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L^2 Semigroup and Linear Stability for Riemann Solutions of Conservation Laws

Xiao-Biao Lin

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ABSTRACT. Riemann solutions for the systems of conservation laws $u_\tau + f(u)_\xi = 0$ are self-similar solutions of the form $u = u(\xi/\tau)$. Using the change of variables $x = \xi/\tau, t = ln(\tau)$, Riemann solutions become stationary to the system $u_t + (Df(u) - xI)u_x = 0$. For the linear variational system around the Riemann solution with n-Lax shocks, we introduce a semigroup in the Hilbert space with weighted L^2 norm. We show that (A) the region $\Re\lambda > -\eta$ consists of normal points only. (B) Eigenvalues of the linear system correspond to zeros of the determinant of a transcendental matrix. They lie on vertical lines in the complex plane. There can be *resonance values* where the response of the system to forcing terms can be arbitrarily large, see Definition 6.2. Resonance values also lie on vertical lines in the complex plane. (C) Solutions of the linear system are $O(e^{\gamma t})$ for any constant γ that is greater than the largest real parts of the eigenvalues and the coordinates of *resonance lines*. This work can be applied to the linear and nonlinear stability of Riemann solutions of conservation laws and the stability of nearby solutions of the Dafermos regularizations $u_t + (Df(u) - xI)u_x = \epsilon u_{xx}$.

CONTENTS

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1. Introduction

The purpose of this paper is to introduce a $L^2 \cap C^0$ semigroup of the linear variational system around Riemann solutions of the hyperbolic conservation laws

$$u_\tau + f(u)_\xi = 0. \tag{1.1}$$

A Riemann solution is an initial value problem for (1.1) with piecewise constant initial data

$$u(\xi, 0) = u^\ell \text{ if } \xi < 0,\ u(\xi, 0) = u^r \text{ if } \xi > 0.$$

We assume that the Riemann solutions have n-Lax shock waves $\Lambda^i, i = 1, \ldots, n$, with speeds $\bar{s}^i, i = 1, \ldots, n$. Let $\bar{s}^0 = -\infty$ and $\bar{s}^{n+1} = \infty$, then

$$u(\xi, \tau) = \bar{u}^i \quad \text{if } \bar{s}^i < \xi/\tau < \bar{s}^{i+1}.$$

The system is assumed to be strictly hyperbolic with respect to the solution, and the Rankine-Hugoniot jump condition is satisfied at each Λ^i. Under some general conditions, this type of Riemann solutions is structurally stable, as shown in [**17**].

A direct application of such semigroup is to determine the linear stability of Riemann solutions of (1.1). Sufficient conditions for the linear stability have been obtained by many authors [**1, 6, 7, 8, 18**] in BV and L^1 spatial spaces. Two kinds of conditions have been proposed by Lewicka and Schohat:

(i) There exist some positive weights to each of the characteristic modes such that the weighted norms of the waves scattering from the shocks are weaker than the norms of the impinging waves hitting the shocks.

(ii) The spectral radius of the matrix expressing such scattering is less than one.

Lewicka showed that the two conditions proposed above are equivalent [**7**], and if the scattering matrix is positive the conditions for BV and L^1 stability correspond to that the real parts of eigenvalues of the linear variational system are less than 0 and -1 respectively [**8**]. These conditions are satisfied for a system of two equations with two Lax Shocks.

Our original conjecture, inspired by Lewicka's work, was that the stability of solutions should depend only on the location of eigenvalues. This conjecture turns out to be over-simplified. Due to the complicated interaction of characteristic waves scattered from large multiple shocks, we have found the so called *resonance values* and *resonance lines* to be defined in Definition 6.2. Together with eigenvalues, they determine the stability and growth rate of solutions.

Let E be a Banach space and $f \in L^2(\mathbb{R}^+, E)$. We say $f(t)$ is $O(e^{\gamma t})$ in L^2 norm if

$$\int_0^\infty |e^{-\gamma t} f(t)|_E^2 dt < \infty.$$

If $|f(t)|_E = O(e^{\gamma t})$, then we say $f(t)$ is $O(e^{\gamma t})$ in sup norm.

In the similarity coordinates to be introduced later, we can prove that if γ is greater than the largest real parts of the eigenvalues and the coordinates of *resonance lines*, then the "L^2 in time solutions" for the linear system are $O(e^{\gamma t})$ in L^2 norm and the "H^1 in time solutions" are $O(e^{\gamma t})$ in sup norm. In §7, we show that the solutions constructed in this paper are in fact continuous in time and in this sense, the semigroup is also C^0. But it is in the L^2 norm we find the optimal growth/decay rate of solutions.

Our second motivation of this paper comes from the study of solutions of the Dafermos regularization of (1.1). Unlike the usual (non-Dafermos) regularization

$$u_\tau + f(u)_\xi = \epsilon u_{\xi\xi}, \tag{1.2}$$

where the perturbation $\epsilon u_{\xi\xi}$ typically destroys similarity solutions of the form $u = u(\xi/\tau)$, the Dafermos regularization of (1.1):

$$u_\tau + f(u)_\xi = \epsilon\tau u_{\xi\xi}, \tag{1.3}$$

can have similarity solutions of the form $u(\xi/\tau)$, which is a well-known type of solution to the conservation laws (1.1). Among them are Riemann solutions consisting of multiple shock or rarefaction waves. See [**3, 19**] for introduction of the Dafermos regularization.

Using the change of variables $x = \xi/\tau, \quad t = \ln\tau$, the Dafermos regularization (1.3) becomes

$$u_t + (Df(u) - xI)u_x = \epsilon u_{xx}. \tag{1.4}$$

The same change of variables brings the system of conservation laws to

$$u_t + (Df(u) - xI)u_x = 0. \tag{1.5}$$

In the (ξ, τ) variables, Riemann solutions of (1.1) are usually non-stationary. But in (x, t), they are stationary solutions to (1.5). In particular the location of the ith shock Λ^i is $x = x^i = \bar{s}^i$, and the regions between shocks are called regular layers. The Riemann solution becomes

$$u(x,t) = \bar{u}^i, \quad \text{if } x \in R^i = (x^i, x^{i+1}), \quad i = 0, \dots, n.$$

At each shock, jump conditions must be imposed to (1.5). These conditions can be derived from the Rankine-Hugoniot conditions of (1.1) as follows. Assume that the shock position for (1.1) are $\xi = \xi^i(\tau)$ for the ith shock. Then

$$f(u(\xi^i+,\tau)) - f(u(\xi^i-,\tau)) = \frac{d}{d\tau}\xi^i(\tau)(u(\xi^i+,\tau) - u(\xi^i-,\tau)).$$

In the (x, t) coordinate, the shock positions are $x^i(t) = \xi^i(\tau)/\tau$. Since

$$\frac{d\xi^i}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau}\tau + x^i(t) = \dot{x}^i(t) + x^i(t),$$

we have

$$f(u(x^i+,t)) - f(u(x^i-,t)) = (\dot{x}^i(t) + x^i(t))(u(x^i+,t) - u(x^i-,t)).$$

Consider the linear variational system to (1.4) where U is the variation of u, U_0 is the perturbation to the initial data and k is the perturbation to the equation:

$$U_t + (Df(\bar{u}^i) - xI)U_x - \epsilon U_{xx} = k(x,t), \quad U(x,0) = U_0(x). \tag{1.6}$$

For the purpose of constructing a semigroup, it suffice to consider the system with $k = 0$. We include the nonhomogeneous term $k(x, t)$ for completeness.

The singular limit system for U is precisely the linear variational system of (1.5):

$$U_t + (Df(\bar{u}^i) - xI)U_x = k(x,t), \quad U(x,0) = U_0(x). \tag{1.7}$$

Let $\Delta^i = \bar{u}^i - \bar{u}^{i-1}$. Let and X^i be the variation of the shock position x^i. For a stationary solution $x^i(t) = x^i$ is constant and $\dot{x}^i(t) = 0$. Linearize around the jump condition, we have

$$Df(\bar{u}^i)U(x^i+,t) - Df(\bar{u}^{i-1})U(x^i-,t) = (\dot{X}^i(t)+X^i(t))\Delta u^i + x^i(U(x^i+,t)-U(x^i-,t)).$$

The desired jump condition for (1.7) at each shock is:

$$[Df(\bar{u}^i) - x^i I]U(x^i+,t) - [Df(\bar{u}^{i-1}) - x^i I]U(x^i-,t) = (\dot{X}^i(t) + X^i(t))\Delta u^i. \tag{1.8}$$

It has been shown that in suitable function spaces, the second order system (1.6) is sectorial and generates an analytic semigroup [**9**]. After the Laplace transform, we have

$$s\hat{U} + (Df(\bar{u}^i) - xI)\hat{U}_x - \epsilon\hat{U}_{xx} = \hat{k}(x,s) + U_0(x). \tag{1.9}$$

Using exponential dichotomies, it has been proved that for large s, system (1.9) is invertible. Estimates on solutions of (1.9) have been obtained ensuing the existence of an analytic semigroup with a possibly large growth rate. Based on this, local existence of solutions to the nonlinear system can be proved by the standard method.

To determine the stability of the solution, we need to study the resolvent for small s that is near the critical eigenvalues. In contrast to the "fast eigenvalues" of (1.6) which are of $O(\frac{1}{\epsilon})$ and come from the dynamics of the singular layers near the shocks of the conservation laws, the critical eigenvalues are so called "slow eigenvalues" in [**14, 16, 10**] which are of $O(1)$ and are the eigenvalues of the inviscid system (1.7).

The Laplace transform of (1.7) is:

$$s\hat{U} + (Df(\bar{u}^i) - xI)\hat{U}_x = \hat{k}(x,s) + U_0(x), \tag{1.10}$$

If the real part of s is greater than the largest real parts of the eigenvalues of (1.7), then the first order system (1.10) is invertible. If the real parts is also greater than the coordinates of any resonance lines, certain estimates of solutions can be obtained that is uniformly valid for $\Re s > \gamma$. With additional assumptions on solutions in singular layers, we can find solutions to (1.9) and its inverse Laplace transform. The growth or decay rate for solutions of (1.6) is $O(e^{\gamma t})$ by the Paley-Wiener theorem of the one-sided Laplace transforms [Lemma 3.1].

A comment on the nonlinear stability of (1.5) is in order. The $L^2 \to L^2$ regularity to the linear system is not strong enough to guarantee the existence of solutions of the nonlinear problem and their stability. However, if we can prove the existence of global solutions u^ϵ of the initial value problem of (1.4), as described in the previous paragraph, and if $u^\epsilon \to u^0$ as $\epsilon \to 0$, then we have a global solution u^0 to the initial value problem of (1.5). If the solutions u^ϵ are stable, so is u^0. Liu and Yang developed an L^1 semigroup theory for systems of hyperbolic conservation laws [**11, 12**]. Zumbrun and collaborators used extensively the spectral method to study the stability of viscous shock waves [**21, 22**].

We would like to compare some alternative approaches to construct a semigroup. According to the Hille-Yosida theorem, to show the existence of a C^0 semigroup for a given "infinitesimal generator" $\mathcal{A}$, it is sufficient to show that

$$\|(\lambda - \mathcal{A})^{-k}\| \le \frac{C}{(\lambda-\gamma)^k}, \quad \lambda > \gamma$$

For system (1.7), it is technically hard to prove such an estimate with the constant C independent of k due to the complicated scattering of characteristics from the shocks.

Another way to construct a C^0 semigroup is to show directly in (x, t) coordinates that the linear system has a unique solution for every initial data U_0 in the domain $D(\mathcal{A})$, see [**13**]. This can be done using the characteristic method as in [**6, 7, 8, 18**]. However, it is not easy to get the exact growth rate for the solutions from such method.

In the standard theory of semigroups, the infinitesimal generator $\mathcal{A}$ is densely defined in a base Banach space E. For the conservation laws with shocks, some jump conditions must be satisfied by the solutions of the linearized system. If the function space consists of piecewise continuous or BV functions, the same kind of jump conditions must be imposed on functions in E. In particular, we must assume the forcing terms for the linearized system also satisfy the same jump conditions, an unnatural constrain in our view. This will not be a problem if L^p norms are used in the function space E.

Although constructed by the Laplace transform on functions that are L^2 in time, we will show that the solution $U(x, t)$ is continuous in time as $t \to U(\cdot, t)$ defines a continuous map $\mathbb{R}^+ \to E$. In this sense, the initial data is satisfied and the semigroup is a $C^0 \cap L^2$ semigroup. The compelling reason to use L^2 norm is that in this norm the optimal growth/decay rate is obtained. For our L^2 solution, we will also show that $x \to U(x, \cdot)$ defines a continuous map $R^i \to \mathbb{R}^+$ with one sided limits at x^i. In this sense, the jump conditions are satisfied.

A brief preview of the rest of the paper. In §2, we define the function space and present the basic settings of the paper.

In §3, we apply the Laplace transform to the linearized first order system. We observe that the system in the dual variable has an algebraic dichotomy in each R^i. By taking advantage of the algebraic dichotomies, we express the solutions as the solutions of a system of integral equations with undetermined boundary values that correspond to the characteristic modes entering R^i.

Since system (1.10) is the singular limit of (1.9), according to the geometric singular perturbation theory [**4, 5**], (1.10) models the flow on the so-called slow manifold (center manifold), and the entire phase space is foliated by fast fibers on which the flow exhibits fast exponential growth or decay. In this regard, the algebraic dichotomies on the slow manifold completed the dynamics of system (1.9).

An important part of this paper are the estimates on the integral equations proved in §4. We show that one of the integral term is similar to the convolution and derive an inequality similar to Young's inequality. The other integral term can be interpreted as the Fourier transform, and an equality parallel to the Plancherel's equality has been proved. They are the main tools in proving the estimates.

In §5, under the assumptions that the initial data and forcing terms are L^2 functions, we give an explicit formulation for the L^2 solutions of the linear non-homogeneous system (1.10). Its inverse Laplace transform is the weak solution of the linearized system (1.7). A convergent power series expansion of our formula can be derived. This can be interpreted as the characteristic method used by [**6, 7, 8, 18**].

