converges uniformly to a limit is call the regular layer. Letting $b \to 0$ in (3.4)-(3.6), we find that in the regular layer, the limiting system becomes,

$$c\dot{v} + c^2(v - v_-) + (p - p_-) = 0,$$
$$w(\lambda, v) = 0, \quad \mu = 0.$$

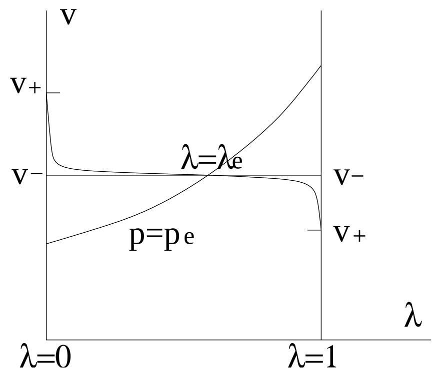

FIGURE 3.1. Singular traveling wave solutions in the (λ, v)-plane

The algebraic equation $w(\lambda, v) = 0$ yields three branches of solutions

$$p(\lambda, v) = p_e, \quad \text{or } \lambda = 0, \quad \text{or } \lambda = 1.$$

Definition 3.1. The union of the solutions of the algebraic equations

$$w(\lambda, v) = 0, \mu = 0$$

in the (λ, μ, v) space is called the slow manifold of the singular limit system. It consists of three smooth branches, denoted by:

$$\begin{aligned} \mathcal{S}_0 &:= \{\lambda = 0,\ \mu = 0.\}, \\ \mathcal{S}_1 &:= \{\lambda = 1,\ \mu = 0.\}, \\ \mathcal{S}_e &:= \{p(\lambda, v) = p_e, \mu = 0\} \end{aligned}$$

The motion on the slow manifolds is determined by $c\dot{v} + c^2(v - v_-) + (p - p_-) = 0$. Using (2.8), it is equivalent to

$$c\dot{v} + c^2(v - v_+) + (p(\lambda, v) - p_+) = 0.$$

On the slow manifold $p(\lambda, v) = p_e = p_-$, the flow is determined by

$$c\dot{v} + c^2(v - v_-) = 0.$$

Since $c > 0$, the equilibrium $v = v_-$ is stable. No solution can leave the equilibrium $v = v_-$ along the slow manifold $p = p_e$. If we look for the traveling wave solution from $p = p_e$ to $\lambda = 0$ or 1, the singular traveling wave solution must start with a fast jump from $\mathcal{S}_e$ to $\mathcal{S}_0$ or $\mathcal{S}_1$.

After the fast jump, the flow on the slow manifold $\mathcal{S}_1$ is determined by the initial value problem

$$\begin{aligned} &c\dot{v} + c^2(v - v_+) + (p(1, v) - p_+) = 0, \\ &v(0) = v_- > v_+. \end{aligned}$$

We see that $c\dot{v} = -H(, \lambda_+, v)$ where $H(\lambda, v)$ is defined in the proof of Theorem 2.7. We have shown that $H(\lambda_+, v) > 0$ for $v > v_+$. Thus v_+ is a stable on $\mathcal{S}_1$ with respect to the points $v > v_+$.

In an analogous manner, we can prove that on the slow manifold $\mathcal{S}_0$, the equilibrium solution $(\lambda, v) = (0, v_+)$ is stable and from attracts all the points $v < v_+$ ($\dot{v} > 0$ for all $v < v_+$).

The fast system and heteroclinic solutions connecting the slow manifolds: The only way a traveling wave leaves E_- is to fast jump along the strong unstable manifold transverse to the slow manifold. We now study the fast jump.

Let $b = 0$ in the fast system (3.3), we have

$$v' = 0.$$

Thus in the singular layer v is a constant. Since the jump starts at E_-, $v = v_-$. The rest of the equations (3.1), (3.2) become

$$\lambda' = \mu, \tag{3.7}$$

$$\mu' = -c\mu - (ab)w(\lambda, v_-). \tag{3.8}$$

Observe that the slow manifolds consist of equilibrium points for (3.7), (3.8). We will show that $\mathcal{S}_e$ consists of saddle points and $\mathcal{S}_0, \mathcal{S}_1$ consist of stable equilibrium points.

