Construction and Stability of Internal Layer Solutions for Singular Perturbed Problems

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Abstract

$\sim$ Singularly perturbed equations naturally occur in many areas of engineering, physical and biological sciences.
$\sim$ Due to different time scales involved in the equations, the system can have internal layer solutions that exhibit fast and slow behaviors.
$\sim$ Fundamental problems are the construction, asymptotic expansion and stability of the layered solutions.

$\sim$ Lin: Construction and asymptotic stability of structurally stable internal layer solutions, Trans. AMS 2001
$\sim$ Hale & Lin: Multiple internal layer solutions generated by spatially oscillatory perturbations, JDE 1999

Introduction Systems of parabolic equations in fast-slow form,

$$u_t = \epsilon^2 u_{xx} + f(u,v), \quad u \in \mathbb{R}^m,\ v \in \mathbb{R}^n,$$
$$v_t = v_{xx} + g(u,v), \quad 0 < x < 1, \tag{1}$$
Boundary conditions at $x = 0,\ 1$.

Example 1 The x-dependent scalar equation

$$u_t = \epsilon^2 u_{xx} + (1-u^2)(u - a(x)), \quad 0 < x < 1,$$
$$u_x = 0, \quad x = 0, 1,$$

$a(x) \in C^\infty[0,1]$ with $a(x^i) = 0$, $a'(x^i) \neq 0$ at points $0 < x^1 < x^2 < \cdots < x^r < 1$.

P. Fife 1974, Hale & Sakamoto 1988. Angenent, Mallet-Paret & Peletier 1987.

Look for stationary solution, $u_t = 0$. Near $x = x^i$, $a(x) \approx 0$, the solution has internal layers connecting $u = 0$ and $u = 1$. Let $x = x^i + \epsilon\xi$, $u' = du/d\xi$:

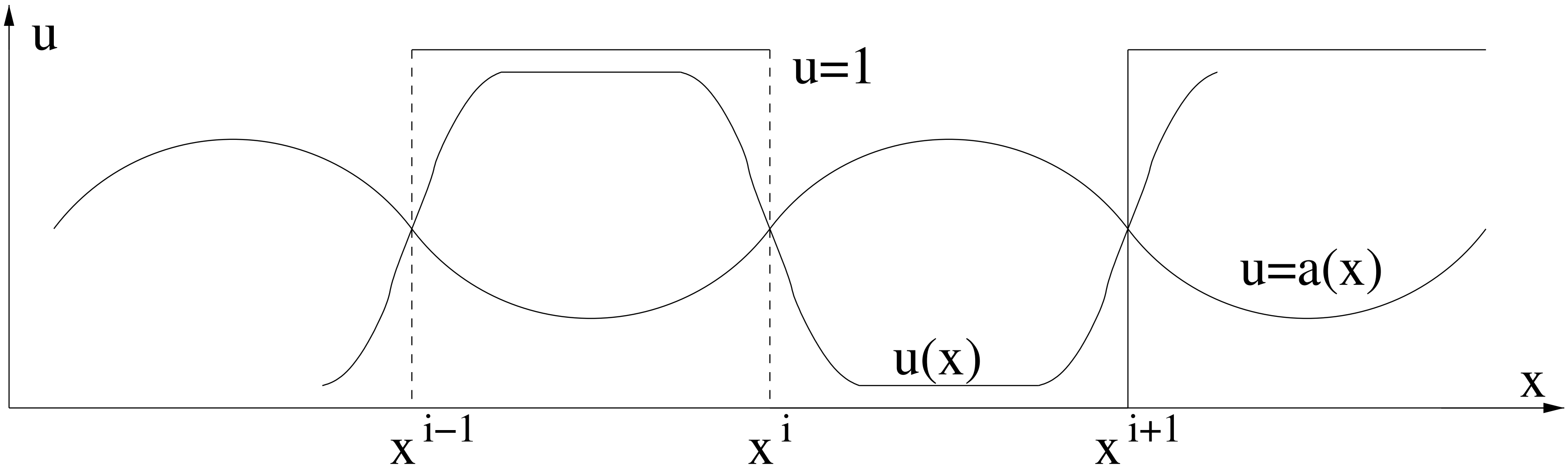
$$0 = u'' + (1-u^2)u, \quad u' = \epsilon u_x.$$

Angenent, Mallet-Paret & Peletier:

Total number of internal layer solutions: 2^r

Number of stable internal layer solutions:

The rth Fibonacci number.



By letting $v = x$ which satisfies

$$v_t = v_{xx}, \quad v(0) = 0,\ v(1) = 1.$$

The equation can be converted into (1) with $g(u, v) = 0$.

Example 2 The activator-inhibitor model.

$$
\begin{aligned}
u_t &= \epsilon^2 u_{xx} + u - u^3 - v, \\
v_t &= v_{xx} + a_0 u - a_1 v.
\end{aligned} \tag{2}
$$

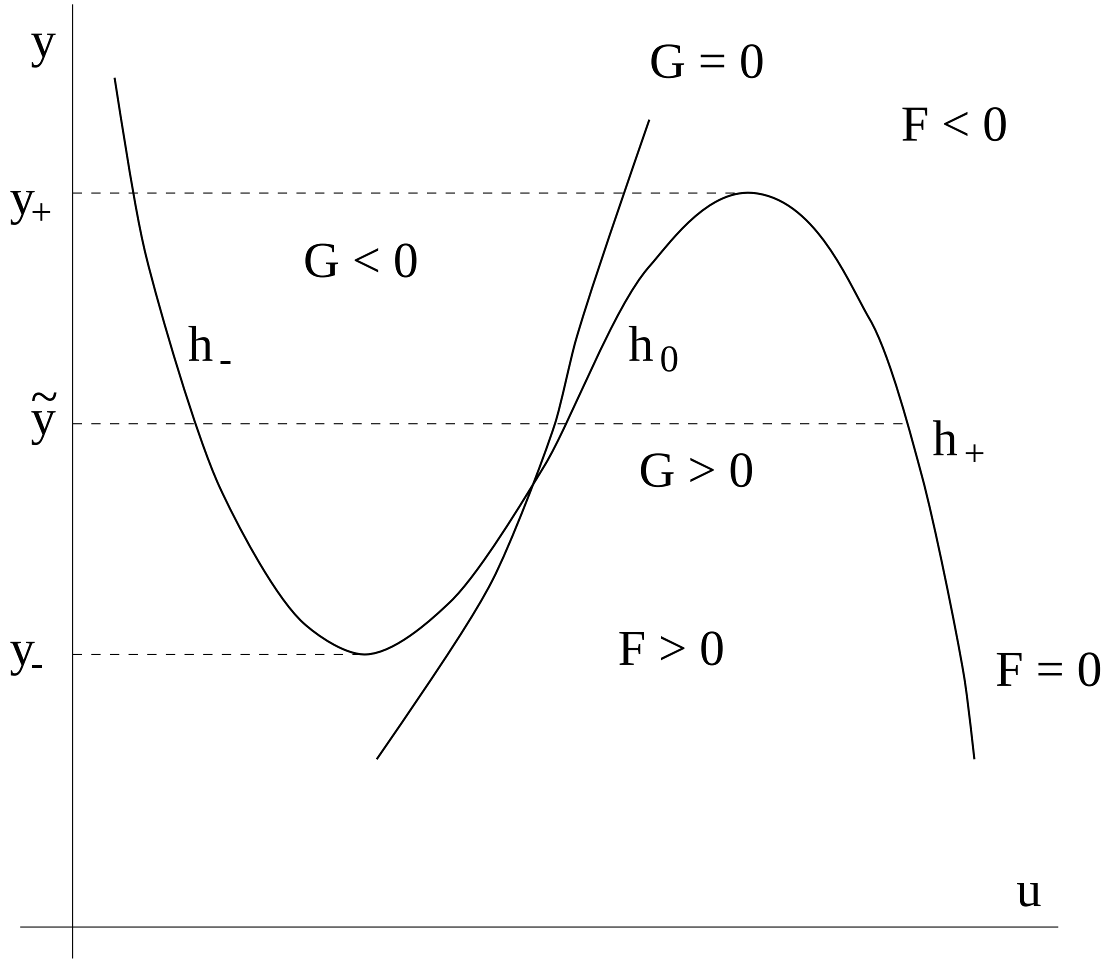
The stationary internal layer solutions satisfy

$$
\begin{aligned}
0 &= \epsilon^2 u_{xx} + f(u,v), \\
0 &= v_{xx} + g(u,v), \quad 0 < x < 1.
\end{aligned} \tag{3}
$$

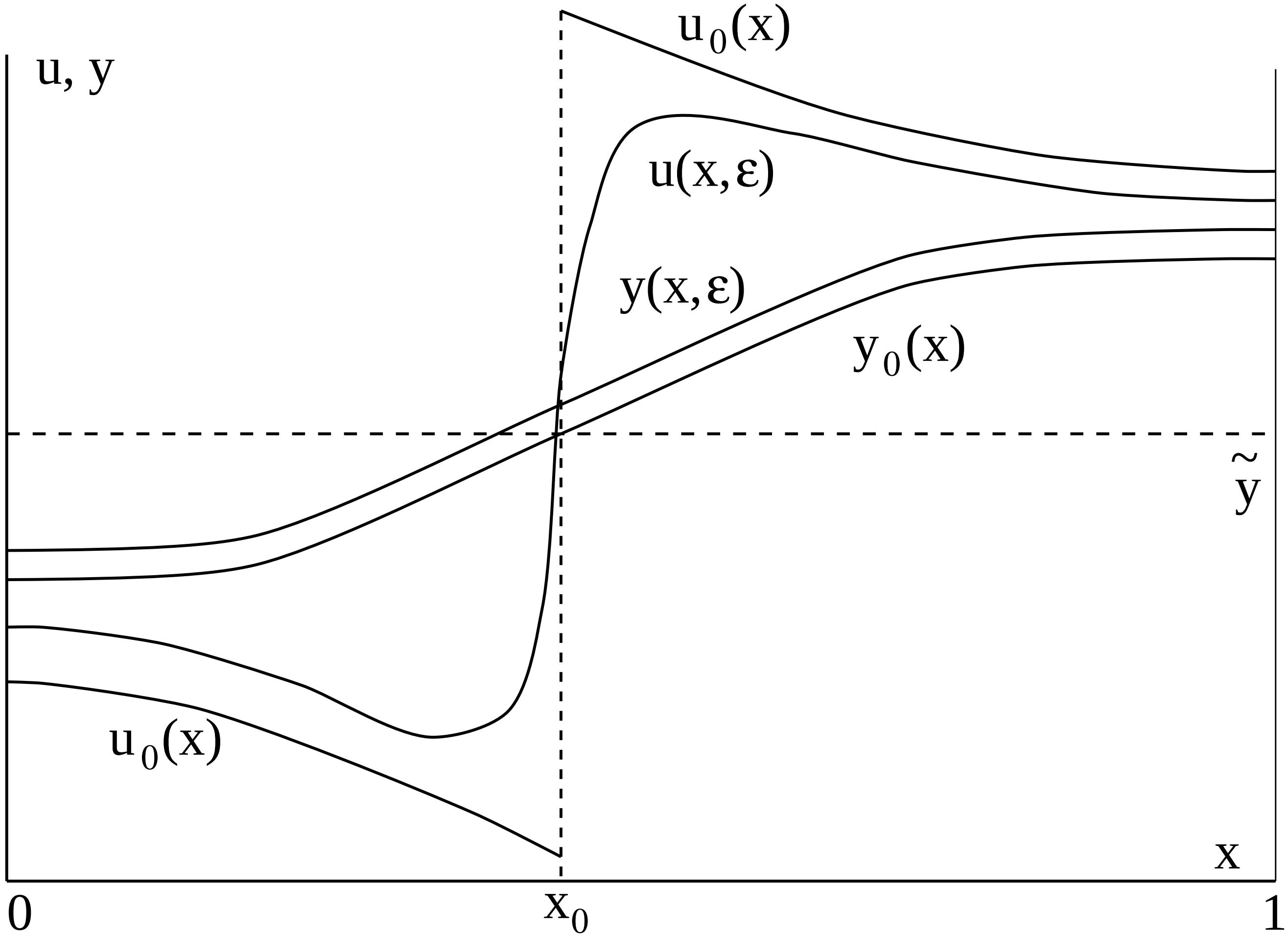
A1. $f(u,v) = 0$ has three solutions $u = h_\pm(v)$, $u = h_0(v)$ at $I_\pm$, I_0.
A2. $J(v) = \int_{h_-(y)}^{h_+(y)} f(s,y)ds$ has an isolated zero $\tilde{y}$:

$$J(\tilde{y}) = 0, \quad dJ(\tilde{y})/dy < 0.$$

A3 $f_u < 0$ on I_- and I_+.
A4 $g < 0$ on I_- and $g > 0$ on I_+. $\frac{d}{dy} g(h_\pm(y), y) < 0$.
A5 $g_y \leq 0$ on $I_\pm$.
There exist one mono layer solution and many Multi layer solutions. The stability of both mono and multi-layered solutions was first proved by Nishiura and Fujii 1987.



The nullclines of F and G



Mono layer solutions and their singular limits

Describe the internal layer solutions for the two models:

$\sim$ $f(u, v) = 0$ has three branches of solutions $u = h_{\pm}(v),\ h_0(v)$. The branches $u = h_{\pm}(v)$ is stable for the reaction equation,

$$u_t = f(u, v), \text{ where } v \text{ is a parameter.}$$

The branch $u = h_0(v)$ is unstable for the reaction equation.

$\sim$ An internal layer solution $(u(x), v(x))$ stays near the two slow manifolds $u = h_{\pm}(v)$ for most of the points x. These x form the regular layers.

$\sim$ Near a finite sequence $\{x^i\}_1^r$, where the solution jumps between the two slow manifolds. These points form the internal layers. They are also the places where $\epsilon^2 u_{xx}$ is no longer negligible.

A regular layer is an interval where the solutions $u(x, \epsilon)$ converge uniformly to a limit as $\epsilon \to 0$.

A singular layer is an interval where the solutions do not converge uniformly to a limit as $\epsilon \to 0$.

In the singular layer near x^i, the stretched variable $\xi = (x - x^i)/\epsilon$ is introduced. With the new variable $u(\xi, \epsilon)$ converges uniformly to a limit $u(\xi, 0)$.
If $\epsilon = 0$, the u-equation becomes

$$u_{\xi\xi} + f(u, \bar{v}) = 0. \tag{4}$$

The condition on the parameter $\bar{v} = v(x^i)$ is that (4) must have a heteroclinic solution connecting the two slow manifolds $u = h_\pm(\bar{v})$.

In the regular layers, when $\epsilon = 0$, v satisfies

$$v_{xx} + g(h(v), v) = 0, \quad h(v) = h_\pm(v), \tag{5}$$

with boundary conditions at $x = 0, 1$.
Let $x^0 = 0$ and $x^{r+1} = 1$. Then $v \in C^1[0, 1]$ and $v \in C^2(x^i, x^{i+1})$.
v_{xx} has a jump at x^i because $h(v)$ switches between $h_-(v)$ and $h_+(v)$.

Bifurcation of Mono-layer solutions

$\sim$ In the activator-inhibitor model, there is only one mono-layer solution which jumps from near $u = h_-(v)$ to near $u = h_+(v)$.

$\sim$ In the x-dependent model, there can be r mono-layer solutions.

$\sim$ Internal layer solutions in both models are structurally stable – solutions persist under small perturbations of f and g.

$\sim$ Bifurcation of structurally unstable internal layer solutions, Hale & Lin, 1999:

$$\begin{aligned} 0 &= \epsilon^2 u_{xx} + f(u, \alpha y + \beta a(x)), \\ 0 &= y_{xx} + g(u, y). \\ g(u, y) &= a_0 u - a_1 y \\ f(u, y) &= (1 - u^2)(u - y) \text{ or } u - u^3 - y. \end{aligned} \tag{6}$$

By setting $v = (y, x)$, it has two slow variables.

α and β are parameters, $0 \le \alpha \le 1$, $\beta = 1 - \alpha$.

When $\alpha = 1$, unique mono-layered solution.

When $\alpha = 0$, can have multiple mono-layered solution.

The number of solutions changes through saddle-node bifurcations.

Notations and basic lemmas

$$u^{Ri}(x,\epsilon) = u(x,\epsilon), \text{ for } x \in R^i$$
$$u^{Si}(\xi,\epsilon) = u(\epsilon\xi + x_0^i,\epsilon), \text{ for } x \in S^i,$$
$$u^{Ri}(x,\epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{Ri}(x),$$
$$u^{Si}(\xi,\epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{Si}(x)$$

Let $C_{bu}^m(I) = \{u | u, u', \dots u^{(m)} \in C_{bu}(I)\}$ with the norm

$$\|u\|_{C_{bu}^m} = \sum_{i=0}^{m} \|u^{(i)}\|_{C_{bu}}.$$

We consider functions satisfying $|u(\xi)| \le C(1+|\xi|^j)e^{-\gamma\xi}$.
We use the weight function

$$w(\xi) = (1+|\xi|^j)e^{-\gamma\xi}, \quad \gamma \ge 0,\ j \ge 0. \tag{7}$$

Define the Banach spaces of functions with the weight $w(\xi)$:

$$E_{\mathbb{R}}(w) = \{u : \mathbb{R} \to \mathbb{R}^n | u(\cdot)/w(\cdot) \in C_{bu}(\mathbb{R}, \mathbb{R}^n)\}.$$
$$\|u\|_{E(w)} = \sup\{|u(\xi)/w(\xi)|, \xi \in \mathbb{R}\}.$$
$$E^m_{\mathbb{R}}(w) = \{u | u, \dots, u^{(m)} \in E(w)\}.$$
$$\|u\|_{E^m(w)} = \textstyle\sum_{j=0}^m \|u^{(j)}\|_{E(w)}.$$

Similarly, $E^m_{\mathbb{R}^+}(w)$ and $E^m_{\mathbb{R}^-}(w)$ are defined on $\mathbb{R}^+$ and $\mathbb{R}^-$.

Equation (8) is equivalent to system (9).

$$u_{\xi\xi} + f(u) = 0, \quad u \in \mathbb{R}^n. \tag{8}$$

$$u_\xi = v, \quad v_\xi = -f(u), \quad u, v \in \mathbb{R}^n. \tag{9}$$

The phase space for (8) is $(u, u_\xi) \in \mathbb{R}^{2n}$.
We say
(1) p is a hyperbolic equilibrium for (8) if $(p, 0)$ is a hyperbolic equilibrium for (9);
(2) $u(\xi)$ is a heteroclinic solution of (8) if $(u(\xi), u_\xi(\xi))$ is a heteroclinic solution for (9).
We say equation

$$u_{\xi\xi} + A(\xi)u = 0 \tag{10}$$

has an exponential dichotomy on an interval $I \subset \mathbb{R}$ if the system

$$u_\xi = v \quad v_\xi = -A(\xi)u \tag{11}$$

has an exponential dichotomy on I. Here $A(\cdot) : I \to \mathbb{R}^{n\times n}$ is a continuous matrix valued function. The stable and unstable subspaces and the projections associated to such spaces of (10) are the ones associated to that of (11).

Lemma 1 *If*

$$f(p) = 0, \;\; Re\{\sigma Df(p)\} < -\sigma_0 < 0, \tag{12}$$

Then

$$u_{\xi\xi} + Df(p)u = 0 \tag{13}$$

has an exponential dichotomy on $\mathbb{R}$*,*
has n*–dimensional stable and unstable spaces.*
Then the exponential decay rate of solutions on the stable (or unstable) subspace is $\sqrt{\sigma}$*.*

Example: $u'' - k^2 u = 0, \; u_1 = e^{kt}, \; u_2 = e^{-kt}$.

Let W^s and W^u denote the stable and unstable subspaces of

$$u_{\xi\xi} + Df(p)u = 0.$$

W^s and W^u are transversal to the Neumann boundary condition:

$$\begin{aligned} (u,v) \in W^s \cap \{v = 0\} \quad &\Rightarrow u = 0, \\ (u,v) \in W^u \cap \{v = 0\} \quad &\Rightarrow u = 0. \end{aligned}$$

Let $q(\xi)$ be a heteroclinic solution connecting saddle to saddle. Then

$$u_{\xi\xi} + Df(q(\xi))u = 0 \tag{14}$$

has exponential dichotomies on $\mathbb{R}^-$ or $\mathbb{R}^+$ respectively with n-dimensional stable and unstable subspaces.

$$(\dot{q}(0), \ddot{q}(0)) \in \mathcal{R}P_u(0^-) \cap \mathcal{R}P_s(0^+).$$

Lemma 2 *(Linear system in $\mathbb{R}^{\pm}$)*
Assume that
(1) $q(\xi)$ approaches a hyperbolic saddle as $\xi \to \pm\infty$.
(2) $u = 0$ is the only solution to the B.V.P:

$$u_{\xi\xi} + Df(q)u = 0, \quad u_\xi(0) = 0.$$

(3) Let X be $E^m_{\mathbb{R}+}(w)$ or $E^m_{\mathbb{R}-}(w)$, and $g \in X$.
Then there exists a unique solution $u \in E^{m+2}_{\mathbb{R}+}(w)$ or $E^{m+2}_{\mathbb{R}^-}(w)$ to the Neumann boundary value problem:

$$\begin{aligned} &u_{\xi\xi} + Df(q)u = g, \quad \xi \geq 0 \\ &u_\xi(0) = \phi, \end{aligned}$$

Moreover,

$$\|u\|_{E^{m+2}(w)} \leq C(\|g\|_{E^m(w)} + \|\phi\|_{\mathbb{R}^n}).$$

Let $q(\xi)$ be a heteroclinic solution to (8) connecting two saddles p^1 and p^2. Let $X = E^m_{\mathbb{R}}(w)$.
Define $L_q : X \to X$ with $D(L_q) = E^{m+2}_{\mathbb{R}}(w)$ by

$$L_q u = u_{\xi\xi} + Df(q(\xi))u. \tag{15}$$

Lemma 3 *(Linear systems in $\mathbb{R}$)*
L_q is a Fredholm operator with Fredholm index zero. Assume that $\dim Ker(L_q) = 1$*, then $Ker(L_q) = span\{\dot{q}\}$ and $Range(L_q) = \{\psi\}^{\perp}$. Here ψ is the unique nontrivial bounded solution for the adjoint equation, up to a scalar multiple,*

$$L_q^* \psi \stackrel{def}{=} \psi_{\xi\xi} + Df^{\tau}(q(\xi))\psi = 0.$$

$$\{\psi\}^{\perp} \stackrel{def}{=} \{u \in X | \int_{-\infty}^{\infty} \psi^{\tau}(\xi)u(\xi)d\xi = 0\}.$$

$$u_{\xi\xi} + Df(q(\xi))u = g, \quad g \in E^m_{\mathbb{R}}(w)$$

has a solution $|u| \leq C((1 + |\xi|^j)e^{-\gamma\xi})$ if and only if

$$\int_{-\infty}^{\infty} \psi^{\tau}(\xi)g(\xi)d\xi = 0.$$

Lemma 4 *(Fredholm property in finite intervals)*
Assume the same conditions of Lemma 3. Let $\Psi = (-\dot{\psi}, \psi)^\tau$ *where* ψ *as in Lemma 3. Let* $\xi_1 < 0 < \xi_2$, $g \in [\xi_1, \xi_2]$, *and* $\phi_s \in \mathcal{R}P_s(\xi_1)$, $\phi_u \in \mathcal{R}P_u(\xi_2)$ *be two given vectors. Consider*

$$u_{\xi\xi} + Df(q(\xi))u = g, \quad P_s(\xi_1)\begin{pmatrix} u \\ v \end{pmatrix} = \phi_s, \quad P_u(\xi_2)\begin{pmatrix} u \\ v \end{pmatrix} = \phi_u.$$

The boundary value problem has a solution in $[\xi_1, \xi_2]$ *if and only if*

$$\Psi^\tau(\xi_1)\phi_s(\xi_1) - \Psi^\tau(\xi_2)\phi_2(\xi_2) + \int_{\xi_1}^{\xi_2} \psi^\tau(\xi)g(\xi)d\xi = 0.$$

If also $< \dot{q}, u > + < \ddot{q}, v >= 0$, *then the solution is unique and satisfies*

$$|u| \le C(|\phi_s| + |\phi_u| + |g|),$$

where C *does not depend on* ξ_1 *or* ξ_2.

Constructing multi-layered solutions

$$
\begin{aligned}
&u_t = \epsilon^2 u_{xx} + f(u,v), \quad 0 < x < 1 \\
&v_t = v_{xx} + g(u,v), \quad u \in \mathbb{R}^m,\ v \in \mathbb{R}^n \\
&u_x(0) = u_x(1) = 0, \\
&A_j v_x(j) + B_j v(j) = \beta_j, \quad j = 0,\ 1.
\end{aligned}
\tag{16}
$$

The Robin type boundary conditions on v: A_j and B_j are $n \times n$ diagonal matrices satisfying $A_0 B_0 \leq 0$, $A_1 B_1 \geq 0$ and $A_j^2 + B_j^2 = I$.

