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Multiple transverse homoclinic solutions near a degenerate homoclinic orbit

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Abstract

Consider an autonomous ordinary differential equation in $\mathbb{R}^n$ that has a homoclinic solution asymptotic to a hyperbolic equilibrium. The homoclinic solution is degenerate in the sense that the linear variational equation has 2 bounded, linearly independent solutions. We study bifurcation of the homoclinic solution under periodic perturbations. Using exponential dichotomies and Lyapunov–Schmidt reduction, we obtain general conditions under which the perturbed system can have transverse homoclinic solutions and nearby periodic or chaotic solutions.
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1. Introduction

Consider the system of autonomous differential equations

$$\dot{y}(t) = f(y(t)), \tag{1.1}$$

where $y \in \mathbb{R}^N$. We make the following assumptions:

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(H1) $f \in C^3$.
(H2) $f(0) = 0$ and the eigenvalues of $Df(0)$ lie off the imaginary axis.
(H3) Eq. (1.1) has a homoclinic solution $\gamma(t)$ asymptotic to the equilibrium $y = 0$. That is,

$$\dot{\gamma}(t) = f(\gamma(t)) \text{ and } \lim_{t\to\pm\infty} \gamma(t) = 0.$$

The variational equation of (1.1) along the homoclinic solution γ is

$$\dot{u}(t) = Df(\gamma(t))u(t). \tag{1.2}$$

Since $\dot{\gamma}$ is a bounded solution of (1.2), system (1.2) has $d \geq 1$ linearly independent bounded solutions. In this paper, we study bifurcation of $\gamma(t)$ to nearby transverse homoclinic solutions under the periodic perturbations. The perturbed system is

$$\dot{y}(t) = f(y(t)) + g(y(t), \mu, t), \tag{1.3}$$

where $y \in \mathbb{R}^N$ and $\mu \in \mathbb{R}$ is a parameter. We assume g satisfies

(H4) $g \in C^3$, $g(y, 0, t) = 0$ and $g(y, \mu, t + 2) = g(y, \mu, t)$.

By (H2), $y = 0$ is a hyperbolic equilibrium of (1.1). From $g(x, 0, t) = 0$, generically, Eq. (1.3) has a hyperbolic periodic orbit $\theta(\mu) := O(|\mu|)$ near 0. Using the change of variable $y = x + \theta(\mu)$, (1.3) becomes $\dot{x} = f(x) + \tilde{g}(x(t), \mu, t)$, where $\tilde{g}(x, \mu, t) = f(x + \theta(\mu)) + g(x + \theta(\mu), \mu, t) - g(\theta(\mu), \mu, t) - f(x) - f(\theta(\mu))$. Clearly, $\tilde{g}$ satisfies $\tilde{g}(0, \mu, t) = 0$.

With this as a motivation, we consider the following perturbed problem to (1.1):

$$\dot{x}(t) = f(x(t)) + \tilde{g}(x(t), \mu, t),$$

which satisfies assumptions (H1)–(H4) and $\tilde{g}(0, \mu, t) = 0$.

From $g(x, 0, t) = 0$ in (H4), we can rewrite the perturbation term as $\tilde{g}(x, \mu, t) = \mu\bar{g}(x(t), \mu, t)$. After dropping $\sim$ and $-$ on g, we consider the following problem that is equivalent to (1.3):

$$\dot{x}(t) = f(x(t)) + \mu g(x(t), \mu, t), \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R}. \tag{1.4}$$

The new system satisfies (H1)–(H4) and an additional condition

(H5) $g(0, \mu, t) = 0$.

The advantage of having (H5) on (1.4) is that $x = 0$ will be a hyperbolic equilibrium even after small periodic perturbations. For the autonomous equation when $\mu = 0$, let $W^s(0)$, $W^u(0)$ be the stable and unstable manifolds of the equilibrium 0. Clearly, the homoclinic orbit γ lies on $W^s(0) \bigcap W^u(0)$. If the variational equation of (1.1) along γ has d dimension bounded solutions, then $d = \dim(T_{\gamma(0)}W^s(0) \bigcap T_{\gamma(0)}W^u(0))$.

When $\mu \neq 0$, (1.4) may have bifurcations near γ. The case $d = 1$ has been extensively studied. In [8], Hale proposed to study the degenerate cases where $d \geq 2$. See also [3,9,12,13]. The purpose of the present work is to treat the case $d = 2$. Using the method of Lyapunov–Schmidt reduction, see [1], and exponential dichotomies, we derive a system of bifurcation functions H_j, $j = 1, 2$, the zeros of which correspond to the persistence of homoclinic solutions for (1.4). By the method of codiagonalization of quadratic forms, we show that the quadratic system can have up to 4 solutions. Finally, if the solutions to the quadratic system are nondegenerate, then the corresponding homoclinic orbits are transversal. Using the Shadowing Lemma in [16,17], we prove the existence of chaotic motions near such homoclinic bifurcations [7].

We use codiagonalization as an abbreviation of simultaneous diagonalization (of two matrices or two quadratic forms). Codiagonalization of matrices has been used by Jibin Li and Lin [14] to study systems of coupled KdV equations, and will be one of the main tool used in this paper. Given a symmetric real matrix $A \in \mathbb{R}^{2\times 2}$, then

$$F(a_1, a_2) = (a_1, a_2)A(a_1, a_2)^T$$

is a quadratic form associated to A. If A is diagonalized by a nonsingular matrix M: $M^T AM = \mathrm{diag}(d_1, d_2)$, then

$$F(a_1, a_2) = (b_1, b_2)\mathrm{diag}(d_1, d_2)(b_1, b_2)^T = d_1 b_1^2 + d_2 b_2^2,$$

where $(a_1, a_2)^T = M(b_1, b_2)^T$. The symmetric transformation described above is also called the congruence diagonalization. One should not confuse this with the similarity transformation of A which is defined by $M^{-1}AM$. For example the matrix $\mathrm{diag}(\lambda_1, -\lambda_2)$, $\lambda_j > 0$, can be reduced to $\mathrm{diag}(1, -1)$ by the matrix $M = \mathrm{diag}(1/\sqrt{\lambda_1}, 1/\sqrt{\lambda_2})$, which is a symmetric reduction, not similarity reduction.

In Section 2, we introduce notations to be used in this paper. We also present the reduced bifurcation functions, which to the lowest degree, represent the breaking of the homoclinic orbits under the periodic perturbations. In Section 3, we introduce the methods of codiagonalizing two quadratic forms and use them to study the quadratic bifurcation equations. The case when one equation is elliptic is considered in Section 3.1. The other case when both equations are hyperbolic is considered in Section 3.2. In Section 4, we study the coexistence of homoclinic solutions. In Section 5, we study the transversality of homoclinic solutions obtained in the previous sections.