We then study heteroclinic solutions of (3.7), (3.8), that connects $\mathcal{S}_e$ to $\mathcal{S}_1$ or $\mathcal{S}_0$.

The λ equation for a fixed $v = v_0$ is

$$\lambda'' + c\lambda' + (ab)w(\lambda, v_-) = 0. \tag{3.9}$$

The line $v = v_-$ intersects the graph of $w = 0$ in three points, as seen in Figure 3.1. These are equilibrium solutions for (3.9). We now determine the eigenvalues of (3.9) at the equilibria. The linear variational equation for (3.9) is

$$\Lambda'' + c\Lambda' + (ab)\frac{\partial w}{\partial \lambda}\Lambda = 0. \tag{3.10}$$

$$\frac{\partial w}{\partial \lambda} = \frac{\partial p}{\partial \lambda}\lambda(\lambda-1) < 0, \text{ if } p = p_e,$$
$$\frac{\partial w}{\partial \lambda} = |p - p_e| > 0, \text{ if } \lambda = 0, \ 1.$$

Therefore, for each fixed v_0, the equilibrium point on $\mathcal{S}_e$ a saddle point, and the equilibrium point on $\mathcal{S}_0, \mathcal{S}_1$ is an attractor if $c > 0$ (or a repeller if $c < 0$, which is not our case).

We investigate the heteroclinic solution connecting $p = p_e$ to $\lambda = 1$. Under the conditions $c > 0$, the connection is from a saddle to an attractor, For a fixed $v_0 = v_-$, the graph of $w = w(v_0, \lambda)$ is sigmoid, with three zeros at $\lambda = 0, \lambda_e = \lambda^*(v_0, p_e), \lambda = 1$. See Figure 2.1 for the graph of $w(\lambda, v_0)$ and its potential function

$$W(v_0, \lambda) = \int_0^\lambda w(v_0, \alpha) d\alpha.$$

The system (3.9) has a Fisher, or KPP type nonlinearity. For any $c > 0$, it is known to have a traveling wave solution connecting the saddle point $\lambda = \lambda\_e$ to the attractor $\lambda = 1$ or 0 . Moreover, if (H1) is satisfied, $c \geq 2\sqrt{ab|p(\lambda_+, v) - p_e|}$, then all the eigenvalues at $\lambda = 0, \ 1$ are real and negative. In this case, the heteroclinic solutions are monotone in λ and satisfy $0 \leq \lambda \leq 1$.

3.2. **Singular limit of the traveling waves as $b \to \infty$.** Consider the singularly perturbed system which is derived from (2.4) by letting $d = 1/b$ to be a small parameter while (ab) remains constant.

$$\begin{aligned} \lambda' &= \mu, \\ \mu' &= -c\mu - (ab)w(\lambda, v), \\ dv' &= -c(v - v_-) - \frac{1}{c}(p(\lambda, v) - p_-). \end{aligned} \tag{3.11}$$

Let $d = 0$. The last equation defines the so called slow manifold

$$W^c(0) = \{(\lambda, \mu, v) : \mu \in \mathbb{R}, c^2(v - v_-) + p(\lambda, v) - p_e = 0.\}.$$

Due to $p_\lambda > 0$, the slow manifold exists and can be expressed as $\lambda = \lambda^*(v)$. We make the following additional assumption in this subsection:

$$c^2 + p_v(\lambda, v) > 0, \tag{H4}$$

along the slow manifold $W^c(0)$. We can solve the above equation for v:

$$v = v_0^*(\lambda), \quad \frac{d}{d\lambda}v_0^* < 0. \tag{3.12}$$

The graph of (3.12) a smooth decreasing curve connecting λ_e to v_+ in Figure 3.1 (not plotted).