Stationary internal layer layer solutions:

$$
\begin{aligned}
&0 = \epsilon^2 u_{xx} + f(u,v), \quad 0 < x < 1, \\
&u_x(0) = u_x(1) = 0, \\
&0 = v_{xx} + g(u,v), \\
&A_j v_x(j) + B_j v(j) = \beta_j, \quad j = 0,\ 1.
\end{aligned}
\tag{17}
$$

Preview how the solution looks like:

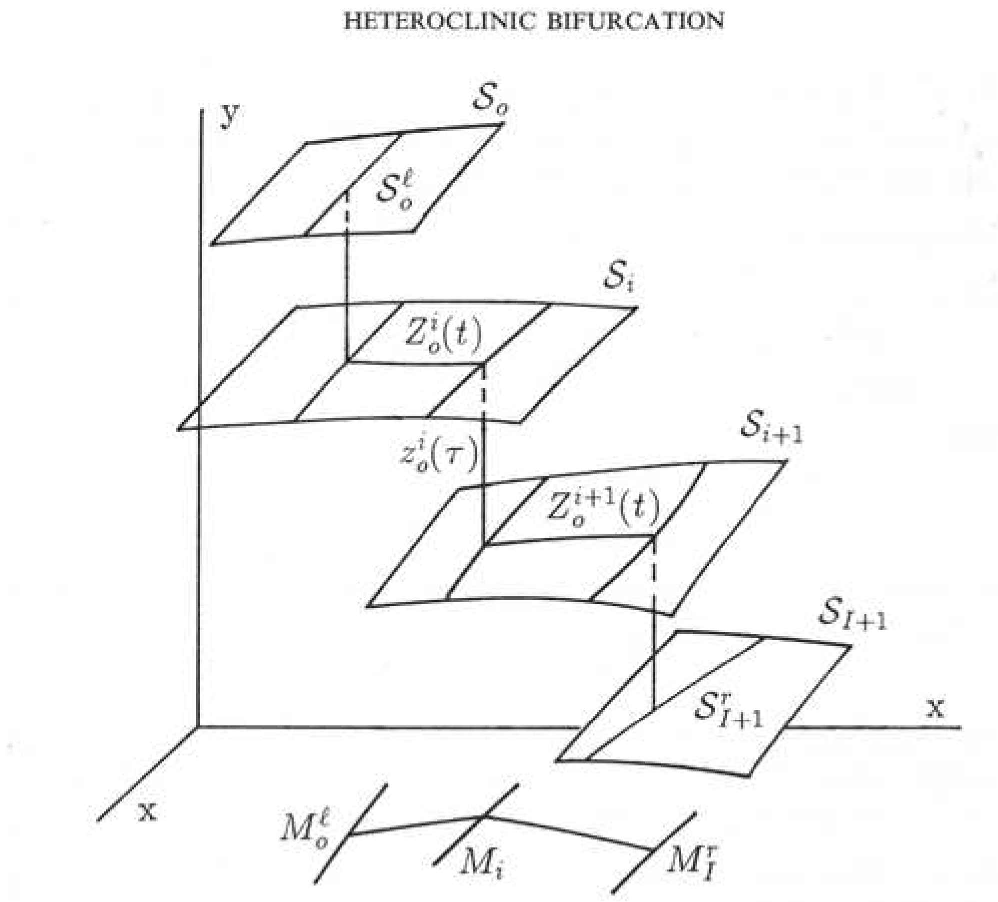


FIGURE 1

Flaherty & O'Malley, loss of boundary conditions in singular perturbation problems

Assumptions and existence of solutions

In regular layers, the $\epsilon^2 u_{xx}$ term in (17) drops. The u-equation simplifies to $f(u,v) = 0$.

(H1) (The hyperbolic slow manifolds)
$f(u,v) = 0$ has several solution manifolds $u = h^i(v)$, $0 \le i \le r$, on which we have

$$\mathrm{Re}\{\sigma f_u(h^i(v), v)\} < 0, \quad 0 \le i \le r.$$

Piecewise smooth solutions for the boundary value problem for v:

$$\begin{aligned} v_{xx} + g(h(v), v) &= 0, \quad 0 < x < 1, \\ A_j v_x(j) + B_j v(j) &= \beta_j, \quad j = 0,\ 1, \end{aligned} \tag{18}$$

where $h(v) = h^i(v)$, $0 \le i \le r$.
The flow on the $r+1$ slow manifolds, each is related to one h^i:

$$\begin{aligned} v_x &= w, \\ w_x &= -g(h^i(v), v), \quad u \in \mathbb{R}^m, v \in \mathbb{R}^n. \end{aligned} \tag{19}$$

$\sim$ There is a sequence $\{x_0^i\}_{i=1}^r$ where switching from the $(i-1)$th slow manifold to the ith can happen.

Using $\xi = (x - x_0^i)/\epsilon$, rewrite

$$\epsilon^2 u_{xx} + f(u,v) = 0,$$
$$u_\xi = \widehat{u},\ \widehat{u}_\xi = -f(u,v).$$

From (H1), the system has hyperbolic equilibria $u = h^i(v)$. Assume that there is a smooth $(n-1)$-dimensional surface Σ^i in $\mathbb{R}^n$ such that

$$W^s(h^{i-1}(v) \cap W^u(h^i(v)) \neq \emptyset$$

iff $v \in \Sigma^i$ and the connection breaks transversely if v moves away from Σ^i.

(H2) (The codimension one take-off surfaces):

(i) For any $\overline{v}^i \in \Sigma^i$, the following equation

$$u_{\xi\xi} + f(u, \overline{v}^i) = 0,$$

has a heteroclinic solution $q^i(\xi)$, $1 \leq i \leq r$ connecting $h^{i-1}(\overline{v}^i)$ to $h^i(\overline{v}^i)$.

(ii) There exists $\gamma_0 > 0$ such that, in the region $\mathrm{Re}\lambda > -\gamma_0$, the only eigenvalue of the linear operator on U

$$U_{\xi\xi} + f_u(q^i(\xi), \overline{v}^i)U, \quad U \in L^2(\mathbb{R})$$

is the simple eigenvalue $\lambda = 0$, corresponding to the eigen function $\dot{q}(\xi)$.

(H2) implies that the eigenspace is spanned by $\dot{q}^i(\xi)$, and there is a unique bounded solution ψ^i, $1 \le i \le r$ to the adjoin equation,

$$\begin{aligned} &\psi_{\xi\xi} + f_u^\tau(q^i(\xi), \bar{v}^i)\psi = 0, \\ &< \psi^i, \dot{q}^i >= 1. \end{aligned}$$

The function ψ^i can be used to measure the gap between the unstable fibers of $u = h^{i-1}(v)$ and the stable fibers of $u = h^i(v)$:

(H3) (Melnikov's method) The following vector

$$\mathbf{n}^i = \int_{-\infty}^{\infty} f_v^\tau(q^i(\xi), \bar{v}^i)\psi^i(\xi)d\xi \neq 0. \tag{20}$$

Using Melnikov's method, if $G^i(v)$ is the gap function between the unstable manifold at $0-$ and the stable manifold at $0+$, then

$$\mathbf{n}^i = \nabla G^i(\xi).$$

Constructing the solutions geometrically

$$
\begin{aligned}
v_{xx} + g(h(v), v) = 0, \quad 0 < x < 1, \\
A_j v_x(j) + B_j v(j) = \beta_j, \quad j = 0,\, 1,
\end{aligned}
$$

(v_0^R, w_0^R, x) satisfies a first order system:

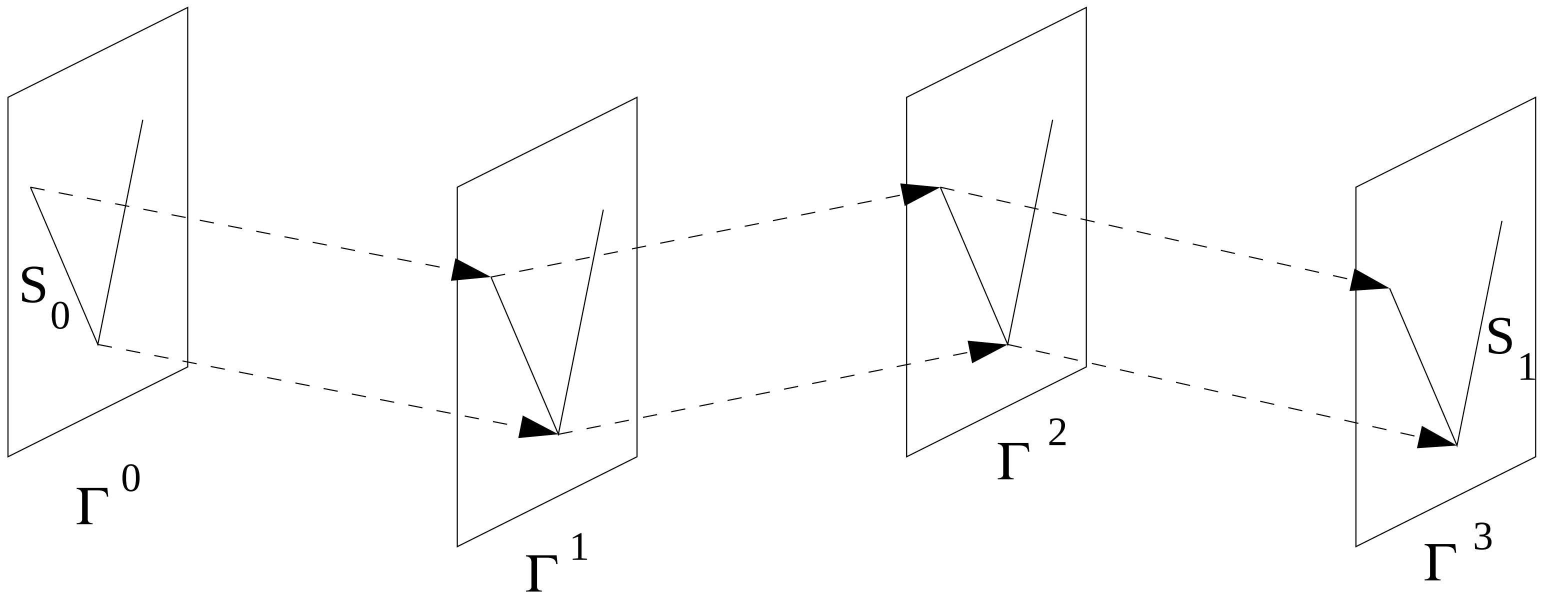
$$
\begin{aligned}
\frac{dv}{dt} &= w, \\
\frac{dw}{dt} &= -g(h^i(v), v), \quad 0 \le i \le r, \\
\frac{dx}{dt} &= 1.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\Gamma^i &= \{(v, w, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, v \in \Sigma^i\}, \quad 1 \le i \le r, \\
\Gamma^0 &= \{(v, w, x) : x = 0\}, \\
\Gamma^{r+1} &= \{(v, w, x) : x = 1\}, \\
\mathcal{S}_0 &= \{(v, w, x) : A_0 w + B_0 v = \beta_0,\, x = 0\}, \\
\mathcal{S}_1 &= \{(v, w, x) : A_1 w + B_1 v = \beta_1,\, x = 1\}.
\end{aligned}
$$

A solution (v, w, x) must start at $\mathcal{S}_0$ and end at $\mathcal{S}_1$ and switch from $u = h^{i-1}(v)$ to $h^i(v)$ at each switching point $\wp^i \stackrel{\text{def}}{=} (v_0^R(x_0^i), w_0^R(x_0^i), x_0^i)$ on Γ^i.

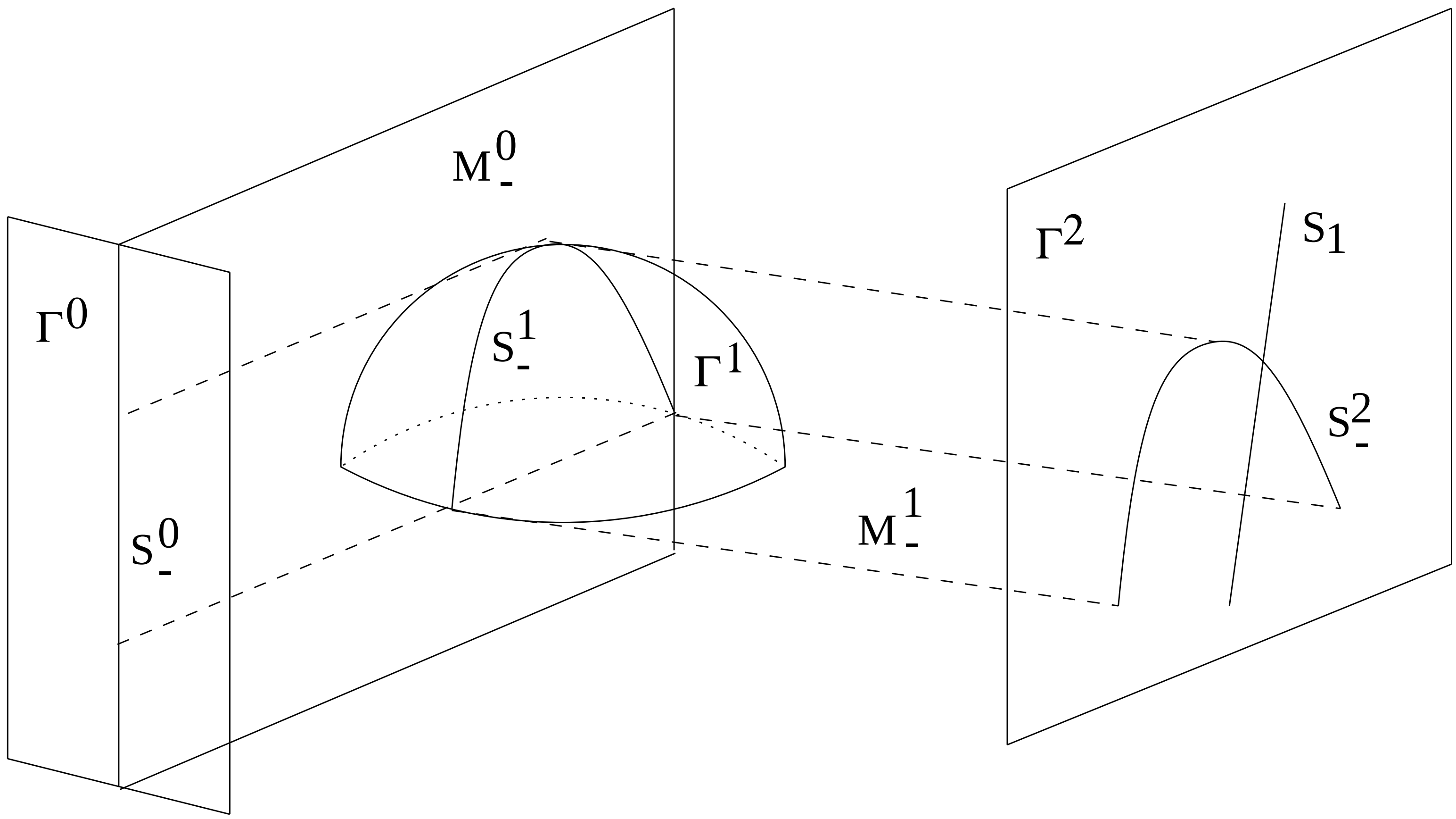
Finding the switching points:

$\sim$ Method 1: Assumed that the slow flow is transverse to each Γ^i. Then a Poincare mapping $\mathcal{P}^i : \Gamma^i \to \Gamma^{i+1}$, $0 \leq i \leq r$ can be defined. A shooting method can be used to find the switching points.



Problem: There are some structurally stable internal layer solutions that cannot be found by the shooting method.

$\sim$ Method 2: Define pseudo Poincare mappings without assuming the transversality of the slow flow to Γ^i:



Let the solution of in the ith slow manifold be $\Phi^i(x)$. Let $\mathcal{S}^0_- = \mathcal{S}_0$.

$$\mathcal{M}^0_- \stackrel{\text{def}}{=} \bigcup\{\Phi^0(x) \cdot \mathcal{S}^0_-,\ x \geq 0\}$$

is an $(n+1)$-dimensional smooth manifold. Assume that $\mathcal{M}^0_-$ intersects Γ^1 transversely. $\mathcal{S}^1_- \stackrel{\text{def}}{=} \mathcal{M}^0_- \pitchfork \Gamma^1$ is an n-dimensional submanifold of Γ^1. The procedure of associate the sets $\mathcal{S}^0_- \to \mathcal{S}^1_-$, denoted by $\mathcal{P}^0$, is a mapping between two sets. The mapping $\mathcal{P}^0$ will be called a pseudo-Poincare mapping.

We now proceed inductively. Assume that an n-dimensional submanifold $\mathcal{S}^i_- \subset \Gamma^i$ has been defined. Assume that:
(H4) The flow $\Phi^i(\cdot)$ is transverse to $\mathcal{S}^i_-$, $1 \leq i \leq r$.
From (H4), $\mathcal{M}^i_- \stackrel{\text{def}}{=} \bigcup\{\Phi^i(x) \cdot \mathcal{S}^i_-,\ x \geq 0\}$ is a smooth $(n+1)$ dimensional manifold. Assume that:
(H5) $\mathcal{M}^i_- \pitchfork \Gamma^{i+1}$.

Assumption (H5) implies that

$$\mathcal{S}_-^{i+1} \stackrel{\text{def}}{=} \mathcal{M}_-^i \cap \Gamma^{i+1}.$$

is an n-dimensional smooth submanifold of Γ^{i+1}.
With (H4) and (H5), the pseudo-Poincare mapping

$$\mathcal{P}^i : \mathcal{S}_-^i \to \mathcal{S}_-^{i+1}$$

is locally uniquely defined. We assume that:

(H6) The image of $\mathcal{S}_-^0$ under the composite mapping:

$$\mathcal{P}^r \cdots \mathcal{P}^1 \cdot \mathcal{P}^0$$

intersects transversely and nonemptily with $\mathcal{S}_1$ in Γ^{r+1}.

$$(v_0^R(1), w_0^R(1)) = \mathcal{S}_1 \cap \mathcal{P}^r \cdots \mathcal{P}^1 \cdot \mathcal{P}^0 \mathcal{S}_-^0.$$

The switching points $\wp^i \in \Gamma^i$ can be obtained by applying the inverse mappings of $\mathcal{P}^r, \cdots, \mathcal{P}^0$ to $\mathcal{S}_-^{r+1} \cap \mathcal{S}_1$ successively.
The solution (v_0^R, w_0^R, x) of (21) can be computed by using $\Phi^i(x)$ between these switching points.

We have found a sequence of points

$$x_0^0 = 0 < x_0^1 < x_0^2 < \cdots < x_0^r < 1 = x_0^{r+1}$$

and a solution $v_0^R \in C^1([0,1])$ of (18). $v_0^R \in C^\infty(x_0^i, x_0^{i+1})$.
v_{xx}^R may have a jump across x_0^i.

The union of the regular and singular solutions,

$$u_0^R = h^i(v_0^R(x)),\ v_0^R = v_0^R(x), \quad \text{for } x_0^i \le x \le x_0^{i+1},\ 0 \le i \le r,$$

$$u_0^{Si} = q^i(\xi),\ v_0^{Si} = \bar{v}^i, \quad \begin{cases} \text{for } \xi \in \mathbb{R}, & 1 \le i \le r, \\ \text{for } \xi \in \mathbb{R}^+, & i = 0, \\ \text{for } \xi \in \mathbb{R}^-, & i = r+1, \end{cases}$$

forms a singular internal layer solution. It is the 0th order expansion for a multiple internal layer solution.
From (H1)-(H6) we can show for $\epsilon > 0$ but small, there exist a true internal layered solution near the singular limit.

$\sim$ Method 3: Consider mono-layer solutoins of the system:

$$
\begin{aligned}
&u_t = \epsilon^2 u_{xx} + F(u, y + \frac{k}{\omega}\sin(\omega x + b)), && 0 < x < 1, \\
&y_t = y_{xx} + \sigma G(u, y), && u,\ y \in \mathbb{R}, \\
&u_x = y_x = 0, && x = 0,\ 1.
\end{aligned} \tag{22}
$$

Let $v = (y, x)$. The stationary solutions of (22) satisfies

$$
\begin{aligned}
&0 = \epsilon^2 u_{xx} + f(u, v), \\
&0 = v_{xx} + g(u, v), \\
&u_x = y_x = 0, \quad x = 0,\ 1, \\
&x(0) = 0,\ x(1) = 1,
\end{aligned}
$$

where

$$
\begin{aligned}
f(u, v) &= F(u, y + \tfrac{k}{\omega}\sin(\omega x + b)), \\
g(u, v) &= \begin{pmatrix} \sigma G(u, y) \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
dy/dx &= z, \\
dz/dx &= -\sigma G(h(y + \tfrac{k}{\omega}\sin(\omega x + b)), y), \\
dx/dx &= 1,
\end{aligned} \tag{23}
$$

where $k = 0$, $h = h_-$ if $x < x_0$, $h = h_+$ if $x > x_0$. Obviously, (y_0^R, z_0^R, x) where $z_0^R = y_{0x}^R$ is a solution of (23). Let

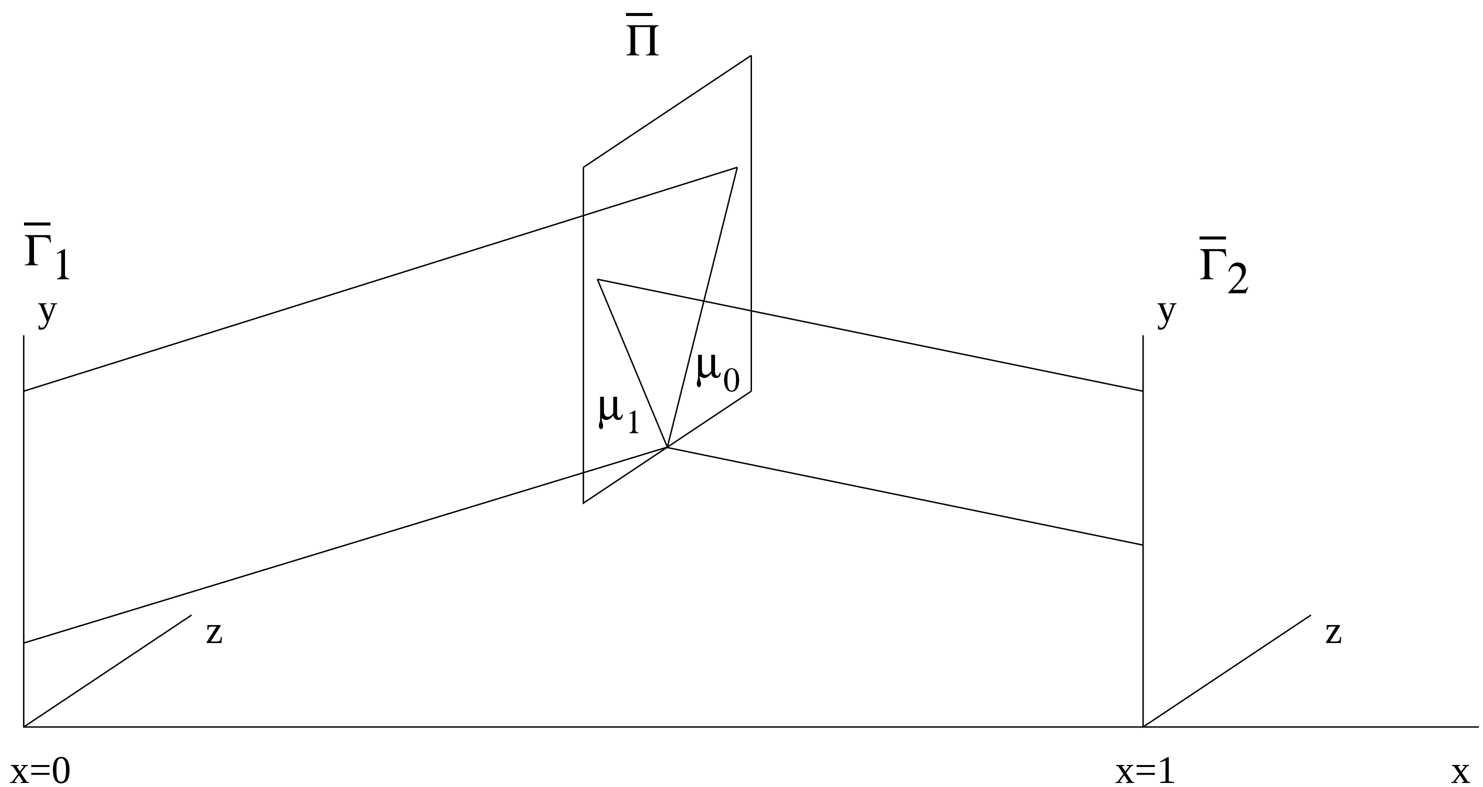
$$
\begin{aligned}
\bar{\Gamma}_0 &= \{(y, z, x) | x = 0\}, \\
\bar{\Gamma}_1 &= \{(y, z, x) | y = \tilde{y}\}, \\
\bar{\Gamma}_2 &= \{(y, z, x) | x = 1\}, \\
\bar{\mathcal{S}}_0 &= \{(y, z, x) | x = 0,\ z = 0\}, \\
\bar{\mathcal{S}}_1 &= \{(y, z, x) | x = 1,\ z = 0\}, \\
\bar{\Pi} &= \{(y, z, x) | x = x_0\}.
\end{aligned} \tag{24}
$$

Denote Φ_- the solution map of (23) with $h = h_-$ for all $0 \le x \le 1$. Denote Φ_+ the solution map of (23) with $h = h_+$ for all $0 \le x \le 1$.