In §6, we study the eigenvalue problem related to the linearized system around the Riemann solutions. We prove that the region $\Re\lambda > -\eta,\ \eta > 0$ consists of normal points - either resolvent points or eigenvalues. The eigenvalues are zeros of the determinant of a transcendental matrix. We show that the zeros of the determinant are equivalent to the zeros of the determinant of the SLEP matrix defined in [**9, 14**]. In general, eigenvalues that are not equal to -1 lie on vertical lines in $\mathbb{C}$. There can also be vertical *resonance lines* containing countably many *resonance values*. At a *resonance line* $\{\sigma+i\omega|\sigma=\sigma_0\}$, there exists ω such that the system responds to forcing terms of frequency ω can be arbitrarily large. Therefore, the growth rates of solutions are not determined by eigenvalues only.

In section §7, we show the the L^2 solutions by the Laplace transform method are continuous functions of t in the space E. The fact that they are continuous functions of x in $L^2(\mathbb{R}^+)$ is proved in §5.

In §8, we show that the solution is differentiable if the initial data and forcing terms are differentiable.

During the writing of this paper, I have benefited from many discussions with K. Jenssen, M. Lewicka, R. Pan and S. Schecter on hyperbolic conservation laws.

2. Basic settings

If $F(x)$ has a simple discontinuity at x^i, then the jump of $F(x)$ at x^i is denoted by

$$[F(x)]_{x^i} := F(x^i+) - F(x^i-).$$

We consider the linear system with jump conditions at $x^i, i=1,\dots,n$:

$$\begin{aligned}&U_t + (Df - xI)U_x = k(x,t), \quad U(x,0) = U_0(x),\\ &[(Df(x) - xI)U]_{x^i} = [\dot{X}^i(t) + X^i(t)]\Delta^i, \text{ where } \Delta^i = \bar{u}^i - \bar{u}^{i-1}.\end{aligned} \tag{2.1}$$

For brevity, we use Df for $Df(u(x))$ or $Df(\bar{u}^i)$ if no confusion should arise. We make the nonsingular change of variables

$$\begin{aligned}V &= e^t(Df - xI)U, & g(x,t) &= e^t(Df - xI)k(x,t),\\ Y^i(t) &= e^t X^i(t), & h(x) &= (Df - xI)U_0(x).\end{aligned}$$

System (2.1) is equivalent to

$$\begin{aligned}V_t + (Df - xI)V_x &= g(x,t), \quad V(x,0) = h(x),\\ [V(x,t)]_{x^i} &= \dot{Y}^i(t)\Delta^i.\end{aligned} \tag{2.2}$$

Remark 2.1. The change of variables $U \to V$ brings a change of the growth rates of solutions. If $(V, Y^i) = O(e^{\gamma t})$ then $(U, X^i) = O(e^{(\gamma-1)t})$.

In particular, we will show that $\lambda = 0$ is an eigenvalue for (2.2). This implies that $\lambda = -1$ is always an eigenvalue for the (2.1).

Assume that the system is strictly hyperbolic. In each R^i, $Df(\bar{u}^i)$ has n distinct eigenvalues $\lambda_j(\bar{u}^i)$ associated to eigenvectors $\mathbf{r}_j(\bar{u}^i)$.

Assume that the ith shock Λ^i is a Lax i-shock, i.e.:

$$\begin{aligned}&\lambda_j(\bar{u}^i) < x^i < x < x^{i+1}, \quad 1 \le j \le i,\\ &x^i < x < x^{i+1} < \lambda_j(\bar{u}^i), \quad i+1 \le j \le n.\end{aligned} \tag{2.3}$$

The relation between the characteristic and shock waves is illustrated in Figure 2.1

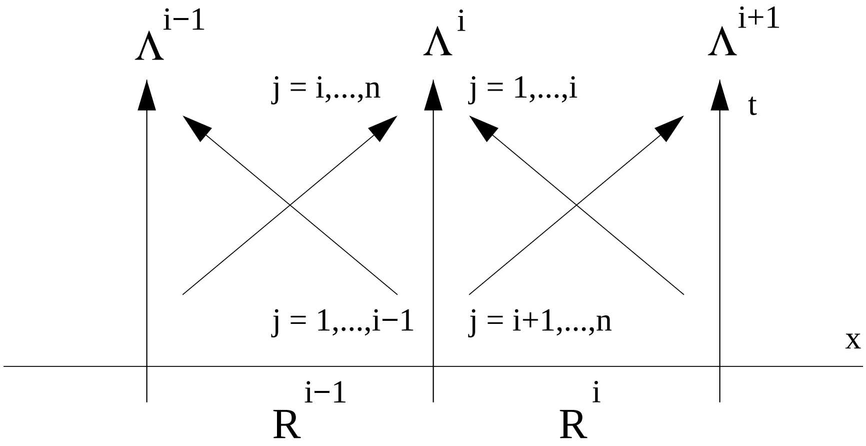

FIGURE 2.1. The left and right going characteristics in R^{i-1} and R^i.

Assume that in R^i,

$$V=\sum_1^n v_j^i(x)\mathbf{r}_j(\bar{u}_i),\quad g=\sum_1^n g_j^i(x,s)\mathbf{r}_j(\bar{u}^i),\quad h=\sum_1^n h_j^i(x)\mathbf{r}_j(\bar{u}^i).$$

We drop the super super-script i if no confusion should arise.

DEFINITION 2.1. Let L_w^2 be the linear space of locally L^2 functions with the following weighted norm being finite: If the restriction of U to R^i is U^i and if $U^i=\sum_1^n u_j^i(x)\mathbf{r}_j(\bar{u}_i)$, then

$$\|U\|=\|U\|_w:=\left(\sum_{i=0}^n\sum_{j=1}^n\|u_j^i\|^2\right)^{1/2},$$

$$\|u_j^i\|:=\left(\int_{R^i}|u_j^i(x)|^2\frac{dx}{|x-\lambda_j(\bar{u}^i)|}\right)^{1/2},\quad \text{if } 1\le i\le n-1,$$

$$\|u_j^0\|:=\left(\int_{R^0}\left|\left(\frac{\lambda_j(\bar{u}^0)-x}{\lambda_j(\bar{u}^0)-x^1}\right)^\eta u_j^0(x)\right|^2\frac{dx}{|x-\lambda_j(\bar{u}^0)|}\right)^{1/2},$$

$$\|u_j^n\|:=\left(\int_{R^n}\left|\left(\frac{\lambda_j(\bar{u}^n)-x}{\lambda_j(\bar{u}^n)-x^n}\right)^\eta u_j^n(x)\right|^2\frac{dx}{|x-\lambda_j(\bar{u}^n)|}\right)^{1/2}.$$

If the weighted norm for the restriction of U to R^i is finite then we say that U^i and the scalar function u_j^i are in $L_w^2(R^i)$.

We assume that the constant $\eta>0$ so that the weight functions

$$\left(\frac{\lambda_j(\bar{u}^0)-x}{\lambda_j(\bar{u}^0)-x^1}\right)^\eta\ge 1,\ \text{in } R^0;\quad \left(\frac{\lambda_j(\bar{u}^n)-x}{\lambda_j(\bar{u}^n)-x^n}\right)^\eta\ge 1,\ \text{in } R^n.$$

Thus as $x\to\pm\infty$, $u_j^0(x)$ and $u_j^n(x)\to 0$ algebraically of order $|x-\lambda_j|^{-\eta}$.

The semigroup will be defined in the Hilbert space $E := L^2_w$. Define the differential operator $\mathcal{A}$ as

$$\mathcal{A}(V) = -(Df - xI)V_x, \quad \text{on each } R^i,$$

with

$$D(\mathcal{A}) := \{V : V,\ V_x \in E,\ [V(x)]_{x^i} \in \operatorname{span}(\Delta^i),\ i = 1, \ldots, n\}.$$

Some comments about the domain of the operator $\mathcal{A}$ is in order.

(1) If $V_0 \in D(\mathcal{A})$ then the compatibility condition is satisfied by the initial condition and the solution is continuous across the characteristics issuing at $t = 0, x = x^i$.

(2) The derivatives V_t, V_x are not continuous across the characteristics, causing the so called weak shocks issuing from each x^i at $t = 0$. To avoid such discontinuity of derivatives, further compatibility conditions must be imposed on the initial condition.

(3) If we only require the solution to be H^1_{loc} in time and space, then the discontinuity of derivatives across the characteristics are allowed. No further condition is needed on the initial condition besides $V_0 \in D(\mathcal{A})$.

Assume the following Majda's stability condition at the shock Λ^i: The vectors

$$\mathbf{r}_1(\bar{u}^{i-1}), \ldots, \mathbf{r}_{i-1}(\bar{u}^{i-1}), \Delta^i, \mathbf{r}_{i+1}(\bar{u}^i), \ldots, \mathbf{r}_n(\bar{u}^i),$$

are linearly independent in $\mathbb{R}^n$, and will be called Majda's basis at Λ^i.

3. Laplace transform and a system of integral equations

A function $y(s)$ is in the Hardy-Lebesgue class $\mathcal{H}(\gamma), \gamma \in \mathbb{R}$, if

(i) $y(s)$ is analytic in $\Re(s) > \gamma$;

(ii) $\{\sup_{\sigma>\gamma}(\int_{-\infty}^{\infty} |y(\sigma + i\omega)|^2 d\omega)^{1/2}\} < \infty$.

$\mathcal{H}(\gamma)$ is a Banach space with the norm defined by the left side of (ii). Based on the Paley-Wiener Theorem of one-sided Laplace transforms [**20**], we have

Lemma 3.1. *If $e^{-\gamma t} z(t) \in L^2(\mathbb{R}^+)$, then its Laplace transform $y(s) = \mathcal{L}z(s) \in \mathcal{H}(\gamma)$.*

The converse of this holds: Let $y(s) \in \mathcal{H}(\gamma)$. Then the boundary function $y(\gamma + i\omega) \in L^2(-\infty, \infty)$ exists in the sense that

$$\lim_{\sigma\to\gamma+} \int_{-\infty}^{\infty} |y(\sigma + i\omega) - y(\gamma + i\omega)|^2 d\omega = 0.$$

Moreover, the inverse Fourier transform

$$z(t) = (2\pi)^{-1/2} \lim_{N\to\infty} \int_{-N}^{N} y(\gamma + i\omega)e^{it\omega} d\omega$$

vanishes for $t < 0$ and $y(s)$ may be obtained as the one-sided Laplace transform of $z(t)$. Further more,

$$\int_{t=0}^{\infty} e^{-2\gamma t} |z(t)|^2 dt = \int_{\omega=-\infty}^{\infty} |y(\gamma + i\omega)|^2 d\omega.$$

Applying the Laplace transform to (2.2), we have

$$s\hat{V} + (Df - xI)\hat{V}_x = \hat{g}(x, s) + h(x),$$
$$[\hat{V}(x, s)]_{x^i} = [s\hat{Y}^i(s) - Y^i(0)]\Delta^i.$$

On the other hand, if we can find a solution $\hat{V} \in \mathcal{H}(\gamma)$ for the above, then the inverse transform shows that $V(x,t)$ is a weak solution with $e^{-\gamma t}V(\cdot,t) \in L^2(\mathbb{R}^+)$, i.e., $V = O(e^{\gamma t})$ in L^2 norm.

To simplify the notations, we will drop the hat on $\hat{V}(x,s)$ and $\hat{g}(x,s)$ if no confusion should arise. The use of the dual variable s already indicates that they are the images of the Laplace transform of $V(x,t)$ and $g(x,t)$. The convention also applies to other time dependent functions and their L-transforms.

We now drop the hat and consider

$$V_x + s(Df - xI)^{-1}V = (Df - xI)^{-1}(g + h). \tag{3.1}$$

The equation for V will be solved for fixed s with $\Re s > -\eta$, except for a set of measure 0, under the condition that $g(x,s)$ and $h(x)$ are L^2 functions of x.

If $V = \sum v_j(x)\mathbf{r}_j(\bar{u}^i)$ satisfies (3.1), then the jth mode $v_j(x)\mathbf{r}_j(\bar{u}^i),\ j = 1,\dots,n$ satisfies

$$v_{jx} + s(\lambda_j - x)^{-1}v_j = (\lambda_j - x)^{-1}(g_j(x,s) + h_j(x)).$$

Observe that if $\Re s > 0$, the system has an algebraic dichotomy in each R^i. See [**2**] for discussions of exponential and non-exponential dichotomies.

Since the region for s is unbounded to the right, the growth or decay is important even in the finite regions $R^i, i = 1,\dots,n-1$. To take advantage of the decay for each wave in certain direction, we will solve the right going waves from x^i to x^{i+1} and the left going waves from x^{i+1} to x^i. This approach is consistent with the characteristic method which requires that each wave must be prescribed on the point where the wave enters R^i.

For clarity, we use $\ell = 1,\dots,i$ and $r = i+1,\dots,n$ for the indices of the left and right going waves in R^i. For the mode $i+1 \le r \le n,\ \lambda_r > x$, using the integration factor $(\lambda_r - x)^{-s}$, we have

$$((\lambda_r - x)^{-s}v_r)_x = (\lambda_r - x)^{-s-1}(g_r + h_r).$$

For the mode $1 \le \ell \le i,\ x > \lambda_\ell$, using the integration factor $(x - \lambda_\ell)^{-s}$, we have

$$((x - \lambda_\ell)^{-s}v_\ell)_x = -(x - \lambda_\ell)^{-s-1}(g_\ell + h_\ell).$$

The solution in R^i satisfies the integral equations:

$$\begin{aligned} v_r(x,s) =& \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s v_r(x^i,s) + \int_{x^i}^{x} \left(\frac{\lambda_r - x}{\lambda_r - y}\right)^s (g_r(y,s) + h_r(y))\frac{dy}{\lambda_r - y}, \\ & x^i \le x \le x^{i+1}, \quad r = i+1,\dots,n, \end{aligned} \tag{3.2}$$

$$\begin{aligned} v_\ell(x,s) =& \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s v_\ell(x^{i+1},s) + \int_{x^{i+1}}^{x} \left(\frac{\lambda_\ell - x}{\lambda_\ell - y}\right)^s (g_\ell(y,s) + h_\ell(y))\frac{dy}{\lambda_\ell - y}, \\ & x^i \le x \le x^{i+1}, \quad \ell = 1,\dots,i. \end{aligned} \tag{3.3}$$

The only unknown variables in the right hand sides are $v_r(x^i,s)$ and $v_\ell(x^{i+1},s)$. As a convention, $x^0 = -\infty, x^{n+1} = \infty$, and the terms involving $v_r(x^0,s)$ and $v_\ell(x^{n+1},s)$ are ignored.