2. Notations and preliminaries

Notations. Since $y = 0$ is a hyperbolic equilibrium point, from [4–6,16], (1.2) has exponential dichotomies on $J = \mathbb{R}^{\pm}$ respectively. In particular, there exist projections to the stable and unstable subspaces, $P_s + P_u = I$, and constants $m > 0$, $K_0 \geq 1$ such that

$$\begin{aligned} &(i) \quad |U(t)P_s U^{-1}(s)| \leq K_0 e^{2m(s-t)}, \quad \text{for } s \leqslant t \text{ on } J, \\ &(ii) \quad |U(t)P_u U^{-1}(s)| \leq K_0 e^{2m(t-s)}, \quad \text{for } t \leqslant s \text{ on } J. \end{aligned} \tag{2.1}$$

For the same $m > 0$, define the Banach space

$$\mathcal{Z} = \{z \in C^0(\mathbb{R}, \mathbb{R}^n) : \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|} < \infty\},$$

with the norm $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|}$. The linear variational system

$$Lu := \dot{u} - Df(\gamma)u = h \tag{2.2}$$

will be considered in $\mathcal{Z}$. The adjoint operator for L is

$$L^*\psi := \dot{\psi} + (Df(\gamma))^*\psi. \tag{2.3}$$

The domains of (2.2) and (2.3) are the dense subset of $\mathcal{Z}$, defined as

$$D(L) := \{u : u, u_t \in \mathcal{Z}\}, \quad D(L^*) := \{\psi : \psi, \psi_t \in \mathcal{Z}\}.$$

From the theory of homoclinic bifurcation [16,17,9], $L : \mathcal{Z} \to \mathcal{Z}$ is a Fredholm operator with index 0. The range of L is orthogonal to the null space of L^*. That is

$$h \in R(L) \text{ iff } \int_{-\infty}^{\infty} \langle \psi(t), h(t)\rangle dt = 0, \text{ for all } \psi \in N(L^*). \tag{2.4}$$

From $d = 2$, $N(L)$ is two dimensional. Let (u_1, u_2) be an orthonormal unit basis of $N(L)$ and (ψ_1, ψ_2) be an orthonormal unit basis of $N(L^*)$.

We define some Melnikov types of integrals [15] that will be used in the future. For integers i, p, q from the set $\{1, 2\}$, define

$$b_{pq}^{(i)} = \int_{-\infty}^{+\infty} \langle \psi_i(t), D^2 f(\gamma(t))u_p(t)u_q(t)\rangle dt, \; p, q = 1, 2,$$

$$\tilde{a}_i = \int_{-\infty}^{+\infty} \langle \psi_i(t), g(\gamma(t), 0, t)\rangle dt.$$

Define the 2×2 matrices $B^{(1)} = (b_{pq}^{(1)})$, $B^{(2)} = (b_{pq}^{(2)})$, and $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$.

By changing ψ_i to $-\psi_i$, we can change $B^{(i)}$ to $-B^{(i)}$ without altering the result of the paper. Without the loss of generality, we assume the following conditions are satisfied:

(H6) If the eigenvalues of $B^{(i)}$ satisfy $\lambda_1\lambda_2 > 0$, then $\lambda_1 > 0, \; \lambda_2 > 0$. If the eigenvalues of $B^{(i)}$ satisfy $\lambda_1\lambda_2 = 0$, then $\lambda_1 > 0, \lambda_2 = 0$. If the eigenvalues of $B^{(i)}$ satisfy $\lambda_1\lambda_2 < 0$, then $\lambda_1 > 0 > \lambda_2$.

We look for conditions so that (1.4) can have homoclinic solutions near γ. Define the reduced bifurcation functions $M^{(i)} : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2$ as follows:

$$M^{(i)}(\boldsymbol{\beta}, \mu) = \tilde{a}_i\mu + \frac{1}{2}\sum_{p,q=1}^{2} b_{pq}^{(i)}\beta_p\beta_q, \; i = 1, 2. \tag{2.5}$$

We shall show that, to the lowest degree, (2.5) describes the jump discontinuity along the direction of ψ_i in Section 4.

We need to solve the following system of bifurcation equations

$$\boldsymbol{\beta}^T B^{(i)} \boldsymbol{\beta} = a_i \mu, \quad i = 1, 2, \tag{2.6}$$

where $a_i = -2\tilde{a}_i$. Geometric method based on circular and hyperbolic rotation will be used to codiagonalize the quadratic system (2.6), which can significantly simplify the system.

The following lemma and its corollary show how codiagonalization of matrices are related to solutions of quadratic systems.

Lemma 2.1. *Let $F(b_1, b_2) := (b_1, b_2)A(b_1, b_2)^T$ be the quadratic form associated to a symmetric matrix $A \in \mathbb{R}^{2\times 2}$. If there exist $\beta_1 \neq 0$, $\beta_2 \neq 0$ such that either (i) $F(\beta_1, \beta_2) = F(-\beta_1, \beta_2)$, or (ii) $F(\beta_1, \beta_2) = F(\beta_1, -\beta_2)$, then A is a diagonal matrix.*

Proof. Assume $A = (a_{ij})$ with $a_{12} = a_{21}$. In both cases, we have

$$a_{11}\beta_1^2 + 2a_{12}\beta_1\beta_2 + a_{22}\beta_2^2 = a_{11}\beta_1^2 - 2a_{12}\beta_1\beta_2 + a_{22}\beta_2^2.$$

Thus $a_{12} = 0$. $\square$

Let $B^{(1)}, B^{(2)} \in \mathbb{R}^{2\times 2}$ be symmetric, nonzero matrices as in (2.6).

Corollary 2.2 *(Converse to the existence of 4 solutions). Assume that Eq. (2.6) has 4 solutions that form a parallelogram, and there exists a nonsingular 2×2 matrix X that transfers the parallelogram into a rectangle that is symmetric about the α_1 and α_2-axes. Then*

$$X^T B^{(1)} X = \Lambda_1 \text{ and } X^T B^{(2)} X = \Lambda_2,$$

where Λ_1, Λ_2 are diagonal matrices.

Proof. Suppose that $P_1 = (a, b)$, $P_2 = (c, d)$, $P_3 = (-a, -b)$, $P_4 = (-c, -d)$ are the vertexes of the parallelogram. Without loss of generality, let X be a nonsingular 2×2 matrix such that after the change of variables $(\beta_1, \beta_2)^T = X(\alpha_1, \alpha_2)^T$, then P_1, P_2 are symmetric about α_1-axis in (α_1, α_2) coordinates, that is $X^{-1}(a, b)^T = (a_0, b_0)^T$ and $X^{-1}(c, d)^T = (a_0, -b_0)^T$. The quadratic form $\boldsymbol{\beta}^T B^{(i)} \boldsymbol{\beta} = \boldsymbol{\alpha}^T X^T B^{(i)} X \boldsymbol{\alpha}$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$ satisfies the conditions in Lemma 2.1. Therefore Λ_1 is a diagonal matrix.

Similarly, $\Lambda_2 = X^T B^{(2)} X$ is a diagonal matrix. $\square$

3. Codiagonalization and the bifurcation of the homoclinic solution

We say that the quadratic equation $\boldsymbol{\beta}^T B \boldsymbol{\beta} = h$, $h \neq 0$ is of elliptic (or hyperbolic, or line) type if the graph of the equation is an ellipse (or two hyperbolas, or two lines). We consider the graph of two symmetric lines as a special case of two hyperbolas, where the distance to two lines replaces the real semiaxis of the hyperbolas.

3.1. Codiagonalization and solutions of (2.6) if one equation is elliptic

It is well-known that two symmetric matrices can be simultaneously diagonalized if one of the matrices is positive definite, [10,11,2]. For 2×2 matrices, the proof can be carried out by a circular rotation which may serve as a motivation to the hyperbolic rotation to be used in the case when both equations are hyperbolic.