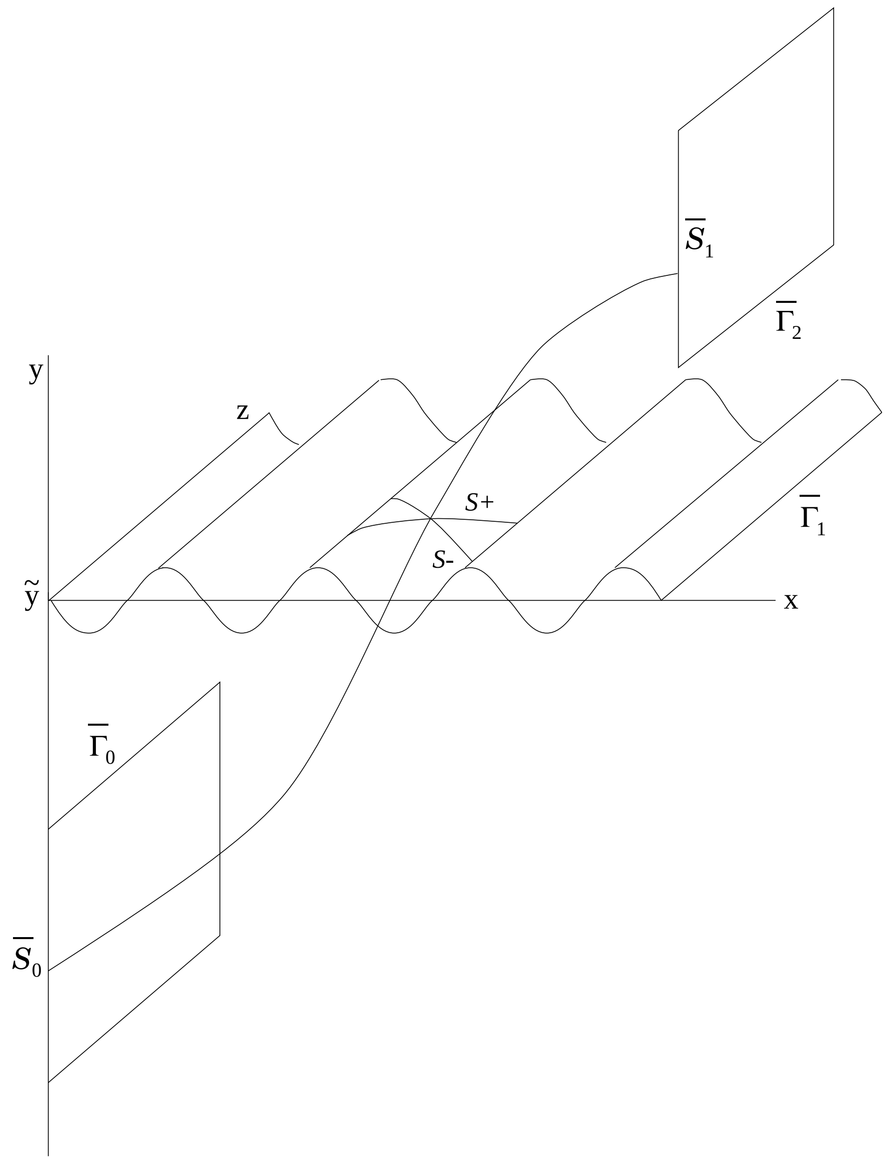
Let

$$\mathcal{M}_- = \cup\{\Phi_-(x,0)\bar{\mathcal{S}}_0 | 0 \le x \le 1\},$$
$$\mathcal{M}_+ = \cup\{\Phi_+(x,1)\bar{\mathcal{S}}_1 | 0 \le x \le 1\}.$$

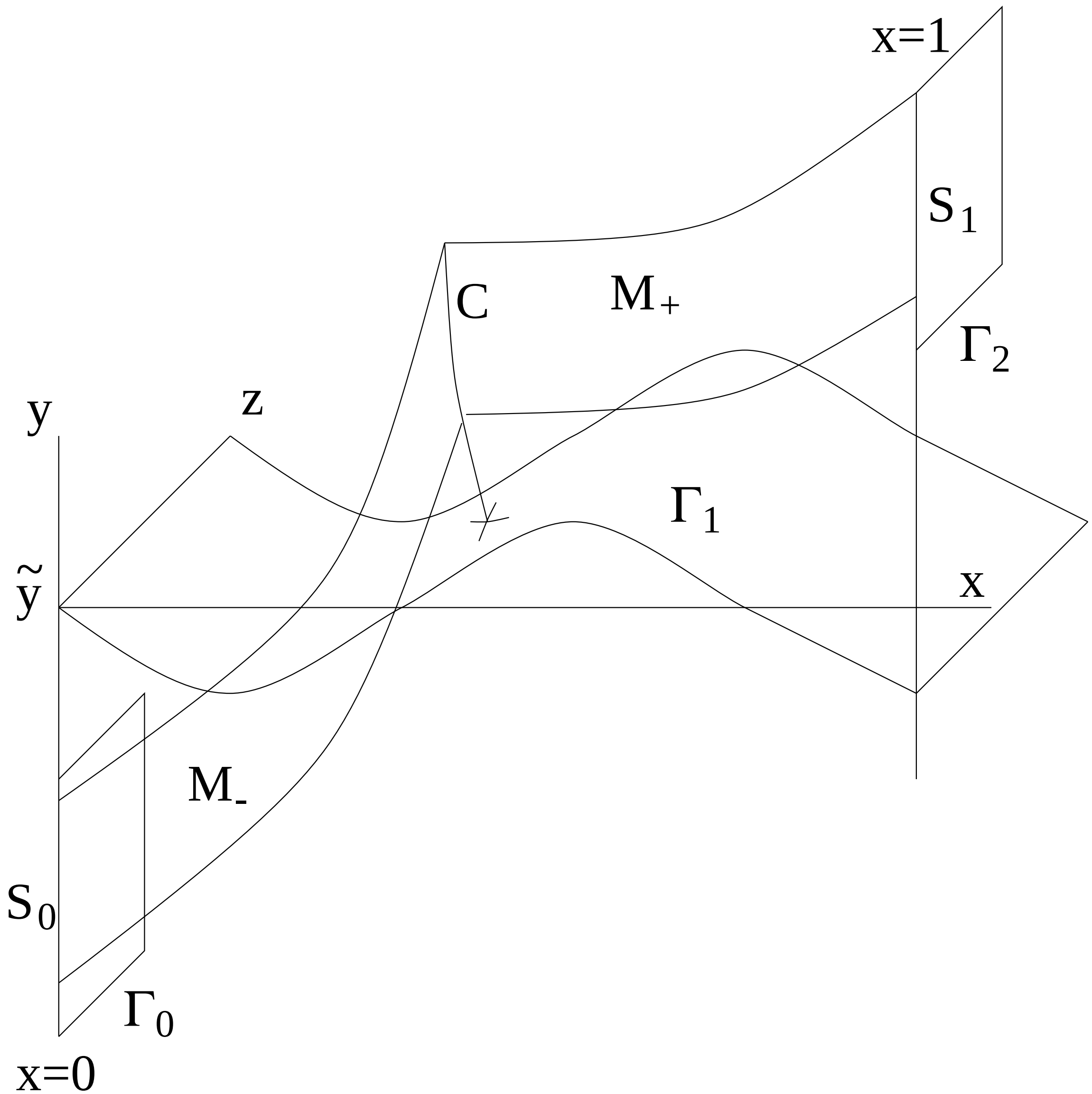
Let $\mu_0 = \mathcal{M}_- \cap \bar{\Pi}$, $\mu_1 = \mathcal{M}_- \cap \bar{\Pi}$. Note that the matching point $\wp = (y_0^R(x_0), z_0^R(x_0), x_0) \in \mu_0 \cap \mu_1$.



The transversal intersection of μ_0 and μ_1



Poincare mappings induced by the slow flow
It is difficult to find the sub-manifolds S_- and S_+.



The switching point is determined by the intersection of a slow-switching curve and a fast jumping surface.

Determine the switching point for small $k \neq 0$

The slow manifolds are graphs of

$$u = h_{\pm}(y + \frac{k}{\omega}\sin(\omega x + b)).$$

$k \neq 0$ and $h = h_-$ if $x < x^\dagger$, or h_+ if $x > x^\dagger$, where $x^\dagger$ is part of the unknowns. The switching points are determined by the intersection of a **fast jump surface** and a **slow switching curve**.

$\sim$ The fast jump surface for $k \neq 0$ is

$$\bar{\Gamma}_1 = \{(y, z, x) | y + \frac{k}{\omega}\sin(\omega x + b) = \tilde{y}\}.$$

Due to the fact $J(\tilde{y}) = 0$, if $y + \frac{k}{\omega}\sin(\omega x + b) = \tilde{y}$ then equation

$$u_{\xi\xi} + F(u, y + \frac{k}{\omega}\sin(\omega x + b)) = 0,$$

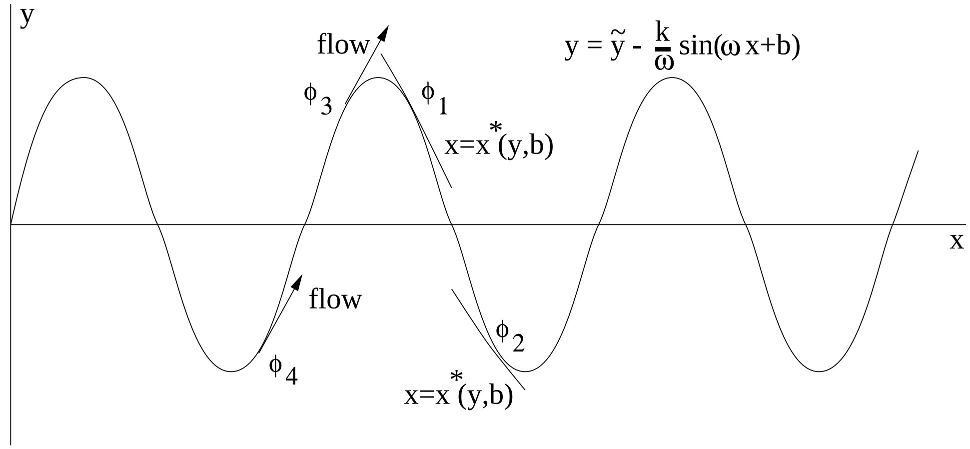
has a heteroclinic solution q.

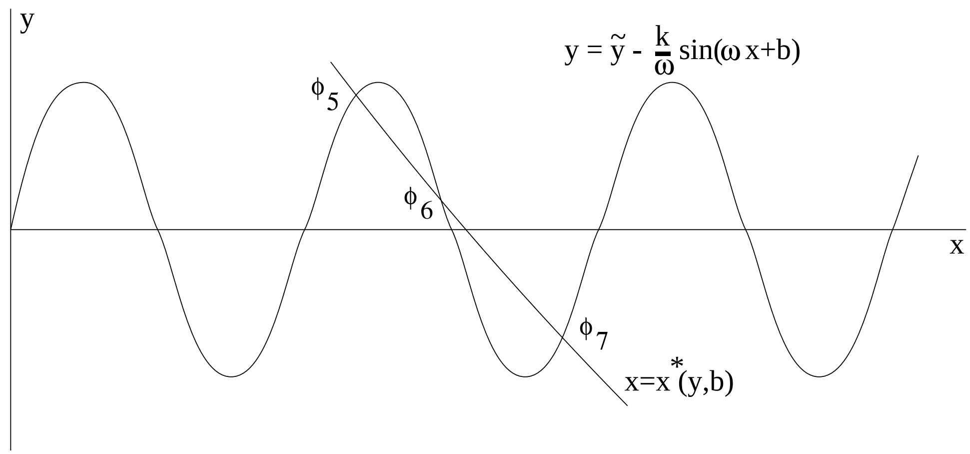
$\sim$ The slow switching curve $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{M}_- \cap \mathcal{M}_+$. The slow flow has to switch from $u = h^0(v)$ to $h^1(v)$ at some $\wp \in \mathcal{C}$ in order to satisfy boundary conditions at $x = 0, 1$.

Express $\mathcal{C}$ as C^1 functions

$$\mathcal{C} = \cup\{(y,z,x)|z = z^*(y,b), x = x^*(y,b),\ \tilde{y} - T < y < \tilde{y} + T\}.$$

Since $\bar{\Gamma}_1$ is flat in the z direction, prot it in (x,y) plane.
Several possible intersections of $\mathcal{C}$ and $\bar{\Gamma}_1$ are depicted in the figure. It follows from Lemma 14 that if k/ω is sufficiently large, then $\frac{\partial}{\partial y}x^*(y,b) < 0$ for all $y \in (\tilde{y} - T, \tilde{y} + T)$. The non transverse intersection of Γ_1 and $\mathcal{C}$ can occur at the part of Γ_1 that is decreasing.





(H4)-(H6) are not satisfied at the tangential intersections corresponding to ϕ_1, ϕ_2 but are satisfied at ϕ_3, ϕ_4 where the flow is tangent to Γ_1. The internal layer solution is stable at ϕ_5, ϕ_7 but unstable at ϕ_6.

$\sim$ First assume that k is sufficiently small so that

$$|\frac{\partial}{\partial y}x^*(y,b)| < \frac{1}{k},$$

for all $\tilde{y}-T < y < \tilde{y}+T$. Since the maximum of the slope of $\Pi\Gamma_1$ is k, $\Pi\mathcal{C}$ intersects $\Pi\Gamma_1$ transversely at a unique point $\wp = (y^\dagger, z^\dagger, x^\dagger)$ for any $b \in \mathbb{R}$.

$\sim$ Next, assume that k is sufficiently large so that

$$|\frac{\partial}{\partial y}x^*(y,b)| > \frac{1}{k},$$

for all $\tilde{y} - T < y < \tilde{y} + T$. $\Pi\mathcal{C}$ can intersect with $\Pi\Gamma_1$ at multiple points.

The intersections of $\mathcal{C}$ and $\bar{\Gamma}_1$ correspond to solutions of the equation

$$\frac{k}{\omega}\sin(\omega x^*(y,b) + b) + y = \tilde{y}.$$

Let $\phi = \omega x^*(y,b) + b$. With Φ as a parameter on $\Pi\Gamma_1$, the intersections correspond to zeros of the function

$$\mathfrak{E}(\phi, b) \stackrel{\text{def}}{=} \omega x^*(\tilde{y} - \frac{k}{\omega}\sin\phi, b) + b - \phi = 0.$$

For each $\phi \in \mathbb{R}$, there exists a unique $b = b^*(\phi)$ such that $\mathfrak{E} = 0$. Moreover, b^* is a C^1 function of ϕ with

$$\frac{\partial b^*}{\partial \phi} = (1 + \omega \frac{\partial x^*}{\partial b})^{-1}(1 + k\frac{\partial x^*}{\partial y}\cos\phi). \tag{25}$$

Condition $\mathcal{C} \pitchfork \bar{\Gamma}_1$ becomes $\partial b^*/\partial\phi \neq 0$, or equivalently,

$$1 + k\frac{\partial x^*}{\partial y}\cos\phi \neq 0. \tag{26}$$

Consider one period $\phi \in [-3\pi/2, \pi/2]$ for the time being. For $\phi \in [-3\pi/2, -\pi/2)$, $\cos\phi \leq 0$, (26) is valid. The left hand side of (26) is positive if $\phi = -\pi/2$, but is negative if $\phi = 0$. Therefore, it is easy to see that there exist $\phi_1 \in (-\pi/2, 0), \phi_2 \in (0, \pi/2)$ such that

$$1 + k\frac{\partial x^*}{\partial y}\cos\phi_j = 0, \quad j = 1, 2. \tag{27}$$

Using the fact that $\frac{\partial x^*}{\partial y}$ is almost a constant, it is easy to verify that ϕ_1, ϕ_2 are the only points in $[-3\pi/2, \pi/2]$ that do not satisfy (26).

The eigenvalue is $\lambda(\epsilon) = \epsilon^j \lambda_j$ with $\lambda_0 = 0$.

$$\begin{aligned}\lambda_1 &= \|\dot{q}\|^{-2} J'(\tilde{y})(Y^c(x) - z_0^R(x) + Xk\cos(\omega x + b) - k\cos(\omega x + b)) \\ &= \|\dot{q}\|^{-2} J'(\tilde{y})(Y^c(x) - z_0^R(x) - k\cos\phi).\end{aligned}$$

$\sim$ When k is sufficiently small, then $\lambda_1 < 0$. Our result agrees with that of Nishiura & Fujii 1987 where $k = 0$.

$\sim$ When k is sufficiently large, the stability depends on the parameter ϕ. If $\phi_1 + 2\nu\pi < \phi < \phi_2 + 2\nu\pi$, $\nu \in \mathbb{Z}$, the internal layer solution solution is unstable ($\lambda_1 > 0$); If $\phi_2 + 2\nu\pi < \phi < \phi_1 + 2(\nu + 1)\pi$, then the solution is stable ($\lambda_1 < 0$).

Theorem 5 *$\lambda_1 = \mathbf{n}^1 \cdot \Delta v$ where $\mathbf{n}^1$ is the normal of the surface Σ^1 as in §3 and $(\Delta v, \Delta w, -1)$ is a tangent vector of $\mathcal{M}_- \cap \mathcal{M}_+$ at $(v_0^R(x_0), w_0^R(x_0), x_0)$. Let $\mathbf{N} = (\mathbf{n}^1, 0, 0)$ be a normal of Γ^1. The result can also be expressed as*

$$\lambda_1 = \mathbf{t} \cdot \mathbf{N}.$$

(i) When $b = b^*(\phi_j) + 2\pi\nu$, $j = 1, 2$, the intersection of $\mathcal{C}$ and $\bar{\Gamma}_1$ is nontransversal, and saddle-node type bifurcations may occur.
$\sim$ Analysis of bifurcations caused by moving b through critical values is completed in Hale & Lin 1997.
(ii) If $\mathcal{C}$ is oriented with the positive direction pointing to the decreasing of x, then $\lambda_1 < 0$ if $\mathcal{C}$ passes through $\bar{\Gamma}_1$ from below; $\lambda_1 > 0$ if $\mathcal{C}$ passes through $\bar{\Gamma}_1$ from above. This interpretation agrees with Theorem 10.
(iii) From Figure **??**, we see that at the intersections corresponding to $\phi = \phi_3, \phi_4$, the flow of (23) is tangent to $\bar{\Gamma}_1$, however, the internal layer solution is structurally stable due to $\mathcal{C} \pitchfork \bar{\Gamma}_1$. These solutions will be missed if one insists that the flow must be transverse to $\bar{\Gamma}_1$.

Critical eigenvalue and eigenfunctions

SLEP method, Nishiura and Fujii 1987:

Stability is determined by r critical eigenvalues $\lambda(\epsilon) = \epsilon\lambda_1 + O(\epsilon^2)$, where r is the number of internal layers,

λ_1 is an eigenvalue of the so called SLEP matrix.

Eigenvalue problem of the internal layer solution:

$$\lambda U = \epsilon^2 U_{xx} + f_u U + f_v V, \tag{28}$$

$$\lambda V = V_{xx} + g_u U + g_v V, \tag{29}$$

Lemma A (Nishiura and Fujii): The critical eigenvalues of the operator

$$\epsilon^2 D_{xx} + f_u, \quad \mu = \sum \epsilon^j \mu_j,\ \mu_0 = 0,$$

is not equal to the critical eigenvalues of (28), (29).

More precisely, $\lambda_1 \neq \mu_1$.

By Lemma A, U can be solved from (28) and substituted into (29). This yields the reduced eigenvalue problem

$$\lambda V = V_{xx} + g_u(\lambda - \epsilon^2 D_{xx} - f_u)^{-1} f_v V + g_v V. \tag{30}$$

The SLEP matrix is derived from the above. $\lambda_1 < 0$ is proved by studying the eigenvalues of the inverse of the SLEP matrix.

Problem: There are examples where $\mu_1 = \lambda_1$.

$\sim$ Asymptotic method can be used to compute the critical eigenvalues without using the Lemma A. Our approach can be used on some systems not covered by the SLEP method due to Nishiura and Fujii 1987, Nishiura 1994, Sakamoto 1990.

The critical eigenvalue and eigenfunctions are determined by three factors:
(1) A system of differential equations;
(2) boundary conditions in boundary layers;
(3) and the matching conditions.

(1) In regular layers,

$$\lambda(\epsilon)U(\epsilon) = \epsilon^2 U(\epsilon)_{xx} + f_u U(\epsilon) + f_v V(\epsilon), \tag{31}$$
$$\lambda(\epsilon)V(\epsilon) = V(\epsilon)_{xx} + g_u U(\epsilon) + g_v V(\epsilon). \tag{32}$$

In singular layers, using the stretched variable $\xi = (x - x^i(\epsilon))/\epsilon$,

$$\lambda(\epsilon)U(\epsilon) = U(\epsilon)_{\xi\xi} + f_u U(\epsilon) + f_v V(\epsilon), \tag{33}$$
$$\epsilon^2 \lambda(\epsilon)V(\epsilon) = V(\epsilon)_{\xi\xi} + \epsilon^2 (g_u U(\epsilon) + g_v V(\epsilon)).$$

Let $W = V_x$. Convert the V equation into a first order system:

$$V_\xi(\epsilon) = \epsilon W(\epsilon), \tag{34}$$
$$W_\xi(\epsilon) = -\epsilon g_u U(\epsilon) - \epsilon g_v V(\epsilon) + \epsilon \lambda(\epsilon) V(\epsilon). \tag{35}$$

Denote the expansions in both regular and singular layers by

$$U(\epsilon) = \sum_{j=0}^{\infty} \epsilon^j U_j, \quad V(\epsilon) = \sum_{j=0}^{\infty} \epsilon^j V_j,$$

$$W^S(\xi,\epsilon) = V^S_\xi/\epsilon = \sum_{j=0}^{\infty} \epsilon^j W^S_j(\xi).$$

We can prove $V^S_0(\xi) = 0$, thus no ϵ^{-1} term in the expansion of

(2) The boundary conditions in the boundary layers are

$$U_x(x,\epsilon) = 0, \quad \text{for } x = 0, 1,$$
$$A_j W(j,\epsilon) + B_j V(j,\epsilon) = 0, \quad j = 0, 1,$$

where $W = V_x$. Expanding in the powers of ϵ, we find for all $j \geq 0$:

$$\begin{aligned} &U^{Si}_{j\xi}(0) = 0, \quad i = 0, r+1, \\ &A_0 W^{S0}_j(0) + B_0 V^{S0}_j(0) = 0, \\ &A_1 W^{S,r+1}_j(0) + B_1 V^{S,r+1}_j(0) = 0, \end{aligned} \tag{36}$$

Using $W^S(\xi,\epsilon) = V^S_\xi(\xi,\epsilon)/\epsilon$, we have $W^S_j = V^S_{j+1,\xi}$ in the above.

(3) Exponential matching principles

Inner expansion of outer layers

Let U^R be the outer solution in one of the regular layers adjacent to x_0^i. The inner expansion of U^R is denoted by $\tilde{U}^R$.

$$\sum_0^\infty \epsilon^j \tilde{U}_j^R(\xi) = U^R(\sum_0^\infty \epsilon^j x_j^i + \epsilon\xi, \epsilon),$$
$$\sum_0^\infty \epsilon^j \tilde{V}_j^R(\xi) = V^R(\sum_0^\infty \epsilon^j x_j^i + \epsilon\xi, \epsilon).$$

The exponential matching principle

$$|\tilde{U}_j^R(\xi) - U_j^S(\xi)| + |\tilde{U}_{j\xi}^R(\xi) - U_{j\xi}^S(\xi)| \leq C(1 + |\xi|^j)e^{-\gamma|\xi|}, \tag{37}$$
$$|\tilde{V}_j^R(\xi) - V_j^S(\xi)| + |\tilde{V}_{j\xi}^R(\xi) - V_{j\xi}^S(\xi)| \leq C(1 + |\xi|^j)e^{-\gamma|\xi|}. \tag{38}$$

Let $\sum \epsilon^j \tilde{W}_j^R(\xi)$ denote the inner expansion of W^R. (38) is equivalent to

$$|\tilde{V}_j^R(\xi) - V_j^S(\xi)| + |\tilde{W}_j^R(\xi) - W_j^S(\xi)| \leq C(1 + |\xi|^j)e^{-\gamma|\xi|}. \tag{39}$$

(1) The ϵ^0-th order expansion: Since $\lambda(\epsilon)$ is critical, $\lambda_0 = 0$.