Note that

$$0 < \frac{\lambda_j - x}{\lambda_j - y} < 1 \quad \text{if } \begin{cases} x^i < y < x & \text{for } i+1 \le j \le n, \\ x < y < x^{i+1} & \text{for } 1 \le j \le i. \end{cases}$$

In the region $\Re s > -\eta$, the integral terms in (3.2) and (3.3) are bounded for $x^i < x < x^{i+1}$ uniformly with respect to s.

In each R^i, define the "propagator" $\Phi^i_j(x, x^i, s)$ as

$$\text{if } j = i+1, \dots, n, \quad \Phi^i_j(x,y,s) = \begin{cases} (\frac{\lambda_j - x}{\lambda_j - y})^s & \text{if } y \le x, \\ 0 & \text{if } x < y, \end{cases}$$

$$\text{if } j = 1, \dots, i, \quad \Phi^i_j(x,y,s) = \begin{cases} (\frac{\lambda_j - x}{\lambda_j - y})^s & \text{if } x \le y, \\ 0 & \text{if } y < x. \end{cases}$$

Let

$$z^i_j = \begin{cases} x^i & \text{if } i+1 \le j \le n, \\ x^{i+1} & \text{if } 1 \le j \le i. \end{cases}$$

The integral equations (3.2), (3.3) in R^i can be written as:

$$v_j(x,s) = \Phi^i_j(x, z^i_j, s) v_j(z^i_j, s) + \int_{x^i}^{x^{i+1}} \Phi^i_j(x,y,s)(h_j(y) + (g_j(y,s))\frac{dy}{|\lambda_j - y|}. \tag{3.4}$$

4. Estimates of the integral terms

In this section, we derive some estimates for the integral terms of (3.2) and (3.3).

To simplify the notation, in the following definitions we make use of the information carried in the names of variables, e.g., x is the spatial, t is the time variable and $s = \sigma + i\omega$ is the dual to t after the Laplace transform.

Definition 4.1. We say $V(x,t)$ is in $L^2_w(x)$ if $V(\cdot, t) \in L^2_w$ for a fixed t. We say $V(x,t)$ is in $L^2(t)$ if it is in $L^2(\mathbb{R}^+)$ for a fixed x. We say $V(x,t)$ is in $L^2_w(x,t)$ if it is locally a L^2 function for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, and for almost every t, $V(\cdot,t) \in L^2_w$ with

$$\int_0^\infty \|V(\cdot,t)\|^2_w dt < \infty.$$

After the Laplace transform, $V(x,t)$ becomes $V(x,s)$ with $s = \sigma + i\omega$. We say $V(x,s)$ is in $L^2_w(x)$ if $V(\cdot,s) \in L^2_w$ for a fixed s. We say $V(x,s)$ is in $L^2(\omega)$ if $V(x,s)$ is in $L^2(\mathbb{R})$ for a fixed x and σ. We say $V(x,s)$ is in $L^2_w(x,\omega)$ if for a fixed σ, it is locally a L^2 function for $(x,\omega) \in \mathbb{R}^2$, and for almost every ω, $V(\cdot,s) \in L^2_w$ with

$$\int_{-\infty}^\infty \|V(\cdot,s)\|^2_w d\omega < \infty.$$

These definitions also extend to functions defined only in one regular layer $R^i, i = 0, \dots, n$.

Let $s = \sigma + i\omega$ with $\sigma > -\eta$. In each R^i, define

$$F_r(x,s) := \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}} h_r(y) dy, \quad r \ge i+1,$$

$$F_\ell(x,s) := \int_{x^{i+1}}^x \frac{(x - \lambda_\ell)^s}{(y - \lambda_\ell)^{s+1}} h_\ell(y) dy, \quad \ell \le i.$$

$$G_r(x,s) := \int_{x^i}^x \left(\frac{\lambda_r - x}{\lambda_r - y}\right)^s g_r(y,s) \frac{dy}{\lambda_r - y}, \quad r \ge i+1,$$

$$G_\ell(x,s) := \int_{x^{i+1}}^x \left(\frac{\lambda_\ell - x}{\lambda_\ell - y}\right)^s g_\ell(y,s) \frac{dy}{\lambda_\ell - y}, \quad \ell \le i.$$

LEMMA 4.1. *Assume that* $h \in L^2_w$, *i.e. in* $R^i, i = 0, \dots, n$, *the weighted norms* $\|h_j\|$ *of* $h_j, j = 1, \dots, n$, *as in Definition 2.1, are finite.*

Then for $\sigma > -\eta$, $F_j(x,s) \in L^2_w(x,\omega)$ *with:*

(1) For each $x \in R^i$, $i = 1, \dots, n-1$, $F_j(x,s) \in L^2(\omega)$ *with*

$$\int_{\omega=-\infty}^{\infty} |F_j(x,s)|^2 d\omega \le C(\eta)\|h_j\|^2,$$

where $C(\eta)$ *depends only on* η.

In R^0 *and* R^n, $F_r(x^1,s)$ *and* $F_\ell(x^n,s) \in L^2(\omega)$ *with*

$$\int_{\omega=-\infty}^{\infty} |F_r(x^1,s)|^2 d\omega \le C(\eta)\|h_r\|^2,$$

$$\int_{\omega=-\infty}^{\infty} |F_\ell(x^n,s)|^2 d\omega \le C(\eta)\|h_\ell\|^2.$$

Moreover, $x \to F_j(x,s)$ *is a continuous function from* R^i *to* $L^2(\omega)$ *with one-sided limits at* $x = x^i$.

(2) For almost every ω, $F(\cdot,s) \in L^2_w$. *Moreover, in* $R^i, i = 1, \dots, n-1$,

$$\int_{\omega=-\infty}^{\infty} \|F_j(\cdot,s)\|^2 d\omega \le C(\eta)\|h_j\|^2.$$

In R^0 *and* R^n,

$$\int_{\omega=-\infty}^{\infty} \|F_j(\cdot,s)\|^2 d\omega \le \frac{1}{\sigma+\eta}\|h_j\|^2.$$

LEMMA 4.2. *Assume that for* $\sigma > -\eta$, $g(x,s) \in L^2_w(x,\omega)$. *That is, the weighted norm* $\|g_j(\cdot,s)\|$ *of* g_j *in* R^i, *as in Definition 2.1 is finite for almost every* ω *and*

$$|||g_j||| := \left(\int_{\omega=-\infty}^{\infty} \|g_j(\cdot,s)\|^2 d\omega\right)^{1/2} < \infty$$

Then for $\sigma > -\eta$, $G_j(x,s)$ *is in* $L^2_w(x,\omega)$ *with:*

(1) For each $x \in R^i$, $i = 1, \dots, n-1$,

$$\int_{\omega=-\infty}^{\infty} |G_j(x,s)|^2 d\omega \le C(\eta)|||g_j|||^2, \quad j = 1, \dots, n,$$

where $C(\eta)$ *only depends on* η.

In R^0 *and* R^n, *we have*

$$\int_{\omega=-\infty}^{\infty} |G_r(x^1,s)|^2 d\omega \le \frac{1}{\eta+\sigma}|||g_r|||^2,$$

$$\int_{\omega=-\infty}^{\infty} |G_\ell(x^n,s)|^2 d\omega \le \frac{1}{\eta+\sigma}|||g_\ell|||^2.$$

(2) For almost every ω, $G(\cdot,s) \in L^2_w$. *Moreover, in* R^0 *and* R^n,

$$|||G_j|||^2 \le \frac{1}{\eta+\sigma}|||g_j|||^2.$$

In the bounded regions R^i, $i = 1, \dots, n-1$,

$$|||G_j|||^2 \le C(\eta)|||g_j|||^2.$$

(3) For $\sigma > -\eta$, $x \to G_j(x,s)$ *is a continuous function from* R^i *to* $L^2(\omega)$ *with one-sided limits at* x^i.

(4) If $(Df - xI)g(x,s) \in L^2_w(x,\omega)$, then $(Df - xI)G(x,s)$ and $sG(x,s)$ are in $L^2_w(x,\omega)$ and their norms are bounded by of $|||(Df - xI)g|||$.

(5) If $(Df - xI)g(x,s) \in L^2_w(x,\omega)$, then $(Df - xI)G(x,s)$ and $sG(x,s)$ depend continuously on $x \in R^i$ in $L^2(\omega)$.

The rest of the section is dedicated to the proof of Lemmas 4.1 and 4.2.

First observe the expressions of (3.2) and (3.3) as well as $F_j(x,s)$ and $G_j(x,s)$ in Lemma 4.1 and Lemma 4.2 are much simpler if we make the following change of spatial variable in each R^i that depends on the mode number j.

Definition 4.2. We define the logarithmic change of spatial variables as follows: To the right going characteristics $\lambda_r(\bar{u}^i)$ in R^i, note that $\lambda_r > x$, and let

$$
\begin{aligned}
X_r &= \ln(\lambda_r - x), & x &= \lambda_r - e^{X_r}, \\
X_r^i &= \ln(\lambda_r - x^i), & dX_r &= \frac{-dx}{\lambda_r - x}, \\
\tilde{\phi}(X_r) &= \phi(\lambda_r - e^{X_r}) = \phi(x), & \partial_X \tilde{\phi} &= -(\lambda_r - x)\partial_x \phi.
\end{aligned}
$$

To the left going characteristics $\lambda_\ell(\bar{u}^i)$ in R^i, note that $x > \lambda_\ell$ and let

$$
\begin{aligned}
X_\ell &= \ln(x - \lambda_\ell), & x &= \lambda_\ell + e^{X_\ell}, \\
X_\ell^i &= \ln(x^i - \lambda_\ell), & dX_\ell &= \frac{-dx}{\lambda_\ell - x}, \\
\tilde{\phi}(X_\ell) &= \phi(\lambda_\ell + e^{X_\ell}) = \phi(x), & \partial_X \tilde{\phi} &= (x - \lambda_\ell)\partial_x \phi.
\end{aligned}
$$

The new variables X and X_j^i are called the log variables in R^i. The subscript will often be dropped for simplicity

The order of variables are listed below for your convenience:

$$
\begin{aligned}
&x^i < y < x < x^{i+1} < \lambda_r, \\
&X_r^{i+1} < X < Y < X_r^i, \quad \text{for right going waves;} \\
&\lambda_\ell < x^i < x < y < x^{i+1}, \\
&X_\ell^i < X < Y < X_\ell^{i+1}, \quad \text{for left going waves.}
\end{aligned}
$$

As a convention, we set $X_r^0 = \infty$ and $X_\ell^{n+1} = \infty$. Note that in the new variables, we always have $X < Y$, i.e., equations are always solved from right to left, regardless the original waves are left or right going waves.

In the log variable X, we find that $v_j, j = 1, \ldots, n$, satisfies

$$
sv_j - v_{jX} = g_j + h_j. \tag{4.1}
$$

This means that all the modes are unstable for large $\sigma = \Re s$. This is consistent with the observation that all the equations in X must be solved from right to left.

For brevity, we will use the same notation for the same function after changing the variable from x to X. We will drop the index j for X_j^i if no confusion should arise.

The following systems are equivalent to (3.2), (3.3):

$$v_r(X,s) = e^{s(X-X^i)} v_r(X^i,s) - \int_{X^i}^{X} e^{s(X-Y)}(g_r(Y,s)+h_r(Y))dY \\ X^{i+1} \le X \le X^i, \quad r=i+1,\dots,n \tag{4.2}$$

$$v_\ell(X,s) = e^{s(X-X^{i+1})} v_\ell(X^{i+1},s) - \int_{X^{i+1}}^{X} e^{s(X-Y)}(g_\ell(Y,s)+h_\ell(Y))dY \\ X^i \le X \le X^{i+1}, \quad \ell=1,\dots,i. \tag{4.3}$$

The norms defined in Definition 2.1 can be expressed in the log variables:

$$\begin{aligned} \|u_j^i\| &:= \left| \int_{X^i}^{X^{i+1}} |u_j^i(Y)|^2 dY \right|^{1/2}, \quad \text{if } 1 \le i \le n-1, \\ \|u_j^0\| &:= \left(\int_{X^1}^{\infty} (e^{\eta(Y-X^1)} |u_j^i(Y)|)^2 dY \right)^{1/2}, \\ \|u_j^n\| &:= \left(\int_{X^n}^{\infty} (e^{\eta(Y-X^n)} |u_j^n(Y)|)^2 dY \right)^{1/2}. \end{aligned} \tag{4.4}$$

The functions $F_j(x,s)$ and $G_j(x,s)$ expressed in X are:

$$\begin{aligned} F_r(X,s) &= -\int_{X^i}^{X} e^{s(X-Y)} h_r(Y) dY, \quad r=i+1,\dots,n, \\ F_\ell(X,s) &= -\int_{X^{i+1}}^{X} e^{s(X-Y)} h_\ell(Y) dY, \quad \ell=1,\dots,i. \end{aligned} \tag{4.5}$$

$$\begin{aligned} G_r(X,s) &= -\int_{X^i}^{X} e^{s(X-Y)} g_r(Y,s) dY, \quad r=i+1,\dots,n, \\ G_\ell(X,s) &= -\int_{X^{i+1}}^{X} e^{s(X-Y)} g_\ell(Y,s) dY, \quad \ell=1,\dots,i. \end{aligned} \tag{4.6}$$

Using the log variables, Lemmas 4.1 and 4.2 are translated into the following Lemmas 4.3 and 4.4 respectively.