Introducing the fast variable $\tau = \xi/d$, we have the so called fast system

$$\begin{aligned} \lambda_\tau &= d\mu, \\ \mu_\tau &= -d(c\mu + (ab)w(\lambda, v)), \\ v_\tau &= -c(v - v_-) - \frac{1}{c}(p(\lambda, v) - p_-). \end{aligned} \tag{3.13}$$

When $d = 0$, the fast system has a two-dimensional center manifold that consists of equilibrium points

$$W^c(0) = \{(\lambda, \mu, v) | \lambda \in \mathbb{R}, \mu \in \mathbb{R}, v = v_0^*(\lambda)\}.$$

Linearizing the last equation of (3.13),

$$V_\tau = -(c + p_v/c)V,$$

we find that the manifold is normally stable, due to (H4). Therefore the center manifold persists for small d, denoted by $W^c(d)$.

$$W^c(d) = \{(\lambda, \mu, v) | \lambda \in \mathbb{R}, \mu \in \mathbb{R}, v = v_d^*(\lambda, \mu)\}.$$

The function v_d^* is $O(d)$ close to v_0^* in C^2 norm. Also $W^c(d)$ should contain the two equilibrium points $E_\pm$ which are independent of d.

Consider the first two equations of (3.11) restricted to $W^c(d)$:

$$\begin{aligned} \lambda_\tau &= \mu, \\ \mu_\tau &= -c\mu - (ab)w(\lambda, v_d^*(\lambda, \mu)). \end{aligned} \tag{3.14}$$

At the equilibrium $\lambda = 0$ or 1, we have $w_v = 0$, so $\frac{dw}{d\lambda} = w_\lambda > 0$. Together with $c > 0$, we conclude that the equilibrium $(\lambda, \mu) = (\lambda_+, 0)$ is a hyperbolic attractor. Therefore for small d the equilibrium is still a hyperbolic attractor.

At the equilibrium $\lambda = \lambda_e$, if $d = 0$, we have

$$w_\lambda < 0, \quad w_v > 0, \quad \frac{d}{d\lambda} v_0^*(\lambda) < 0.$$

Therefore

$$\frac{dw}{d\lambda} = w_\lambda + w_v \cdot \frac{d}{d\lambda} v_0^*(\lambda) < 0.$$

This implies that the equilibrium $(\lambda, \mu) = (\lambda_e, 0)$ is a hyperbolic saddle.

For $d = 0$, the restricted flow on the slow manifold, system (3.14), has a fisher type traveling wave if $c > 0$ that is a saddle-attractor connection. For $d > 0$ but small, the slow manifold and the traveling wave connection should persist.

4. Numerical Results on the Large Time Behavior of Collapsing and Explosion Waves

From Theorem 2.8, we see that for a given (v_+, λ_+), there are infinitely many collapsing or explosion waves connecting to it with different speeds. If two of collapsing (or explosion) waves can appear simultaneously in Riemann solvers, nonuniqueness of solutions will occur. Thus, we want to study numerically which traveling wave appears when $t > 0$ is large enough in the solution of (1.1-3) with Riemann initial data

$$(u, v, \lambda)(x, 0) = \begin{cases} (u_+, v_+, \lambda_+), & x > 1 \\ (u_-, v_-, \lambda_-), & x < 1. \end{cases} \tag{4.1}$$

We take

$$p(v, \lambda) = \frac{(1+\lambda)^2}{4v^2}, \tag{4.2}$$

with the equilibrium pressure

$$p_e = 1. \tag{4.3}$$

We fix the parameters of (1.1) as $a = \epsilon/\gamma = 2$, $b = \beta/\epsilon = 1$ in our numerical computations. We use WENO3-4 scheme with 3rd order Runge-Kuta explicit time stepping, [8]. For the purpose of double checking, we also use Lax-Friedrichs' central scheme to compute the same problem.

Example 4.1. Choose $(u_+, v_+, \lambda_+) = (0, 1/\sqrt{2}, 1)$, $(u_-, v_-, \lambda_-) = (-0.5, 0.75, 0.5)$ in (4.1). We compute the solution of (1.1, 4.1, 4.2) using WENO3-4 with $\Delta x = 0.002$, $\Delta t = 0.0001$. The solution at $t = 1$ is displayed as Figure 4.1. The right most wave at around $x = 3.5$ is a collapsing wave. To compute the speed c of the collapsing wave, we plugging in the values of (v, λ) at the two sides of the wave, $x = 3$ and $x = 4$, into (1.6). Then, we calculate the difference

$$\sqrt{4ab|p(1, v(4,1)) - p_e|} - c = 2\sqrt{2} - c = 0.0085. \tag{4.4}$$

From Theorem 2.8, we know that the slowest speed of a collapsing wave connecting $(v(4,1), 1)$ on the right is $2\sqrt{2}$. Thus the result (4.4) suggests that the speed of the collapsing wave we see in the solution should be $2\sqrt{2}$, the minimum speed for all collapsing waves connecting $(v, \lambda) = (v(4,1), 1)$ on the right side of the wave.