In regular layers, from (31), (32),

$$\begin{aligned} & f_u U_0 + f_v V_0 = 0, \\ & V_{0xx} + g_u U_0 + g_v V_0 = 0. \\ \text{Therefore,} \quad & U_0 = -f_u^{-1} f_v V_0, \\ & V_{0xx} - (g_u f_u^{-1} f_v - g_v) V_0 = 0. \end{aligned}$$

In singular layers, from (34), (35)

$$\begin{aligned} & U_{0\xi\xi} + f_u U_0 + f_v V_0 = 0, \\ & V_{0\xi} = 0, \\ & W_{0\xi} = 0. \end{aligned} \tag{40}$$

The last two equations imply that V_0^S and W_0^S are constants in singular layers. Form the matching principle,

$$[V_0^R](x_0^i) = [W_0^R](x_0^i) = 0, \quad 1 \leq i \leq r.$$

The boundary conditions for V_0^R are

$$A_j V_x(j) + B_j V(j) = 0, \quad j = 0, 1. \tag{41}$$

We need the following hypothesis,
(H7) If $V \in C^1([0,1]) \cap C^2((x_0^i, x_0^{i+1}), 0 \le i \le r$, then $V = 0$ is the only solution for the following boundary value problem:

$$\begin{aligned} &V_{xx} - (g_u f_u^{-1} f_v - g_v)V = 0, \\ &A_j V_x(j) + B_j V(j) = 0, \quad j = 0, 1. \end{aligned} \tag{42}$$

We comment the if (H7) is not satisfied, then the regular eigenvalues, which solve the reduced eigenvalue problem (71), will have $\lambda = 0$ as a root. In this case, asymptotic expansions of critical eigenvalues are quite different and will not be touched in this paper. In §7, a stronger assumption (H9), which implies (H7), will be imposed to ensure that the regular eigenvalues are in the region $\mathrm{Re}\lambda \le -\gamma < 0$.

From (H7), we can prove the following lemma:

Lemma 6 *Assume that V satisfies*

$$\begin{aligned}
&V_{xx} - (g_u f_u^{-1} f_v - g_v)V = E_1,\\
&[V](x_0^i) = E_2,\\
&[V_x](x_0^i) = E_3,\\
&A_j V_x(j) + B_j V(j) = E_{4j},\ j = 0, 1.
\end{aligned}$$

Here E_2, E_3, $E_{4j} \in \mathbb{R}^n$, $E_1 \in C((x_0^i, x^{i+1}))$, $0 \le i \le r$ and has one-sided limits at the boundary points. Then there exists a unique piecewise C^2 solution $V \in C^1([0,1]) \cap C^2((x_0^i, x_0^{i+1}))$, $0 \le i \le r$.

(H7) $\Rightarrow V_0^R = 0$ on $[0,1]$, $\Rightarrow V_0^S = 0$ in all the singular layers.
$V_0^S = 0 \Rightarrow U_0^S = c_0^i \dot{q}^i$ in the ith singular layer.
When $i = 0$ or $r+1$, $U_0^{Si} = \dot{q}^i = 0$, which satisfies the Neumann boundary conditions.

To summarize,

$$
\begin{aligned}
\lambda_0 = 0, &\quad \text{critical eigenvalue,} \\
V_0^R = 0,\, U_0^R = 0, &\quad \text{in regular layers,} \\
V_0^S = 0,\, U_0^S = c_0^i \dot{q}^i, &\quad \text{in the } i\text{th singular layers.}
\end{aligned}
$$

Set $c_0^0 = c_0^{r+1} = 1$, but $\{c_0^i\}_1^r$ remain to be determined.

(2) The ϵ^1-th order expansion:

In the regular layers, since $\lambda_0 U_1^R + \lambda_1 U_0^R = 0 = \lambda_0 V_1^R + \lambda_1 V_0^R$. We have,

$$\begin{aligned} & f_u U_1^R + f_v V_1^R = 0, \\ & V_{1xx}^R + g_u U_1^R + g_v V_1^R = 0. \\ \text{Therefore,}\quad & U_1^R = -f_u^{-1} f_v V_1^R, \\ & V_{1xx}^R - (g_u f_u^{-1} f_v - g_v) V_1^R = 0. \end{aligned}$$

In the ith singular layer, the equations for (U_1^S, V_1^S, W_1^S) become

$$\lambda_1 c_0^i \dot{q}^i = U_{1\xi\xi}^S + f_u U_1^S + f_v V_1^S + c_0^i (f_{uu} \dot{q}^i u_1^S + f_{uv} \dot{q}^i v_1^S), \tag{43}$$

$$V_{1\xi}^S = W_0^S = 0, \tag{44}$$

$$W_{1\xi}^S = -g_u U_0^S - g_v V_0^S = -g_u c_0^i \dot{q}^i. \tag{45}$$

From (44), (45),

$$\begin{aligned} & V_1^S = \text{constant} = V_1^R(x_0^i), \\ & W_1^S(\infty) - W_1^S(-\infty) = -c_0^i \int_{-\infty}^{\infty} g_u \dot{q}^i(\xi)\, d\xi,\ 1 \le i \le r. \end{aligned}$$

Let $\mathcal{M}^i = g(q^i(-\infty), \bar{v}^i) - g(q^i(\infty), \bar{v}^i)$. By the matching principle,

$$\begin{aligned} &[V_1^R](x_0^i) = 0, \\ &[V_{1x}^R](x_0^i) = c_0^i \mathcal{M}^i. \end{aligned} \tag{46}$$

In the boundary layers, $\dot{q}^i = 0$, $i = 0, r+1$. Thus (V_1^S, W_1^S) are constants solutions in the boundary layers. V_1^R satifies $A_j V_x(j) + B_j V(j) = 0, \quad j = 0, 1$.

Motivation: The V_1^R is driven by the jump of V_x at x_0^i.
Define V_c^i, $1 \le i \le r$ to be the solution of (42) that satisfies

$$\begin{aligned} &[V](x_0^\nu) = 0, \quad \text{for all } \nu, \\ &[V_x](x_0^\ell) = 0, \quad \text{for all } \ell \neq i, \\ &[V_x](x_0^i) = \mathcal{M}^i. \end{aligned} \tag{47}$$

$$V_1^R = \sum_1^r c_0^i V_c^i. \tag{48}$$

by the superposition principle.

To find a solution $U_1^S = O(1+|\xi|)$ for (43),

$$\lambda_1 c_0^i \dot{q}^i = U_{1\xi\xi}^S + f_u U_1^S + f_v V_1^S + c_0^i (f_{uu} \dot{q}^i u_1^S + f_{uv} \dot{q}^i v_1^S),$$

the nonhomogeneous terms must be in the range of a Fredholm operator.

Define $L_q u = u_{\xi\xi} + Df(q(\xi))u$.
L_q is a Fredholm operator with Fredholm index zero. Assume that $\dim Ker(L_q) = 1$, then $Ker(L_q) = \text{span}\{\dot{q}\}$ and $Range(L_q) = \{\psi\}^\perp$. Here ψ is the unique nontrivial bounded solution for the adjoint equation, up to a scalar multiple,

$$L_q^* \psi \overset{def}{=} \psi_{\xi\xi} + Df^\tau(q(\xi))\psi = 0.$$

$$\{\psi\}^\perp \overset{def}{=} \{u \in X | \int_{-\infty}^{\infty} \psi^\tau(\xi) u(\xi) d\xi = 0\}.$$

$$\lambda_1 c_0^i < \psi^i, \dot{q}^i > = < \psi^i, f_v V_1^S + c_0^i (f_{uu} \dot{q}^i u_1^S + f_{uv} \dot{q}^i v_1^S) > .$$

The above can be simplified using integration by parts as follows:

$$\begin{aligned}
& f_{uu}\dot{q}^i u_1^S + f_{uv}\dot{q}^i v_1^S \\
=& \partial_\xi(f_u u_1^S + f_v v_1^S) - f_u u_{1\xi}^S - f_v v_{1\xi}^S \\
=& \partial_\xi(-u_{1\xi\xi}^S) - f_u u_{1\xi}^S - f_v v_{1\xi}^S \\
=& -\{(u_{1\xi}^S)_{\xi\xi} + f_u(u_{1\xi}^S)\} - f_v v_{1\xi}^S.
\end{aligned}$$

The term in the $\{\}$ is in the range of a Fredholm operator, thus

$$<\psi^i, \{(u_{1\xi}^S)_{\xi\xi} + f_u(u_{1\xi}^S)\}>= 0.$$

Therefore,

$$<\psi^i, (f_{uu}\dot{q}^i u_1^S + f_{uv}\dot{q}^i v_1^S)>= -<\psi^i, f_v v_{1\xi}^S> . \tag{49}$$

$$c_0^i \lambda_1 <\psi^i, \dot{q}^i>=<\psi^i, f_v(V_1^S - c_0^i v_{1\xi}^S)> .$$

Recall that $<\psi^i, \dot{q}^i>= 1$, $\mathbf{n}^i =<\psi^i, f_v>$ and V_1^S and $v_{1\xi}^S$ are constants, we have $c_0^i\lambda_1 = \mathbf{n}^i \cdot (V_1^S - c_0^i v_{1\xi}^S)$. Using $v_{1\xi}^S = w_0^S = w_0^R(x_0^i) = v_{0x}^R(x_0^i)$, $V_1^{Si} = V_1^R(x_0^i)$, we have

$$c_0^i\lambda_1 = \mathbf{n}^i \cdot (V_1^R(x_0^i) - c_0^i v_{0x}^R(x_0^i)). \tag{50}$$

From $V_1^R = \sum_1^r c_0^i V_c^i$, equation (50) becomes:

$$\lambda_1 c_0^i = \mathbf{n}^i \cdot (\sum_{\ell=1}^{r} c_0^\ell V_c^\ell(x_0^i) - c_0^i w_0^R(x_0^i)).$$
$$i = 1, 2, \cdots, r.$$

Define the **coupling matrix** $A = (a_{i\ell})_{r\times r}$ by

$$a_{i\ell} = \mathbf{n}^i \cdot (V_c^\ell(x_0^i) - \delta_{i\ell} v_{0x}^R(x_0^i)). \tag{51}$$

A is precisely the SLEP matrix by Nishiura & Fujii derived using the SLEP method.

We see that λ_1 is an eigenvalue while $(c_0^1, c_0^2, \cdots, c_0^r)^\tau$ is an eigenvector for A.

$$\lambda_1 \begin{pmatrix} c_0^1 \\ \vdots \\ c_0^r \end{pmatrix} = A \begin{pmatrix} c_0^1 \\ \vdots \\ c_0^r \end{pmatrix}.$$

To construct higher order expansions, we may use any of the r eigenvalues and the corresponding eigenvector of A. With such λ_1 and $(c_0^1, c_0^2, \cdots, c_0^r)$, (43) has a solution $|U| \leq C(1 + |\xi|)$ which can be written as

$$U_1^S = Z_1^S + c_1^i \dot{q}^i.$$

Here $< \dot{q}^i, Z_1^i >= 0$, and the parameters $\{c_1^i\}_1^r$ remain to be determined.

Finally, in the boundary layers, (43) becomes

$$U_{1\xi\xi} + f_u U_1 + f_v V_1 = 0.$$

With V_1^S already obtained, there exists a unique solution $U_1^S = O(1 + |\xi|)$ in the boundary layers. See Lemma 2.

Quiz: If $U_j(\xi) = O(1 + |\xi|^j) \to \infty$ as $|\xi| \to \infty$, then how can

$$\sum_{j=0}^{\infty} \epsilon^j U_j$$

be an asymptotic series?

We summarize our result in the following theorem:

Theorem 7 *Assume that (H1)–(H7) are satisfied, than the asymptotic expansion of critical eigenvalues λ and eigenfunctions (U, V) can be obtained up to ϵ^1. λ_1 is an eigenvalue for the coupling matrix A. The associated eigenvector $\{c_0^i\}_1^n$ provides informations about the eigenfunction (U, V), which satisfies $U_0^R = 0$, $V_0^R = 0$ and $V_1^R = \sum c_0^i V_c^i$ in regular layers; and $U_0^{Si}(\xi) = c_0^i \dot{q}^i(\xi)$ and $V_0^S = 0$ in the singular layer at x_0^i.*

Assume that (H8) is also satisfied:
(H8) *λ_1 is a simple eigenvalue for the matrix $\lambda I - A$. (The Jordan blocks of $\lambda I - A$ corresponding to λ_1 are of order 1.)*

Then the higher order expansion of critical eigenvalues and the corresponding eigenfunctions can be obtained by a recursive procedure to any power of ϵ.

(3) The ϵ^j-th order expansions, $j \geq 2$:

Assume that we have computed

$$\lambda_0, \lambda_1, \cdots, \lambda_{j-1}.$$

We have obtained in regular layers:

$$\begin{aligned} &U_0, U_1, \cdots, U_{j-1}, \\ &V_0, V_1, \cdots, V_{j-1}, \\ &W_0, W_1, \cdots, W_{j-1}. \end{aligned}$$

In singular layers, we have computed all the above except for U_{j-1}^S which, in the ith internal layer, has the form

$$U_{j-1}^{Si} = c_{j-1}^i \dot{q}^i + Z_{j-1}^i, \quad < \dot{q}^i, Z_{j-1}^i >= 0.$$

Assume that Z_{j-1}^i has been determined but c_{j-1}^i is still a free parameter. In the ϵ^j-th expansion, we will determine λ_j, $\{c_{j-1}^i\}_1^r$, V_j, W_j and U_j^R. We will determine U_j^S up to $c_j^i \dot{q}^i$.

Definition An eigenfunction $(U(\epsilon), V(\epsilon))$ is called a normalized eigenfunction if the corresponding parameters $\{c_j^i\}_1^r$ satisfy

$$\sum_{i=1}^{r} (c_0^i)^2 = 1,$$
$$(c_\ell^1, c_\ell^2, \cdots, c_\ell^r) \perp (c_0^1, c_0^2, \cdots, c_0^r), \quad \ell \geq 1.$$

It is not hard to verify that if $(U(\epsilon), V(\epsilon))$ is a normalized eigenfunction, and if $\alpha(\epsilon) = \sum \epsilon^j \alpha_j$ is a scalar series, then $(\alpha(\epsilon)U(\epsilon),\ \alpha(\epsilon)V(\epsilon))$ is the general form of all the eigenfunctions. In the sequel, we will assume that the eigenfunctions are normalized.

In the regular layer, since $\lambda_0 = 0$ and $U_0^R = V_0^R = 0$,

$$\lambda_j U_0^R + \cdots + \lambda_0 U_j^R = \ell \cdot o \cdot t,$$
$$\lambda_j V_0^R + \cdots + \lambda_0 V_j^R = \ell \cdot o \cdot t.$$

Therefore,

$$\begin{aligned}
&f_u U_j^R + f_v V_j^R = \ell \cdot o \cdot t, \\
&V_{jxx}^R + g_u U_j^R + g_v V_j^R = \ell \cdot o \cdot t, \\
&U_j^R = -f_u^{-1} f_v V_j^R + \ell \cdot o \cdot t, \\
&V_{jxx}^R - (g_u f_u^{-1} f_v - g_v) V_j^R = \ell \cdot o \cdot t. \qquad (52)
\end{aligned}$$

From (H7), V_j^R can be uniquely solved for if the boundary conditions at $x = 0$, 1 and the jumps across $\{x_0^i\}_{i=1}^r$ can be found. The jumps can be found by matching the internal and regular layers.

In the ith internal layer, since $U^S_{j-1} = c^i_{j-1}\dot{q}^i + Z^i_{j-1}$ and $V^S_0 = 0$,

$$\begin{aligned}
&\lambda_1 U^S_{j-1} + \cdots + \lambda_j U^S_0 = c^i_{j-1}\lambda_1\dot{q}^i + c^i_0\lambda_j\dot{q}^i + \ell\cdot o\cdot t,\\
&\lambda_0 V^S_j + \cdots \lambda_j V^S_0 = \ell\cdot o\cdot t.
\end{aligned}$$

$$\begin{aligned}
& c^i_{j-1}\lambda_1\dot{q}^i + c^i_0\lambda_j\dot{q}^i\\
= \;& U^S_{j\xi\xi} + f_u U^S_j + f_v V^S_j + f_{uu} c^i_{j-1}\dot{q}^i u_1 + f_{uv} c^i_{j-1}\dot{q}^i v_1 + \ell\cdot o\cdot t.
\end{aligned} \tag{53}$$

$$V^S_{j\xi} = W^S_{j-1}, \tag{54}$$

$$\begin{aligned}
W^S_{j\xi} &= -g_u U^S_{j-1} - g_v V^S_{j-1} + \ell\cdot o\cdot t,\\
&= -c^i_{j-1} g_u \dot{q}^i + \ell\cdot o\cdot t,
\end{aligned} \tag{55}$$

$$V^S_j(\xi) = V^S_j(0) + \int_0^\xi \ell\cdot o\cdot t, \tag{56}$$

$$W^S_j(\xi) = W^S_j(0) + \int_0^\xi (-c^i_{j-1} g_u \dot{q}^i) + \ell\cdot o\cdot t, \tag{57}$$

Notice that (V_j^S, W_j^S) behaves like a polynomial of degree j as $|\xi| \to \infty$. The matching of higher powers of ξ can be proved by induction, see Lemma **??**. We only have to match the constant terms. Integrating from $\xi = -\infty$ to ∞, and applying the matching principles, similar to (**??**) and (**??**), we conclude that for $1 \leq i \leq r$,

$$[V_j^R](x_0^i) = \ell \cdot o \cdot t,$$
$$[W_j^R](x_0^i) = c_{j-1}^i \mathcal{M}^i + \ell \cdot o \cdot t.$$

In the boundary layers, since $\dot{q}^i = 0$, $i = 0, r+1$, from (56), (57), and the matching of outer and inner layers, we have

$$V_j^R(0+) = V_j^{S0}(0) + \ell \cdot o \cdot t,$$
$$W_j^R(0+) = W_j^{S0}(0) + \ell \cdot o \cdot t.$$

Therefore the boundary condition at $x = 0$ can be obtained,

$$A_0 W_j^R(0) + B_0 V_j^R(0) = \ell \cdot o \cdot t.$$

Similarly, at $x = 1$,

$$A_1 W_j^R(1) + B_1 V_j^R(1) = \ell \cdot o \cdot t.$$

With all the boundary and jump conditions, based on (H7), we can solve for V_j^R from (52). Using the superposition principle and the basis functions $\{V_c^i\}$, we can express the solution as a function of $\{c_{j-1}^i\}_1^r$.

$$V_j^R = \sum_{i=1}^{r} c_{j-1}^i V_c^i + \ell \cdot o \cdot t. \tag{58}$$

In order to have a solution $|U_j^{Si}| \leq C(1 + |\xi|^j), 1 \leq i \leq r$ for (53), the nonhomogeneous terms must be in the range of a Fredholm operator, see Lemma 3. This leads to

$$c_0^i \lambda_j + c_{j-1}^i \lambda_1 = < \psi^i, f_v V_j^S + c_{j-1}^i (f_{uu} \dot{q}^i u_1 + f_{uv} \dot{q}^i v_1) > + \ell \cdot o \cdot t.$$

Using integration by parts similar to (49),

$$c_0^i \lambda_j + c_{j-1}^i \lambda_1 = < \psi^i, f_v (V_j^S - c_{j-1}^i w_0^R(x_0^i)) > + \ell \cdot o \cdot t. \tag{59}$$

From (56), we then have,

$$c_0^i \lambda_j + c_{j-1}^i \lambda_1 = < \psi^i, f_v (V_j^S(0) - c_{j-1}^i w_0^R(x_0^i)) > + \ell \cdot o \cdot t. \tag{60}$$

By the matching principle and (56),

$$V_j^S(0) = V_j^R(x_0^i+) + \ell \cdot o \cdot t.$$

Thus

$$c_0^i \lambda_j + c_{j-1}^i \lambda_1 =< \psi^i, f_v(V_j^R(x_0^i+) - c_{j-1}^i w_0^R(x_0^i)) > +\ell \cdot o \cdot t. \quad (61)$$

Using (58) we have

$$c_0^i \lambda_j + c_{j-1}^i \lambda_1 = \mathbf{n}^i \cdot (\sum_{\ell=1}^{r} c_{j-1}^\ell V_c^\ell(x_0^i) - c_{j-1}^i w_0^R(x_0^i)) + \ell \cdot o \cdot t.$$

In the matrix form,

$$(\lambda_1 I - A) \begin{pmatrix} c_{j-1}^1 \\ \vdots \\ c_{j-1}^r \end{pmatrix} = \lambda_j \begin{pmatrix} c_0^1 \\ \vdots \\ c_0^r \end{pmatrix} + \ell \cdot o \cdot t. \quad (62)$$

We need the following hypothesis.

(H8) λ_1 is a pole of order one for the matrix $\lambda I - A$. (The Jordan blocks of $\lambda I - A$ corresponding to λ_1 are of order 1.)

Remark 1 *If (H8) is not satisfied, then $\lambda(\epsilon)$ may not be expanded as integer powers of ϵ. A discussion of asymptotic expansions for eigenvalues of an ϵ dependent matrix can be found in [**?**].*

Condition (H8) is always satisfied if all the eigenvalues of the coupling matrix are distinct, which is certainly true if mono-internal layer solutions are considered.

Based on (H8), $(c_0^1, \cdots, c_0^r)$ is not in the range of $\lambda_1 I - A$. (62) uniquely determines λ_j and $\{c_{j-1}^i\}_1^r$, due to the normalization

$$(c_{j-1}^1, \cdots, c_{j-1}^r) \perp (c_0^1, \cdots, c_0^r).$$

It is clear with such λ_j and $\{c_{j-1}^i\}_1^r$, we can uniquely find $Z_j^i = O(1 + |\xi|^j)$, $< \dot{q}^i, Z_j^i >= 0$, such that the solution for (53) has the form in the ith internal layers:

$$U_j^S = Z_j^i + c_j^i \dot{q}^i,$$

V_j^R then comes from (58). V_j^S comes from (56). After obtaining V_j^S, in the boundary layers, since $\dot{q}^i = 0$, $i = 0, r+1$, the equation for U_j^S becomes

$$U^S_{j\xi\xi} + f_u U^S_j = \ell \cdot o \cdot t.$$

Since the right hand side is of $O(1 + |\xi|^j)$, the above equation with Neumann boundary conditions can be uniquely solved for a solution $U = O(1 + |\xi|^j)$ in $\mathbb{R}^+$ or $\mathbb{R}^-$ respectively for $i = 0$, or $i = r + 1$. See Lemma 2.

Mono layer solutions and a geometric method

We first introduce a geometric method to determine mono-layer solutions. We show that the geometric method also determines λ_1, hence, the stability of the mono-layer solution. In the end of this section, we comment on the relation of our approach with the geometric singular perturbation theory.

Let Φ_- and Φ_+ be the solution maps of (21) where $h = h^0$ and $h = h^1$ respectively for all x. Since Φ_- and Φ_+ are transverse to Γ^0 and Γ^2, the following are $(n+1)$-dimensional manifolds,

$$\mathcal{M}_- = \cup\{\Phi_-(x)\mathcal{S}_0 | 0 \le x \le 1\},$$
$$\mathcal{M}_+ = \cup\{\Phi_+(x-1)\mathcal{S}_1 | 0 \le x \le 1\}.$$

Lemma 8 *If (H4)-(H6) are satisfied, then $\mathcal{M}_-$ intersects with $\mathcal{M}_+$ transversely. The intersection $\mathcal{C}$ is a smooth one-dimensional curve that satisfies $\mathcal{C} \pitchfork \Gamma^1$.*

Conversely, if $\mathcal{M}_- \pitchfork \mathcal{M}_+$ and the intersection $\mathcal{C}$ satisfies $\mathcal{C} \pitchfork \Gamma^1$, then (H4)-(H6) are satisfied.

proof Denote $\wp = (v^\dagger, w^\dagger, x^\dagger)$ the intersection of $\mathcal{C}$ and Γ^1. Since Γ^1 is of codimension one, if $T_\wp\mathcal{M}_- \cap T_\wp\mathcal{M}_+$ is two dimensional, then there exists a nonzero vector

$$\mathbf{a} \in T_\wp\mathcal{M}_- \cap T_\wp\mathcal{M}_+ \cap T_\wp\Gamma^1.$$

Therefore, $\mathbf{a} \in T_\wp\mathcal{S}^1_-$. From (H5), the flow at $\wp$ is transverse to $\mathcal{S}^1_-$. The derivative of the Poincare mapping $\mathcal{P}^1$ will send $\mathbf{a}$ to a vector in the tangent spaces of both $\mathcal{S}^2_-$ and $\mathcal{S}_1$. From (H6), it must be a zero vector. The contradiction shows that $\mathcal{M}_- \pitchfork \mathcal{M}_+$.