LEMMA 4.3. *Assume that $h \in L^2_w$, i.e., the norms $\|h_j\|$ of h_j are finite.*
Then for $\sigma > -\eta$ $F_j(x,s) \in L^2_w(X,\omega)$ with:
(1) $X \to F_j(X,s)$ is continuous from $R^i \to L^2(\omega)$ with one-sided limits at X^i. In particular, for each $X \in R^i, 1 \le i \le n-1$, $F_j(X,s) \in L^2(\omega)$ with

$$\int_{\omega=-\infty}^{\infty} |F_j(X,s)|^2 d\omega \le C(\eta)\|h_j\|^2.$$

In R^0 and R^n, $F_r(X^1,s)$ and $F_\ell(X^n,s)$ are in $L^2(\omega)$ with

$$\int_{\omega=-\infty}^{\infty} |F_r(X^1,s)|^2 d\omega \le C(\eta)\|h_r\|^2,$$
$$\int_{\omega=-\infty}^{\infty} |F_\ell(X^n,s)|^2 d\omega \le C(\eta)\|h_\ell\|^2.$$

(2) For almost every ω, $F_j(X,s) \in L^2_w$. Moreover, in $R^i, 1 \le i \le n-1$,

$$\int_{\omega=-\infty}^{\infty} \|F_j(\cdot,s)\|^2 d\omega \le C(\eta)\|h_j\|^2.$$

In R^0 and R^n,

$$\int_{\omega=-\infty}^{\infty} \|F_r(\cdot,s)\|^2 d\omega \le \frac{1}{\sigma+\eta}\|h_r\|^2,$$
$$\int_{\omega=-\infty}^{\infty} \|F_\ell(\cdot,s)\|^2 d\omega \le \frac{1}{\sigma+\eta}\|h_\ell\|^2.$$

PROOF. Extend the domain of $h_j(X)$ from R^i to $\mathbb{R}$ such that $h_j(X)=0$ for $X \notin R^i$.

Proof of (1): We will only consider the right going waves in R^0 since the other cases are similar. Express F_r as the Fourier-Laplace transform of h_r,

$$\begin{aligned} F_r(X,s) &= -\int_{\infty}^{X} e^{-s(Y-X)} h_r((Y-X)+X)dY \\ &= \int_{\xi=0}^{\infty} e^{-s\xi} h_r(\xi+X)d\xi \\ &= \int_{\xi=-\infty}^{\infty} e^{-i\omega\xi}[e^{-\sigma\xi}H(\xi)h_r(\xi+X))]d\xi. \end{aligned}$$

Since $-\sigma<\eta$, $e^{-\sigma\xi}H(\xi)h_r(\xi+X)$ is in $L^2(\xi)$. Plancherel's formula shows that $F_r(X,s)$ is in $L^2(\omega)$ for such fixed σ. The shift operator $X \to h_r(.+X)$ is a continuous mapping of $\mathbb{R} \to L^2$. Therefore, $F_r(X,s)$ is in $L^2(\omega)$ that depends continuously on X. This proves (1).

Proof of (2): Again we will prove the case of the right going waves in R^0 only.

Let $\mathfrak{h}$ be the Hilbert space of $L^2_{loc}(X_1,\infty)$ functions with the finite norm

$$\|k\|_{\mathfrak{h}} = (\int_{X_1}^{\infty} |e^{\eta(X-X^1)}k(X)|^2 dX)^{1/2}.$$

For almost every $\xi>0$, the function $k(\xi) := h_r(\xi+\cdot)$ is in $\mathfrak{h}$. Moreover

$$\begin{aligned} \|k(\xi)\|^2_{\mathfrak{h}} &= e^{-2\eta\xi}\int_{X^1}^{\infty} e^{2\eta(X+\xi-X^1)}|h_r(\xi+X)|^2 dX \\ &\le e^{-2\eta\xi}\|h_r\|^2_{\mathfrak{h}}. \end{aligned}$$

For $\sigma>-\eta$, $-\sigma<\eta$, the function $e^{-\sigma\xi}H(\xi)k(\xi)$ is in $L^2(\xi)$ and

$$F_r(\cdot,\sigma+i\omega) = \mathcal{F}(e^{-\sigma\xi}H(\xi)k(\xi)).$$

From the Plancherel's theorem which is also valid for Fourier transforms with values in Hilbert spaces, we have

$$\begin{aligned} \int_{\omega=-\infty}^{\infty} \|F_r(\cdot,\sigma+i\omega)\|^2_{\mathfrak{h}} d\omega &= \int_{\xi=0}^{\infty} \|e^{-\sigma\xi}k(\xi)\|^2_{\mathfrak{h}} d\xi \\ &\le \int_0^{\infty} e^{-2(\eta+\sigma)\xi}\|h_r\|^2_{\mathfrak{h}} d\xi \\ &\le \frac{1}{\sigma+\eta}\|h_r\|^2_{\mathfrak{h}} \end{aligned}$$

□

LEMMA 4.4. *Assume that for $\sigma>-\eta$, $g(X,s)\in L^2(X,\omega)$, i.e.,*

$$|||g_j||| := \left(\int_{\omega=-\infty}^{\infty} \|g_j\|^2 d\omega\right)^{1/2} < \infty.$$

Then for $\sigma > -\eta$, $G(X,s) \in L^2_w(X,\omega)$ with:
(1) For each $X \in R^i$, $i = 1, \ldots, n-1$,

$$\int_{\omega=-\infty}^{\infty} |G_j(X,s)|^2 d\omega \le C(\eta) |||g_j|||^2, \quad j = 1, \ldots, n,$$

where $C(\eta)$ only depends on η. In R^0, and R^n we have

$$\int_{\omega=-\infty}^{\infty} |G_r(X^1,s)|^2 d\omega \le \frac{1}{\eta+\sigma} |||g_r|||^2,$$
$$\int_{\omega=-\infty}^{\infty} G_\ell(X^n,s)|^2 d\omega \le \frac{1}{\eta+\sigma} |||g_\ell|||^2.$$

(2) For almost every ω, $G(X,s) \in L^2_w(X)$. Moreover, in R^0 and R^n,

$$|||G_j||| \le \frac{1}{\eta+\sigma} |||g_j|||^2.$$

In $R^i, 1 \le i \le n-1$,

$$|||G_j||| \le C(\eta) |||g_j|||^2.$$

Here and $C(\eta)$ only depends on η.
(3) $X \to G_j(X,s)$ is continuous from R^i to $L^2(\omega)$ with one-sided limits at X^i.
(4) If $\partial_X g_j(X,s) \in L^2(X,\omega)$, then $\partial_X G_j(X,s)$ and $sG_j(X,s)$ are in $L^2(X,\omega)$. Their norms are bounded by $|||\partial_X g_j|||$ in that space.
(5) If $\partial_X g_j(X,s) \in L^2(X,\omega)$, then $\partial_X G_j(X,s)$ and $sG_j(X,s)$ depend continuously on $X \in R^i$ in the space $L^2(\omega)$.

Proof. Proof of (1): In $R^i, 1 \le i \le n-1$, using the fact that the interval R^i is finite, since $\sigma > -\eta$, regardless $X^i > X^{i+1}$ or $X^{i+1} > X^i$,

$$|\int_{X^{i+1}}^{X^i} |e^{\sigma(X^{i+1}-Y)}|^2 dY| + |\int_{X^i}^{X^{i+1}} |e^{\sigma(X^i-Y)}|^2 dY| \le C(\eta).$$

Using the Cauchy's inequality, we have

$$|G_r(X^{i+1},s)|^2 \le C(\eta) \int_{X^{i+1}}^{X^i} |g_r(Y,s)|^2 dY, \tag{4.7}$$

$$|G_\ell(X^i,s)|^2 \le C(\eta) \int_{X^i}^{X^{i+1}} |g_\ell(Y,s)|^2 dY \tag{4.8}$$

Observe that the L^2 norms of $e^{(\sigma+\eta)(Y-X^1)}$ in R^0 and $e^{(\sigma+\eta)(Y-X^n)}$ in R^n are bounded by $1/\sqrt{2(\sigma+\eta)}$. In R^0, applying Cauchy's inequality to

$$\begin{aligned} |G_r(X^1,s)|^2 &\le (\int_{X^1}^{\infty} e^{(\sigma+\eta)(X^1-Y)} \cdot |e^{\eta(Y-X^1)} g_r(Y,s)|dY)^2 \\ &\le \frac{1}{2(\sigma+\eta)} \int_{X^1}^{\infty} |e^{\eta(Y-X^1)} g_r(Y,s)|^2 dY. \end{aligned} \tag{4.9}$$

Similarly, in R^n, applying Cauchy's inequality we have

$$|G_\ell(X^n,s)|^2 \le \frac{1}{2(\sigma+\eta)} \int_{X^n}^{\infty} |e^{\eta(Y-X^n)} g_\ell(Y,s)|^2 dY. \tag{4.10}$$

Integrating both sides of (4.7)-(4.10) in ω from $-\infty$ to ∞, the desired result follows.

Proof of (2): The $L^2(X)$ norm of $G_r(X,s)$ can be obtained from the convolution inequality (Young's inequality). Consider R^0 where $X^1 < X < Y < \infty$,

$$e^{\eta(X-X^1)}\int_X^\infty e^{-\sigma(Y-X)}|g_r(Y,s)|dY = \int_X^\infty e^{-(\eta+\sigma)(Y-X)}|e^{\eta(Y-X^1)}g_r(Y,s)|dY.$$

$$\begin{aligned}&(\int_{X^1}^\infty |e^{\eta(X-X^1)}G_r(X,s)|^2dX)^{1/2}\\ &\le (\int_{X^1}^\infty |e^{\eta(Y-X^1)}g_r(Y,s)|^2dY)^{1/2}(\int_0^\infty |e^{-(\eta+\sigma)Y}|dY)\\ &\le \frac{1}{\eta+\sigma}(\int_{X^1}^\infty |e^{\eta(Y-X^1)}g_r(Y,s)|^2dY)^{1/2} = \frac{1}{\eta+\sigma}\|g_j\|,\end{aligned}$$

for $\sigma > -\eta$. The estimate in (2) follows by integrating both sides in ω.

The proof for the cases of $X \in R^i, i = 1,\dots,n$ is similar and will be omitted.

Proof of (3): This ought come from part (1) if we can approximate L^2 functions by a sequence of C_c^∞ functions and using the fact that the limit of a uniformly convergent sequence of continuous functions is continuous. For clarity, we will give a direct proof for the right going waves in R^i, $1 \le i \le n-1$ only, leaving other cases to the readers.

In R^i, let $X^{i+1} \le X_1 < X_2 \le X^i$. Without loss of generality, assume that g_r is a smooth function.

$$\begin{aligned}&|G_r(X_1,s) - G_r(X_2,s)|\\ &\le |\int_{X_1}^{X_2} e^{s(X_1-Y)}g_r(Y,s)dY| + |\int_{X^2}^{X^i}[e^{s(X_1-Y)} - e^{s(X_2-Y)}]g_r(Y,s)dY| = I_1 + I_2.\end{aligned}$$

The term I_1 satisfies $\int_{-\infty}^\infty I_1^2 d\omega \le C|X_1-X_2|\int_{-\infty}^\infty \|g_r(\cdot,s)\|^2 d\omega \to 0$ as $|X_1-X_2| \to 0$.

The second term I_2 satisfies

$$\int_{-\infty}^\infty |I_2|^2 d\omega \le \int_{-\infty}^\infty \left(\int_{X_2}^{X^i}|e^{s(X_1-Y)} - e^{s(X_2-Y)}|^2dY \cdot |\int_{X_2}^{X^i}|g_r(Y,s)|^2dY\right)d\omega.$$

There exists $K > 0$, independent of X_1, X_2 such that

$$\int_{X_2}^{X^i}|e^{s(X_1-Y)} - e^{s(X_2-Y)}|^2dY \le K.$$

For any $\epsilon > 0$ and $\omega \in \mathbb{R}$, there exists $\Omega > 0$ such that

$$K\int_{|\omega|>\Omega}\int_{X_2}^{X^i}|g_r(Y,s)|^2dYd\omega < \epsilon.$$

Therefore, it suffice to consider the domain $|\omega| \le \Omega$.

Notice that

$$\begin{aligned}&\int_{X^2}^{X^i}|e^{s(X_1-Y)} - e^{s(X_2-Y)}|^2dY = o(|X_1-X_2|), \quad \text{uniformly for } |\omega| \le \Omega,\\ &\int_{|\omega|\le\Omega}\int_{X_2}^{X^i}|g_r(Y,s)|^2dY \le \|g_r\|_{L^2}.\end{aligned}$$

This proves that as $|X_1 - X_2| \to 0$, $\int_{-\infty}^{\infty} |I_2|^2 d\omega \to 0$. Hence,

$$\int_{-\infty}^{\infty} |G_r(X_1, s) - G_r(X_2, s)|^2 d\omega \to 0.$$

Proof of (4): We show that $\partial_X G(X, s)$ and $sG(X, s)$ are in $L^2(X, \omega)$. We will prove the case of right going waves in $R^i, i = 1, \ldots, n-1$ only, since the proof of other cases is similar.

$$\begin{aligned} sG_r(X, s) &= \int_{X-X^i}^{0} se^{sY} g_r(X - Y, s) dY \\ &= \int_{X-X^i}^{0} (\frac{d}{dY} e^{sY}) g_r(X - Y, s) dY \\ &= g_r(X, s) - e^{s(X-X^i)} g_r(X^i, s) + \int_{X-X^i}^{0} e^{sY} \partial_X g_r(X - Y, s) dY. \end{aligned}$$

$$\partial_X G_r(X, s) = -e^{s(X-X^i)} g_r(X^i, s) + \int_{X-X^i}^{0} e^{sY} \partial_X g_r(X - Y, s) dY.$$

Of course, this verifies that $sG_r - \partial_X G_r = g_r(X, s)$. Since each of the terms in the representations of sG_r and $\partial_X G_r$ is in $L^2(X, \omega)$. This shows that both sG_r and $\partial_X G_r$ are in $L^2(X, \omega)$.