To further test the above numerical result, we take finer $\Delta x = 0.0002$, $\Delta t = 0.000002$. The difference actually increases to

$$\sqrt{4ab|p(1, v(4,1)) - p_e|} - c = 0.024. \tag{4.5}$$

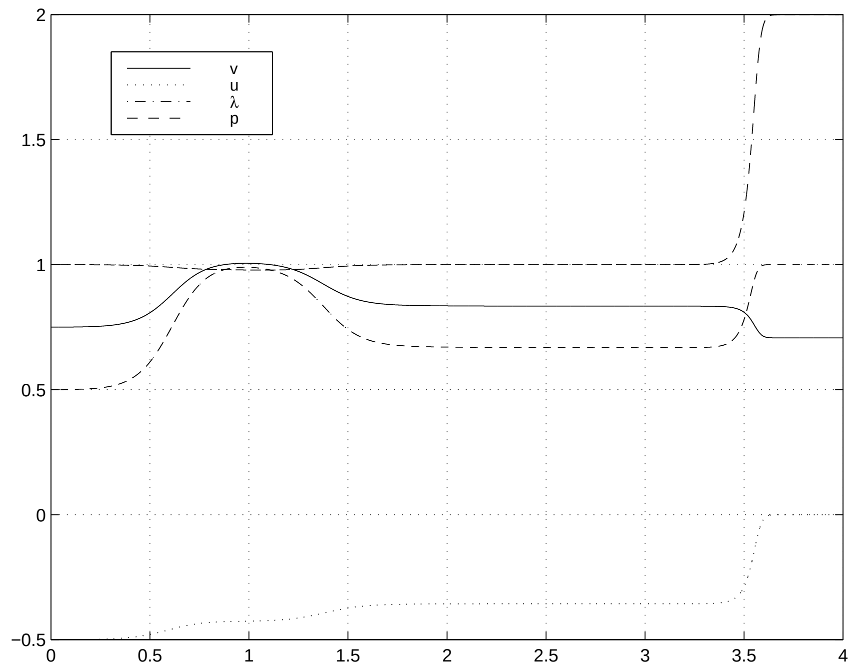

FIGURE 4.1. The solution of (1.1, 4.1-3) at $t = 1$.

Although this still suggests that the collapsing wave in the solution should be the one with the slowest speed, it is contrary to our expectation that the difference should be smaller than that in (4.4). We use Lax-Friedrichs' central scheme to compute the same problem. the result is similar to (4.5). One factor contributing to this error is the truncation error in front of the wave. The diffusion term in $(1.1)_3$ pulls λ down from $\lambda_+ = 1$. This deviation from the unstable equilibrium $\lambda = 1$ repels λ into $\lambda < 1$ exponentially fast, and is one of the major driving force of the wave. However, with Δt and diffusion effect so small away from the front, such small deviation from $\lambda = 1$ is truncated in computation, resulting in slowing down of the wave. The shrinking of the diffusion effect can be measured by the change of the distance from the center of collapsing wave to the first occurrence of $\lambda = \lambda_+ = 1$, where the center of the wave is defined as the place x where the pressure is at the middle between that at the front and back of the wave. We observe that at $t = 1$, the distance is 0.3567 for $\Delta x = 0.002$, $\Delta t = 0.0001$, while with $\Delta x = 0.0002$, $\Delta t = 0.000002$, the distance is 0.2264.

5. Traveling Wave Admissibility Criterion for Riemann Problems of (1.12)

In [3], we showed that the Riemann problems for (1.12) has a unique solution under the following kinetic relation on phase boundaries of positive speeds.