We now show that $\mathcal{C} \pitchfork \Gamma^1$. Assume a vector $\mathbf{a} \in T_\wp\mathcal{C} \subset T_\wp\Gamma^1$, then as the above, $\mathbf{a} \in \mathcal{S}^1_- \cap \mathcal{M}_+$. Thus, as before, $\mathbf{a} = 0$. This shows that $\mathcal{C} \pitchfork \Gamma^1$.

The converse of the lemma can be proved by a similarly elementary argument and will not be given here.

QED

The curve $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{M}_- \cap \mathcal{M}_+$ is called a **slow switching curve** since the slow flow has to switch from $u = h^0(v)$ to $h^1(v)$ at some $\wp \in \mathcal{C}$ in order to satisfy boundary conditions at $x = 0, 1$. $\mathcal{C}$ is not a solution curve of (21) if the slow equation has a jump causing by $h^0 \neq h^1$.

We have obtained the following theorem.

Theorem 9 *Assume that (H1)–(H3) are satisfied and $\mathcal{C} \pitchfork \Gamma^1$ at some nonempty point $\wp = (v^\dagger, w^\dagger, x^\dagger)$. Then there exists a singular mono-internal layer solution with the internal layer at $x = x^\dagger$ and $(v(x^\dagger), w(x^\dagger)) = (v^\dagger, w^\dagger)$. Moreover, the asymptotic expansions of the mono-internal layer solution to any powers of ϵ can be calculated recursively as in §3.*

Let $(\Delta v, \Delta w, \Delta x)$ be a nonzero tangent vector of $\mathcal{C}$ at $\wp$. We can show that $\Delta x \neq 0$. For otherwise using $(\Delta v, \Delta w) \neq 0$ as an initial condition at $x = x_0$, the linear system

$$\begin{aligned} V_x &= W, \\ W_x &= (g_u f_u^{-1} f_v - g_v) V, \end{aligned}$$

where $h = h^0$ if $x < 0$, $h = h^1$ if $x > x_0$, has a nontrivial solution that is C^1 on $[0, 1]$. This is a contradiction to (H7).

After rescaling, assume that $\mathbf{t} = (\Delta v, \Delta w, -1)$ is a tangent vector of $\mathcal{C}$ at $\wp$. We have the following simple result.

Theorem 10 $\lambda_1 = \mathbf{n^1} \cdot \Delta v$ *where* $\mathbf{n^1}$ *is the normal of the surface* Σ^1 *as in* §*3 and* $(\Delta v, \Delta w, -1)$ *is a tangent vector of* $\mathcal{M}_- \cap \mathcal{M}_+$ *at* $(v_0^R(x_0), w_0^R(x_0), x_0)$*. Let* $\mathbf{N} = (\mathbf{n}^1, 0, 0)$ *be a normal of* Γ^1*. The result can also be expressed as*

$$\lambda_1 = \mathbf{t} \cdot \mathbf{N}.$$

proof Since there is only one internal layer, we drop the super-index $i = 1$ for the layer. Let (V_c, W_c) be a solution of the system (42) with

$$[V](x_0) = 0,$$
$$[W](x_0) = g(q(-\infty), \bar{v}) - g(q(\infty), \bar{v}).$$

Then

$$(V_c(x_0), W_c(x_0-), 0) \in T_\wp \mathcal{M}_-,$$
$$(V_c(x_0), W_c(x_0+), 0) \in T_\wp \mathcal{M}_+.$$

The following two vectors are equal,

$$\begin{aligned}(V_c(x_0) - w_0^R(x_0), W_c(x_0-) + g(q(-\infty), \bar{v}), -1)\\ = (V_c(x_0) - w_0^R(x_0), W_c(x_0+) + g(q(\infty), \bar{v}), -1).\end{aligned} \quad (63)$$

It is clear that

$$(-w_0^R(x_0), g(q(-\infty), \bar{v}), -1) \in T_\wp \mathcal{M}_-,$$
$$(-w_0^R(x_0), g(q(\infty), \bar{v}), -1) \in T_\wp \mathcal{M}_+,$$

since they are flows of Φ_- and Φ_+ respectively. Therefore the common vector in (63) is in

$$T_\wp \mathcal{C} = T_\wp \mathcal{M}_- \cap T_\wp \mathcal{M}_+.$$

Thus, we must have $V_c(x_0) - w_0^R(x_0) = \Delta v$. The desired result now follows from Theorem 7 where

$$A = (a_{11}) = \mathbf{n}^1 \cdot (V_c(x_0) - w_0^R(x_0)) = \mathbf{n}^1 \cdot \Delta v.$$

QED

A similar theorem can be stated for the existence of a singular heteroclinic solution which has an internal layer. Let $(p^i, 0)$ be a hyperbolic equilibrium for the reduced system $v' = w$, $w' = -g(h^i(v), v)$ with $i = 0, 1$. Assume that $\dim W^u((p^0, 0)) - \dim W^u((p^1, 0)) = 1$ where the stable unstable manifolds of $(p^i, 0)$ are referred to the vector fields with $h = h^i$. Assume the nonempty transversal intersection of $W^u((p^0, 0))$ and $W^s((p^1, 0))$ on $\mathbb{R}^{2n}$. Then $\mathcal{C} \stackrel{\text{def}}{=} W^u(p^0) \cap W^s(p^1)$ is a smooth one dimensional curve. Define Γ^1

to be the set of points $(v, w) \in \mathbb{R}^{2n}$ where $u'' + f(u, v) = 0$ has a heteroclinic solution connecting $u = h^0(v)$ to $h^1(v)$.

We can show if $\mathcal{C} \pitchfork \Gamma^1$ at a nonempty point, then there exists a singular internal layer solution connecting $(u, v) = (h^0(p^0), p^0)$ to $(u, v) = (h^1(p^1), p^1)$. The singular heteroclinic solution has an internal layer based at $\mathcal{C} \cap \Gamma^1$. Moreover, asymptotic expansions of internal layer solutions can be obtained to any order of ϵ. The critical eigenvalue can also be determined by the angle of intersection of $\mathcal{C}$ and Γ^1. This is most useful if $\mathcal{C}$ and Γ^1 has multiple intersection points, for it shows that generically the stability index of these mono-layered solutions changes alternatively. See the example in §6.3.

There is a close relation between our approach to the geometric singular perturbation theory. According to Fenichel [**?**], there exist smooth stable and unstable manifolds in $\mathbb{R}^{2m+2n}$ of the normally hyperbolic slow manifolds $u = h^i(v)$, $i = 0, 1$. These manifolds admit smooth foliations by strongly stable and unstable fibers respectively. Let $\mathfrak{M}^-$ be the union of unstable fibers passing through $(u, 0, v, w)$ with $u = h^0(v)$, $(v, w) \in W^u((p^0, 0))$ and let $\mathfrak{M}^+$ be the union of stable fibers passing through $(u, 0, v, w)$ with $u = h^1(v)$, $(v, w) \in W^s((p^1, 0))$. Using the geometric singular perturbation theory, if $\mathfrak{M}^-$ intersects transversely with $\mathfrak{M}^+$ at $\epsilon = 0$, then they also do so at small ϵ. The internal layer solution is determined by this intersection.

It can be shown that the transverse intersection of $\mathfrak{M}^-$ and $\mathfrak{M}^+$ is equivalent to the condition $\mathcal{C} \pitchfork \Gamma^1$. Details are left to the readers. We have found a simple way to check Fenichel's transversal condition in $\mathbb{R}^{2m+2n}$ by reducing it to a lower dimensional space $\mathbb{R}^{2n}$.

Suitable changes can also be made for the case of a singular traveling wave solution by included the wave speed as a phase variable.

Let us return to the original boundary value problem with boundary conditions at $x = 0, 1$. Again, the slow manifolds are normally hyperbolic. Let $\mathfrak{M}^-$ be the union of strongly unstable fibers passing through $(u, 0, v, w, x)$ with $u = h^0(v)$, $(v, w, x) \in \mathcal{M}_-$ and let $\mathfrak{M}^+$ be the union of strongly stable fibers passing through $(u, 0, v, w, x)$ with $u = h^1(v)$, $(v, w, x) \in \mathcal{M}_+$. We prove that $\mathcal{C} \pitchfork \Gamma^1$ is equivalent to the transversal intersection of $\mathfrak{M}^-$ and $\mathfrak{M}^+$ as follows.

Let us write $u_{\xi\xi} + f(u, v) = 0$ into a system $u_\xi = \hat{u}$, $\hat{u}_\xi + f(u, v) = 0$. At the singular limit $\epsilon = 0$, we pick a point $p = (u, \hat{u}, v, w, x) \in \mathfrak{M}^- \cap \mathfrak{M}^+$ where $\wp = (v, w, x)$ is on $\mathcal{M}_- \cap \mathcal{M}_+$ and $(u, \hat{u}) = (q(0), \dot{q}(0))$ is on the heteroclinic solution $(q, \dot{q})$ connecting $h^0(v)$ to $h^1(v)$. Let

$$(\Delta u, \Delta \hat{u}, \Delta v, \Delta w, \Delta x) \in T_p\mathfrak{M}^- \cap T_p\mathfrak{M}^+.$$

Then $(\Delta v, \Delta w, \Delta x) \in T_\wp \mathcal{C}$. On the other hand, since moving along $(\Delta v, \Delta w, \Delta x)$ does not break the heteroclinic solution, we must have $(\Delta v, \Delta w, \Delta x) \in T_\wp \Gamma^1$.

If $\Gamma^1 \pitchfork \mathcal{C}$, from the above argument, we have $(\Delta v, \Delta w, \Delta x) = 0$, and the tangent vector $(\Delta u, \Delta \hat{u}, 0, 0, 0)$ is on $T_\wp W^u \cap T_\wp W^s$. But the strongly unstable fiber $W^u(\wp)$ has an one-dimensional intersection with the strongly stable fiber $W^s(\wp)$. This shows that $(\Delta u, \Delta \hat{u}) = C(\dot{q}(0), \ddot{q}(0))$ where C is a scalar, and $T_p\mathfrak{M}^- \cap T_p\mathfrak{M}^+$ is one-dimensional. Since $\dim T_p\mathfrak{M}^- = \dim T_p\mathfrak{M}^+ = 2n + 1 + m$ and the intersection occurs in a $2n + 2m + 1$ dimensional space, thus $\mathfrak{M}^- \pitchfork \mathfrak{M}^+$. The converse is also true.

Examples

A x–dependent system.

When the matrix coupling A is diagonal, then there is no coupling among the internal layers through the slow field up to $O(\epsilon)$, and the r eigenvalues, λ_1, are determined locally layer by layer. This happens if the jumps $\mathcal{M}^i \stackrel{\text{def}}{=} g(q^i(-\infty), \bar{v}^i) - g(q^i(\infty), \bar{v}^i) = 0$ for $1 \le i \le r$.

As a special case, consider the following x dependent system

$$\begin{aligned} &\epsilon^2 u_{xx} + f(u,x) = 0, \quad u \in \mathbb{R}^m,\ 0 < x < 1, \\ &u_x = 0, \quad x = 0, 1. \end{aligned}$$

Letting $v = x$, we have (3) with $g = 0$, whence $\mathcal{M}^i = 0$ for all i. Therefore, $(V_c^i, W_c^i) = 0$ and $V_1^R = 0$. See (**??**). The coupling matrix has the simplest form

$$A = -\text{diag}(\mathbf{n}^i \cdot w_0^R(x_0^i))_{i=1}^r.$$

Using $w_0^R = v_{0x}^R = 1$ and (20), we have the r eigenvalues

$$\lambda_1 = - < \psi^i, \, f_x(q^i(\xi), x_0^i) >, \quad 1 \le i \le r.$$

The above formula for λ_1 is valid when u and f are in $\mathbb{R}^m, m \ge 1$. For scalar equations, observing that the linear equation $U_{\xi\xi} + f_u(\dot{q}^i(\xi), x_0^i)U = 0$ is self adjoint in $L^2(\mathbb{R})$, we must have $\psi^i = \dot{q}^i/|\dot{q}^i|^2$. See (H2). Therefore,

$$\begin{aligned}\lambda_1 &= -|\dot{q}^i|^{-2} \int_{-\infty}^{\infty} \dot{q}^i(\xi) f_x(q^i(\xi), x_0^i) d\xi \\ &= -|\dot{q}^i|^{-2} \frac{d}{dx} \int_{q^i(-\infty)}^{q^i(\infty)} f(u, x_0^i) du.\end{aligned}$$

Following Fife [?], let $J^i(x) = \int_{h^{i-1}(x)}^{h^i(x)} f(u, x) du$. Then

$$\lambda_1 = -|\dot{q}^i|^{-2} \frac{d}{dx} J^i(x_0^i), \quad 1 \le i \le r.$$

The existence of a heteroclinic solution at x^i is equivalent to $J^i(x_0^i) = 0$ (equal area rule), while (H3) is equivalent to $\frac{d}{dx} J^i(x_0^i) \neq 0$.

In the original AMP model, $f(u,x) = (1-u^2)(u-a(x))$. If q^i connects $u=-1$ to $u=1$, then $J^i(x) = -\frac{4}{3}a(x)$. Thus $J^i(x_0^i) = 0 \Leftrightarrow a(x_0^i) = 0$, and $\lambda_1 = \frac{4}{3}|\dot{q}^i|^{-2}a'(x_0^i)$. It is known that $q^i(\xi) = \tanh(\frac{\xi}{\sqrt{2}})$ and $|\dot{q}^i|^2_{L^2} = \frac{2\sqrt{2}}{3}$. Therefore, $\lambda_1 = \sqrt{2}a'(x_0^i)$.

Similarly, for the internal layer jumping from near $u=1$ to $u=-1$, we can show that $\lambda_1 = -\sqrt{2}a'(x_0^i)$.

We summarize the results in the following

Theorem 11 *For the x-dependent system, A is diagonal, with $\lambda_1 = - < \psi^i, f_x(q^i(\xi), x_0^i) >$, $1 \le i \le r$. In particular, for the AMP model $\lambda_1 =$ sign$\{\dot{q}^i\}\sqrt{2}a'(x_0^i)$.*

The stability index of the multi-layered solution derived from above agrees with the result in [**?**].

Coupled Ginzburg-Landau equations

Consider (1) with $f(u,v) = u - u^3 - \frac{1}{3}v$, $g(u,v) = \sigma(v - v^3)$ and $u_x = v_x = 0$ at $x = 0, 1$. The stationary solution of this system is a pair of Duffing oscillators with a unilateral coupling in the fast equation. We will show that Lemma A in §1 is not satisfied in this example. This example is highly special since the slow equation does not contain u. In the end of this subsection, we will give another example where coupling terms appear in both equations.

As in the examples in §1, $f(u,v) = 0$ has three branches of solution manifolds $u = h_0(v)$ and $u = h_\pm(v)$. At $\bar{v} = 0$, (4) has a heteroclinic loop, $q(\xi)$ and $q(-\xi)$, connecting $u = h_-(0)$ and $u = h_+(0)$. Write the second equation of (3) as a system

$$\begin{aligned} v_x &= w, \\ w_x &= -\sigma(v - v^3). \end{aligned} \tag{64}$$

For any constant $\sigma > 0$, (64) has three equilibria $(v,w) = (0,0)$ and $(\pm 1, 0)$ and has two heteroclinic orbits connecting the hyperbolic equilibria $(\pm 1, 0)$. Notice that the interval $[-1, 1]$ is contained in the domains of $h_\pm(v)$. The region bounded by the heteroclinic loop is filled up with periodic solutions that surround the center $(0,0)$. For every point $(\eta,0), 0<\eta<1$ on the v axis, there passes a unique periodic orbit whose period will be denoted $d(\eta)$. Using an elliptic integral one can show that $d'(\eta)>0$ and there exist one sided limits $d(0^+)=\frac{2\pi}{\sqrt{\sigma}}$ and $d(1)=\infty$. For any $m\in\mathbb{N}^+$, let σ be sufficiently large so that $\frac{2\pi}{\sqrt{\sigma}}<\frac{2}{m}$. Fix that σ. It is clear from the above that there exists a unique $0<\eta_0<1$ such that $d(\eta_0)=2/m$. Let $(v(x),w(x))$ be the period $2/m$ solution that satisfies $v(0)=\eta_0, w(0)=0$. Let $(v_0^R(x),w_0^R(x))$ be the restriction of $(v(x),w(x))$ to $x\in[0,1]$. v_0^R satisfies the following.

$$\begin{aligned}
&v_x(\ell/m)=0, \quad \ell=0,1,\dots,m,\\
&v(\ell/m)=(-1)^\ell\eta_0, \quad \ell=0,1,\dots,m,\\
&v(\frac{1}{2m}+\frac{\ell}{m})=0, \quad \ell=0,1,\dots,m-1.
\end{aligned} \tag{65}$$

Let $\{x^1<x^2<\cdots<x^r\}$ be a subset of $\{\frac{1}{2m}+\frac{\ell}{m}:\ell=0,1,\dots,m-1\}$. It is clear that $v_0^R(x^i)=0, 1\le i\le r$. Let $x^0=0$ and $x^{r+1}=1$. For $x\in(x^{i-1},x^i)$, define $u_0^R(x)=h_+(v_0^R(x))$ if i is odd, and $u_0^R(x)=$

$h_-(v_0^R(x))$ if i is even. The function (u_0^R, v_0^R) is the 0th order expansion of a r-layered solution in regular layers. Let

$$u_0^{Si}(\xi) = \begin{cases} q^i(\xi) \equiv q(-\xi), & \text{if } i \text{ is odd,} \\ q^i(\xi) \equiv q(\xi), & \text{if } i \text{ is even.} \end{cases}$$

Let $v_0^{Si}(\xi) = 0$. The union of $(u_0^R(x), v_0^R(x))$ in regular layers $(x^{i-1}, x^i), i = 1, 2, \ldots, r+1$, and $(u^{Si}(\xi), v_0^{Si}(\xi))$ in singular layers at $x^i, i = 1, 2, \ldots, r$ is a singular internal layer solution. We can verify that Hypotheses (H1)-(H7) are satisfied by this solution.

It is trivial to verify (H1)-(H3) since the u-equation is the same as the activator-inhibitor model in §1. The **Transversality Hypothesis** in §1 is satisfied since the fast jump surface $\Gamma^i = \{(v, w, x) : v = 0\}$ is transverse to the flow at each x^i, due to $v_{0x}^R(x^i) \neq 0$. From the transversality hypothesis, (H4) and (H5) are satisfied. We only need to prove (H6) and (H7). Let $\mathcal{S}_0 = \{(v, w, x) : x = 0, w = 0\}$, $\mathcal{S}_1 = \{(v, w, x) : x = 1, w = 0\}$ as in §3.

Let $\Phi(x)(v_0, w_0)$ be the solution map for (64) with $\Phi(0)(v_0, w_0) = (v_0, w_0)$. Assume that $v_0^R(x)$ consists of m monotonic paths with

m being even. (The case m is odd can be considered similarly.) Let $(v_0^R(0), w_0^R(0)) = (\eta_0, 0)$. Then $(v_0^R(1), w_0^R(1)) = (\eta_0, 0)$ since m is even. Let $\Delta\eta$ be a small variation of η_0. The periodic solution with the initial data $(\eta_0 + \Delta\eta, 0)$ has the period $d(\eta_0 + \Delta\eta) = 2(1 + \Delta x)/m$ where Δx is small. This leads to $\Phi(1 + \Delta x)(\eta_0 + \Delta\eta, 0) = (\eta_0 + \Delta\eta, 0)$. Since $d'(\eta) > 0$, we have

$$\frac{d\Delta x}{d\Delta\eta} > 0. \tag{66}$$

A tangent vector on $(\mathcal{P}^r \ldots \mathcal{P}^1\mathcal{P}^0)\mathcal{S}_0$ can be obtained by taking the limit as $\Delta\eta \to 0$ on the following vector

$$\begin{aligned} &\frac{\Phi(1)(\eta_0 + \Delta\eta, 0) - \Phi(1)(\eta_0, 0)}{\Delta\eta} \\ &= \frac{\Phi(1)(\eta_0 + \Delta\eta, 0) - \Phi(1 + \Delta x)(\eta_0 + \Delta\eta, 0)}{\Delta\eta} \\ &+ \frac{\Phi(1 + \Delta x)(\eta_0 + \Delta\eta, 0) - \Phi(1)(\eta_0, 0)}{\Delta\eta}. \end{aligned}$$

As $\Delta\eta \to 0$, the first quotient in the right hand side has the limit $\frac{d\Delta x}{d\Delta\eta}(w_0^R(1), -\sigma(v_0^R(1)-(v_0^R(1))^3))$, due to equation (64), and the second has the limit $(1,0)$ since $\Phi(1+\Delta x)(\eta_0+\Delta\eta, 0) = (\eta_0+\Delta\eta, 0)$ and $\Phi(1)(\eta_0, 0) = (\eta_0, 0)$. Thus, $(1, -\frac{d\Delta x}{d\Delta\eta}\sigma(v_0^R(1) - (v_0^R(1))^3))$ is a tangent vector on $(\mathcal{P}^r \ldots \mathcal{P}^1\mathcal{P}^0)\mathcal{S}_0$. By (66) and $-\sigma(v_0^R(1)-(v_0^R(1))^3) \neq 0$, the tangent vector is not on $T\mathcal{S}_1$. This proves both (H6) and (H7).

The results of §3 and §4 can be used on our system since (H1)-(H7) are valid. We conclude that there is a matched asymptotic expansion of internal layer solution $(\sum \epsilon^j u_j, \sum \epsilon^j v_j)$. Due to the special form of our system, $v = v_0^R$ is independent of ϵ. There also exist asymptotic expansions of critical eigenvalues $\sum \epsilon^j \lambda_j$ and corresponding eigenfunctions $(\sum \epsilon^j U_j, \sum \epsilon^j V_j)$ both in internal and regular layers.

From (H7), we see that the eigenvalues for the problem $\lambda V = V_{xx} + g_v(v_0^R(x))V$ is nonzero. Let $\lambda(\epsilon)$, be a critical eigenvalue for (28) and (29). Using $\lambda_0 = 0$, we infer that the eigenfunction (U, V) satisfies $V \equiv 0$. Substitute into (28), the critical eigenvalue satisfies

$$\lambda U = \epsilon^2 U_{xx} + f_u U, \quad U_x(0) = U_x(1) = 0.$$

From the above, the critical eigenvalue is precisely the eigenvalue of the operator $\epsilon^2 D_{xx} + f_u$ in the function space $H^2_N(I)$, see [**?**]. Therefore, the system does not satisfies Lemma A.

However, using the method of §4, we can calculate expansions of λ to any order of ϵ. In particular, since $V_1^R = 0$, the coupling matrix $A_{r\times r}$ is diagonal. From (50), the ith critical eigenvalue satisfies $\lambda_1 = -\mathbf{n}^i \cdot v^R_{0x}(x^i)$, similar to the case in the AMP model.

We now briefly describe another example where both equations contain coupling terms. The example is adapted from the AMP model. Consider (1) again with $f(u, v) = (1-u^2)(u-\frac{1}{2}v)$, $g(u, v) = \sigma(v-v^3)+\gamma u^2 v$, and $u_x = v_x = 0$ at $x = 0, 1$. The roots of $f(u, v) = 0$ consist of three branches: $u = h_\pm(v)$ and $u = h_0(v)$ where

$h_+(v) = 1$, $h_-(v) = -1$ and $h_0(v) = \frac{1}{2}v$. (4) has a heteroclinic loop $q(\xi)$ and $q(-\xi)$ connecting the equilibria $u = \pm 1$ if $\bar{v} = 0$. In regular layers, inserting $u = \pm 1$ into the second equation of (3), we have a reduced system

$$v_{xx} + \sigma(v - v^3) + \gamma v, \quad v_x(0) = v_x(1) = 0.$$

For any $m \in \mathbb{N}^+$, as before, we can find $\sigma > 0, \gamma > 0$ so that the above has an oscillatory solution $v_0^R(x)$ that satisfies (65). For any $r \leq m$, a singular r-layered solution can be defined as in the previous example. One can verify that (H1)-(H7) are satisfied, so the method of §3 and §4 can be used to obtain asymptotic expansions for internal layer solutions and critical eigenvalues and eigenfunctions. Since $g(1, 0) = g(-1, 0)$, $\mathcal{M}^i = 0$ for all $1 \leq i \leq r$, cf. (46). Therefore, the eigenfunction (U, V) for a critical eigenvalue satisfies $V_1^R = 0$. The coupling matrix is diagonal. The critical eigenvalues for (28), (29) and for the operator $\epsilon^2 D_{xx} + f_u$ in $H_N^2(I)$ agree up to ϵ^1.