Let $M, N \in \mathbb{R}^{2\times 2}$ be symmetric matrices, where N is positive definite. Let the eigenvalues of N be $\lambda_1, \lambda_2 > 0$ with the corresponding eigenvectors x_1 and x_2. Assume that the eigenvectors are normalized and let $X = (x_1, x_2)$. Let

$$Y = X \begin{pmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{pmatrix}.$$

Then

$$Y^T N Y = I. \tag{3.1}$$

The matrix $Y^T MY$ is still symmetric. In the next lemma, we will further diagonalize $Y^T MY$ without changing the normalized $Y^T NY$. Notice if M is positive-definite (or indefinite), then the graph of the corresponding quadratic form is an ellipse (or hyperbola).

The circular rotation $T(\theta)$ is defined by the 2×2 matrix

$$T(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

If we use the change of variables defined by $T(\theta)$, Eq. (3.1) remains unchanged.

Lemma 3.1. *If M is positive definite (or indefinite), then there exists a circular rotation with the angle θ_0, such that the major (or real) axis of the matrix $Y^T MY$, after the rotation, is on the c_1 axis, i.e., $M' = T(-\theta_0)Y^T MYT(\theta_0)$ is diagonal, and $T(-\theta_0)Y^T NYT(\theta_0)$ is still normalized. That is*

$$T(-\theta_0)Y^T MYT(\theta_0) = diag(d_1, d_2),$$
$$T(-\theta_0)Y^T NYT(\theta_0) = I.$$

Proof. We assume that the major (or real) semiaxis of

$$(b_1, b_2)Y^T MY(b_1, b_2)^T = 1$$

is at (b_1^0, b_2^0). Let $\theta_0 = \arctan b_2^0/b_1^0$. Then

$$T(-\theta_0)(b_1^0, b_2^0)^T = (\sqrt{(b_1^0)^2 + (b_2^0)^2}, 0)^T.$$

After the circular rotation $T(-\theta_0)$, the major (or real) axis is on the c_1-axis. Consequently the minor (or imaginary) axis is on the c_2-axis.

The quadratic form $(b_1, b_2)Y^T MY(b_1, b_2)^T = (c_1, c_2)M'(c_1, c_2)^T$ if $M' = T(-\theta_0)Y^T MYT(\theta_0)$ and $(c_1, c_2)^T = T(-\theta_0)(b_1, b_2)^T$. Obvious M' is a diagonal matrix. □

Corollary 3.2 *(LE-type). When $|M| = 0$ and $M \neq 0$, the quadratic form $(b_1, b_2)Y^T MY(b_1, b_2)^T$ can be expressed as $(cb_1 + db_2)^2$ for some $c, d \in \mathbb{R}$. Let $\theta_0 = \arctan(d/c)$. Then*

$$\begin{aligned} &T(-\theta_0)Y^T MYT(\theta_0) = diag(d_1, 0), \quad d_1 = \sqrt{c^2 + d^2}, \\ &T(-\theta_0)Y^T NYT(\theta_0) = I. \end{aligned} \tag{3.2}$$

Proof. After the circular rotation, the normal direction of the two lines defined by

$$(c_1, c_2)T(-\theta_0)Y^T MYT(\theta_0)(c_1, c_2)^T = 1$$

is on the c_1 axis. This proves the first equation of (3.2). □

Let us go back to consider (2.6) where $B^{(1)} = (b^{(1)}_{pq})$, $B^{(2)} = (b^{(2)}_{pq})$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$. Assume that $B^{(2)}$ is positive definite. From Lemma 3.1, there exists a 2×2 matrix X such that $X^T B^{(2)} X = I$ and $X^T B^{(1)} X = \Lambda = \mathrm{diag}(d_1, d_2)$.

Let us write the second equation of (2.6) as

$$(X^{-1}\boldsymbol{\beta})^T X^T B^{(2)} X(X^{-1}\boldsymbol{\beta}) = a_2\mu.$$

We assume that $a_2\mu > 0$, otherwise (2.6) has no solution. Let $\boldsymbol{\alpha} = \frac{1}{\sqrt{a_2\mu}} X^{-1}\boldsymbol{\beta}$. Then by $X^T B^{(2)} X = I$, the second equation of (2.6) becomes

$$\alpha_1^2 + \alpha_2^2 = 1.$$

The trajectories of the solutions form a unit circle in the (α_1, α_2) plane. The first equation of (2.6) can be written as

$$(X^{-1}\boldsymbol{\beta})^T X^T B^{(1)} X(X^{-1}\boldsymbol{\beta}) = a_1\mu.$$

By the same substitution $\boldsymbol{\alpha} = \frac{1}{\sqrt{a_2\mu}} X^{-1}\boldsymbol{\beta}$, and from $X^T B^{(1)} X = \Lambda$, the first equation becomes

$$d_1\alpha_1^2 + d_2\alpha_2^2 = \frac{a_1}{a_2}.$$

Hence we have proved the following result.

Lemma 3.3. *If $B^{(2)}$ in (2.6) is positive definite, then there exists a change of variable $\boldsymbol{\alpha} = \frac{1}{\sqrt{a_2\mu}} X^{-1}\boldsymbol{\beta}$ such that (2.6) becomes*

$$\begin{aligned} d_1\alpha_1^2 + d_2\alpha_2^2 &= \frac{a_1}{a_2}, \\ \alpha_1^2 + \alpha_2^2 &= 1. \end{aligned} \tag{3.3}$$

Depending on $B^{(1)}$ in (2.6) is positive definite, indefinite, or degenerate, the graph of the first equation of (3.3) is an ellipse, a hyperbola, or two lines. Then system (2.6) will be called the elliptic–elliptic, hyperbolic–elliptic or line–elliptic type, denoted by *EE*, *HE*, *LE* for brevity. When system (2.6) is of (*EE*) type, from Lemma 3.3 the system (2.6) becomes (3.3), without loss of generality, we assume $d_2 > d_1 > 0$. We now study in detail all the sub-cases if the matrix $B^{(2)}$ in (2.6) is positive definite.

Theorem 3.4. *Assume that the matrix $B^{(2)}$ in (2.6) is positive definite, and the matrix $B^{(1)}$ satisfies the condition (H6). After the codiagonalization as in Lemma 3.3, system (2.6) becomes (3.3). Then under the following conditions, system (2.6) has four simple zeros.*

(1) If system (2.6) is of (EE) type, then $0 < d_1 < a_1/a_2 < d_2$;

(2) if system (2.6) is of (HE) type, then $d_2 < 0 < d_1$ and $0 < a_1/a_2 < d_1$;

(3) if system (2.6) is of (LE) type, then $d_2 = 0$ and $0 < a_1/a_2 < d_1$.

Proof. By the discussions above Lemma 3.3, we see the major (or real) axis of the first equation of (3.3) is on α_1-axis.

Proof of case (1), (*EE*)-type: Since $d_2 > d_1 > 0$ and $a_1/a_2 > 0$, then the first equation of (3.3) is elliptic and the major axis is on the α_1-axis. The semi-major and semi-minor axes of the ellipse are $r_2 = \sqrt{a_1/d_1a_2} > r_1 = \sqrt{a_1/d_2a_2}$. If $0 < r_1 < 1 < r_2$, that is

$$0 < d_1 < a_1/a_2 < d_2,$$

then system (2.6) has four solutions. The proof for the (*EE*) case has been completed.