(i) For each liquid state $(v_-, \lambda_- = 0)$ with $p(v_-, 0) > p_e$, there is only one vapor state $(v_+, \lambda_+ = 1)$ that can be connected to $(v_-, 0)$ by a liquefaction wave. The speed of the liquefaction wave $s = s(v_-)$ is a decreasing C^1 function satisfying $s^2 + p_v(\lambda_\pm, v_\pm) \leq 0$.

(ii) For each vapor state $(v_-, \lambda_- = 1)$ with $p(1, v_-) < p_e$, there is only one liquid state $(v_+, \lambda_+ = 0)$ that can be connected to $(v_-, 1)$ by an evaporation wave. The speed of the liquefaction wave $s = s(v_-)$ is an increasing C^1 function satisfying $s^2 + p_v(\lambda_\pm, v_\pm) \leq 0$.

The above kinetic relation is motivated by the behavior liquefaction and evaporation traveling waves. When these waves exist, their slowest speed satisfy the kinetic relation (i-ii). In [3], the collapsing and explosion waves are excluded.

In this section, we shall include the collapsing and explosion waves in the kinetic relations. In the sequel, when we say the collapsing (or explosion) wave, we refer to the collapsing (explosion) wave of the slowest positive speed among all collapsing (explosion) wave connecting the same (v_+, λ_+) on the right side.

We know that collapsing, explosion, liquefaction and evaporation traveling waves does not exist if their speed, c, satisfy $c^2 < 4ab|p_+ - p_e|$. But, collapsing and explosion waves are supersonic while liquefaction and evaporation waves are subsonic. Thus, we can be sure that for any given $(v_+, \lambda_+ = 0 \text{ or } 1)$, only one kind of wave, among collapsing, explosion waves, liquefaction and evaporation waves can connect to it. But, this may not exclude the possibility of having two solutions to the same Riemann problem. We shall show, in the following, that it is possible to have two Riemann solvers for the same Riemann data, one has a collapsing wave and the other has a liquefaction or evaporation wave.

Consider the Riemann initial data of (1.12)

$$(u, v, \lambda)(x, 0) = \begin{cases} (u_+, v_+, \lambda_+), & x > 0, \\ (u_-, v_-, \lambda_-), & x < 0, \end{cases} \tag{5.1}$$

where $\lambda_+ = 1$, $\lambda_- = 0$ and $p_\pm = p(\lambda_\pm, v_\pm) > p_e$, . Assume that there is a collapsing wave of slowest speed connecting (v_+, λ_+) to some (v_e, λ_e). Typically, this requires $p_+ - p_e > 0$ to be away enough from 0. For simplicity, we assume that

$$p_{vv} > 0. \tag{5.2}$$

we can further choose (5.1) so that there is a Riemann solver for (5.1)

$$(u_-, v_-, \lambda_-) \to (u_2, v_2, 1) \to (u_+, v_+, 1), \tag{5.3}$$

with $\lambda_- = 0$ as depicted in Figure 5.1. The Rankine-Hugoniot condi-

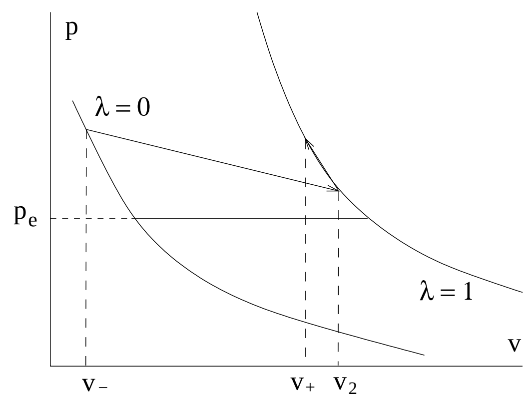

FIGURE 5.1. The solution (5.3) illustrated on (v, p)-plane

tions for (5.3) are

$$\begin{aligned} & - s(v_2 - v_-) = u_2 - u_-, \\ & - \int_{v_2}^{v_+} \sqrt{-p_v(v,1)}dv = u_+ - u_2, \\ & v_2 > v_\pm. \end{aligned} \tag{5.4}$$