Multiple existence of mono-layer solutions.

This is the longest example and partially motivates the entire paper. As in the introduction, we consider a homotopy between the AMP and the NF types system. Assume that $a(x) = \sin(\omega x + b)$ and $\alpha \approx 1$, $\beta \approx 0$ so that the system can be treated as a perturbation to the NF type system. After rescaling, assume that $\alpha = 1$ and $0 \leq \beta \leq \beta_0$ where $\beta_0 > 0$ is independent of ϵ and is to be determined in the sequel. Only mono-layer solutions will be considered. Our goal is to show that by choosing (β, ω, b), the system may have any prescribed number of mono-layer solutions. Moreover, some of these solutions can only be found by the new shooting method using pseudo Poincare mappings.

We will only consider mono-layers that jumps upwards, so the superscript $i = 1$ which is used to index internal layers will be dropped. For the convenience, let $\beta = \frac{k}{\omega}$ where $0 \leq k \leq \beta_0 \omega$.

The assumptions on F and G are listed in A1–A5 below. The nullclines of F and G are plotted in Figure **??**. These assumptions are identical to those used in [?] and are qualitatively similar to the activator inhibitor model (1), (2).

A1. The nullcline of F is sigmoidal and consists of three curves

$$\begin{aligned} R^- &= \{(u,y) : u = h_-(y),\ y \in I_-\}, \\ R^0 &= \{(u,y) : u = h_0(y),\ y \in I_0\}, \\ R^+ &= \{(u,y) : u = h_+(y),\ y \in I_+\}, \end{aligned}$$

where

$$I_- = (y_-, \infty), \quad I_0 = (y_-, y_+), \quad I_+ = (-\infty, y_+).$$

A2. Define $J(y) = \int_{h_-(y)}^{h_+(y)} F(s,y)ds$. $J(y)$ has an isolated zero $\tilde{y} \in (y_-, y_+)$:

$$J(\tilde{y}) = 0, \quad dJ(\tilde{y})/dy < 0.$$

A3. $F_u < 0$ on R^- and R^+.

A4. $G < 0$ on R^- and $G > 0$ on R^+. $\frac{d}{dy} G(h_\pm(y), y) < 0$ for $y \in I_\pm$. The latter is equivalent to $-G_u F_u^{-1} F_y + G_y < 0$.

A5. $G_y|_{R^\pm} \leq 0$.

To use the method in our paper, we verify (H1)-(H8). For a mono-layer solution, (H8) is always satisfied. We first verify that when $k = 0$, the singular limit solution actually satisfy (H1)–(H7). We then use a perturbation method to show that (H1)–(H7) are satisfied when k/ω is small and when certain conditions are posed on parameters (k, ω, b).

The unperturbed system: $k = 0$

Under A1–A5, it is well known that there exist $\sigma_0, \epsilon_0 > 0$ such that (22) has a unique stationary mono-internal layer solution $(u(x,\epsilon), y(x,\epsilon))$ if $0 < \sigma < \sigma_0$ and $0 < \epsilon < \epsilon_0$ [**?**, **?**, **?**, **?**]. This solution jumps upwards from near $u = h_-(y)$ to near $u = h_+(y)$ at $x \approx x_0$. As $\epsilon \to 0$, this solution has a limit $(u_0^R(x), y_0^R(x))$ in two regular layers separated by $x_0 \in (0,1)$. $(u_0^R(x), y_0^R(x))$ has a jump discontinuity at the internal layer x_0. Using a stretched variable $\xi = (x - x_0)/\epsilon$, there exists the limit in the internal layer $(u(x_0 + \epsilon\xi, \epsilon), y(x_0 + \epsilon\xi, \epsilon)) \to (u_0^S(\xi), y_0^S(\xi))$ as $\epsilon \to 0$. The mono-layer solution $(u(x,\epsilon), y(x,\epsilon))$ and the limit in regular layers (u_0^R, y_0^R) are plotted in Figure **??**. In particular, the jump point x_0 satisfies $y_0^R(x_0) = \tilde{y}$ where $\tilde{y}$ as in A2, and y_0^R is concave up for $x < x_0$ and concave down for $x > x_0$; and $y_{0x}^R > 0$, $0 < x < 1$.

There is a another mono-layer solution that jumps from near $u = h_+(y)$ to near $u = h_-(y)$. But this will not be used in this section. By mono-layer solution, we always mean the one that jumps upwards.

While the existence of the mono-internal layer solution is well known, the existence of matched expansions of this solution, or the existence of matched expansions of critical eigenvalue and eigenfunctions has not been proved before. To this end, we will verify (H1)–(H7).

From A1, the slow manifolds are $R^+ := \{u = h_+(y)\}$ and $R^- := \{u = h_-(y)\}$. From A3, $f_u < 0$ on $R^+ \cup R^-$. Thus (H1) is satisfied. It is clear that $v_0^R = (y_0^R, x)^\tau$ satisfies (18) with $x_0^1 = x_0$ and $h^0 = h_-$, $h^1 = h_+$. At $x = x_0$, $\bar{v} = \begin{pmatrix} \tilde{y} \\ x_0 \end{pmatrix}$, equation

$$u_{\xi\xi} + f(u, \bar{v}) = u_{\xi\xi} + F(u, \tilde{y}) = 0$$

has a heteroclinic solution $q(\xi)$ due to the fact $J(\tilde{y}) = 0$, (the equal area rule, see A2). Thus the surface in (H2) is $\Sigma = \{(y, x) | y = \tilde{y}, x \in \mathbb{R}\}$.

The function $\dot{q}$ is clearly an eigenfunction corresponding to the eigenvalue $\lambda = 0$. Using the fact

$$U_{\xi\xi} + f_u U = 0, \tag{67}$$

has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$, we see that $(q(0), \dot{q}(0))$ is in the intersection of the unstable subspace at $0-$ and stable subspace at $0+$, both are one dimensional. Thus, the eigenspace is spanned by $\dot{q}$. Equation (67) is self adjoint. Let $\psi = \dot{q}/\|\dot{q}\|_{L^2}$.

Then $< \psi, \dot{q} >= 1$ and $\dot{q}$ is not in the range of the operator $\partial_{\xi\xi} + f_u \cdot I$. Condition (H2) is satisfied.

The normal of Σ is

$$\begin{aligned}\mathbf{n} &= \int_{-\infty}^{\infty} f_v^{\tau}(q(\xi), \bar{v}), \psi(\xi) d\xi \\ &= \|\dot{q}\|^{-2} \int_{h_-(\tilde{y})}^{h_+(\tilde{y})} F_v(u, \tilde{y}) du \\ &= \|\dot{q}\|^{-2} J_v(\tilde{y}) \\ &= \|\dot{q}\|^{-2} J'(\tilde{y}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.\end{aligned}$$

since $w_0^R(x_0) = \begin{pmatrix} y_{0x}^R(x_0) \\ 1 \end{pmatrix}$, we have $\mathbf{n} \cdot w_0^R(x_0) = J'(\tilde{y}) \cdot y_{0x}^R(x_0) \neq 0$. Therefore, (H3) is satisfied. See A2. Also the flow is transverse to Σ.

The major job is to verify (H4)-(H7). We use a geometric method similar to that used in Theorem 10. Since the flow of the x variable is trivial, it is reasonable to consider a reduced system that is equivalent to (21). (Equation (21) is 5–dimensional.)

Lemma 12 *For the unperturbed system, $k = 0$, (H7) is satisfied. Also, $\mu_0 \pitchfork \mu_1$ at $\wp$ in $\bar{\Pi}$. See Figure* **??**.

proof We first prove (H7). From (H7), it is easy to see that $\mu_0 \pitchfork \mu_1$ at $\wp$ in $\bar{\Pi}$.

Let (Y^-, Z^-) be the solution of the linear variational system of the of (23) around the 0th order expansion, with $k = 0$,

$$\begin{aligned}
&Y_x = Z, \\
&Z_x = (g_u f_u^{-1} f_v - g_v) Y, \\
&Y(0) = 1, \\
&Z(0) = 0,
\end{aligned}$$

If we recall that $u^R = h(y^R)$ where $h = h_-$ if $x < x_0$ and $h = h_+$ if $x > x_0$, then using $g_u f_u^{-1} f_v - g_v = \sigma \frac{d}{dy} G(h(y), y) < 0$ for $x \neq x_0$, it is easy to show that $Y^{-}(x_0) > 1$, $Z^{-}(x_0) > 0$. Similarly, the solution (Y^{+}, Z^{+}) of the linear variational system

$$
\begin{aligned}
&Y_x = Z,\\
&Z_x = (g_u f_u^{-1} f_v - g_v)Y,\\
&Y(1) = 1,\\
&Z(1) = 0,
\end{aligned}
$$

satisfies $Y^{+}(x_0) > 1$, $Z^{+}(x_0) < 0$.

Assume that (H7) is not satisfied for the mono-layer solution. Then there exists a nonzero C^1 solution V to (42). Without loss of generality, assume that $V(x_0) > 0$. Then there exist $\gamma_1, \gamma_2 > 0$ such that $V(x_0) = \gamma_1 Y^{-}(x_0) = \gamma_2 Y^{+}(x_0)$. However, we have a contradiction $V_x(x_0) = \gamma_1 Z^{-}(x_0) > 0$ and $V_x(x_0) = \gamma_2 Z^{+}(x_0) < 0$.

QED

Since $\frac{d}{dx} y_0^R(x_0) > 0$, $\bar{\Gamma}_1$ is a cross section of Φ_- and Φ_+, regular Poincare mapping $\mathcal{P}^i : \bar{\Gamma}_i \to \bar{\Gamma}_{i+1}$, $i = 0, 1$ can be defined. Hypotheses (H4)-(H5) are clearly satisfied.

Lemma 12 implies that $\mathcal{M}_-$ intersects $\mathcal{M}_+$ transversely. The intersection is an one-dimensional curve $\mathcal{C}$ passing through $\wp$. As in §5, it is called a **slow switching curve** since the slow flow has to switch from $u = h_-(y)$ to $h_+(y)$ in order to satisfy boundary conditions for y at $x = 0, 1$.

Let (Y^c, Z^c) be a solution to the following linear system:

$$\begin{aligned} &dY/dx = Z, \\ &dZ/dx = -\sigma \tfrac{d}{dy} G(h(y), y)Y, \\ &Z(0) = 0,\ Z(1) = 0, \\ &[Y](x_0) = 0, \\ &[Z](x_0) = \sigma(G(h_-(\tilde{y}), \tilde{y}) - G(h_+(\tilde{y}), \tilde{y})). \end{aligned} \tag{68}$$

From Lemma 12, such solution (Y^c, Z^c) uniquely exists. Similar to §5, we can show that the vector

$$\begin{aligned} &(Y^c(x_0) - z_0^R(x_0), Z^c(x_0-) + \sigma G(h_-(\tilde{y}), \tilde{y}), -1) \\ =&(Y^c(x_0) - z_0^R(x_0), Z^c(x_0+) + \sigma G(h_+(\tilde{y}), \tilde{y}), -1) \end{aligned}$$

is a tangent vector of $\mathcal{C}$ at $(y_0^R(x_0), z_0^R(x_0), x_0)$. Nishiura & Fujii in [**?**] proved that $Y^c(x_0) - z_0^R(x_0) > 0$. Thus, $\mathcal{C}$ intersects $\bar{\Gamma}_1$

transversely. Suppose that Φ_- maps $\bar{\mathcal{S}}_0$ onto $\mathcal{S}_-$ in $\bar{\Gamma}_1$ and Φ_+ maps $\bar{\mathcal{S}}_1$ onto $\mathcal{S}_+$ in $\bar{\Gamma}_1$. It is now easy to show that

$$\mathcal{S}_- \pitchfork \mathcal{S}_+ \quad \text{on} \quad \bar{\Gamma}_1.$$

In fact, since $\mathcal{C}$ intersects $\bar{\Gamma}_1$ transversely, the tangent spaces of $\mathcal{M}_\pm$ have a common subspace $T_\wp\mathcal{C}$ which is not on $T_\wp\bar{\Gamma}_1$. If the tangent spaces of $\mathcal{S}_-$ and $\mathcal{S}_+$ coincide, then $T_\wp\mathcal{M}_- = T_\wp\mathcal{M}_+$. This is contradictory to $\mathcal{M}_- \pitchfork \mathcal{M}_+$. From here we deduce that (H6) is satisfied.

Perturbed internal layer solution: $k \neq 0$

We now show that (H1)-(H7) are still valid if k/ω is small.

We use our geometric method to construct singular limit solutions of (23) and show that (H2)-(H7) are satisfied for these solutions. Since (23) is piecewise C^∞, it is convenient to find the matching point $\wp = (y^\dagger, z^\dagger, x^\dagger)$ first. With $\wp$ as an initial point at $x = x^\dagger$, a solution can be obtained by solving (23) in $[0, x^\dagger]$ and $[x^\dagger, 1]$. Notice that when $k = 0$, $\wp = (\tilde{y}, z_0^R(x_0), x_0)$.

As in the case $k = 0$, let Φ_- and Φ_+ be respectively solution maps of (23) with $h = h_-$ and $h = h_+$ throughout $x \in [0, 1]$. Define $\mathcal{M}_-$ and $\mathcal{M}_+$ as before.

The intersection of the three manifolds $\mathcal{M}_\pm$ and $\bar{\Gamma}_1$ determines the matching point $\wp$. Since it is difficult to study the intersection of $\bar{\Gamma}_1$ with $\mathcal{M}_-$ or $\mathcal{M}_+$, we study the intersection of $\mathcal{M}_-$ and $\mathcal{M}_+$ first.

Lemma 13 *The distances between $\mathcal{M}_\pm$ and the corresponding ones with $k = 0$ are $O(\frac{k}{\omega^2})$ in C^0 metric and are and are $O(\frac{k}{\omega})$ in C^1 metric. When $k \neq 0$ but k/ω is small, $\mathcal{M}_-$ and $\mathcal{M}_+$ intersect transversely. The intersection $\mathcal{C} = \mathcal{M}_- \cap \mathcal{M}_+$ is a C^1 curve, and its distance from the one with $k = 0$ is $O(k/\omega^2)$ in C^0 metric. and is $O(k/\omega)$ in C^1 metric.*

proof With the initial data $(y,z,x)=(\eta,0,0)$, $\mathcal{M}_-$ can be expressed as

$$\mathcal{M}_- = \cup\{(y,z,x)|(y,z,x)=\Phi_-(x;\eta,0,0;k), \quad 0\le x\le 1,\, \eta\in\mathbb{R}\},$$

where Φ_- is the solution map of (23) with $h=h_-$. Also, $\frac{\partial\Phi_-}{\partial k}$ satisfies the linear variational system

$$\begin{aligned}
&(y_k)'=z_k,\\
&(z_k)'=-\sigma\frac{d}{dy}G(h(y+\frac{k}{\omega}\sin(\omega x+b)),y)y_k-\sigma\frac{\partial}{\partial u}G(h,y)h'\frac{1}{\omega}\sin(\omega x+b),\\
&(x_k)'=0.
\end{aligned} \tag{69}$$

The forcing term for (69) is of $O(\frac{1}{\omega})$, thus, in general $(y_k,z_k,x_k)=O(\frac{1}{\omega})$ only. However, since $\sin(\omega x+b)$ is fast oscillatory, using a standard method in the theory of averaging, we have

$$(y_k,z_k,x_k)=O(\frac{1}{\omega^2}).$$

This proves that

$$\Phi_-(x;\eta,0,0;k)-\Phi_-(x;\eta,0,0;0)=O(\frac{k}{\omega^2}).$$

Thus, the distance between $\mathcal{M}_-$ and the one with $k = 0$ is of $O(k/\omega^2)$ in the C^0 metric. Using the same method, one can show

$$\frac{\partial}{\partial \eta}\{\Phi_-(x;\eta,0,0;k) - \Phi_-(x;\eta,0,0;0)\} = O(\frac{k}{\omega}).$$

From the right hand side of system (69), we have

$$\frac{\partial^2}{\partial k \partial x}\Phi_-(x;\eta,0,0;k) = O(1/\omega).$$

Therefore, the distance between $\mathcal{M}_-$ and the one with $k = 0$ is of $O(k/\omega)$ in the C^1 metric.

The statements about $\mathcal{M}_+$ can be proved similarly.

The assertions concerning $\mathcal{C}$ can be proved using the implicit function theorem, or a contraction mapping principle and will not be given here. QED

When $k = 0$,

$$(Y^c(x_0) - z_0^R(x_0), Z^c(x_0-) + \sigma G(h_-(\tilde{y}), \tilde{y}), -1)$$

is a tangent vector of $\mathcal{C}$ at $\wp$ with the y component being positive. Therefore, locally the curve $\mathcal{C}$ can be expressed as

$$x = x^*(y, b), \quad z = z^*(y, b), \quad \tilde{y} - T \leq y \leq \tilde{y} + T, \tag{70}$$

where x^* and z^* are C^1 functions and $T > 0$ is a constant.

Lemma 14 *$\frac{\partial x^*}{\partial y}(\tilde{y}, b) = -(Y^c(x_0) - z_0^R(x_0))^{-1}$ if $k = 0$. If $k \neq 0$ then,*

$$\frac{\partial}{\partial b} x^*(y, b) = O(k/\omega^2),$$
$$\frac{\partial}{\partial b}(\frac{\partial x^*(y, b)}{\partial y}) = O(k/\omega).$$

proof The assertion for $k = 0$ is obvious.

To prove the assertion about $\frac{\partial x^*}{\partial b}$, we can use a linear variational system to show that $\Phi_\pm$ is a C^1 function of b and $\frac{\partial \Phi_\pm}{\partial b} = O(k/\omega^2)$ in C^0 metric and is of $O(k/\omega)$ in C^1 metric. Similar to the proof of Lemma 13, the fast oscillatory property of $\sin(\omega x + b)$ is important in the proof.

Final remarks and stability of internal layer solutions

1. Our methods of constructing asymptotic series for the internal layer solutions and the critical eigenvalue-eigenfunctions are actually related, although one uses the pseudo-Poincare mapping or the (BVPIC), the other uses the coupling matrix (SLEP matrix). An intuitive reason is that the unknown shift $\{\Delta x^i\}$ in the (BVPIC) can also be formulated by adding multiples of $\dot{q}^i$ in the ith internal layer as in the coupling matrix. The following lemma asserts that asymptotic expansions for internal layer solutions can be obtained if the coupling matrix is nonsingular:

Lemma 15 *If in addition to (H4)-(H6) as in Lemma* **??**, *condition (H7) is also valid, then (BVPIC) has a unique solution if and only if the coupling matrix is nonsingular.*

proof Let V_c^i and $\mathcal{M}^i$ be as in (47). Define $\bar{V} := \sum_1^r V_c^i \Delta x^i$ which satisfies the first two equations in (BVPIC) with zero right sides, and

$$
\begin{aligned}
&[\bar{V}](x_0^i) = 0, \\
&[\bar{V}_x](x_0^i) = \mathcal{M}^i \Delta x^i = \Delta x^i [w_{0x}^R](x_0^i), \quad 1 \le i \le r.
\end{aligned}
$$

Let $\tilde{V} = V + \bar{V}$, V as in the (BVPIC), then

$$
\begin{aligned}
&\tilde{V}_{xx} - (g_u f_u^{-1} f_v - g_v)\tilde{V} = E_1, \\
&A_j \tilde{V}_x(j) + B_j \tilde{V}(j) = E_{2j}, \quad j = 0, 1, \\
&[\tilde{V}](x_0^i) = E_4^i, \\
&[\tilde{V}_x](x_0^i) = E_5^i.
\end{aligned}
$$

According to Lemma 6, the above has a unique solution $\tilde{V}$. Substituting $V = \tilde{V} - \bar{V}$ into the third equation of (BVPIC), $\mathbf{n}^i \cdot (w_0^R(x_0^i)\Delta x^i + V(x_0^i+)) = E_3^i$, we have

$$
\mathbf{n}^i \cdot (w_0^R(x_0^i)\Delta x^i - \sum_{\ell=1}^{r} V_c^\ell \Delta x^\ell) = E_3^i - \mathbf{n}^i \tilde{V}(x_0^i+).
$$

The linear system for Δx^i has a unique solution if the coefficient matrix, the negation of the coupling matrix, is nonsingular. QED

2. The name “critical eigenvalue” used in this paper is not precise. Following Nishiura and Fujii, we have only considered critical eigenvalues whose eigenfunction (U, V) has a jump in V_x across x_0^i. These eigenvalues will be called "singular" critical eigenvalues. There may be "regular" critical eigenvalues that satisfy the reduced eigenvalue problem (30) with $V \in C^1[0,1]$. Calculation of regular critical eigenvalues is quite different from the procedure given in this paper. From (H1) there exists $\gamma_1 > 0$ such that if $\mathrm{Re}\lambda > -\gamma_1$ then $(f_u - \lambda)^{-1}$ exists. To avoid regular critical eigenvalues, we assume that

(H9) There exists $\gamma > 0$ such that for $\mathrm{Re}\lambda > -\gamma$, the following equation

$$V_{xx} - (\lambda + g_u(f_u - \lambda)^{-1} f_v - g_v)V = 0, \tag{71}$$

with boundary conditions (41) does not have any piecewise smooth, nonzero solution that is in $C^1[0,1]$.

Notice that (71) comes from (30) by setting $\epsilon = 0$. With (H9), it is easy to show that there is no regular critical eigenvalue in the region $\mathrm{Re}\lambda > -\gamma$.

For system (22) with $k = 0$, Nishiura and Fujii showed that (H9) is satisfied [**?**]. If k/ω is sufficiently small, (H9) can be verified easily as a small perturbation to the one with $k = 0$.

3. To use the expansions of critical eigenvalues in the stability analysis, we need to prove that, in the region $\text{Re}\lambda \geq -\beta_0 > -\gamma$, all the eigenvalues are the singular critical eigenvalues obtained in §4.

Consider an asymptotic series $\mu(\epsilon) = \sum_j \epsilon^j \mu_j$ with $\mu_0 > -\beta_0$, Assume that $\mu(\epsilon)$ is not equal to any of the critical eigenvalues obtained in this paper. That is, for any critical eigenvalue $\lambda^i(\epsilon), 1 \leq i \leq r$, there exists an integer j_0^i such that $\mu_j = \lambda_j^i, j < j_0^i$ but $\mu_{j0} \neq \lambda_{j0}^i$. Let

$$j_0 = \max\{j_0^i, 1 \leq i \leq r\}.$$

We want to show that $\mu(\epsilon)$ is a regular value. Note that if the corresponding eigenfunction of $\mu(\epsilon)$ has an asymptotic expansion in ϵ, then we know that $\mu(\epsilon)$ is not an singular critical eigenvalue due to results of the previous sections. However, since we can not assume that the corresponding eigenfunction has an asymptotic expansion in ϵ, the result needs to be proved separately.

Assume that $h(x,\epsilon) = (h^u(x,\epsilon), h^v(x,\epsilon))$ is C^∞ and admits asymptotic expansions in the same regular and singular layers defined by the internal layer solution $(u(x,\epsilon), v(x,\epsilon))$. Consider the resolvent problem

$$\mu U = \epsilon^2 U_{xx} + f_u U + f_v V + h^u,$$
$$\mu V = V_{xx} + g_u U + g_v V + h^v,$$

with suitable boundary conditions at $x = 0, 1$. We look for a matched formal series solution $(U(\epsilon), V(\epsilon))$. Denote the above as

$$\mu \Xi - \mathfrak{A}(\epsilon)\Xi = h(\epsilon), \quad \Xi = (U, V), h = (h^u, h^v). \tag{72}$$

We show formally that the inverse of $\mu - \mathfrak{A}$ exists, with $(\mu - \mathfrak{A})^{-1} = O(\epsilon^{-j_0})$.

Theorem 16 *Assume that* $\mu(\epsilon) = \sum_j \epsilon^j \mu_j$, $\mu_0 > -\beta_0 = \min\{\gamma_0, \gamma\}$, *is an asymptotic series that is not equal to any of the critical* *eigenvalues obtained in this paper. Let j_0 be the largest of the powers as above. Then for any $h(\epsilon) = \sum_j \epsilon^j h^j$ with $h_j = 0$ for $j < j_0$ and $h_{j_0} \neq 0$, the eigenvalue problem (72) has a unique matched formal series solution $\Xi = \sum_{j=0}^{\infty} \epsilon^j \Xi_j$.*

proof **Case 1.** $j_0 = 0$, i.e. $\mu_0 \neq 0$.