Proof of case (2), (*HE*)-type: When $d_2 < 0 < d_1$, the first equation of (3.3) is hyperbolic. Based on the fact of $a_1/a_2 > 0$, we know its real axis is on the α_1-axis. The hyperbola intersects the real axis at $C = (\pm\sqrt{a_1/(d_1a_2)}, 0)$. If $|OC| < 1$, the hyperbola and the ellipse in (3.3) has four intersections. That is, if

$$\frac{a_1}{a_2} < d_1, \tag{3.4}$$

then the system (2.6) has four solutions. The proof of (*HE*) has been completed.

Proof of case (3), (*LE*)-type: When $d_2 = 0$ and $d_1 > 0$, the first equation of (3.3) are two straight lines if $a_1/a_2 > 0$. The two vertical lines intersect the α_1-axis at C. If (3.4) holds, then the points C are inside the unit circle. Therefore system (2.6) has four solutions. □

3.2. Codiagonalization and solutions of (2.6) if both equations are hyperbolic

We first introduce the method of hyperbolic rotation that can codiagonalize two symmetric indefinite matrices under some general conditions. The hyperbolic rotation $H(\theta)$ with angle θ is defined by the 2×2 matrix

$$H(\theta) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Let $M, N \in \mathbb{R}^{2\times 2}$ be nonsingular, symmetric and indefinite matrices. Let the eigenvalues for N be $\lambda_1 > 0 > -\lambda_2$ with the corresponding eigenvectors x_1 and x_2, and let $X = (x_1, x_2)$ be a 2×2 matrix. Assume that the eigenvectors are normalized. Then it is well-known that

$$X^T N X = \Lambda, \text{ where } \Lambda = \mathrm{diag}(\lambda_1, -\lambda_2). \tag{3.5}$$

From (3.5), we can further normalize N by

$$Y^T N Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ where } Y = X \begin{pmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{pmatrix}. \tag{3.6}$$

After the normalization, the matrix $Y^T M Y$ is still nonsingular, symmetric and indefinite. Without affecting $Y^T N Y$, in Lemmas 3.6 and 3.7, we show under some general conditions, the hyperbolic rotation can be used to further reduce $Y^T M Y$ to a diagonal form. Moreover, the real semiaxis of the transformed hyperbola is determined by Corollary 3.8. First we show the existence of asymptotes for any nonsingular symmetric indefinite matrix.

Lemma 3.5. *To each nondegenerate, symmetric, indefinite 2×2 matrix A, there are two asymptotes (lines) L_1, L_2 such that $(b_1, b_2)A(b_1, b_2)^T = 0$ iff $(b_1, b_2)^T \in L_j$, $j = 1, 2$.*

Proof. There exists a matrix of eigenvectors Z such that

$$Z^T A Z = \mathrm{diag}(\mu_1, -\mu_2), \quad \mu_1, \mu_2 > 0.$$

Let $(b_1, b_2)^T = Z(c_1, c_2)^T$. Then (c_1, c_2) satisfies

$$(c_1, c_2) Z^T A Z (c_1, c_2)^T = \mu_1 c_1^2 - \mu_2 c_2^2 = 0.$$

The last equation defines two asymptotes in (c_1, c_2) coordinates. The asymptotes in (b_1, b_2) coordinates are

$$L_1 := \{(b_1, b_2) | \langle (\sqrt{\mu_1}, \sqrt{\mu_2})^T, Z^{-1}(b_1, b_2)^T \rangle = 0\},$$
$$L_2 := \{(b_1, b_2) | \langle (\sqrt{\mu_1}, -\sqrt{\mu_2})^T, Z^{-1}(b_1, b_2)^T \rangle = 0\}. \qquad \square$$

Definition 3.1. Let L_j, $j = 1, 2$, be the asymptotes for the nondegenerate, symmetric, indefinite 2×2 matrix A. Then L_j, $j = 1, 2$, divide $\mathbb{R}^2$ into four sectors. We say (b_1, b_2) is in the positive (or negative) sector if $(b_1, b_2)A(b_1, b_2)^T > 0$ (or $(b_1, b_2)A(b_1, b_2)^T < 0$).

The slopes of the two asymptotes for $(b_1, b_2)Y^T N Y (b_1, b_2)^T = 0$ is clearly $k = \pm 1$. For the matrix $Y^T M Y$, the slope of the asymptote L_j can be expressed as $k_j = b_2^{(j)}/b_1^{(j)}$ for a nonzero $(b_1^{(j)}, b_2^{(j)}) \in L_j$.

We now assume the sectors $(b_1, b_2)Y^T M Y (b_1, b_2)^T > 0$ are in the interior of $b_1^2 - b_2^2 > 0$, so the conditions $-1 < k_1 < k_2 < 1$ of the following lemma are satisfied.

Lemma 3.6. *If $-1 < k_1 < k_2 < 1$, then there exists a hyperbolic rotation with the angle θ_0, such that the image $H(-\theta_0)L_j$ becomes symmetric about the b_1 axis. The same hyperbolic rotation also diagonalizes the matrix Y^TMY, i.e., $M' = H(\theta_0)Y^TMYH(\theta_0)$ is diagonal.*

$$H(\theta_0)Y^TMYH(\theta_0) = diag(d_1, d_2), \quad d_1d_2 < 0$$
$$H(\theta_0)Y^TNYH(\theta_0) = diag(1, -1).$$

Proof. The condition $-1 < k_1 < k_2 < 1$ ensures that the asymptote L_j, $j = 1, 2$, intersects with the hyperbola $b_1^2 - b_2^2 = 1$. Let (b_{j1}, b_{j2}) be the intersection. Then $(b_{j1}, b_{j2}) = (\cosh(\theta_j), \sinh(\theta_j))$. Let $\theta_0 = (\theta_1 + \theta_2)/2$. Then

$$\begin{aligned} H(-\theta_0)(b_{j1}, b_{j2})^T &= (\cosh(\theta_j - \theta_0), \sinh(\theta_j - \theta_0))^T \\ &= \begin{cases} (\cosh(\theta_1 - \theta_2)/2, \sinh(\theta_1 - \theta_2)/2)^T, & \text{if } j = 1, \\ (\cosh(\theta_2 - \theta_1)/2, \sinh(\theta_2 - \theta_1)/2)^T, & \text{if } j = 2. \end{cases} \end{aligned}$$

The quadratic form $(b_1, b_2)Y^TMY(b_1, b_2)^T = (c_1, c_2)M'(c_1, c_2)^T$ if $M' = H(\theta_0)Y^TMYH(\theta_0)$ and $(c_1, c_2)^T = H(-\theta_0)(b_1, b_2)^T$. The quadratic equation after hyperbolic rotation, $(c_1, c_2)M'(c_1, c_2)^T = 0$ has two solutions $(1, k)$ and $(1, -k)$ with $k > 0$. This shows M' is a diagonal matrix. □

If the sectors $(b_1, b_2)Y^TMY(b_1, b_2)^T > 0$ are in the interior of $b_1^2 - b_2^2 < 0$, then $|k_1|, |k_2| > 1$ as in the following lemma hold.