By the behavior of liquefaction traveling waves, the speed s is positive. Adding the right hand side of the two equations of (5.4), we get

$$F_1(v_2) := -s(v_2 - v_-) - \int_{v_2}^{v_+} \sqrt{-p_v(v,1)}dv. \tag{5.5}$$

It is easy to see that the necessary and sufficient condition for (4.3) to have a solution (v_2, u_2) is that

$$F_1(v_2) = u_+ - u_-, \quad v_2 > v_\pm \tag{5.6}$$

has a solution

Now, we shall construct another solution of (1.12) with the same initial data (5.1) satisfying (5.6), using collapsing waves. Consider such a solution with the structure

$$(u_-, v_-, \lambda_- = 0) \to (u_1, v_1, 0) \to (u_e, v_e, e) \to (u_+, v_+, 1), \tag{5.7}$$

see Figure 5.2. Similar to (5.4)-(5.6), such a solution exists if and only

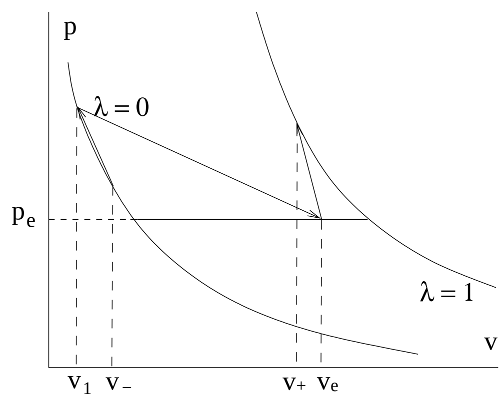

FIGURE 5.2. The solution (5.7) on (v, p)-plane.

if there is a solution v_1 for the equation

$$\begin{aligned} F_2(v_1) :=& - \chi(-\infty < v_1 < v_-) s_1(v_1 - v_-) \\ & + \chi(v_- \le v_1 \le m) \int_{v_-}^{v_1} \sqrt{-p_v(v,0)}dv \\ & - s_2(v_e - v_1) - s(v_+ - v_e) = u_+ - u_-, \end{aligned} \tag{5.8}$$

where the s is the speed of collapsing wave connecting (v_e, λ_e) to $(v_+, \lambda_+ = 1)$. Since v_1 can vary from $-\infty$ to m, the range of $F_2(v_1)$ at least include the interval

$$(F_2(-\infty), F_2(m)] = \left(-\infty, \int_{v_-}^{m} \sqrt{-p_v(v,0)}dv - s(v_+ - v_e)\right]. \tag{5.9}$$

To prove (5.8) also has a solution v_1, it suffices to show that the range of $F_1(v_2)$ can be included in (5.9), by adding assumptions on $p(\lambda, v)$ if necessary. To this end, we see that

(5.10)

$$
\begin{aligned}
F_1(v_2) &:= -s(v_2 - v_-) - \int_{v_2}^{v_+} \sqrt{-p_v(v,1)}dv \\
&= -\sqrt{(p_- - p_2)(v_2 - v_-)} + \int_{v_+}^{v_2} \sqrt{-p_v(v,1)}dv \\
&\le -\sqrt{(p_- - p_2)(v_2 - v_-)} + \left(\int_{v_+}^{v_2} -p_v(v,1)dv\right)^{1/2} \left(\int_{v_+}^{v_2} dv\right)^{1/2} \\
&= -\sqrt{(p_- - p_2)(v_2 - v_-)} + \sqrt{(p_+ - p_2)(v_2 - v_+)}
\end{aligned}
$$

If the left hand side of (5.10) is < 0, then the range of F_1 is contained in the interval $(-\infty, 0]$ and hence in the range of $F_2(v_1)$. Thus we have the following result.

Lemma 5.1. *There is also a solution of (1.12), (4.1) that contains a collapsing wave if*

$$\sqrt{(p(1,v_+) - p(1,v_2))(v_2 - v_+)} \le \sqrt{(p(0,v_-) - p(1,v_2))(v_2 - v_-)}.$$

and if the following three conditions hold.
(i) There is a collapsing wave of slowest speed connecting $(v_+, \lambda_+ = 1)$ *to some* (v_e, λ_e) *with* $p(\lambda_e, v_e) = p_e$.
(ii) The data $v_-, u_\pm$ *are such that there is a solution of the Riemann problem (1.12), (4.1) consisting of a positive speed liquefaction wave and a faster moving rarefaction (or a shock) wave.*
(iii) There is a liquefaction wave connecting $(v_-, \lambda_- = 0)$ *to a* $(v_2, \lambda_2 = 1)$.