Proof of (5), since $\partial_X g_j(X, s) \in L^2(X, \omega)$, the mapping $X \to g_j(X, s)$ is continuous from X to $L^2(\omega)$. Based on this, the proof of (5) follows by inspecting the terms in the expressions of sG_r and $\partial_X G_r$.

$\square$

Remark 4.1. Unlike the case for $G(X, s)$ the following is not true for $F(X, s)$: "If $\partial_X h_j(X)$ is in $L^2(X)$, then $\partial_X F(X, s)$ and $sF(X, s)$ are in $L^2(X, \omega)$. Moreover, $\partial_X F(X, s)$ and $sF(X, s)$ depends continuously on X in $L^2(\omega)$."

5. L^2 solutions via the Laplace transform

In this section, we derive an explicit formula for the solution $V(x, s)$ of equations (3.2), (3.3). We show that these solutions are in $L^2_w(x, \omega)$ for $\sigma > \gamma$ where γ is a constant to be specified and $s = \sigma + i\omega$. Moreover, for $\sigma > \gamma$, the solution $V(x, s)$ is a continuous function of $x \in R^i$ in $L^2(\omega)$ with one-sided limits at $x = x^i$. The jump condition

$$V(x^i+, s) - V(x^i-, s) = (s\hat{Y}^i(s) - Y^i(0))\Delta^i \tag{5.1}$$

at the ith shock is satisfied in the sense that both sides are functions in $L^2(\omega)$.

Based on this, the inverse Laplace transform $V(x, t)$ of $V(x, s)$ is a weak solution in the sense of distribution. The function $e^{-\gamma t} V(x, t)$ is in $L^2_w(x, t)$ as in Definition 4.1. Moreover, $V(x, t)$ is in $L^2(t)$ and is continuous with respect to x in each region R^i with one-sided limits at $x = x^i$. The value of the jump at x^i is understood as a function in $L^2(t)$.

If $V(x^i+, s)$ and $V(x^i-, s)$ along the shock Λ^i are specified then the above can be used to determine $s\hat{Y}^i(s) - Y^i(0)$. If the initial condition $Y^i(0)$ is given, we can compute $\hat{Y}^i(s)$ and $Y^i(t)$. From now on, we require that the jumps of $V(x, t)$ and hence $\hat{V}(x,s)$ are along the direction of Δ^i but ignore the values of the jumps. The jump conditions thus simplify to

$$[V(x,s)]_{x^i} = 0, \ \bmod \Delta^i. \tag{5.2}$$

The rest of the section is devoted to solving the system (3.2), (3.3) and (5.2).

5.1. An non-homogeneous algebraic system with jump conditions. We consider an algebraic system for $V(x,s) = \sum_1^n v_j(x,s)\mathbf{r}_j(\bar{u}^i)$:

(5.3)

$$\begin{aligned}
v_r(x,s) &= \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s v_r(x^i,s) + H_r(x,s), \quad x^i \le x \le x^{i+1}, \quad r = i+1,\dots,n,\\
v_\ell(x,s) &= \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s v_\ell(x^{i+1},s) + H_\ell(x,s), \quad x^i \le x \le x^{i+1}, \quad \ell = 1,\dots,i.\\
[V(x,s)]_{x^i} &= 0, \quad \bmod \Delta^i, \quad i = 1,\dots,n.
\end{aligned}$$

In R^0 and R^n, we set $x^0 = -\infty$ and $x^{n+1} = \infty$, and the terms involving $v_r(x^0,s)$ and $v_\ell(x^{n+1},s)$ drop out from (5.3).

The terms H_j satisfy

H 5.1. There is a constant $\gamma \in \mathbb{R}$, to be specified below, such that for $\sigma > \gamma$, $H(x,s) \in L^2_w(x,\omega)$. That is, with the fixed σ, for almost every ω, $H(\cdot,s) \in L^2_w$ with

$$\int_{\omega=-\infty}^{\infty} \|H(\cdot,s)\|_w^2 d\omega < \infty.$$

Moreover $x \to H(x,s)$ is a continuous function from R^i to $L^2(\omega)$ with one-sided limits at $x = x^i$.

¿From the first two equations of (5.3), we have

$$\begin{aligned}
v_r(x^{i+1}-,s) &= \Phi^i_r(x^{i+1},x^i,s)v_r(x^i+,s) + H_r(x^{i+1},s) \\
& \qquad x^i \le x \le x^{i+1}, \quad r = i+1,\dots,n
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
v_\ell(x^i+,s) &= \Phi^i_\ell(x^i,x^{i+1},s)v_\ell(x^{i+1}-,s) + H_\ell(x^i,s) \\
& \qquad x^i \le x \le x^{i+1}, \quad \ell = 1,\dots,i.
\end{aligned} \tag{5.5}$$

In R^0 and R^n, recall that $x^0 = -\infty$ and $x^{n+1} = \infty$, and the terms involving $v_r(x^0+,s)$ and $v_\ell(x^{n+1},s)$ drop out from (5.4) and (5.5).

We now solve $(v_r(x^i+,s),\ v_\ell(x^{i+1}-,s))$ from (5.4), (5.5) and (5.2).

Following Lewicka [**7**], we place the left going wave $v_\ell(x^i-)$ and then the right going waves $v_r(x^i+)$ in an $(n-1)$ dimensional vector χ_i. See Figure 5.1. Next, we define χ as a block structured vector:

$$\chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix}, \quad \chi_i = \begin{pmatrix} v_1(x^i-) \\ \vdots \\ v_{i-1}(x^i-) \\ v_{i+1}(x^i+) \\ \vdots \\ v_n(x^i+) \end{pmatrix}.$$

In particular, χ_1 contains only the right going modes $v_r(x^1+), r = 2,\dots,n$, and χ_n contains only the left going modes $v_\ell(x^n-), \ell = 1,\dots,n-1$ while $\chi_i, 2 \le$

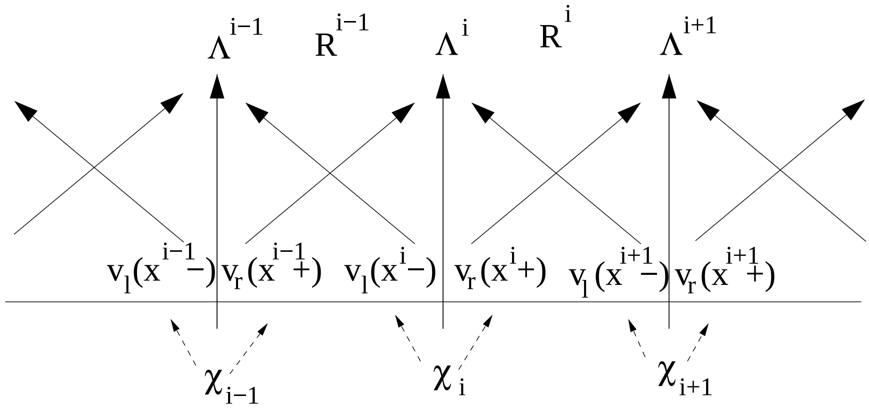

FIGURE 5.1. χ_i consists of the left and right going characteristics leaving Λ^i: $v_\ell(x^i-), \ell = 1, \ldots, i-1$ and $v_r(x^i+), r = i+1, \ldots, n$.

$i \leq n-1$, contains both the left and right going modes leaving Λ^i. To simplify the notations, we have dropped s in $v_j(x^i, s)$.

We can also express χ as the union of n dimensional vectors in $R^i, i = 1, \ldots, n-1$:

$$\chi = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n-1} \end{pmatrix}, \quad \zeta_i = \begin{pmatrix} v_{i+1}(x^i+) \\ \vdots \\ v_n(x^i+) \\ v_1(x^{i+1}-) \\ \vdots \\ v_i(x^{i+1}-) \end{pmatrix}. \tag{5.6}$$

Each ζ_i contains boundary values of waves entering R^i from Λ^i or Λ^{i+1}. To ensure that the two expressions of χ are identical, we put the right going modes in R^i before the left going modes, because the right going modes are issued from Λ^i while the left going modes are issued from Λ^{i+1}.

Define the following matrices $D = \operatorname{diag}(D_1 \ldots D_n)$ where

$$D_i = \operatorname{diag}\left(\left(\frac{\lambda_\ell(\bar{u}^{i-1}) - x^{i-1}}{\lambda_\ell(\bar{u}^{i-1}) - x^i} \right)_{\ell=1}^{i-1}, \left(\frac{\lambda_r(\bar{u}^i) - x^{i+1}}{\lambda_r(\bar{u}^i) - x^i} \right)_{r=i+1}^{n} \right), \quad 1 \leq i \leq n.$$

System (5.4) and (5.5) can be expressed as for all $1 \leq i \leq n$,

$$\begin{pmatrix} v_1(x^{i-1}+) \\ \vdots \\ v_{i-1}(x^{i-1}+) \\ v_{i+1}(x^{i+1}-) \\ \vdots \\ v_n(x^{i+1}-) \end{pmatrix} = D_i^s \chi_i + \begin{pmatrix} H_1(x^{i-1}+, s) \\ \vdots \\ H_{i-1}(x^{i-1}+, s) \\ H_{i+1}(x^{i+1}-, s) \\ \vdots \\ H_n(x^{i+1}-, s) \end{pmatrix}. \tag{5.7}$$

The last column vector shall be denoted by $\mathcal{H}_i$.

For $i = 0$ or $n+1$, the formulas are modified to

$$\begin{pmatrix} v_1(x^1-) \\ \vdots \\ v_n(x^1-) \end{pmatrix} = \begin{pmatrix} H_1(x^1-, s) \\ \vdots \\ H_n(x^1-, s) \end{pmatrix} = \mathcal{H}_0, \tag{(5.7)$_0$}$$

$$\begin{pmatrix} v_1(x^n+) \\ \vdots \\ v_n(x^n+) \end{pmatrix} = \begin{pmatrix} H_1(x^n+, s) \\ \vdots \\ H_n(x^n+, s) \end{pmatrix} = \mathcal{H}_{n+1}. \tag{(5.7)$_{n+1}$}$$

Using Majda's basis, any $\mathbf{u} \in \mathbb{R}^n$ can be expressed uniquely as:

$$\mathbf{u} = \sum_{j=1}^{i-1} \alpha_j \mathbf{r}_j(\bar{u}^{i-1}) + \alpha_i \Delta^i + \sum_{j=i+1}^{n} \alpha_j \mathbf{r}_j(\bar{u}^i).$$

Let B_i be the matrix of which the columns are the Majda basis. Then

$$(\alpha_1, \ldots, \alpha_n)^\tau = B_i^{-1}\mathbf{u}.$$

Let

$$\bar{E}_i = \begin{pmatrix} I_{(i-1)\times(i-1)} & 0 & 0 \\ 0 & 0 & -I_{(n-i)\times(n-i)}. \end{pmatrix}$$

Then

$$(\alpha_1, \ldots, \alpha_{i-1}, -\alpha_{i+1}, \ldots, -\alpha_n)^\tau = \tilde{E}_i B_i^{-1}\mathbf{u},$$

Let the left and right going waves in R^i and R^{i-1} be

$$V^{left}(x) = \sum_1^i v_\ell(x) r_\ell(\bar{u}^i), \qquad V^{right}(x) = \sum_{i+1}^n v_r(x) r_r(\bar{u}^i), \quad x \in R^i,$$

$$V^{left}(x) = \sum_1^{i-1} v_h(x) r_h(\bar{u}^{i-1}), \qquad V^{right}(x) = \sum_i^n v_k(x) r_k(\bar{u}^{i-1}), \quad x \in R^{i-1}.$$

¿From the jump condition at x^i, we have

$$V^{right}(x^i+) + V^{left}(x^i+) = V^{right}(x^i-) + V^{left}(x^i-) + S^i\Delta^i.$$

Written in the coordinates,

$$\begin{aligned} &\sum_1^{i-1} v_h(x^i-) r_h(\bar{u}^{i-1}) - \sum_{i+1}^n v_r(x^i+) r_r(\bar{u}^i) \\ =&\sum_1^i v_\ell(x^i+) r_\ell(\bar{u}^i) - \sum_i^n v_k(x^i-) r_k(\bar{u}^{i-1}) + S^i\Delta^i. \end{aligned}$$

Applying $\tilde{P}_i := \tilde{E}_i B_i^{-1}$ to both sides of the equation, we have

$$\begin{aligned} &(v_1(x^i-), \ldots, v_{i-1}(x^i-), v_{i+1}(x^i+), \ldots, v_n(x^i+))^\tau \\ =&\tilde{P}_i(\sum_1^i v_\ell(x^i+) r_\ell(\bar{u}^i) - \sum_i^n v_k(x^i-) r_k(\bar{u}^{i-1})). \end{aligned}$$

For $1 \le i \le n$, define the $(n-1)\times(n-i+1)$ and $(n-1)\times i$ matrices

$$\begin{aligned} M_i^l &= -\tilde{P}_i(r_i(\bar{u}^{i-1}), \ldots, r_n(\bar{u}^{i-1}), \\ M_i^r &= \tilde{P}_i(r_1(\bar{u}^i), \ldots, r_i(\bar{u}^i)). \end{aligned}$$

Here M_i^l and M_i^r represent the projections of impinging waves to the departing waves at the shock Λ^i. The waves leaving Λ^i can be expressed by the waves hitting Λ^i from the left and right as

$$\begin{pmatrix} v_1(x^i-) \\ \vdots \\ v_{i-1}(x^i-) \\ v_{i+1}(x^i+) \\ \vdots \\ v_n(x^i+) \end{pmatrix} = M_i^l \begin{pmatrix} v_i(x^i-) \\ \vdots \\ v_n(x^i-) \end{pmatrix} + M_i^r \begin{pmatrix} v_1(x^i+) \\ \vdots \\ v_i(x^i+) \end{pmatrix}. \tag{5.8}$$

Note that in the right hand side of the above, $(v_i(x^i-),\dots,v_n(x^i-))^\tau$ comes from the lower half of (5.7) with i replaced by $i-1$, and $(v_1(x^i+,\dots,v_i(x^i+))^\tau$ comes from the upper half of (5.7) with i replaced by $i+1$. In other words, the waves hitting Λ^i come from Λ^{i-1} and Λ^{i+1}. This motivates the definition of matrix $\tilde{M}$ with the following block structure:

$$\tilde{M} = \begin{pmatrix} M_1^l & [\Theta] & [M_1^r\Theta] & & & & & \\ & [M_2^l] & [\Theta] & [M_2^r\Theta] & & & & \\ & & [\Theta M_3^l] & [\Theta] & [M_3^r\Theta] & & & \\ & & & & \vdots & \vdots & & \\ & & & & & [\Theta M_n^l] & [\Theta] & M_n^r \end{pmatrix}.$$

In the above, the ith block of rows represents the scattering of impinging waves hitting Λ^i from the left and right to the outgoing wave leaving Λ^i. M_1^l and M_r are $(n-1)\times n$ matrices that represents the contribution of $\mathcal{H}_0$ and $\mathcal{H}_{n+1}$ to χ_1 and χ_n respectively. The zero matrix $[\Theta]$ on the main diagonal is of size $(n-1)\times(n-1)$. The matrix Θ to the left of M_i^l has $(i-2)$ columns and Θ to the right of M_i^r has $(n-i-1)$ columns so that the combined size of matrices in each [ ] is of $(n-1)\times(n-1)$.