Lemma 3.7. *If $|k_1|, |k_2| > 1$, then there exists a hyperbolic rotation with the angle θ_0, such that the image $H(-\theta_0)L_j$ becomes symmetric about the b_2 axis. The same hyperbolic rotation also diagonalizes the matrix Y^TMY, i.e., $M' = H(\theta_0)Y^TMYH(\theta_0)$ is diagonal.*

$$H(\theta_0)Y^TMYH(\theta_0) = diag(d_1, d_2), \quad d_1d_2 < 0$$
$$H(\theta_0)Y^TNYH(\theta_0) = diag(1, -1).$$

Proof. The condition $|k_1|, |k_2| > 1$ ensures that the line L_j, $j = 1, 2$, intersects with the hyperbola $b_2^2 - b_1^2 = 1$. Let (b_{j1}, b_{j2}) be the intersection. Then $(b_{j1}, b_{j2}) = (\sinh(\theta_j), \cosh(\theta_j))$. Let $\theta_0 = (\theta_1 + \theta_2)/2$. Then

$$\begin{aligned} H(-\theta_0)(b_{j1}, b_{j2})^T &= (\sinh(\theta_j - \theta_0), \cosh(\theta_j - \theta_0))^T \\ &= \begin{cases} (\sinh(\theta_1 - \theta_2)/2, \cosh(\theta_1 - \theta_2)/2)^T, & \text{if } j = 1, \\ (\sinh(\theta_2 - \theta_1)/2, \cosh(\theta_2 - \theta_1)/2)^T, & \text{if } j = 2. \end{cases} \end{aligned}$$

The quadratic form $(b_1, b_2)Y^TMY(b_1, b_2)^T = (c_1, c_2)M'(c_1, c_2)^T$ if $M' = H(\theta_0)Y^TMYH(\theta_0)$ and $(c_1, c_2)^T = H(-\theta_0)(b_1, b_2)^T$. The quadratic equation after hyperbolic rotation, $(c_1, c_2)M'(c_1, c_2)^T = 0$ has two solutions $(1, k)$ and $(-1, k)$ with $k > 1$. This shows M' is a diagonal matrix. □

Corollary 3.8. *(1) Assume the same conditions as Lemma 3.6 are satisfied. Let $\theta_0 = (\theta_1 + \theta_2)/2$ as in Lemma 3.6. If the ray*

$$(r\cosh(\theta_0), r\sinh(\theta_0)),\ \ r > 0,$$

intersects with the hyperbola $(b_1, b_2)Y^T MY(b_1, b_2)^T = h, h > 0$ at $r = r_0$. Then after the hyperbolic rotation and in (c_1, c_2) coordinates, $c_1 = r_0$ is the real semiaxis for the transformed hyperbola.

(2) Assume the same conditions as Lemma 3.7 are satisfied. Let $\theta_0 = (\theta_1 + \theta_2)/2$ as in Lemma 3.7. If the ray

$$(r\sinh(\theta_0), r\cosh(\theta_0)),\ \ r > 0,$$

intersects with the hyperbola $(b_1, b_2)Y^T MY(b_1, b_2)^T = h, h > 0$ at $r = r_0$. Then $c_2 = r_0$ is the real semiaxis of the hyperbola after the hyperbolic rotation.

We now study (2.6) where $B^{(1)} = (b^{(1)}_{pq})$, $B^{(2)} = (b^{(2)}_{pq})$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$. Since $B^{(2)}$ is indefinite, there exists a 2×2 matrix X such that $X^T B^{(2)} X = \mathrm{diag}(1, -1)$ and $X^T B^{(1)} X = \mathrm{diag}(d_1, d_2)$. Similar to the elliptic case, let us write the second equation of (2.6) as

$$(X^{-1}\boldsymbol{\beta})^T X^T B^{(2)} X(X^{-1}\boldsymbol{\beta}) = a_2\mu.$$

By changing ψ_i to $-\psi_i$ if necessary, we can assume $a_2\mu > 0$. Let $\boldsymbol{\alpha} = \frac{1}{\sqrt{a_2\mu}} X^{-1}\boldsymbol{\beta}$. Then by $X^T B^{(2)} X = \mathrm{diag}(1, -1)$, the second equation of (2.6) becomes

$$\alpha_1^2 - \alpha_2^2 = 1.$$

The trajectories of the solutions form a unit hyperbola in the (α_1, α_2) plane. The first equation of (2.6) can be written as

$$(X^{-1}\boldsymbol{\beta})^T X^T B^{(1)} X(X^{-1}\boldsymbol{\beta}) = a_1\mu.$$

By the same substitution, the first equation becomes

$$d_1\alpha_1^2 + d_2\alpha_2^2 = \frac{a_1}{a_2}, \quad d_1 d_2 < 0.$$

We have proved the following result.

Lemma 3.9. *If with $M = B^{(1)}$, $N = B^{(2)}$, conditions in Lemmas 3.6 and 3.7 are satisfied, then there exists a change of variable $\boldsymbol{\alpha} = \frac{1}{\sqrt{a_2\mu}} X^{-1}\boldsymbol{\beta}$ such that (2.6) becomes*

$$\begin{aligned} d_1\alpha_1^2 + d_2\alpha_2^2 &= \frac{a_1}{a_2}, \quad d_1 d_2 < 0, \\ \alpha_1^2 - \alpha_2^2 &= 1. \end{aligned} \tag{3.7}$$

We now study (3.7). Let

$$F_1(\alpha_1, \alpha_2) := (\alpha_1, \alpha_2)\mathrm{diag}(d_1, d_2)(\alpha_1, \alpha_2)^T,$$
$$F_2(\alpha_1, \alpha_2) := (\alpha_1, \alpha_2)\mathrm{diag}(1, -1)(\alpha_1, \alpha_2)^T.$$

Define $h = a_1/a_2$, then system (3.7) can be recast as

$$F_1(\alpha_1, \alpha_2) = h, \quad F_2(\alpha_1, \alpha_2) = 1. \tag{3.8}$$

The number of solutions for (2.6) depends on the relative positions of the asymptotes and the positive–negative sectors separated by the asymptotes. Listed below in cases (0)–(iii).

Case (0): The two asymptotes of $F_1(\alpha_1, \alpha_2) = h$ are alternatively in the positive and negative sectors of $F_2(\alpha_1, \alpha_2) = \alpha_1^2 - \alpha_2^2$.

Next, assume that the two asymptotes for $F_1(\alpha_1, \alpha_2)$ are adjacent to each other and are either both in the sector where $F_2 > 0$ or in the sector $F_2 < 0$. Then we have three sub-cases determined by the positive sectors of the two matrices.

Case (i): The two positive sectors of F_1 are in the interior of the two sectors where $F_2 > 0$ respectively.
Case (ii): The two negative sectors of F_1 are in the interior of two sectors where $F_2 > 0$ respectively.
Case (iii): The two positive sectors of F_1 are in the interior of the two sectors where $F_2 < 0$ respectively.
Case (iv): The two negative sectors of F_1 are in the interior of two sectors where $F_2 < 0$.

We now state a precise theorem on the number of solutions for system (2.6).

Theorem 3.10. *Let r_1 be the real semiaxis of the hyperbola $F_1(\alpha_1, \alpha_2) = h$. The number of solutions are determined by the asymptotes and positive–negative sectors separated by the asymptotes as follows:*

Case(0): In this case, for any $h \neq 0$, system (3.8) has two simple zeros. See Fig. 3.1.

Case (i): The system has 4 solutions provided that the real semiaxis of $F_1(\alpha_1, \alpha_2) = h$, $h > 0$ satisfies $r_1 < 1$. See Fig. 3.2.