Lemma 5.1 points out a way to construct two solutions for a Riemann problem. We shall construct an example later. Now we consider which solution is admissible in the sense that it can serve as the $\epsilon \to 0$ limit of the solution of (1.1) for the same initial data. When two solutions depicted in Fig 5.1 and Fig 5.2 exist simultaneously, two mechanisms are competing at $t = 0$ around the point $x = 0$ where liquid and metastable vapor contact. The first mechanism is to reduce the pressure first via rarefaction wave, without phase change, followed by a liquefaction wave. This gives us the solution (5.3). Another mechanism is to undergo the phase change and reduce the pressure immediately, resulting in the collapsing wave in the second solution (5.7). The second mechanism is faster, indicated by the supersonic speed of the collapsing wave versus the sonic and subsonic wave speeds in the solution (5.3). It is reasonable to expect the second mechanism wins, resulting a solution of the type (5.7). To confirm this expectation, let us do some numerical experiments. We use the same functions as in (4.2), (4.3), and the same WENO3-4 method as in last section. Our numerical experiments are divided into the following examples

Example 5.1. A solution of the type (5.3). We choose $v_- = 1/(2\sqrt{1.5})$,

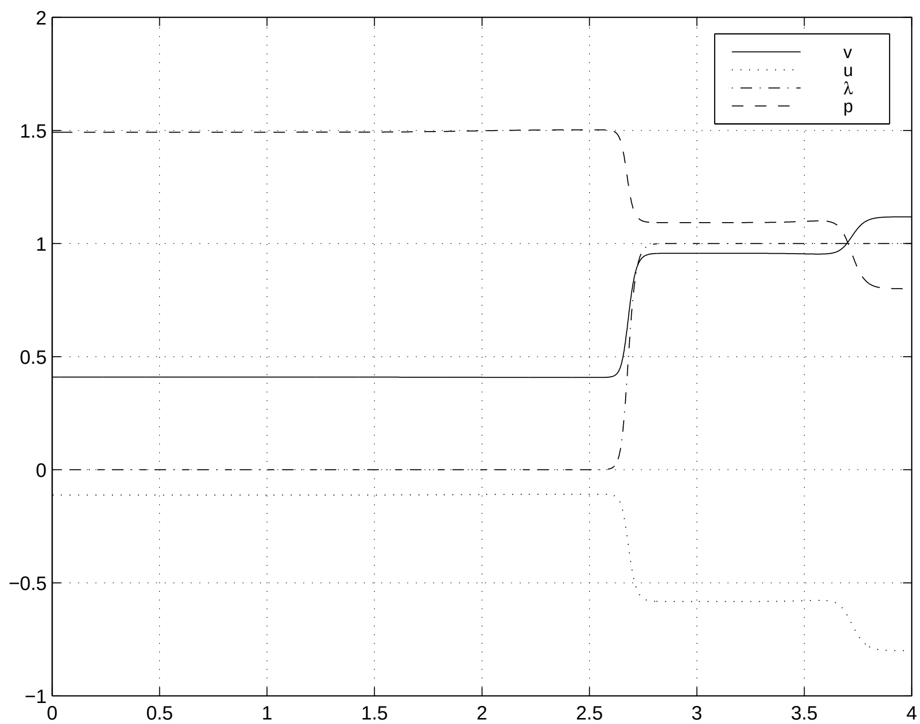

FIGURE 5.3. A liquefaction wave is between $x = 2$ and $x = 3$.