Let M be the $(n-1)n\times(n-1)n$ matrix resulting from removing the first and the last n columns of $\tilde{M}$. Using the matrix M, from (5.7) and (5.8), we derived the following equation for $\{\chi_i\}_{i=1}^n$:

$$\begin{gathered} \chi = MD^s\chi + \tilde{M}\mathcal{H}, \\ \text{or } (I-MD^s)\chi = \tilde{M}\mathcal{H}, \end{gathered} \tag{5.9}$$

where D^s is the power of the diagonal matrix D, $\mathcal{H}$ is from the right hand side of (5.7), $(5.7)_0$, and $(5.7)_{n+1}$.

$$\mathcal{H} := (\mathcal{H}_0^\tau,\dots,\mathcal{H}_{n+1}^\tau)^\tau.$$

Definition 5.1. Let $\Xi(s) := \det(I-MD^s)$ and

$$\sigma_M = \sup\{\sigma : \inf_\omega |\Xi(\sigma+i\omega)| = 0\}.$$

In the next section, we show that the roots of $\Xi(s)$ correspond to eigenvalues of the linearized system. Moreover, this condition is equivalent to that s is the root of the SLEP function (determinant of the corresponding SLEP matrix [**15**]) as defined in [**10**].

LEMMA 5.1. *Let γ be any constant that satisfies $\gamma > \max\{-\eta, \sigma_M\}$. Then the inverse matrix $(I - MD^s)^{-1}$ exists and*

$$\|(I - MD^s)^{-1}\| \leq C(\gamma)$$

uniformly in the region $\sigma \geq \gamma$.

PROOF. There exists $N > 0$ such that if $\sigma > N$ then $|MD^s| < 1/2$, and thus $|(I - MD^s)^{-1}| < 2$. So what left is the region $\Sigma := \{\gamma \leq \sigma \leq N\}$.

Using the log variables the entries in D^s can be expressed as

$$\left(\frac{\lambda_r(\bar{u}^i) - x^{i+1}}{\lambda_r(\bar{u}^i) - x^i}\right)^s = e^{s(X^{i+1} - X^i)}, \quad \left(\frac{\lambda_\ell(\bar{u}^{i-1}) - x^{i-1}}{\lambda_\ell(\bar{u}^{i-1}) - x^i}\right)^s = e^{s(X^{i-1} - X^i)}.$$

Each entry of MD^s is uniformly bounded with respect to $s = \sigma + i\omega \in \Sigma$. Using minors to express the inverse matrix $(I - MD^s)^{-1}$, the numerators are bounded with respect to s. The denominators are $\Xi(s) := \det(I - MD^s)$. Since $\gamma > \sigma_M$, we have that $\Xi(s) \neq 0$ for $s \in \Sigma$.

Assume that there is a sequence $\{s_n\}_1^\infty = \{\sigma_n + i\omega_n\}_1^\infty \subset \Sigma$ such that $\Xi(s_n) \to 0$ as $n \to \infty$. Since the real parts of s_n are bounded, without loss of generality, assume that $\sigma_n \to \sigma_0, \gamma \leq \sigma_0 \leq N$. Let $\tau_n = \sigma_0 + i\omega_n$. Since $\Xi(s)$ is uniformly continuous with respect to σ, we find that $\Xi(\tau_n) \to 0$ as $n \to \infty$. This is a contradiction to $\Re\tau_n = \sigma_0 > \sigma_M$. Therefore, there exists $C_1(\gamma) > 0$ such that $|\Xi(s)| > C_1(\gamma)$ for $s \in \Sigma$.

This proves that $|(I - MD^s)^{-1}|$ is uniformly bounded by a constant $C(\gamma)$. $\square$

For $\sigma > \gamma := \max\{-\eta, \sigma_M\}$, from Lemma 5.1 the inverse matrix $(I - MD^s)^{-1}$ exists and is bounded by $C(\gamma)$. System (5.9) has a unique solution

$$\chi(s) = (I - MD^s)^{-1}\tilde{M}\mathcal{H}.$$

The solution $(v_r(x, s), v_\ell(x, s))$ of (3.2) and (3.3) can be obtained if we extract the ith block vector $\zeta_i = (v_r(x^i+, s), v_\ell(x^{i+1}-, s))$ from χ, see (5.6), and substitute them into (3.2) and (3.3). Since χ satisfies (5.9), the jump conditions are satisfied at each x^i.

¿From the assumption that the vectors $\mathcal{H}_i, i = 0, \dots, n$, are in $L^2(\omega)$ bounded by

$$\int_{\omega=-\infty}^{\infty} \sum_{i=0}^{n} |\mathcal{H}_i(s)|^2 d\omega \leq C[\int_{\omega=-\infty}^{\infty} (\sum_{i=1}^{n}\sum_{\ell=1}^{i} |H_\ell(x^i+, s)|^2 + \sum_{i=1}^{n}\sum_{r=i-1}^{n} |H_r(x^i-, s)|^2) d\omega].$$

Since the matrix $(I - MD^s)^{-1}$ is uniformly bounded with respect to s, we found that $\chi(s)$ is in $L^2(\omega)$, so is the vector $\zeta_i(s) = (v_r(x^i+, s), v_\ell(x^{i+1}-, s))$.

Using $\zeta_i(s) = (v_r(x^i+, s), v_\ell(x^{i+1}-, s))$, we can compute $v_j(x, s)$ in each R^i. It remains to show that the solution so obtained is in $L^2_w(x, \omega)$. To this end, we consider the right or left going modes separately in each R^i.

For $i = 0$ or n, $v_j(x, s) = H_j(x, s)$. Therefore, all the desired properties on v_j are satisfied.

For $1 \leq i \leq n - 1$, let $v_j(x.s) = \psi_j(x, s) + H_j(x, s)$ where

$$\psi_r(x, s) = \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s v_r(x^i, s),\ r \geq i+1, \quad \psi_\ell(x, s) = \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s v_\ell(x^{i+1}, s),\ \ell \leq i.$$

Since the regions R^i are finite and s is bounded to the left in $\mathbb{C}$, there exists $C(\gamma) > 0$ such that

$$\| \left(\frac{\lambda_r - x}{\lambda_r - x^i} \right)^s \| \le C(\gamma), \quad \| \left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}} \right)^s \| \le C(\gamma)$$

if $\sigma > \gamma$. from this it follows that

$$\int_{\omega=-\infty}^{\infty} \sum_{j=1}^{n} \|\psi_j(\cdot, s)\|^2 d\omega \le C(\gamma) \int_{\omega=-\infty}^{\infty} |\zeta_i(s)|^2 d\omega.$$

Based on this, we have that $V(x, s)$ is in $L_w^2(x, \omega)$.

It is easy to verify that $V(x, s)$ is a continuous function of x into $L^2(\omega)$ for $x \in R^i$. The proof shall be omitted.

We have proved the following lemma:

LEMMA 5.2. *Assume that* $\gamma > \max\{-\eta, \sigma_M\}$ *where* η *is the constant in the definition of* L_w^2 *and* σ_M *is as in the Definition 5.1. For the forcing terms* H_j *of* (5.3), *assume that the Hypothesis H 5.1 is satisfied.*

Then there exists a unique solution $V(x, s)$ *to the system* (5.3) *that is in* $L_w^2(x, \omega)$. *Moreover, for the fixed* γ,

$$\begin{aligned} \int_{\omega=-\infty}^{\infty} \|V(\cdot, s)\|_w^2 d\omega \quad &< \quad C(\gamma)[\int_{\omega=-\infty}^{\infty} (\|H(\cdot, s)\|_w^2 + \sum_{i=1}^{n} \sum_{\ell=1}^{i} |H_\ell(x^i+, s)|^2 \\ &+ \quad \sum_{i=1}^{n} \sum_{r=i}^{n} |H_r(x^i-, s)|^2) d\omega]. \end{aligned}$$

The mapping $x \to V(x, s)$ *is continuous from* R^i *to* $L^2(\omega)$ *with one-sided limits at* x^i.

5.2. Solving the system of integral equations. To solve $v_j(x, s)$ from (3.2), (3.3) and (5.2), all we need is to write the integral terms as $H_j(x, s)$ and use Lemma 5.2.

The main result of this section is the following theorem:

THEOREM 5.3. *Assume that* $h(x) \in L_w^2(x)$ *and* $e^{-\gamma t} g(x, t) \in L_w^2(x, t)$, *i.e., for almost every* t, $g(x, t) \in L_w^2(x)$ *with* $\int_{t=0}^{\infty} e^{-2\gamma t} \|g(\cdot, t)\|^2 dt < \infty$. *Then for any constant* $\gamma > \max\{-\eta, \sigma_M\}$, *there exists a unique solution* $V(x, s) \in L_w^2(x, \omega)$ *to* (3.2), (3.3) *and* (5.2). *The mapping* $x \to V(x, s)$ *is continuous from* R^i *to* $L^2(\omega)$. *Moreover*

$$\int_{\omega=-\infty}^{\infty} \|V(\cdot, s)\|_w^2 d\omega \le C(\gamma)(\int_{\omega=-\infty}^{\infty} \|g(\cdot, s)\|_w^2 d\omega + \|h\|_w^2).$$

The inverse transform $V(x, t) = \mathcal{L}^{-1} V(x, s)$ *satisfies that* $e^{-\gamma t} V(x, t) \in L_w^2(x, t)$ *with*

$$\int_{t=0}^{\infty} e^{-2\gamma t} \|V(\cdot, t)\|^2 dt \le C(\gamma)(\int_{t=0}^{\infty} e^{-2\gamma t} \|g(\cdot, t)\|_w^2 dt + \|h\|_w^2).$$

The function $V(x, t)$ *is a weak solution in the sense of distribution. Moreover, the mapping* $t \to V(x, t)$ *is continuous from* $\mathbb{R}^+ \to L_w^2(x)$; *and the mapping* $x \to e^{-\gamma t} V(x, t)$ *is continuous from* $x \to L^2(t)$. *In this sense the initial value and the jump conditions are satisfied in* L^2 *spaces.*

Finally there exists a C^0 semigroup $T(t) : L_2^2 \to L_w^2$ such that the mapping $(h, g) \to V$ can be expressed as

$$V(\cdot, t) = T(t)h(\cdot) + \int_0^t T(t-\tau)g(\cdot, \tau)d\tau.$$

PROOF. In R^i, let $g_j(x, s)$ be the Laplace transform of $g_j(x, t)$. Then $g_j(x, s)$ is in $L_w^2(x, \omega)$ for $\sigma > \gamma$. In deed, it is in the Hardy-Lebesgue class $\mathcal{H}(\gamma)$ with values in the Hilbert space $L_w^2(x)$. Therefore, $g \in L_w^2(x, \omega)$ for $\sigma > \gamma$. Let

$$H_r(x,s) := \int_{x^i}^{x} (\frac{\lambda_j - x}{\lambda_j - y})^s (g_r(y,s) + h_r(y)) \frac{dy}{\lambda_j - y}, \quad x^i \le x \le x^{i+1}, \quad r = i+1, \dots, n,$$
$$H_\ell(x,s) := \int_{x^{i+1}}^{x} (\frac{\lambda_j - x}{\lambda_j - y})^s (g_\ell(y,s) + h_\ell(y)) \frac{dy}{\lambda_j - y}, \quad x^i \le x \le x^{i+1}, \quad \ell = 1, \dots, i.$$

As a convention, $x^0 = -\infty$, $x^n = \infty$. Rewrite $H_j = F_j + G_j$, where F_j and G_j are defined in §4.

For $\sigma > \gamma := \max\{-\eta, \sigma_M\}$, we show that $H_j, 1 \le j \le n$ satisfy conditions in Lemma 5.2.

(i) From Lemmas 4.1 and 4.2, for every $1 \le i \le n$, the vectors

$$F_r(x^i+, s), \ F_\ell(x^i-, s), \ G_r(x^i+, s), \ G_\ell(x^i-, s)$$

are in $L^2(\omega)$ and their norms are bounded by

$$C(\gamma)((\int_{\omega=-\infty}^{\infty} \|g(\cdot, s)\|_w^2 d\omega)^{1/2} + \|h\|_w).$$

These properties pass to the vectors $H_r(x^i+, s)$ and $H_\ell(x^i-, s)$.

(ii) From Theorem 4.2 and 4.1 again, $F_j(x, s)$ and $G_j(x, s)$ are are in $L_w^2(x, \omega)$. Moreover, in Theorem 4.2 and 4.1, it is shown that these terms are continuous function of x into $L^2(\omega)$ for $x \in R^i$ with one-sided limits at x^i. These properties are passed to $H_j(x, s)$. The norms in $L_w^2(x, \omega)$ again are bounded by

$$C(\gamma)((\int_{\omega=-\infty}^{\infty} \|g(\cdot, s)\|_w^2 d\omega)^{1/2} + \|h\|_w).$$

The existence of $V(x, s)$ now follows from Lemma 5.2. The continuous dependence of $V(x, t)$ on the time $t \ge 0$ will be proved in §7.