Case (ii): The system has 4 solutions provided that the real semiaxis of $F_1 = h$, $h < 0$ satisfies $r_1 < 1$. See Fig. 3.2.

Case (iii): The system has 4 solutions provided that the real semiaxis of $F_1(\alpha_1, \alpha_2) = h$, $h < 0$ satisfies $r_1 > 1$. See Fig. 3.3.

Case (iv): The system has 4 solutions provided that the real semiaxis of $F_1 = h$, $h > 0$ satisfies $r_1 > 1$. See Fig. 3.3.

Case (v): The system always has 4 solutions (not depicted) if in cases (i) and (iii), $h < 0$; or in cases (ii) and (iv), $h > 0$.

The proof of Theorem 3.10 is straightforward from those figures and will be omitted. Notice that in cases (i) to (v), the two hyperbolic types of equations can be codiagonalized.

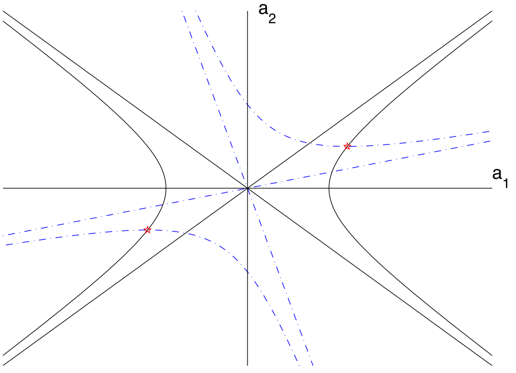

Fig. 3.1. If the asymptotes of $F_1 = 0$ and $F_2 = 0$ are alternating, then there always exist exactly two solutions.

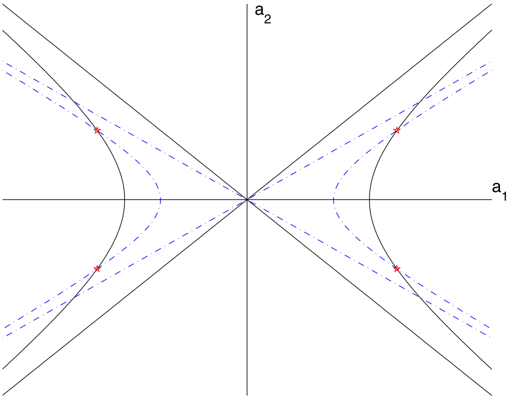

Fig. 3.2. In case (i), if $h > 0$ and $r_1 < 1$, or in case (ii), if $h < 0$ and $r_1 < 1$, then the system has 4 solutions.

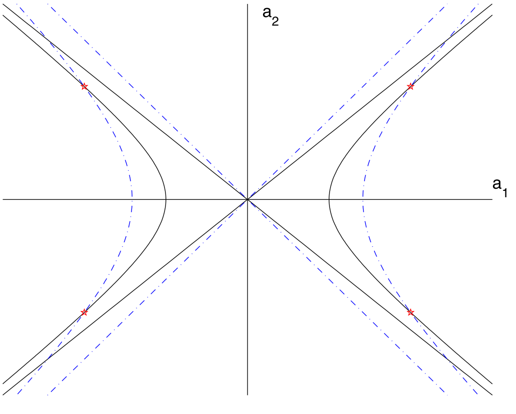

Fig. 3.3. In case (iii), if $h < 0$ and $r_1 > 1$, or in case (iv), if $h > 0$ and $r_1 > 1$, then the system has 4 solutions.

To be consistent with Theorem 3.4, we give more details on the two cases using diagonalized system as follows. We also provide conditions for the (LH) case to have solutions.

Using Lemmas 3.6 and 3.7, we assume the second equation of (2.6) is normalized and the first equation is in the diagonal form $d_1\alpha_1^2 + d_2\alpha_2^2 = h$ with $d_1 d_2 < 0$. We rewrite the system as:

$$\begin{aligned} \varrho_1\alpha_1^2 + \varrho_2\alpha_2^2 &= 1, \\ \alpha_1^2 - \alpha_2^2 &= 1, \end{aligned} \tag{3.9}$$

where $\varrho_i = d_i/h$. Now we are ready to solve (3.9). By the definitions of ϱ_i, d_i, it is clear that $\varrho_1/\varrho_2 = d_1/d_2$.

Theorem 3.11. *Assume that the matrix $B^{(2)}$ in system (2.6) is indefinite, and the matrix $B^{(1)}$ satisfies the condition $(H6)$. After the codiagonalization as in Lemma 3.9, system (2.6) becomes (3.9). Then under the following conditions, system (2.6) has four simple zeros.*

(1) *If system is of (HH) type, then one of the following holds:*
   (i) $0 < \varrho_1 < 1$ *and* $|\varrho_1/\varrho_2| > 1$*;*
   (ii) $\varrho_1 > 1$ *and* $|\varrho_1/\varrho_2| < 1$*;*
(2) *If system is of* (LH) *type, then* $\varrho_2 = 0$ *and* $0 < \varrho_1 < 1$ *or* $\varrho_1 = 0$ *and* $\varrho_2 > 0$.

Proof. Clearly, the slopes of the two asymptotes of the first equation of (3.9) are $\pm\sqrt{-\varrho_1/\varrho_2}$.

Proof of case (1), (HH)-type: (i) Since $\varrho_1 > 0$, then the real axis of the first equation of (3.9) is the α_1-axis. The real axis α_1 intersects the first and second equations of (3.9) at A_1 and A_2, where

$$A_1 = (\pm 1, 0), \quad A_2 = (\pm 1/\sqrt{\varrho_1}, 0).$$

Note that $|\varrho_1/\varrho_2| > 1$. Then the two asymptotes of the first equation is out of the ones of the second equation (3.9). Let $|OA_1|$, $|OA_2|$ be the distance to the origin. If $|OA_2| > |OA_1| = 1$, that is $\varrho_1 < 1$, then (2.6) has four solutions.

(ii) Since $\varrho_1 > 1$, then the real axis of the first equation of (3.9) is the α_1. Since $|\varrho_1/\varrho_2| < 1$, then the two asymptotes of the first equation are between the ones of the second equation (3.9). If $|OA_2| < |OA_1| = 1$, that is $\varrho_1 > 1$, then (2.6) has four solutions.

Proof of case (2), (LH)-type: Since $\varrho_2 = 0$, then Eq. (3.9) becomes:

$$\begin{aligned} \varrho_1\alpha_1^2 &= 1, \\ \alpha_1^2 - \alpha_2^2 &= 1. \end{aligned}$$

The first equation is two lines which are vertical to α_1 axis for $\varrho_1 > 0$. The distance from the origin to the lines is $1/\sqrt{\varrho_1}$. Hence, for $0 < \varrho_1 < 1$, (2.6) has four solutions.

By similar reason, we can prove the case $\varrho_1 = 0$. □

When both the matrices $B^{(1)}$, $B^{(2)}$ are degenerate, Eq. (2.6) can have 4 solutions.