$\lambda_- = 0$. There is a liquefaction wave connecting this $(v_-, \lambda_-0 = 0)$. To generate a liquefaction wave, we computed the numerical solution of (1.1) with Riemann initial data

$$(u, v, \lambda)(x, 0) = \begin{cases} (0, \frac{1}{2\sqrt{1.4}}, 0), & x < 0.8 \\ (-0.8, 1/\sqrt{0.8}, & x > 0.8. \end{cases}$$

The solution at $t = 2.14$ is shown in Figure 5.3. The liquefaction is the jump between $2 < x < 3$ in Figure 5.3. The two sides of the liquefaction wave are read from the numerical solution at $x = 2.2$ and $x = 3$ as $(v_-, \lambda_- = 0) = (1/(2\sqrt{1.5}), 0)$ and $(v_2, 1) = (0.9568, 1)$. The speed of this wave is

$$s_2 = \sqrt{-\frac{p(1, v_2) - p(0, v_-)}{v_2 - v_-}} = 0.8619.$$

Now, we construct a Riemann problem containing the liquefaction wave we found in Figure 5.3. We choose $v_+ = 1/\sqrt{2}$. According (5.5), the solution of type (5.3) can be constructed by choosing

$$(u_- = 0.7709, v_- = 1/(2\sqrt{1.5}), \lambda_- = 0),$$
$$(u_+ = 0, v_+ = 1/\sqrt{2}, \lambda_+ = 1), \tag{5.11}$$
$$u_+ - u_- = -s(v_2 - v_-) - \int_{v_2}^{v_+} \sqrt{-p_v(v,1)}dv = -0.7709.$$

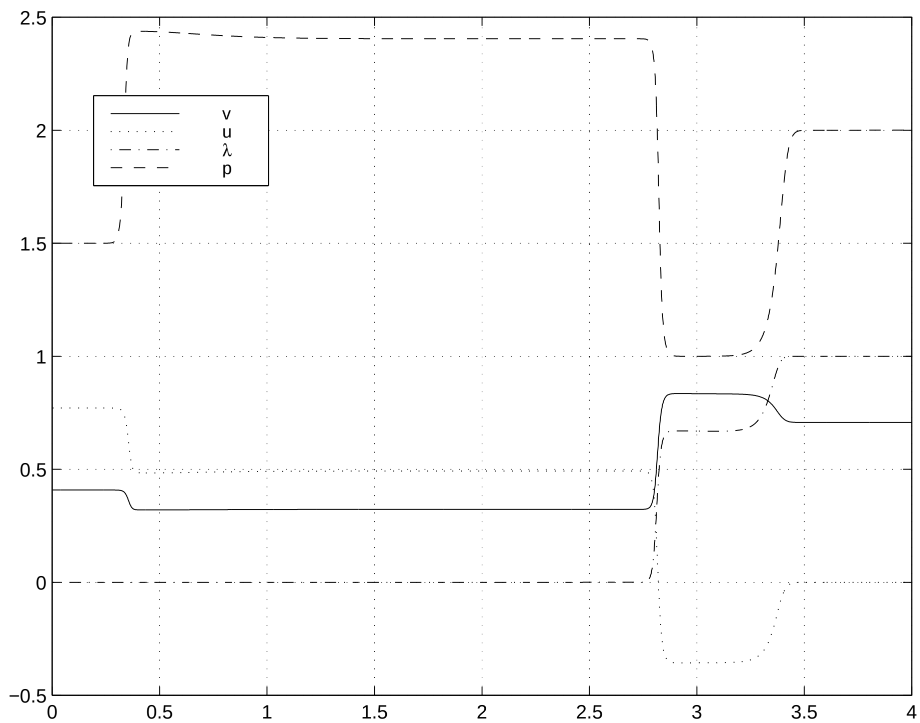

FIGURE 5.4. The solution of (1.1) with initial data (5.11) is of type (5.7).

Example 5.2. For $(u_\pm, v_\pm, \lambda_\pm)$ chosen in Example 5.1, there is another solution of type (5.7) for the same Riemann problem (5.11). To this end, we numerically compute the solution of (1.1) with the Riemann initial data. Parameters a, b, ϵ used are the same as those in Example 4.1. The solution at time $t = 0.5108$ is shown in Figure 5.4. The solution consists, from left to right, a ordinary Lax shock of negative speed, a liquefaction wave compressing liquid/vapor mixture into liquid, a collapsing wave. This solution is of type (5.7). This example not only shows that both type (5.3) solution and type (5.7) solution exist for the same Riemann data, but also point out that type (5.7) is admissible under vanishing viscosity criterion, confirming our expectation.

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Department of Mathematics, Georgetown University, Washington, DC 20057

E-mail address: fan@math.georgetown.edu

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205

E-mail address: xblin@math.ncsu.edu