Consider the ϵ^0th expansion:

In Regular layers,

$$\begin{aligned}
\mu_0 U_0^R &= f_u U_0^R + f_v V_0^R + h_0^u, \\
U_0^R &= (\mu_0 - f_u)^{-1}(f_v V_0^R + h_0^u), \\
\mu_0 V_0^R &= V_{0xx}^R + [g_u(\mu_0 - f_u)^{-1} f_v + g_v] V_0^R + g_u(\mu_0 - f_u)^{-1} h_0^u + h_0^v.
\end{aligned}$$

From (H1) and (H9), if we know the jumps (V_0^R, V_{0x}^R) at x_0^i, we can solve for (U_0^R, V_0^R).

In the ith internal layer,

$$V_{0\xi}^S = 0,\ W_{0\xi}^S = 0, \quad V_0^S = \text{constant},\ W_0^S = \text{constant}.$$

It means that there is no jump for (V_0^R, W_0^R) across x_0^i. Thus we can solve for V_0^R. We also have $V_0^S = V_0^R(x_0^i)$.

$$\mu_0 U_0^S = U_{0\xi\xi}^S + f_u U_0^S + f_v V_0^S + h_0^u.$$

From (H2), μ_0 is not an eigenvalue for the above. One can uniquely solve for U_0^S.

In the ϵ^jth expansion, we can solve for (U_j, V_j) both in regular and singular layers much like the same way for (U_0, V_0).

Case 2. $j_0 = 1$. In this case $\mu_0 = 0$, $\mu_1 \neq \lambda_1^i$ for any $1 \leq i \leq r$ and $h_0 = 0$ but $h_1 \neq 0$.

In the ϵ^0-th expansion, since $\lambda_0 = 0$, $h_0 = 0$, we have

$$\begin{aligned} U_0^R = 0,\ V_0^R = 0, &\quad \text{in regular layers,} \\ V_0^{Si} = 0,\ U_0^{Si} = d_0^i \dot{q}^i, &\quad \text{in the } i\text{-th singular layer.} \end{aligned}$$

d_0^i remains to be determined.

Consider the ϵ^1th expansion:

In the regular layer, since $\mu_0 = 0$, $U_0^R = 0$, $V_0^R = 0$, then

$$
\begin{aligned}
0 &= f_u U_1^R + f_v V_1^R + h_1^u, \\
U_1^R &= -f_u^{-1}(f_v V_1^R + h_1^u), \\
V_{1xx}^R - (g_u f_u^{-1} f_v - g_v) V_1^R &= g_u f_u^{-1} h_1^u - h_1^v.
\end{aligned}
$$

From (H1) and Lemma 6, if we know the jump of (V_1^R, V_{1x}^R) across each x_0^i, we can solve for (U_1^R, V_1^R).

In the i-th singular layer, since $V_0^S = 0$, $\mu_0 V_1^S + \mu_1 V_0^S = 0$, then

$$
\begin{aligned}
V_{1\xi}^S &= W_0^S = 0, \\
W_{1\xi}^S &= -g_u d_0^i \dot{q}^i.
\end{aligned}
$$

Integrate from $\xi = -\infty$ to ∞, and use the matching principle,

$$
\begin{aligned}
V_1^R(x_0^i+) - V_1^R(x_0^i-) &= 0, \\
W_1^R(x_0^i+) - W_1^R(x_0^i-) &= d_0^i \mathcal{M}^i.
\end{aligned}
$$

We can now solve for (V_1^R, W_1^R). They are of the form

$$V_1^R = \sum_1^r d_0^i V_c^i + \ell \cdot o \cdot t, \quad W_1^R = \sum_1^r d_0^i W_c^i + \ell \cdot o \cdot t.$$

Here $\ell \cdot o \cdot t$ involves h_1. Using matching, $V_1^S = V_1^R(x_0^i)$. Plug into

$$\mu_1 d_0^i \dot{q}^i = U_{1\xi\xi}^S + f_u U_1^S + f_v V_1^S + d_0^i (f_{uu} \dot{q}^i u_1 + f_{uv} \dot{q}^i v_1) + h_1^u,$$

we can solve for U_1^S if the nonhomogeneous terms are orthogonal to ψ^i. Integration by parts as in (49),

$$d_0^i \mu_1 = \sum_{\ell=1}^r d_0^\ell \mathbf{n}^i \cdot V_c^\ell(x_0^i) - d_0^i \mathbf{n}^i \cdot w_0^R(x_0^i) + < \psi^i, h_1^u > + \ell \cdot o \cdot t.$$

$$\mu_1 \begin{pmatrix} d_0^1 \\ \vdots \\ d_0^r \end{pmatrix} = A \begin{pmatrix} d_0^1 \\ \vdots \\ d_0^r \end{pmatrix} + \begin{pmatrix} < \psi^1, h_1^{u1} > \\ \vdots \\ < \psi^r, h_1^{ur} > \end{pmatrix} + \ell \cdot o \cdot t.$$

Since $\mu_1 \neq \lambda_1, 1 \leq i \leq r$, it is not an eigenvalue. The above has a unique solution $(d_0^1, \cdots, d_0^r)$.

Assume that the ϵ^{j-1}th expansion has been obtained. We have $U_{j-1}^{Si} = d_{j-1}^i \dot{q}^i + Z_{j-1}^i$ where Z_{j-1}^i has uniquely been determined but d^i_{j-1} has not. In the ϵ^j-th expansion, we can similarly show that

$$d^i_{j-1}\mu_1 = \sum_{\ell=1}^{r} d^\ell_{j-1}\mathbf{n}^i \cdot V^\ell_c(x^i_0) - d^i_{j-1}\mathbf{n}^i \cdot w^R_0(x^i_0) + < \psi^i, h^u_j > + \ell \cdot o \cdot t.$$

$$\mu_1 \begin{pmatrix} d^1_{j-1} \\ \vdots \\ d^r_{j-1} \end{pmatrix} = A \begin{pmatrix} d^1_{j-1} \\ \vdots \\ d^r_{j-1} \end{pmatrix} + \begin{pmatrix} < \psi^1, h^{u1}_j > \\ \vdots \\ < \psi^r, h^{ur}_j > \end{pmatrix} + \ell \cdot o \cdot t.$$

From this, we can solve for $(d^1_{j-1}, \cdots, d^r_{j-1})$.

Case 3. $j_0 > 1$. In this case there exists an critical eigenvalue $\lambda^i(\epsilon)$ such that $\mu_j = \lambda^i_j$, $j < j_0$, $\mu_{j_0} \neq \lambda^i_{j_0}$. We shall solve

$$\mu\Xi - \mathfrak{A}(\epsilon)\Xi = h(\epsilon) \tag{73}$$

where $h_j = 0$ for $j < j_0$, $h_{j_0} \neq 0$.

An important observation is that it suffices to find asymptotic series for (73) up to the ϵ^{j_0}th expansion. Let the normalized eigenfunction corresponding to $\lambda^i(\epsilon)$ be

$$(\bar{U}^i(\epsilon), \bar{V}^i(\epsilon)) = (\sum \epsilon^j \bar{U}^i_j, \sum \epsilon^j \bar{V}^i_j).$$

In the future, we drop the index i on λ^i_j and $(\bar{U}^i, \bar{V}^i)$. For expansions to the order ϵ^j, $j < j_0$, since $h_j = 0$, and $\mu_j = \lambda_j$, we have the same equations as the eigenvalue/eigenfunction equations. Therefore, we set

$$U_j = k_0 \bar{U}_j, \quad V_j = k_0 \bar{V}_j, \quad W_j = k_0 \bar{W}_j, \quad j < j_0,$$

except for $U^S_{j_0-1} = k_0 \bar{U}^S_{j_0-1} + d^i \dot{q}^i$ in the ith internal layer. The parameters $(d^1, \cdots, d^r)$ remain to be determined.

Consider the ϵ^{j_0}-th expansion. Let

$$\begin{aligned} U_{j_0} &= k_0 \bar{U}_{j_0} + U, \\ V_{j_0} &= k_0 \bar{V}_{j_0} + V, \\ W_{j_0} &= k_0 \bar{W}_{j_0} + W. \end{aligned}$$

Here $U = U^S, V = V^S, W = W^S$ in internal layers, and $U = U^R, V = V^R, W = W^R$ in regular layers.

In regular layers, since $(\bar{U}(\epsilon), \bar{V}(\epsilon))$ satisfies the eigenvalue equations, then all the terms multiplied by k_0 should cancel. In the ϵ^{j_0}-th expansion, after the cancellation, we have

$$\begin{aligned}
0 &= f_u U^R + f_v V^R + h^u_{j_0}, \\
0 &= V^R_{xx} + g_u U^R + g_v V^R + h^v_{j_0}, \\
U^R &= -f_u^{-1} f_v V^R - f_u^{-1} h^u_{j_0}, \\
0 &= V^R_{xx} - g_u (f_u^{-1} f_v V^R - f_u^{-1} h^u_{j_0}) + g_v V^R + h^v_{j_0}.
\end{aligned}$$

We can solve for (U, V) if jumps of (V^R, V^R_x) across x^i_0 are obtained.

In the i-th internal layer, we again can cancel all the terms in both sides of the equation that are also in the eigenvalue equation. Since

$$\mu_o U^S_{j_0} + \cdots + \mu_{j_0} U^S_0 = k_0(\lambda_1 \bar{U}^S_{j_0-1} + \cdots + \lambda_{j_0} \bar{U}^S_0) + \lambda_1 d^i \dot{q}^i + k_0(\mu_{j_0} - \lambda_{j_0}) \bar{U}^S_0,$$

After the cancellation,

$$\lambda_1 d^i \dot{q}^i + k_0(\mu_{j_0} - \lambda_{j_0}) \bar{U}^S_0 = U^S_{\xi\xi} + f_u U^S + f_v V^S + h^u_{j_0}.$$

$$
\begin{aligned}
&V_\xi^S = 0, \\
&W_\xi^S = -d^i g_u \dot{q}^i, \\
&V^R(x_0^i+) - V^R(x_0^i-) = 0, \\
&W^R(x_0^i+) - W^R(x_0^i-) = d^i \mathcal{M}^i.
\end{aligned}
$$

The solution of V^R has the form

$$
V^R = \sum_{\ell=1}^{r} d^\ell V_c^\ell + O(|h_{j_0}|).
$$

Substituting $V^S = V^R(x_0^i)$ into the equation for U^S, using the Fredholm alternative, and integrating by parts as (49), we have

$$
\lambda_1 d^i + k_0(\mu_{j_0} - \lambda_{j_0}) c_0^i = \mathbf{n}^i \cdot [\sum_{\ell=1}^{r} d^\ell V_c^\ell(x_0^i) - d^i w_0^R(x_0^i)] + O(|h_{j_0}|).
$$

$$
\lambda_1 \begin{pmatrix} d^1 \\ \vdots \\ d^r \end{pmatrix} = A \begin{pmatrix} d^1 \\ \vdots \\ d^r \end{pmatrix} - k_0(\mu_{j_0} - \lambda_{j_0}) \begin{pmatrix} c_0^1 \\ \vdots \\ c_0^r \end{pmatrix} + O(|h_{j_0}|).
$$

Since $(c_0^1, \ldots, c_0^r)^\tau$ is in the kernel of $(\lambda_1 - A)$, it is not in the range of $\lambda_1 I - A$. Since since $\mu_{j_0} \neq \lambda_{j_0}$, there exists a unique k_0 that allows the equation for $(d^1, \ldots, d^r)$ to be solved. Without loss, let $(d^1, \ldots, d^r) \perp (c_0^1, \ldots, c_0^r)$.

After the ϵ^{j_0}th order expansion has been obtained, we can compute the ϵ^{j_0+1}th and other higher order expansion by induction, with the similar method. QED

The series expansion is a formal solution to the resolvent problem. With the help of some contraction mapping and iteration method, similar to the ones outlined in the Appendix, one can show that there exists a small $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon_0$, then $\sum_0^{j_0} \epsilon^j \mu_j$ is a regular value of the internal layer solution. The constant ϵ_0 depends on $|\mu_{j_0} - \lambda_{j_0}|$.

Denote the critical eigenvalues by $\lambda^{(\ell)}(\epsilon) = \sum_0^\infty \epsilon^j \lambda_j^{(\ell)}$, $1 \le \ell \le r$. A critical eigenvalue $\lambda^{(\ell)}(\epsilon)$ is said to be stable if Re$\lambda_1^{(\ell)} < 0$. It is said to be unstable if Re$\lambda_j^{(\ell)} > 0$. We can show the following

Theorem 17 *With (H9), there exists a constant* $\epsilon_0 > 0$ *such that if* $0 < \epsilon \leq \epsilon_0$*, then the internal layer solution is unstable if there exists at least one unstable critical eigenvalue; the internal layer solution is stable if all the critical eigenvalues are stable.*

proof Only the idea of the proof is given. First if $\lambda^{(\ell)}(\epsilon)$ is an unstable eigenvalue with $\text{Re}\lambda_1^\ell > 0$, then from Theorem 21 there exists a true eigenvalue of the internal layer solution in the right half complex plane if ϵ is sufficiently small. Thus the internal layer solution is unstable.

Next assume that all the critical eigenvalues are stable. There exist $\epsilon_0 > 0$ such that all the truncated eigenvalues $\lambda^\ell(\epsilon) = \epsilon\lambda_1^{(\ell)}$ lie in the left half complex plane provided that $0 < \epsilon < \epsilon_0$. A cone centered at $\lambda^\ell(\epsilon)$ is defined as $\{\epsilon\lambda : |\lambda - \lambda_1^\ell| \leq \delta\}$ and is called a $\ell - \delta$ cone. We choose $\delta > 0$ so that all such cones lie in the left half complex plane for $0 < \epsilon \leq \epsilon_0$.

Let μ be a complex number with Re$\mu \geq 0$. Then for some sufficiently small δ, μ is not in any of the $\ell - \delta$ cone. From Theorem 16, formally μ is a regular value. Using contraction mapping argument, we can show rigorously that there exists a small ϵ_0 such that μ is a regular value if $0 < \epsilon < \epsilon_0$. Care must be taken to ensure that a common ϵ_0 can be found for all such μ. Details will be omitted due to the length of the paper. We have shown that all the eigenvalues are in the left plane Re$\lambda < 0$, therefor the internal layer solution is stable. QED

The existence of the layer solutions and the critical eigenvalue-eigenfunctions

The iteration method as stated in Lemma 18 will be used throughout this section. Let $\mathcal{L}$ be a bounded linear operator from Banach spaces E_1 to E_2. We say $\mathcal{S} : E_2 \to E_1$ is an approximate right inverse of $\mathcal{L}$ if $|I - \mathcal{L}\mathcal{S}| < 1$.

Lemma 18 *If $\mathcal{L}$ has an approximate right inverse $\mathcal{S}$, then the abstract equation $\mathcal{L}x = y$ has a (non unique) solution $x = \mathcal{S}\sum_0^{\infty}(I - \mathcal{L}\mathcal{S})^j y$. If moreover, $\mathcal{S}$ is invertible, then the solution is unique.*

In practice, E_1 is the space of solutions and E_2 is the space of forcing functions plus the space of boundary and jump terms related to a system of differential equations. The operator $\mathcal{S}$ is usually the inverse of a simplified operator $\mathcal{L}_1$ derived from $\mathcal{L}$ by dropping some coefficients, changing the forcing terms, jump terms or the deforming the domain of solutions. If $\mathcal{S} = \mathcal{L}_1^{-1}$, $\mathcal{L}_1 x_1 = y$, then $|I - \mathcal{L}\mathcal{S}| = C_1 < 1$ means

$$|(\mathcal{L}_1 - \mathcal{L})(x_1)| < C|y|, \quad \text{for all } x_1 \in E_1. \tag{74}$$

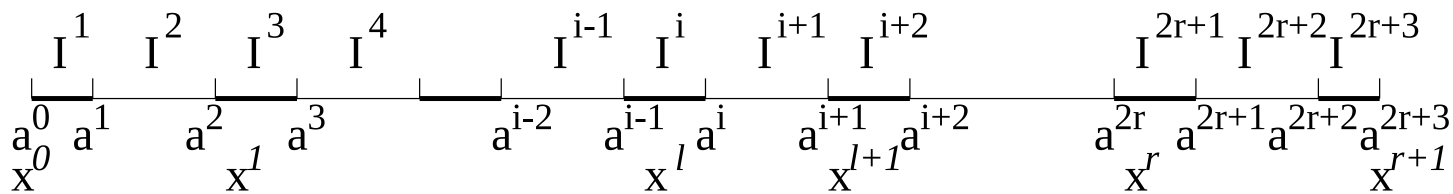
Condition (74) can be checked, a posteriori, without using the exact solution $x = \mathcal{L}^{-1}y$. To solve a difficult abstract equation, we may need to find a finite chain of operators: $\mathcal{L}_j, 1 \leq j \leq k$, satisfying (74) for any two adjacent operators. The last equation $\mathcal{L}_k x_1 = y$ must be easy to solve.

For any integer $m \geq 0$, let $x_{ap}^{\ell}(\epsilon) = \sum_0^m \epsilon^j x_j^{\ell}$, $1 \leq \ell \leq r$ be an approximation of the position of the ℓth internal layer . We look for Δx^{ℓ} so that $x_{\Delta}^{\ell} = x_{ap}^{\ell}(\epsilon) + \Delta x^{\ell}$ is the exact layer position. For convenience, we set $\Delta x^{\ell} = 0$ for $\ell = 0, r+1$. Thus, $x_{\Delta}^0 = x_{ap}^0 = 0$ and $x_{\Delta}^{r+1} = x_{ap}^{r+1} = 1$. Let ϵ^{β}, $0 < \beta < 1$, be an "intermediate variable". Define a sequence of points $a^i, 0 \leq i \leq 2r+3$. Except for $a^0 = 0$ and $a^{2r+3} = 1$, points a^1 to a^{2r+2} are defined as

$$\begin{aligned} a^{2i} &= x_{\Delta}^i - \epsilon^{\beta}, \quad 1 \leq i \leq r+1, \\ a^{2i+1} &= x_{\Delta}^i + \epsilon^{\beta}, \quad 0 \leq i \leq r. \end{aligned}$$

The interval $[0,1]$ is divided by $\{a^i\}_{i=0}^{2r+3}$ into $2r+3$ subintervals that alternatively house singular and regular layers, see Figure .

$$I^i = \{x | a^{i-1} < x < a^i, 1 \le i \le 2r+3\},$$
$I^{2\ell+1}$, $0 \le \ell \le r+1$, are for the $(r+2)$-singular layers,
$I^{2\ell}$, $1 \le \ell \le r+1$, are for the $(r+1)$-regular layers.



The partition of singular and regular layers
where $i = 2\ell+1$, $x^\ell \in I^i = I^{2\ell+1}$, the ℓth internal layer.

Let $\xi^i = a^i/\epsilon$ and $\xi = x/\epsilon$. Then in the stretched variable, $I^i = \{\xi | \xi^{i-1} < \xi < \xi^i\}$. The width of a singular layer is $O(\epsilon^\beta)$ in the x-variable, but is $O(\epsilon^{\beta-1}) >> 1$ in the ξ variable. The interval I^i is customary also called rregular or singular layers if $i = 2\ell$ or $2\ell+1$.

If $\Delta x^\ell = 0$ for $\ell = 1, \ldots, r$, then the corresponding unperturbed sequences of points and intervals are denoted by a_0^i, ξ_0^i and I_0^i.

For any integer $m \geq 0$, define the ϵ^mth approximations of eigenvalues, internal layer solutions with $W = (u, v)$, and eigenfunctions with $W = (U, V)$ by truncating the asymptotic series as follows:

$$\begin{aligned}
\lambda_{ap}^k &= \sum_0^m \epsilon^j \lambda_j^k, && k = 1, \ldots, r, \\
W_{ap}(x, \epsilon) &= \sum_0^m \epsilon^j W_j^R(x), && x \in I^i,\ i = 2\ell, 1 \leq \ell \leq r+1, \\
W_{ap}(x, \epsilon) &= \sum_0^m \epsilon^j W_j^{S\ell}((x - x_\Delta^\ell)/\epsilon, \epsilon), && x \in I^i,\ i = 2\ell+1, 0 \leq \ell \leq r+1.
\end{aligned}$$

A function W in I^i will be denoted W^i if necessary.

Although the interval I^i changes with Δx^i and ϵ, W_{ap} is still well defined. In regular layers, using the differential equations, the domain of $W_j^R(x)$ can be extended from $x \in (R)^\ell = (x_0^{\ell-1}, x_0^\ell)$ to an open interval O^i containing $(R)^\ell$. Therefore, if ϵ and $\max_\ell\{\Delta x^\ell\}$ are sufficiently small, then $I^i \subset O^i$ so that $W_{ap}(x,\epsilon)$ is defined in I^i. The width of a singular layer is fixed and $W_{ap}(x,\epsilon)$ is only shifted in the x direction when $\Delta x^\ell \neq 0$.

Formal approximation of internal layer solutions

(u_{ap}, v_{ap}) defined above is a formal approximation in the sense that after substituting into (31), the residual errors in all layers, boundary errors at $x = 0, 1$, and jump errors between adjacent singular and regular layers are small.

If we let $(-F^i, -G^i)$ be the residual error of the approximation in I^i, then

$$\begin{aligned} u_{ap,\xi\xi} + f(u_{ap}, v_{ap}) &= -F^i, \\ v_{ap,xx} + g(u_{ap}, v_{ap}) &= -G^i, \quad 1 \le i \le 2r+3. \end{aligned}$$

It is easy to verify that $|F^i| + |G^i| = O(\epsilon^{m+1})$ in regular layers. In singular layers, the Taylor expansion of f and g involves polynomial growth terms of ξ. Since the layer width in ξ is of $\epsilon^{\beta-1}$, the residual error due to truncation is of f is $O(\epsilon^{m+1}\xi^{m+1}) = O(\epsilon^{\beta(m+1)}$. In singular layers, only the ϵ^{m-1}th order expansion of g was used due to the extra term ϵ in front of g, thus the truncation error $|G^i| = O(\epsilon^{m\beta})$. In the x scale, the L^1 norm is $O(\epsilon^{\beta(m+1)})$. In conclusion

$$\begin{aligned} |F^i| + |G^i| &= O(\epsilon^{m+1}), \quad \text{in regular layers,} \\ |F^i| + |G^i|_{L^1} &= O(\epsilon^{\beta(m+1)}), \quad \text{in singular layers.} \end{aligned} \tag{75}$$

If we define the jump errors between layers with $\Delta x^\ell = 0, \ell = 1, \ldots, r$ as

$$\begin{aligned} u_{ap}^{i+1}(a_0^i) - u_{ap}^i(a_0^i) &= -J_1^i, \quad u_{ap,x}^{i+1}(a_0^i) - u_{ap,x}^i(a_0^i) = -J_2^i, \\ v_{ap}^{i+1}(a_0^i) - v_{ap}^i(a_0^i) &= -J_3^i, \quad v_{ap,x}^{i+1}(a_0^i) - v_{ap,x}^i(a_0^i) = -J_4^i, \end{aligned}$$

then we have

$$\sum_{i=1}^{4} \sum_{j=1}^{4} |J_j^i| \le C\epsilon^{\beta(m+1)}. \tag{76}$$

For a proof see [**?**, **?**, **?**].

Existence of internal layer solutions

Let $(u_{ap} + u, v_{ap} + v)$ be the exact solution with the exact layer position $x^{\ell}_{ap}(\epsilon) + \Delta x^{\ell}$. The functions (u, v) satisfy the following linear variational equations. In regular layers,

$$
\begin{aligned}
u_{\xi\xi} + f^i_u u + f^i_v v &= F^i(\xi) + M^i(u, v, \epsilon), \\
v_{xx} + g^i_u u + g^i_v v &= G^i(x) + N^i(u, v, \epsilon).
\end{aligned}
$$

In singular layers,

$$
\begin{aligned}
u_{\xi\xi} + f^i_u u + f^i_v v &= F^i(\xi) + M^i(u, v, \epsilon), \\
v_{xx} &= G^i(x) + N^i(u, v, \epsilon).
\end{aligned}
$$

The coefficients are based on linearizing at the ϵ^0th order approximations. For example, in regular layers, $f^i_u = f_u(u^{R\ell}_0(x), v^{R\ell}_0(x)), i = 2\ell$. In singular layers, $f^i_u = f^i_u(u^{S\ell}_0(\xi - x^{\ell}_0/\epsilon), v^{S\ell}_0(\xi - x^{\ell}_0/\epsilon)), i = 2\ell + 1$. Similar definitions apply to f^i_v, g^i_u, g^i_v. If ξ is used in regular layers, let $x = \epsilon x$ and if x is used in singular layers, let $\xi = x/\epsilon$.