Remark 3.1. When $B^{(i)} \neq 0$, $\det(B^{(i)}) = 0, i = 1, 2$, then each equation of (2.6) is two lines. Since $\det(B^{(i)}) = 0$, then we have $b_{11}^{(i)} b_{22}^{(i)} - b_{12}^{(i)^2} = 0$ and hence $b_{11}^{(i)} b_{22}^{(i)} > 0$. If $b_{11}^{(i)}(a_i\mu) < 0$, then the i-th equation of (2.6) has no solutions and hence (2.6) has also no solution. If $b_{11}^{(i)}(a_i\mu) > 0$, (2.6) becomes

$$\begin{aligned}\boldsymbol{\beta}^T B^{(1)}\boldsymbol{\beta} &= (\sqrt{|b_{11}^{(1)}|}\beta_1 + \sqrt{|b_{22}^{(1)}|}\beta_2)^2 = |a_1\mu|,\\ \boldsymbol{\beta}^T B^{(2)}\boldsymbol{\beta} &= (\sqrt{|b_{11}^{(2)}|}\beta_1 + \sqrt{|b_{22}^{(2)}|}\beta_2)^2 = |a_2\mu|.\end{aligned} \tag{3.10}$$

Clearly, (3.10) has four solutions if and only if

$$\begin{vmatrix} b_{11}^{(1)} & b_{22}^{(1)} \\ b_{11}^{(2)} & b_{22}^{(2)} \end{vmatrix} \neq 0.$$

Finally, the four solutions of Theorems 3.4 and 3.11 are simple. It is the following theorem.

Theorem 3.12. *The four solutions of (2.6) obtained in Theorems 3.4 and 3.11 are simple.*

Proof. We only give the proof of (EE) of Theorem 3.4. The proofs of others cases are similar. Let

$$\begin{aligned}G_1(\alpha_1,\alpha_2) &:= d_1\alpha_1^2 + d_2\alpha_2^2 - \frac{a_1}{a_2},\\ G_2(\alpha_1,\alpha_2) &:= \alpha_1^2 + \alpha_2^2 - 1.\end{aligned} \tag{3.11}$$

Under the conditions of (EE) of Theorem 3.4, (3.11) has four zeros $(\alpha_1^{(i)}, \alpha_2^{(i)})$, $i = 1, 2, 3, 4$. We claim that $\alpha_j^{(i)} \neq 0$, $j = 1, 2$. In fact if $\alpha_1^{(i)} = 0$. From $G_2(0, \alpha_2^{(i)}) = 0$, we get that $\alpha_2^{(i)} = \pm 1$. Hence by $G_1(0, 1) = 0$ we have $d_2 = a_1/a_2$. It is impossible since $a_1/a_2 < d_2$ by (EE) of Theorem 3.4.

The normal directions of G_1 and G_2 at $(\alpha_1^{(i)}, \alpha_2^{(i)})$ are $(d_1\alpha_1^{(i)}, d_2\alpha_2^{(i)})$ and $(\alpha_1^{(i)}, \alpha_2^{(i)})$, respective. Clearly, $(d_1\alpha_1^{(i)}, d_2\alpha_2^{(i)})$ and $(\alpha_1^{(i)}, \alpha_2^{(i)})$ are linearly independent. Otherwise $d_1 = d_2$ or $\alpha_j^{(i)} = 0$. It is impossible. Hence

$$\frac{\partial(G_1, G_2)}{\partial(\alpha_1, \alpha_2)}\Big|_{(\alpha_1^{(i)}, \alpha_2^{(i)})} = \begin{vmatrix} d_1\alpha_1^{(i)} & d_2\alpha_2^{(i)} \\ \alpha_1^{(i)} & \alpha_2^{(i)} \end{vmatrix} \neq 0,$$

which implies that $(\alpha_1^{(i)}, \alpha_2^{(i)})$ are simple zeros of $(G_1(\alpha_1, \alpha_2), G_2(\alpha_1, \alpha_2))$. □

4. The coexistence of homoclinic solutions

By (H2), system (1.4) with $\mu = 0$ has a homoclinic solution γ. In this section, we will find conditions such that (1.4), with small $\mu \neq 0$, has homoclinic solution γ_μ satisfying $\|\gamma - \gamma_\mu\| = O(\sqrt{|\mu|})$.

Let $D_i h$ or $D_{ij} h$ denote the derivatives of a multivariate function h with respect to its i-th or the i, j-th variables. With the change of variable $x(t) = \gamma(t) + z(t)$, (1.4) is transformed to

$$\dot{z} = Df(\gamma)z + \widetilde{g}(z, \mu), \tag{4.1}$$

where

$$\widetilde{g}(z, \mu)(t) = f(\gamma(t) + z(t)) - f(\gamma(t)) - Df(\gamma(t))z + \mu g(\gamma(t) + z(t), \mu, t). \tag{4.2}$$

Lemma 4.1. *The function $\widetilde{g}(\cdot, \mu) : \mathcal{Z} \mapsto \mathcal{Z}$ satisfies the following properties:*

$$\begin{aligned}
&(1) \quad \widetilde{g}(0,0) = 0, \ D_1\widetilde{g}(0,0) = 0, \\
&(2) \quad D_{11}\widetilde{g}(0,0) = D^2 f(\gamma), \\
&(3) \quad \frac{\partial \widetilde{g}}{\partial \mu}(0,0) = g(\gamma, 0, t).
\end{aligned}$$

Proof. It is easy to check from (4.2) that (1)–(3) hold. We now prove $\widetilde{g}(\cdot, \mu) : \mathcal{Z} \mapsto \mathcal{Z}$.

Let $\bar{B}_1(0, \delta) \subset \mathbb{R}^n$ and $\bar{B}_2(0, \delta) \subset \mathbb{R}$ be closed balls with radius $\delta > 0$ centered at the origins. For arbitrary $z \in \mathcal{Z}$, we can take a large $\delta > 0$ such that $z(t), \gamma(t), \gamma(t) + z(t) \in \bar{B}_1(0, \delta)$ for $t \in \mathbb{R}$. By (H1) and (H4), there exists a constant A_0 such that

$$|D_1\widetilde{g}(x, \mu)| < A_0, \quad |D_1 g(x, \mu, t)| < A_0$$

for $(x, \mu, t) \in \bar{B}_1(0, \delta) \times \bar{B}_2(0, \delta) \times \mathbb{R}$. Since γ is a homoclinic solution and $z \in \mathcal{Z}$, there is $A_1 > 0$ such that

$$|\gamma(t)| \leqslant A_1 e^{-m|t|}, \quad |z(t)| \leqslant A_1 e^{-m|t|}.$$

Define a map $\sigma : [0, 1] \to \mathcal{Z}$ by $\sigma(s) = \widetilde{g}(sz, \mu) - \mu g((1 - s)\gamma, \mu, t)$. By the smoothness of f, g, we see that $\sigma \in C^1$ and $\sigma(0) = 0$, then

$$\begin{aligned}
\widetilde{g}(z, \mu)(t) = \sigma(1) - \sigma(0) &= \int_0^1 \sigma'(p)dp \\
&= \int_0^1 D_1\widetilde{g}(pz(t), \mu)z(t) + \mu D_1 g((1 - p)\gamma(t), \mu, t)\gamma(t)dp.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\widetilde{g}(z, \mu)(t)| &\leqslant \sup_{x,\mu}\{|D_1\widetilde{g}(x, \mu)|\}|z(t)| + |\mu| \sup_{x,\mu,t}\{|D_1 g(x, \mu, t)|\}|\gamma(t)| \\
&\leqslant A_0 A_1(1 + |\mu|)e^{-m|t|},
\end{aligned} \tag{4.3}$$

which implies that $\widetilde{g}(z, \mu) \in \mathcal{Z}$. The proof is completed. □

Recall that $L(u) = \dot{u} - Df(\gamma)u$ in the Banach space $\mathcal{Z}$. As in [5,16], we define the subspace of $\mathcal{Z}$, which consists the range of L in $\mathcal{Z}$.

$$\widetilde{\mathcal{Z}} = \{h \in \mathcal{Z} : \int_{-\infty}^{\infty} \langle \psi_i(s), h(s) \rangle ds = 0, i = 1, 2\}.$$

Consider a nonhomogeneous equation

$$\dot{z} - Df(\gamma)z = h. \tag{4.4}$$

If $h \in \widetilde{\mathcal{Z}}$, using the variation of constants, with some phase condition, there exists an operator $K : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ such that Kh is a solution of (4.4). Clearly, the general bounded solution of (4.4) is $z(t) = \sum_{p=1}^{2} \beta_p u_p(t) + (Kh)(t)$, where $\beta_p \in \mathbb{R}$.