(iii) Let $Q_j^i : \chi \to \mathbb{R}$ be the projection of χ to $v_j^i(z_j^i, s)$ where $z_j^i = x^i$ if $i + 1 \le j \le n$ and $z_j^i = x^{i+1}$ if $1 \le j \le i$. Then from (3.4), we have

(5.10)

$$\begin{aligned} v_j(x,t) &= \check{\Phi}_j^i(x, z_j^i, t) * \check{v}_j(z_j^i, t) + \int_{x^i}^{x^{i+1}} \check{\Phi}_j^i(x,y,t) * (\delta(0)h_j(y) + g_j(y,t)) \frac{dy}{|\lambda_j - y|} \\ &= \check{\Phi}_j^i(x, z_j^i, t) * \check{v}_j(z_j^i, t) + \int_{x^i}^{x^{i+1}} \check{\Phi}_j^i(x,y,t) h_j(y) \frac{dy}{|\lambda_j - y|} \\ &+ \int_0^t \int_{x^i}^{x^{i+1}} \check{\Phi}_j^i(x,y,t-\tau) g_j(y,\tau) \frac{dy}{|\lambda_j - y|} \end{aligned}$$

where $*$ is the convolution in time t.

The last two integrals in (5.10) are the local terms involving values of h and g in R^i only. The first term in (5.10) is a global term involving values of g and h in all $R^i, i = 0, \dots, n$.

Observe that

$$\check{v}_j^i(z_j^i, t) = Q_j^i \mathcal{L}^{-1}[(1 - MD^s)^{-1}] * (\tilde{M}\check{\mathcal{H}}(t)).$$

Further more,

$$\check{\mathcal{H}} := (\check{\mathcal{H}}_0^\tau, \ldots, \check{\mathcal{H}}_{n+1}^\tau)^\tau,$$

where

$$\check{\mathcal{H}}_i(t) = (\check{H}_1(x^{i-1}+, t), \ldots, \check{H}_{i-1}(x^{i-1}+, t), \check{H}_{i+1}(x^{i+1}-, t), \ldots, \check{H}_n(x^{i+1}-, t))^\tau.$$

From their definitions,

$$\check{H}_r(x^{i+1}, t) := \int_{x^i}^{x^{i+1}} \check{\Phi}_r^i(x^{i+1}, y, t) * (g_r(y, t) + \delta(0)h_r(y)) \frac{dy}{\lambda_r - y}, \quad r = i+1, \ldots, n,$$

$$\check{H}_\ell(x^{i-1}, t) := \int_{x^i}^{x^{i-1}} \check{\Phi}_\ell^{i-1}(x^{i-1}, y, t) * (g_\ell(y, t) + \delta(0)h_\ell(y)) \frac{dy}{\lambda_\ell - y}, \quad \ell = 1, \ldots, i-1.$$

It is now clear the first term in (5.10) can also be expressed by an integral equation with an L^1 kernel applied to h and g in all R^i (nonlocal). The solution is finally expressed by

$$T(t) : (h, g) \to V.$$

□

5.3. Relation with the characteristic method. Since the entries of D are positive and less than one, if $\sigma = \Re s$ is sufficiently large, say $\sigma > \sigma_0 >> 0$, then $\Xi(s) \neq 0$ and $|MD^s| < 1$. The inverse matrix can be expressed as powers of MD^s:

$$(I - MD^s)^{-1} = \sum_{k=0}^{\infty} (MD^s)^k.$$

In this case the inverse Laplace transform of χ can be expressed as

$$\check{\chi} = \mathcal{L}^{-1}(\tilde{M}\mathcal{H}) + \mathcal{L}^{-1}((MD^s)\tilde{M}\mathcal{H}) + \mathcal{L}^{-1}((MD^s)^2\tilde{M}\mathcal{H}) + \cdots + \mathcal{L}^{-1}((MD^s)^k\tilde{M}\mathcal{H}) + \cdots.$$

The term $\mathcal{L}^{-1}(\tilde{M}\mathcal{H})$ represents the scattered waves created by the the initial data h and the forcing term g adjacent to the shock hitting each shock and scattered from the shock. These waves evolve according to (3.4) and hit the shocks again that yield the scattering waves $\mathcal{L}^{-1}(MD^s\tilde{M}\mathcal{H})$. The term $\mathcal{L}^{-1}((D^sM)^k\tilde{M}\mathcal{H})$ represents waves $\mathcal{L}^{-1}\mathcal{H}$ scattered from the shocks k times.

Although written as an infinite series, for any given time $t^* > 0$, there exists a finite $k(t^*) \in \mathbb{N}$ such that the support of $\mathcal{L}^{-1}((MD^s)^k\tilde{M}\mathcal{H})$ is disjoint from $[0, t^*]$ for $k > k(t^*)$. There are only $k(t^*)$ terms involved in computing $\check{\chi}$ up to $0 \leq t \leq t^*$. The condition $|MD^s| < 1$ is not needed for the series to be convergent.

To prove this, consider the jth mode in R^i. For the right mode $r \geq i+1$, using the log change of variable and the inverser Laplace transform,

$$v_r(x^{i+1}-, s) = \left(\frac{\lambda_r - x^{i+1}}{\lambda_r - x^i}\right)^s v_r(x^i+, s)$$

becomes

$$v_r(x^{i+1}-, t) = v_r(x^i+, t - (X_r^i - X_r^{i+1})).$$

Similarly,

$$v_\ell(x^i+, t) = v_\ell(x^{i+1}-, t - (X_\ell^{i+1} - X_\ell^i)).$$

Using the formula $k(t^*)$ times, the conclusion follows from the fact $v_j(x, t) = 0$ for $t < 0$.

We have recovered the characteristic method for computing the initial value problem (2.2).

6. Eigenvalues and resonance values

Let

$$V = e^{\lambda t} V(x), \quad x \in R^i, i = 0, \dots, n,$$
$$Y^i(t) = e^{\lambda t} \bar{Y}^i, \quad i = 1, \dots, n,$$

be a solution for (2.2). Then λ is an eigenvalue with the associated eigenvector $(V(x),\ \{\bar{Y}^i\}_{i=1}^n)$ for the eigenvalue problem:

$$\begin{aligned} V_x + \lambda(Df - xI)^{-1} V &= 0, \\ [V]_{x^i} - \lambda \bar{Y}^i \Delta^i &= 0. \end{aligned} \tag{6.1}$$

The eigenvalue problem is posed on $L^2_w \times \mathbb{R}^n$ with $V \in D(\mathcal{A})$. The first equation comes from $\mathcal{A}V = \lambda V$. Let $V(x) = \sum v^i_j(x) \mathbf{r}_j(\bar{u}^i)$ in R^i.

Lemma 6.1. *Assume that $(V(x),\ \{\bar{Y}^i\}_{i=1}^n)$ is an eigenvector associated to an eigenvalue λ with $\Re\lambda > -\eta$ where η is from the definition of L^2_w. Then $V(x) \equiv 0$ for $x \in R^0 \cup R^n$.*

Proof. In R^0 and R^n, we have

$$v^0_j(x) = \left(\frac{\lambda_j - x}{\lambda_j - x^1} \right)^{\lambda} v^0_j(x^1), \quad -\infty < x < x^1,$$
$$v^n_j(x) = \left(\frac{\lambda_j - x}{\lambda_j - x^n} \right)^{\lambda} v^n_j(x^n), \quad x^n < x < \infty.$$

Since $\Re\lambda > -\eta$, in order to satisfy the conditions

$$\|u^0_j\| < \infty, \quad \|u^n_j\| < \infty,$$

we must have $v_j(x^1-) = v_j(x^n+) = 0$ for all $j = 1, \dots, n$. From this $V(x) = 0$ for $x \in R^0 \cup R^n$. □

Lemma 6.2. *$\lambda = 0$ is alway an eigenvalue. The corresponding eigenspace is the n-dimensional linear subspace of $L^2_w \times \mathbb{R}^n$: $V(x) \equiv 0$, $\{\bar{Y}^i\}_1^n \in \mathbb{R}^n$.*

Proof. Obviously $\lambda = 0$ is an eigenvalue with the said eigenvectors.

On the other hand, if $(V(x), \{\bar{Y}^i\}_1^n)$ is an eigenvector for the zero eigenvalue, then we can show $V = 0$ as follows. For any $\bar{Y}^i \in \mathbb{R}^n$, from (6.1), using $\lambda = 0$, we have $[V]_{x^i} = 0$. From Lemma 6.1, $V(x) = 0$ in R^0. If $V(x) = 0$ in R^i, then from $[V]_{x^{i+1}} = 0$, we have $V(x^{i+1}+) = 0$. Using (6.1), we have $V = 0$ in R^{i+1}. Therefore $V(x) = 0$ in every R^i by induction. □

Remark 6.1. In the U variable, $\lambda = -1$ is alway an eigenvalue which reflect the dynamics of the shock position $x^i(t)$ for the original system (2.1):

$$\dot{X}^i(t) + X^i(t) = 0.$$

The eigenvalue $\lambda = -1$ has a simple interpretation. In the viscous profile, shocks are traveling waves that have 0 as an eigenvalue corresponding to the shift of traveling wave positions (shock positions), say by $\Delta\xi^i$. In the similarity coordinate $x = \xi/\tau$, the shift of shock position decays algebraically like $\Delta x^i = \Delta\xi^i/\tau$. If we use $t = \ln\tau$ as time, the decay becomes exponentially in time with the rate $\lambda = -1$.

We next study the non-zero eigenvalues. We will also show that non-zero eigenvalues are the zeros of the determinant of the "SLEP" matrix as defined in [**14, 10**].

Let $Q(x, y, \lambda)$ be the principal matrix solution for the following system:

$$V_x + \lambda(Df - xI)^{-1}V = 0,$$
$$[V]_{x^i} = 0.$$

First $Q(x, y, \lambda)$ can be defined for the pair (x, y) in each R^i. Then it can be continued using the condition $[V]_{x^i} = 0$ across the boundaries $x = x^i, 1 \leq i \leq n$.

If λ_0 is a non-zero eigenvalue, then let $S^i := \lambda_0 \bar{Y}^i$, we can compute the eigenfunction $V(x)$ based on the jumps $\{S^i\}_{i=1}^n$. For example,

$$V(x^1+) = S^1\Delta^1, \quad V(x) = Q(x, x^1, \lambda_0)V(x^1),\ x \in R^1.$$

The condition $V(x^n+) = 0$ bust be satisfied, based on Lemma 6.1. This leads to the condition on the vector $\{S^i\}_1^n$:

$$\sum_{j=1}^{n} Q(x^n, x^j, \lambda_0)S^j\Delta^j = 0. \tag{6.2}$$

DEFINITION 6.1. The SLEP matrix $\mathcal{M}(\lambda_0)$ is an $n\times n$ matrix whose jth column is the vector $Q(x^n, x^j, \lambda_0)\Delta^j$. The SLEP function $p(\lambda) = \det \mathcal{M}(\lambda)$ [**14**].

THEOREM 6.3. *(1) $\lambda \neq 0$ is an eigenvalue in the region $\Re\lambda > -\eta$ iff $\det(I - MD^\lambda) = 0$. (2) The region $\Re\lambda > -\eta$ contains only normal points of the resolvent equation. (3) The condition $\det(I - MD^\lambda) = 0$ is equivalently to that the SLEP matrix defined in Definition 6.1 is singular.*

PROOF. (1) If $\det(I - MD^\lambda) = 0$, then system $(I - MD^\lambda)\chi = 0$ has a nontrivial solution χ. This means that in R^i, system (5.4), (5.5) has a nontrivial solution $v_r(x^i, \lambda), r = i + 1, \dots, n$ and $v_\ell(x^{i+1}, s), \ell = 1, \dots, i$ with $g_j \equiv 0$ and $h_j \equiv 0$. Then an eigenfunction $V(x)$ corresponding to λ can be constructed using (3.2) and (3.3).

On the other hand if $V(x)$ is an eigenfunction corresponding to a non-zero eigenvalue λ, then the system $(I - MD^\lambda)\chi = 0$ has a non-trivial solution χ. Therefore $\det(I - MD^\lambda) = 0$.

(2) If $\det(I-MD^\lambda) \neq 0$, then $(I-MD^\lambda)^{-1}$ exits for such λ. From the previous section, if $h \in L^2_w$, then we have a unique solution $V \in L^2_w$ for the resolvent equation

$$V_x + \lambda(Df - xI)^{-1}V = (Df - xI)^{-1}h, \quad [V]_{x^i} = 0 \bmod \Delta^i.$$

Certainly $\|V\| \leq C\|h\|$ for some constant C (uniform boundedness theorem in Banach spaces). This shows if λ is not an eigenvalue then it is a resolvent point in $\Re\lambda > -\eta$.

(3) If $\lambda \neq 0$ is an eigenvalue, then $V(x) \neq 0$. This can happen iff (6.2) has a non-trivial solution $\{S^i\}_1^n$. Thus, the SLEP matrix $\mathcal{M}(\lambda)$ must be singular. The converse of this is also true. Therefore

$$\det(I - MD^\lambda) = 0, \quad \Leftrightarrow \quad \det \mathcal{M}(\lambda) = 0.$$

□

REMARK 6.2. The left hand side of (6.1) defines a Fredholm operator on $L^2_w \times \mathbb{R}^n$ [**10**]. This has been used in constructing the eigenvalue/eigenfunctions for the Dafermos regularization of the conservation laws.

Let σ_m be the largest real parts of the zeros of $\Xi(s)$, i.e:

$$\sigma_m = \sup\{\sigma : \text{there exists } \omega \text{ such that } \Xi(\sigma + i\omega) = 0\}.$$

Then $\sigma_m \leq \sigma_M$ and the two can be different.