A direct linearization would yield $v_{xx}+g_u^i u+g_v^i v = G^i(x)+N^i(u,v,\epsilon)$ in singular layers. But since the length of the layer is $O(\epsilon^\beta)$, $g_u^i u + g_v^i v = O(\epsilon^\beta(|u|+|v|))$ and is included in N^i.

The nonlinear terms satisfy,

$$|M^i| + |N^i|_{L^1} \le C(|u^i|^2 + |v^i|^2 + \epsilon^\beta(|u|+|v|)).$$

Let I^i, $i = 2\ell+1$, be a singular layer. Observe that adding Δx^ℓ does not change the values of u^i at the boundaries of I^i. When $\Delta x^\ell \neq 0$ the jump conditions for (u,v) are,

$$\begin{aligned} u^{i+1}(a^i) - u^i(a^i) &= u^i_{ap}(a^i) - u^{i+1}_{ap}(a^i), \\ u^i(a^{i-1}) - u^{i-1}(a^{i-1}) &= u^{i-1}_{ap}(a^{i-1}) - u^i_{ap}(a^{i-1}). \end{aligned}$$

After linearization, we have

$$\begin{aligned} u^{i+1}(a^i) - u^i(a^i) &= J_1^i - u_{0x}^{R,\ell+1}(x_0^\ell+)\Delta x^\ell + K_1^i, \\ u^i(a^{i-1}) - u^{i-1}(a^{i-1}) &= J_1^{i-1} + u_{0x}^{R,\ell}(x_0^\ell-)\Delta x^\ell + K_1^{i-1}. \end{aligned}$$

The nonlinear terms satisfy

$$K_1^i = O(|\Delta x^\ell|^2 + |u^{i+1}|^2 + |u_x^{i+1}|^2 + \epsilon^\beta(|\Delta x^\ell| + |u^{i+1}|^2 + |u_x^{i+1}|^2),$$
$$K_1^{i-1} = O(|\Delta x^\ell|^2 + |u^{i-1}|^2 + |u_x^{i-1}|^2 + \epsilon^\beta(|\Delta x^\ell| + |u^{i-1}|^2 + |u_x^{i-1}|^2)$$

Similar formulas for the jumps of u_x, v, v_x can be written at the junction points.

If we can solve the following system of linear nonhomogeneous equations, then the nonlinear system can be solved by the contraction mapping principle.

In a regular layer $I^i, i = 2\ell, 1 \leq \ell \leq r + 1$,

$$u_{\xi\xi} + f_u^i u + f_v^i v = F^i(\xi), \tag{77}$$
$$v_{xx} + g_u^i u + g_v^i v = G^i(x). \tag{78}$$

In a singular layer $I^i, i = 2\ell + 1, 1 \leq \ell \leq r + 1$,

$$u_{\xi\xi} + f_u^i u + f_v^i v = F^i(\xi), \tag{79}$$
$$v_{xx} = G^i(x). \tag{80}$$

The boundary conditions at $x = 0, 1$ are

$$\begin{aligned} &u_x(0) = u_x(1) = 0, \\ &A_j v_x(j) + B_j v(j) = 0, \quad j = 0, 1. \end{aligned} \tag{81}$$

Denote u, u_x, v, v_x by z, the jump conditions for $i = 2\ell + 1$, I^i a singular layer, are:

$$\begin{aligned} z^{i+1}(a^i) - z^i(a^i) &= J^i_j - z^R_{0x}(x^\ell_0+)\Delta x^\ell, \\ z^i(a^{i-1}) - z^{i-1}(a^{i-1}) &= J^{i-1}_j + z^R_{0x}(x^\ell_0-)\Delta x^\ell, \end{aligned} \tag{82}$$

where $j = 1, 2, 3, 4$ if $z = u, u_x, v, v_x$ respectively.

We can prove the following result

Theorem 19 *The system (77)-(80) with boundary conditions (81) and jump conditions (82) has a unique solution* $(u, v, \{\Delta x^i\}^r_1)$, *that satisfies*

$$|\{\Delta x^i\}| + |u| + |v| \leq C\{\sum |F^i| + \sum |G^i|_{L^1} + \sum_i \sum_{j=1}^{4} |J^i_j|\}.$$

By the superposition principle, the proof is divided into two steps.

(1) STEP ONE: We solve the nonhomogeneous system (77)-(80) in each regular or singular layer, taking care of the boundary conditions (81) but ignoring the jump conditions (82).

(2) STEP TWO: We solve a homogeneous system (77)-(80) with zero (F^i, G^i) and zero boundary conditions, but nonhomogeneous jump conditions, which are modified to accommodate the changing due to the first step. The sum of the solutions in the two steps is the solution of Theorem 19.

System (77)-(82) bears some resemblance of the linear systems in §3. However, in regular layers, the term $u_{\xi\xi} = \epsilon^2 u_{xx}$ can not be dropped to make an algebraic-differential system. Because of this, even the relatively easier STEP ONE is not trivial to carry out. The point is we need to find a solution in each layer that is bounded uniformly by (F^i, G^i) as the length of intervals approaches infinity in the ξ scale when $\epsilon \to 0$. The procedure of performing STEP ONE is discussed in [**?**] and will be skipped in this paper.

To accomplish STEP TWO, based on Lemma 18, we will simplify the system to make it easy to solve. Eventually, the system is reduced to the (BVPIC) which is known to have a solution.

In regular layers, by the change of variable $u = y - (f_u^i)^{-1} f_v^i v$, (78) becomes

$$v_{xx} + g_u^i y + [g_v^i - g_u^i (f_u^i)^{-1} f_v^i] v = 0.$$

The idea is if y=0, we are on the slow manifold of the linear system, so that the deviation y must be small. If we drop $g_u^i y$ then the system to solve in regular layer is

$$u_{\xi\xi} + f_u^i u + f_v^i v = 0, \tag{83}$$

$$v_{xx} + [g_v^i - g_u^i (f_u^i)^{-1} f_v^i] v = 0. \tag{84}$$

In singular layers, we convert $v_{xx} = 0$ into a system $v_x = w, w_x = 0$, and approximate it by $v_x = 0, w_x = 0$. Then in singular layers

$$u_{\xi\xi} + f_u^i u + f_v^i v = 0, \tag{85}$$

$$v_x = 0,\ w_x = 0. \tag{86}$$

Recall that by the iteration method, all we need is to solve the system approximately with small errors. After solving for (u, v, w), we can show a posteriori that $g_u^i y$ is small in L^1 norm, see [**?**] for a proof, and $|w|_{L^1} \leq C\epsilon^\beta |w|_{L^\infty}$ is also small.

We look for solutions of a system consisting of (83)-(86) plus the boundary conditions (81) and the jump conditions (82). The next step is to reduce the system to the (BVPIC) as in §3.

First we solve for v in regular layers from (84). We need jump conditions for v in two ajacent regular layers, one before the other after the ℓth internal layer $I^i = I^{2\ell+1}$. Observe that from (86),

(v^i, w^i) are constants in I^i. If we Recall that $[v_{0x}^R](x_0^\ell) = 0$, from the jump conditions (82), we find that

$$
\begin{aligned}
& v^{i+1}(a^i) - v^{i-1}(a^{i-1}) = J_3^i + J_3^{i-1}, && (87)\\
& w^{i+1}(a^i) - w^{i-1}(a^{i-1}) = J_4^i + J_4^{i-1} + \Delta x^\ell [w_{0x}^R](x_0^\ell), && (88)\\
& A_0 v_x^2(a^1) + B_0 v^2(a^1) = A_0 J_4^1 + B_0 J_3^1, && (89)\\
& A_1 v_x^{2r+2}(a^{2r+2}) + B_1 v^{2r+2}(a^{2r+2}) = -A_1 J_4^{2r+1} - B_1 J_3^{2r+1}. && (90)
\end{aligned}
$$

Let us turn to the u equations in singular and regular layers. Consider (83) and (85) in regular and singular layers with Neumann boundary conditions. For the jump conditions on (u, u_x), consider the ℓth singular layer I^i, $i = 2\ell + 1$.

$$
\begin{aligned}
u^{i+1}(a^i) - u^i(a^i) = H_1^i := J_1^i - u_{0x}^{R,\ell+1}(x_0^\ell +)\Delta x^\ell,\\
u_x^{i+1}(a^i) - u_x^i(a^i) = H_2^i := J_2^i + u_{0x}^{R,\ell}(x_0^\ell -)\Delta x^\ell.
\end{aligned}
\tag{91}
$$

Equations for u have the property that in the two boundary layers and all the regular layers, $u_{\xi\xi} + f_u^i u = 0$ has exponential dichotomies in $I^i = (\xi^{i-1}, \xi^i)$. In each internal layer $I^i = I^{2\ell+1}, \ell = 1, \ldots, r$,

$u_{\xi\xi}+f_u^i u=0$ has exponential dichotomies only on the two half-subintervals of I^i. By having an exponential dichotomy for a second order equation, we mean that the correspponding first order system on (u,u_ξ) has an exponential dichotomy. The constants and exponents of the dichotomies do not depend on ϵ or the length of the intervals, which approaches infinity as $\epsilon\to 0$. Let the projections to stable and unstable spaces in I^i be $P_s^i(\xi)$ and $P_u^i(\xi)$. The projections in internal layers have a jump at the middle of the interval I^i since the dichotomies only exist on half of each I^i.

If $H_j^i, j=1,2$, is given, the system with jump conditions and exponential dichotomies described as above has been studied in [**?**, **?**, **?**, **?**]. The problem to solve is similar to the classical shadowing lemma except for the lack of exponential dichotomies in the whole internal layers. Assuming at ξ^i, $\mathcal{R}P_u^i(\xi^i)\oplus\mathcal{R}P_s^{i+1}(\xi^i)$ which can be verified in our system, we have the unique splitting $(H_1^i,H_2^i)^\tau=\phi_s^{i+1}-\phi_u^i$ where $\phi_s^{i+1}\in\mathcal{R}P_s^{i+1}(\xi^i)$ and $\phi_u^i\in\mathcal{R}P_u^i(\xi^i)$.

Denote $\phi_u^i := Q_u^i(H_1^i, H_2^i)^\tau$, $\phi_s^{i+1} := Q_s^i(H_1^i, H_2^i)^\tau$. The system for u can be approximated by a local boundary value problem in I^i:

$$\begin{aligned} u_{\xi\xi} + f_u^i u + f_v^i v = 0, \\ P_s^i(\xi^{i-1})(u(\xi^{i-1}), u_x(\xi^{i-1})) = \phi_s^i, \\ P_u^i(\xi^i)(u(\xi^i), u_x(\xi^i)) = \phi_u^i, \end{aligned}$$

In regular layers, and in the two boundary layers, the above always has a solution for any continuouse or L^1 function $v(\xi)$ and any vectors (ϕ_s^i, ϕ_u^i). In internal layers, v is constant, If $\Psi = (-\dot{\psi}^\ell, \psi^\ell)$, where ψ^ℓ is the solution to the adjoint eqaution as in §2. To have a solution u in $I^i = I^{2\ell+1}$, which is $(-\epsilon^{\beta-1}, \epsilon^{\beta-1})$ using local coordinate, a Melnikov type condition must be satisfied, see Lemma 4.

$$\begin{aligned} &\int_{-\beta^{-1}}^{\beta^{-1}} \psi^\ell(\xi) f_v v^i d\xi = \Psi^i(\xi^i)\phi_u^i - \Psi^i(\xi^{i-1})\phi_s^i \\ =&\Psi^i(\xi^i) Q_u^i(H_1^i, H_2^i)^\tau - \Psi^i(\xi^{i-1}) Q_s^{i-1}(H_1^{i-1}, H_2^{i-1})^\tau. \end{aligned}$$

It is now clear, base on $\Psi^i(\xi)$ is exponentially small as $\xi \to \infty$, we can drop the Δx^ℓ terms in the definitions of (H_1^i, H_2^i) in (91). The right hand side is approximated by given terms involving only

$(J_1^{i-1}, J_2^{i-1}, J_1^i, J_2^i)$. If we denote $\mathbf{n}_0^\ell := \int_{-\beta^{-1}}^{\beta^{-1}} f_v^i \psi^\ell(\xi) d\xi$ and use the jump condition $v^{i+1}(a^i) - v^i(a^i) = J_3^i - w_0^R(x_0^\ell)\Delta x^\ell$, we have a condition on $v^{i+1}(a^i)$:

$$\mathbf{n}_0^\ell \cdot (v^{i+1}(a^i)) + w_0^R(x_0^\ell)\Delta x^\ell) = \Psi^i(\xi^i) Q_u^i (J_1^i, J_2^i)^\tau - \Psi^i(\xi^{i-1}) Q_s^{i-1}(J_1^{i-1}, J_2^{i-} \tag{92}$$

In the simpified system, the v variable in regular layers must satisfy (92), with jump conditions (87), (88) and boundary conditions (89), (90).

If we shrik the the singular layer to a point $x_0^\ell, 0 \leq \ell \leq r+1$, and move $a^{2\ell}$ and $a^{2\ell+1}$ to x_0^ℓ, and approximate the $\mathbf{n}_0^\ell$ by $\mathbf{n}^\ell = \int_{-\infty}^{\infty} f_v^i \psi^\ell(\xi) d\xi$, then (92) is approximated by

$$\mathbf{n}^\ell \cdot (v^{i+1}(x_0^\ell +) + w_{0x}^R(x_0^\ell +)\Delta x^\ell) = \text{given terms.}$$

This is precisely the third equation in (BVPIC). The boundary conditons become the second equation in (BVPIC) and the jump conditions the last two equations in (BVPIC). Acoording to Lemma **??**, the modefied system has a unique solution. If we solve this

(BVPIC) and map the solution in each $(x_0^{\ell-1}, x_0^{\ell})$ by a near identity map to $(a^{2\ell-1}, a^{2\ell})$, we have a good approximation of the v in regular layers. The error of the approximation approaches zero as $\epsilon \to 0$. By Lemma 18, this means that the system for the v variable in regular layers has a unique solution.

The v in singular layers can be obtained by jump conditions to their neighboring regular layers. Finally, since (92) is satisfied, u with boundary and jump consitons can be obtained.

Once the liear systenm has been solved, the nonlinear variational system can be solved by a contraction mapping principle. We summarize the result below:

Theorem 20 *For any integer* $m \geq 0$, *then there exists* $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$, *there exists a unique internal layer solution*

(u_{exact}, v_{exact}) *near the formal approximation* (u_{ap}, v_{ap}). *The internal layer solution has exact layer positions (determined by some phase condition)* $x^{\ell}_{exact}, 1 \leq \ell \leq r$ *that is near* x^{ℓ}_{ap}. *Moreover,*

$$|u_{exact} - u_{ap}| + |v_{exact} - v_{ap}| + \sum_{\ell} |x^{\ell}_{exact} - x^{\ell}_{ap}| \leq C\epsilon^{\beta(m+1)}, 0 < \beta < 1.$$

Formal approximation of critical eigenvalue and eigenfunctions

By truncating the formal series of eigenvalues and eigenfunctions as above, we can show that $\lambda_{ap}(\epsilon)$ and (U_{ap}, V_{ap}) are approximations of eigenvalue and eigenfunctions with small residual in each I^i and jump errors between layers..

If we let $(-F^i, -G^i)$ be the residual error of the approximation of eigenvalue and eigenfunctions in I^i, then

$$\begin{aligned} &-\lambda_{ap}U_{ap} + U_{ap,\xi\xi} + f^i_u(\text{exact})U + f^i_v(\text{exact})V = -F^i, \\ &-\lambda_{ap}V_{ap} + V_{ap,xx} + g^i_u(\text{exact})U + g^i_v(\text{exact})V = -G^i, \quad 1 \leq i \leq 2r+3. \end{aligned}$$

Here f_u^i(exact) $= f_u(u_{exact}, v_{exact})$ in regular layers, etc.. One can verify that $|F^i|$ and $|G^i|$ satisfy estimates (75) with perhaps different constants C.

When $\Delta^\ell = 0, \ell = 1, \ldots, r$, the jump errors between layers are defined as

$$\begin{aligned} U_{ap}^{i+1}(a_0^i) - U_{ap}^i(a_0^i) &= -J_1^i, \quad U_{ap,x}^{i+1}(a_0^i) - U_{ap,x}^i(a_0^i) = -J_2^i, \\ V_{ap}^{i+1}(a_0^i) - V_{ap}^i(a_0^i) &= -J_3^i, \quad V_{ap,x}^{i+1}(a_0^i) - V_{ap,x}^i(a_0^i) = -J_4^i. \end{aligned}$$

They satisfy (76) with perhaps different constants C.

Existence of critical eigenvalue-eigenfunctions

The existence of a true critical eigenvalue-corresponding eigenfunction near the approximation $(\lambda_{ap}^k(\epsilon), U_{ap}, V_{ap})$ can also be proved by the contraction and iteration methods. For a related system, see [**?**]. We can prove the follwing result:

Theorem 21 *For any integer $m \geq 0$ and $1 \leq k \leq r$, there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, there exists a unique eignevalue-eigenfunction triplet $(\lambda_{exact}, U_{exact}, V_{exact})$ near $(\lambda_{ap}^k(\epsilon), U_{ap}, V_{ap})$. Moreover,*

$$|\lambda_{ap}^k - \lambda_{\text{exact}}| + |U_{ap} - U_{exact}| + |V_{ap} - V_{exact}| = O(\epsilon^{(m+1)\beta}).$$

When we construct U_{ap}, an undetermined term $\epsilon^m c_m^\ell \dot{q}^\ell$ can be added in the ℓth singular layer. The vector $\{c_m^\ell\}_{\ell=1}^r$ will be determined now. Let an exact solution be

$$\begin{aligned}
\lambda_{exact} &= \lambda_{ap}^k + \epsilon^{m+1}\lambda, && \\
U_{exact} &= U_{ap} + \epsilon^{m+1}U, && \text{in regular layers,}\\
U_{exact} &= U_{ap} + \epsilon^m c_m^\ell \dot{q}^\ell + \epsilon^{m+1}U, && \text{in the } \ell\text{th singular layer,}\\
V_{exact} &= V_{ap} + \epsilon^{m+1}V, && \text{in regular and singular layers.}
\end{aligned}$$

In regular layers,

$$\begin{aligned}
U_{\xi\xi} + f_u^i U + f_v^i V &= F^i(\xi) + M^i(U, V, \lambda, \epsilon),\\
V_{xx} + g_u^i U + g_v^i V &= G^i(x) + N^i(U, V, \lambda, \epsilon).
\end{aligned}$$

In the ℓth singular layer,

$$
\begin{aligned}
&-\lambda_1 c_m^\ell \dot{q}^\ell - \lambda c_0^\ell \dot{q}^\ell + U_{\xi\xi} + f_u^i U + f_v^i V + f_{uu}^i c_m^\ell \dot{q}^\ell u_1 + f_{uv}^i c_m^\ell \dot{q}^\ell v_1 \\
&= F^i(\xi) + M^i(U, V, \lambda, c_m^\ell, \epsilon), \\
V_\xi &= \epsilon W, \quad W_\xi = -g_u^i c_m^\ell \dot{q}^\ell + G^i(x) + N^i(U, V, \lambda, c_m^\ell, \epsilon).
\end{aligned}
$$

Th nonlinear terms satisfy

$$
|M^i| + |N^i|_{L^1} \le C(|U|^2 + |V|^2 + |\lambda|^2 + |c_m^\ell|^2 + \epsilon(|U| + |V| + |\lambda| + |c_m^\ell|)).
$$

In regular layers, the nonlinear terms satisfy a similar estimate.

There are also boundary conditions at $x = 0, 1$ and jump conditions at $\{a^i\}$ to be satisfied. If we drop the nonlinear and small terms, we have a linear system.

In regular layers,

$$
\begin{aligned}
U_{\xi\xi} + f_u^i U + f_v^i V &= F^i(\xi), \\
V_{xx} + g_u^i U + g_v^i V &= G^i(x).
\end{aligned}
$$

In the ℓth singular layer,

$$
\begin{aligned}
&-\lambda_1 c_m^\ell \dot{q}^\ell - \lambda c_0^\ell \dot{q}^\ell + U_{\xi\xi} + f_u^i U + f_v^i V + f_{uu}^i c_m^\ell \dot{q}^\ell u_1 + f_{uv}^i c_m^\ell \dot{q}^\ell v_1 = F^i(\xi),\\
&V_\xi = 0, \quad W_\xi = -g_u^i c_m^\ell \dot{q}^\ell + G^i(x).
\end{aligned}
$$

The boundary and jump conditions are,

$$
\begin{aligned}
U_x^1(a^0) = U_x^{2\ell+3}(a^{2\ell+3}) &= 0, & A_j V_x(j) + B_j V(j) &= 0,\\
U^{i+1}(a^i) - U^i(a^i) &= J_1^i, & U_x^{i+1}(a^i) - U_x^i(a^i) &= J_2^i,\\
V^{i+1}(a^i) - V^i(a^i) &= J_3^i, & V_x^{i+1}(a^i) - V_x^i(a^i) &= J_4^i.
\end{aligned}
$$

Integrating in the ℓth singular layer, we have

$$
\begin{aligned}
&V^i(a^i) - V^i(a^{i-1}) = 0,\\
&W^i(a^i) - W^i(a^{i-1}) = c_m^\ell(g(q^\ell(-\epsilon^{\beta-1}), v) - g(q^\ell(\epsilon^{\beta-1}), v)) \approx c_m^\ell \mathcal{M}^\ell.
\end{aligned}
$$

Here $\mathcal{M}^\ell := g(q^\ell(-\infty), v) - g(q^\ell(\infty), v)$ as defined in §4. It is now clear that the jump of (V, W) between the two regular layers next to $I^{2\ell+1}$ are approximately

$$
\begin{aligned}
V^{i+1}(a^i) - V^{i-1}(a^{i-1}) &= J_3^i + J_3^{i-1},\\
W^{i+1}(a^i) - W^{i-1}(a^{i-1}) &= J_4^i + J_4^{i-1} + c_m^\ell \mathcal{M}^\ell.
\end{aligned}
$$

Using the change of variable $U = Y - (f_u^i)^{-1} f_v^i V$, in regular layers

$$V_{xx} - [g_u^i (f_u^i)^{-1} f_v^i - g_v^i] V + g_u^i Y = 0.$$

Dropping the small term $g_u^i Y$, also observing a^i and a^{i-1} are ϵ^β close to x_0^ℓ, an approximate system of V has the form

$$\begin{aligned}
&V_{xx} - [g_u^i (f_u^i)^{-1} f_v^i - g_v^i] V = 0, \\
&[V](x_0^\ell) = J_3^i + J_3^{i-1}, \\
&[V_x](x_0^\ell) = J_4^i + J_4^{i-1} + c_m^\ell \mathcal{M}^\ell,
\end{aligned}$$

with homogeneous boundary conditions. The solution can be written as $V = c_m^\ell V_c^\ell +$ given terms, V_c^ℓ as in §4.

To determine c_m^ℓ, plug V into the U equation in internal layers. In order to have a solution in $I^{2\ell+1}$, we have a Melnikov type condition

$$\int_{-\beta^{-1}}^{\beta^{-1}} < \psi^\ell(\xi), -\lambda_1 c_m^\ell \dot{q}^\ell - \lambda c_0^\ell \dot{q}^\ell + f_v^i V + f_{uu}^i c_m^\ell \dot{q}^\ell u_1 + f_{uv}^i c_m^\ell \dot{q}^\ell v_1 - F^i(\xi)$$

=given terms.

Replacing the domain of integral by $(-\infty, \infty)$ and recalling that $\mathbf{n}^\ell = \int_{-\infty}^{\infty} f_v \psi^\ell d\xi$, we finally have

$$\lambda_1 c_m^\ell + \lambda c_0^\ell = \mathbf{n}^\ell (\sum_1^r c_m^i V_c^i - c_m^\ell w_0^R(x_0^\ell)) + \text{given terms.}$$

With A being the coupling matrix, the above can be written as

$$(A - \lambda_1 I)\mathbf{c}_m = \lambda \mathbf{c}_0 + \text{given terms.}$$

Here we denote a r-vector $(c^1, \ldots, c^r)^\tau$ by $\mathbf{c}$. Since λ_1 is a simple eigenvalue and $\mathbf{c}_0$ is not in the range of $A - \lambda_1 I$, there exists a unique λ such that the above can be solved for a unique vector $\mathbf{c}_m$. After that, we can determine a unique $U^{2\ell+1} \perp \dot{q}^\ell$ in each singular layer. Approximations for (U, V) in regular layers can also be solved accordingly. The exact solution of $(U, V, \lambda, \{c_m^\ell\})$ is obtained by the iteration method as in Lemma 18.