For any $\sigma_0 > -\eta$, the function $\alpha(\omega) := \Xi(\sigma_0 + i\omega)$ is quasi-periodic, with the frequencies defined by a finite linear combinations of $|X_j^{i+1} - X_j^i|,\ i = 1, \ldots, n-1,\ j = 1, \ldots, n$.

If the frequencies are rationally related, then $\alpha(\omega)$ is periodic. In this case $\inf_\omega |\Xi(\sigma_0 + i\omega)| = 0$ generally implies that there exists ω_0 such that $\Xi(\sigma_0 + i\omega_0) = 0$. There are countably many eigenvalues lying on the vertical line $\{\sigma = \sigma_0\}$ with equal vertical spacings. This has been verified in examples consisting of two shocks [**14**].

If the frequencies are not rationally related, then it is possible to find $\sigma_0 > \sigma_m$ such that $\inf_\omega |\Xi(\sigma_0 + i\omega)| = 0$.

Definition 6.2. If $s \in \mathbb{C}$ and $0 < |\Xi(s)| \leq \delta$ for a small number $\delta > 0$, then s is called a *resonance value* of order δ, or a δ resonance value. A vertical line $\Re s = \sigma_0$ in the complex plane that contains resonance values of arbitrarily small order is called a *resonance line* and σ_0 is called the *coordinate of the resonance line.*

Note if σ_0 is the coordinate of a resonance line, then

$$\inf_\omega |\Xi(\sigma_0 + i\omega)| = 0.$$

If a resonance line with coordinate greater than σ_m exists, then $\sigma_m < \sigma_M$. The existence of resonance lines has not been verified by model equations.

Example 6.1. (1) For system (2.2) of two equations with two Lax shocks, under general conditions, $\Xi(\sigma + i\omega)$ is periodic in ω. Non-zero eigenvalues are on a unique vertical line, and are equally spaced [**14**].

(2) For systems of three equations with three shocks, careful computation shows that

$$\Xi(s) = 1 - \sum_{j=1}^{8} a_j e^{b_j s}, \quad b_j < 0.$$

There may be resonance values on a vertical line if the frequencies $b_j, j = 1, \ldots, 8$, are not rationally related.

(3) By moving the third shock Λ^3 to infinity, a system of three equations with two Lax shocks can be treated as a special case of example (2). We find that

$$\Xi(s) = 1 - a_1 e^{b_1 s} - a_2 e^{b_2 s}.$$

As a simplified example, consider the function

$$\Xi(s) = 2 + e^{-\pi(\sigma + i\omega)} + e^{-\pi\alpha(\sigma + i\omega)}.$$

If the parameter $\alpha = p/q$ is rational, with $p,\ q$ being odd, then $\sigma = 0$, $\omega = kq$, with k being odd is an eigenvalue. The eigenvalues are equally spaced on the line $\sigma = 0$. If α is irrational, then $\sigma = 0$ is the coordinate of a resonance line. $|\Xi(0 + i\omega)|$ can be arbitrarily small but can never be zero.

7. Continuous dependence on time or space of the L^2 solutions

For the L^2 solution constructed through the L-transform, we have shown that $x \to e^{-\gamma t}V(x,t)$ is a continuous function in $L^2(t)$. In this section, we show that the $t \to V(\cdot,t)$ is a continuous in $L^2_w(x)$.

Since we do not assume that the solution is differentiable, we cannot prove the assertion by the characteristic method. However, using the inverse Laplace transform, we can show that the solution comes from a combination of several shift operators, Based on this, the continuous dependence on time can be proved.

In R^i, $i = 1,\dots,n-1$, consider the mode $V_r(x,t) = \mathcal{L}^{-1}V_r(x,s), r = i+1,\dots,n$. The other modes can be treated similarly. Using the log variables,

$$\begin{aligned}
&v_r(X,t) = \mathcal{L}^{-1}v_r(X,s)\\
=&\mathcal{L}^{-1}[e^{s(X-X^i)}v_r(X^i,s)] - \mathcal{L}^{-1}[\int_{X^i}^{X} e^{s(X-Y)}(h_r(Y)+g_r(Y,s))dY].
\end{aligned}$$

Given that $v_r(X^i+,s) \in L^2(\omega)$, $g_r(X,s) \in L^2_w(X,\omega)$ and $h_r \in L^2_w(X)$, it is not hard to show that each of the term of the inverser Laplace transform is continuous in $L^2(X)$. Indeed, the inverse of e^{sX} is a shift operator which is continuous in $L^2(X)$.

Using

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)H(t-a),$$

where $F(s) = \mathcal{L}f(t)$, we have

$$\begin{aligned}
w_1(X,t) := \mathcal{L}^{-1}[e^{-s(X^i-X)}v_r(X^i,s)] = v_r(X^i, t+X-X^i)H(t+X-X^i),\\
X^i - t \le X \le X^i.
\end{aligned}$$

Since w_1 is a shift of the L^2 function $v_r(X^i, X-X^i)$ to the left by t followed by the truncation at $X = X^i$, w_1 depends continuously in t in the L^2 norm.

If we recall that $h_r(Y) = 0$ for $Y > X^i$, we can rewrite

$$\begin{aligned}
-\int_{X^i}^{X} e^{s(X-Y)}h_r(Y)dY &= \int_0^{X^i-X} e^{-s\xi}h_r(X+\xi)d\xi\\
&= \mathcal{L}h_r(\cdot + X).
\end{aligned}$$

Therefore

$$w_2(X,t) := \mathcal{L}^{-1}(-\int_{X^i}^{X} e^{s(X-Y)}h_r(Y)dY) = h_r(t+X).$$

Thus, $w_2(X,t)$ is also a continuous function of t in $L^2(R^i)$.

$$\begin{aligned}
w_3(X,t) :=& \mathcal{L}^{-1}[-\int_{X^i}^{X} e^{s(X-Y)}g_r(Y,s)dY]\\
=& \int_0^{X^i-X} \mathcal{L}^{-1}[e^{-s\xi}g_r(X+\xi,s)]d\xi\\
=& \int_0^{X^i-X} g_r(X+\xi, t-\xi)d\xi.
\end{aligned}$$

If we replace the integrand $g_r(X+\xi, t-\xi)$ in the last low by a C_0^∞ approximation with a small error in L^2 norm, we can easily show that $\int_0^{X^i-X} g_r(X+\xi, t-\xi)d\xi$ is a continuous function of t in $L^2(X)$.

Some small change will be needed in the proof for V in unbounded regions R^0 and R^{n+1}. Details will be omitted.

Note at $t=0$, $w_1(X,0) = w_3(X,0) = 0$ and $w_2(X,0) = h(X)$. The results are summarized in the following

Lemma 7.1. *The solution $V(x,t)$ constructed as $\mathcal{L}^{-1}V(x,s)$ is a continuous function of t in the space L^2_w. Moreover, $V(x,0) = h(x)$.*

8. Differentiability of solutions for initial data in $D(\mathcal{A})$

Recall that the differential operator $\mathcal{A}$ is defined as

$$\mathcal{A}(V) = -(Df - xI)V_x, \quad \text{on each } R^i,$$

with

$$D(\mathcal{A}) := \{V : V,\ (Df - xI)V_x \in E = L^2_w,\ [V(x)]_{x^i} \in \operatorname{span}(\Delta^i),\ i = 1, \dots, n\}.$$

H 8.1. Assume that $h(x) = \sum h_j(x)\mathbf{r}_j(\bar u^i) \in D(\mathcal{A})$, and $g(x,t) = \sum g_j(x,t)\mathbf{r}_j(\bar u^i)$ satisfies that

$$e^{-\gamma t} g(x,t) \in L^2_w(x,t), \quad e^{-\gamma t}(Df - xI)g_x(x,t) \in L^2_w(x,t).$$

That is, for almost every t

$$g(\cdot,t),\ (Df - xI)g_x(\cdot,t) \in E = L^2_w.$$

with

$$\int_{t=0}^{\infty} e^{-2\gamma t}(\|g(\cdot,t)\|^2 + \|(Df - xI)g_x(\cdot,t)\|^2)dt < \infty.$$

Remark 8.1. (1) Note that the condition $(Df - xI)V_x \in L^2_w$ is not equivalent to that $V_x \in L^2_w$ since the variable x is unbounded.

(2) We do not assume that $g(x,t)$ satisfies jump conditions at the jumps $x = x^i, i = 1, \dots, n$. In general, $g(\cdot,t) \notin D(\mathcal{A})$.

Theorem 8.1. *If h and g satisfy Hypothesis 8.1, then the L^2 solution constructed in §5 is differentiable. That is, for almost every t, $V(\cdot,t) \in D(\mathcal{A})$ and $V_t(\cdot,t) \in L^2_w$ with*

$$\int_{t=0}^{\infty} e^{-2\gamma t}(\|V(\cdot,t)\|^2 + \|V_t(\cdot,t)\|^2 + \|\mathcal{A}V(\cdot,t)\|^2)dt$$
$$\le C(\|h\|^2 + \int_{t=0}^{\infty} e^{-2\gamma t}(\|g(\cdot,t)\|^2 + \|(Df - xI)g_x(\cdot,t)\|^2)dt).$$

Moreover, in each R^i the L^2 functions $e^{-\gamma t}V(x,t)$, $e^{-\gamma t}V_t(x,t)$ and $e^{-\gamma t}\mathcal{A}V$ depend continuously on x in $L^2(t)$.

PROOF. Let F_j, G_j be defined as in §4. For $i+1 \le r \le n$,

$$\begin{aligned} sF_r(x,s) &= \int_{x^i}^x s\frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}} h_r(y)dy \\ &= \int_{x^i}^x (\lambda_r - x)^s \partial_y((\lambda_r - y)^{-s})h_r(y)dy \\ &= h_r(x) - \frac{(\lambda_r - x)^s}{(\lambda_r - x^i)^s} h_r(x^i) - \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^s}\partial_y h_r(y)dy \\ &= h_r(x) - \frac{(\lambda_r - x)^s}{(\lambda_r - x^i)^s} h_r(x^i) - \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}}(\lambda_r - y)\partial_y h_r(y)dy \end{aligned}$$

¿From (3.2), we have

$$sv_r(x,s) = \left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s sv_r(x^i, s) + sG_r(x,s) + sF_r(x,s).$$

Let $z_j(x,s) = sv_j(x,s) - h_j(x)$, then

$$\begin{aligned} z_r(x,s) = &\left(\frac{\lambda_r - x}{\lambda_r - x^i}\right)^s z_r(x^i, s) \\ &+ \int_{x^i}^x \frac{(\lambda_r - x)^s}{(\lambda_r - y)^{s+1}}(\mathcal{A}h)_r(y)dy + sG_r(x,s). \end{aligned} \tag{8.1}$$

Here $(\mathcal{A}h)_j$ is the jth component of vector valued function $\mathcal{A}h$.

Similarly we can show that for $1 \le \ell \le i$,

$$\begin{aligned} z_\ell(x,s) = &\left(\frac{\lambda_\ell - x}{\lambda_\ell - x^{i+1}}\right)^s z_\ell(x^{i+1}, s) \\ &+ \int_{x^{i+1}}^x \frac{(\lambda_\ell - x)^s}{(\lambda_\ell - y)^{s+1}}(\mathcal{A}h)_\ell(y)dy + sG_\ell(x,s). \end{aligned} \tag{8.2}$$

¿From H 8.1, we can apply lemmas 4.1 and 4.2 to $\int_{x^{i+1}}^x \frac{(\lambda_\ell - x)^s}{(\lambda_\ell - y)^{s+1}}(\mathcal{A}h)_\ell(y)dy$ and $sG_\ell(x,s)$ respectively, The second row of (8.1) or (8.2) defines a function $H_j(x,s) \in L^2_w(x,\omega)$ that depends continuously on x in $L^2(\omega)$.

Let $Z(x,s) = \sum_1^n z_j(x,s)\mathbf{r}_j(\bar{u}^i)$, then the jump conditions on V and h imply that

$$[Z(x,s)]_{x^i} = 0 \bmod \Delta^i. \tag{8.3}$$

Applying Lemma 5.2 to the systems (8.1), (8.2) and (8.3), we find the function $Z(x,s) = sV(x,s) - h(x)$ is in $L^2_w(x,\omega)$ and depends continuously on x in $L^2(\omega)$. Note that as shown in Lemma 7.1, $V(x,0) = h(x)$.

Observe that

$$sV(x,s) - h(x) = s\mathcal{L}(V(x,t) - H(t)h(x)).$$

In the Hilbert space L^2_w, the inverse Laplace transform of the right hand side is $\partial_t V(\cdot,t) - \delta(0)h(\cdot)$. From the Plancherel's theorem, $e^{-\gamma t}(\partial_t V(\cdot,t) - \delta(0)h(\cdot))$ is an $L^2(t)$ function in L^2_w.

We now consider the spatial regularity. From (3.2) and (3.3), one easily obtain that

$$(\lambda_j - x)\partial_x v_j(x,s) = -sv_j(x,s) + g_j(x,s) + h_j(x), \quad 1 \le j \le n.$$

Therefore,

$$(Df - xI)V_x(x,s) = g(x,s) - (sV(x,s) - h(x)).$$

Both $g(x,s)$ and $sV(x,s)-h(x)$ are in $L^2_w(x,\omega)$ if $\sigma>\gamma$. Therefore $\mathcal{A}V(x,s)\in L^2_w(x,\omega)$. By inspecting terms in the right hand side we conclude that $(Df-xI)V_x(x,s)$ depends continuously on x in $L^2(\omega)$. Using the inverse Laplace transform, we find that $e^{-\gamma t}\mathcal{A}V(x,t)$ is in $L^2_w(x,t)$ and depends continuously on x in $L^2(t)$.

□

Using Sobolev's embedding theorem, we have

COROLLARY 8.2. *Under the conditions of Theorem 8.1, in the space L^2_w, the solution $V(x,t)$ of* (2.2) *is $O(e^{\gamma t})$ in sup norm.*

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Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695
E-mail address: xblin@math.ncsu.edu