From (2.4), $R(L) \oplus N(L^*) = \mathcal{Z}$. Note that ψ_1, ψ_2 are orthonormal unit basis of $N(L^*)$. Define a map $P : \mathcal{Z} \to \mathcal{Z}$ by

$$(Pz)(t) = \sum_{i=1}^{2} \psi_i(t) \int_{-\infty}^{\infty} \langle \psi_i(s), z(s) \rangle ds.$$

As in [16], one can prove that P satisfies the following properties:

Lemma 4.2. (1) *P and $I - P$ are projections.*

(2) $R(P) \oplus R(L) = \mathcal{Z}$.

(3) $R(I - P) = N(P) = R(L) = \widetilde{\mathcal{Z}}$.

We now use the Lyapunov–Schmidt reduction to solve (4.1). Applying P and $(I - P)$ on (4.1), we find that (4.1) is equivalent to the following system

$$\dot{z} = Df(\gamma)z - (I - P)\widetilde{g}(z, \mu), \tag{4.5}$$

$$P\widetilde{g}(z, \mu) = 0. \tag{4.6}$$

First, we solve (4.5) for $z \in \mathcal{Z}$. Then the bifurcation equations are obtained by substituting the solution z into (4.6).

Lemma 4.3. *There exist open balls $B_1(\delta_0) \subset \mathbb{R}^2$, $B_2(\delta_0) \subset \mathbb{R}$ with radius $\delta_0 > 0$ centered at the origins and a C^2 map $\phi : B_1(\delta_0) \times B_2(\delta_0) \to \mathcal{Z}$, denoted by $\phi(\boldsymbol{\beta}, \mu)$, such that $z = \phi(\boldsymbol{\beta}, \mu)$ is a solution of Eq. (4.5). Moreover $\phi(\boldsymbol{\beta}, \mu)$ satisfies $\phi(0, 0) = 0$ and $(\partial\phi/\partial\beta_p)|_{(0,0)} = u_p$, $p = 1, 2$.*

Proof. Since $R(I - P) = \widetilde{\mathcal{Z}}$ and $K : \widetilde{\mathcal{Z}} \to \mathcal{Z}$, we define a C^2 map: $F : \mathcal{Z} \times \mathbb{R}^2 \times \mathbb{R} \to \mathcal{Z}$ by

$$F(z, \boldsymbol{\beta}, \mu) = \sum_{p=1}^{2} \beta_p u_p + K(I - P)\widetilde{g}(z, \mu), \tag{4.7}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$. Clearly, the fixed point z of (4.7) is a solution of (4.5) in $\mathcal{Z}$.

From (1) of Lemma 4.1, we have

$$F(0, 0, 0) = 0, \quad D_1F(0, 0, 0) = 0. \tag{4.8}$$

By the smoothness of F, given any $\delta > 0$, there exists $c > 0$ such that

$$\|D_2 F\| < c,\ \|D_3 F\| < c,\ \|D_{11} F\| < c,\ \|D_{12} F\| < c,\ \|D_{13} F\| < c,$$

for $(z, \boldsymbol{\beta}, \mu) \in \bar{B}(\delta) \times \bar{B}_1(\delta) \times \bar{B}_2(\delta)$, where $\bar{B}(\delta) \subset \mathcal{Z}$, $\bar{B}_1(\delta) \subset \mathbb{R}^2$, $\bar{B}_2(\delta) \subset \mathbb{R}$ are closed balls of radius δ. Let

$$\delta_1 = \min\{\delta, \frac{1}{4c}\}, \quad \delta_2 = \min\{\delta, \delta_1, \frac{\delta_1}{8c}\}.$$

For any $(z, \boldsymbol{\beta}, \mu) \in \bar{B}(\delta_1) \times \bar{B}_1(\delta_2) \times \bar{B}_2(\delta_2)$, define a map $\varphi_1 : [0, 1] \to \mathcal{L}(\mathcal{Z}, \mathcal{Z})$ by $\varphi_1(s) = D_1 F(sz, s\boldsymbol{\beta}, s\mu)$. By the smoothness of F, we see $\varphi_1 \in C^1$. By (4.8) we know $\varphi_1(0) = 0$, then

$$\begin{aligned}
\|D_1 F(z, \boldsymbol{\beta}, \mu)\| = \|\varphi_1(1) - \varphi_1(0)\| = \|\int_0^1 \varphi_1'(p)dp\| \\
&\leqslant \|D_{11} F(pz, p\boldsymbol{\beta}, p\mu)\| \cdot \|z\| + \|D_{12} F(pz, p\boldsymbol{\beta}, p\mu)\| \cdot \|\boldsymbol{\beta}\| \\
&\quad + \|D_{13} F(pz, p\boldsymbol{\beta}, p\mu)\| \cdot \|\mu\| \\
&\leqslant c \cdot \frac{1}{4c} + c \cdot \frac{1}{4c} + c \cdot \frac{1}{4c} = \frac{3}{4}.
\end{aligned} \tag{4.9}$$

For $(z, \boldsymbol{\beta}, \mu) \in \bar{B}(\delta_1) \times \bar{B}_1(\delta_2) \times \bar{B}_2(\delta_2)$, define a map $\varphi_2 : [0, 1] \to \mathcal{Z}$ by $\varphi_2(s) = F(sz, s\boldsymbol{\beta}, s\mu)$. Clearly $\varphi_2 \in C^1$ and $\varphi_2(0) = 0$, then

$$\begin{aligned}
\|F(z, \boldsymbol{\beta}, \mu)\| = \|\varphi_2(1) - \varphi_2(0)\| = \|\int_0^1 \varphi_2'(p)dp\| \\
&\leqslant \|D_1 F(pz, p\boldsymbol{\beta}, p\mu)\| \cdot \|z\| + \|D_2 F(pz, p\boldsymbol{\beta}, p\mu)\| \cdot \|\boldsymbol{\beta}\| \\
&\quad + \|D_3 F(pz, p\boldsymbol{\beta}, p\mu)\| \cdot \|\mu\| \\
&\leqslant \frac{3}{4}\delta_1 + c \cdot \frac{\delta_1}{8c} + c \cdot \frac{\delta_1}{8c} = \delta_1,
\end{aligned}$$

which implies that $F(\cdot, \boldsymbol{\beta}, \mu)$ maps $\bar{B}(\delta_1)$ into itself.

For $z_1, z_2 \in \bar{B}(\delta_1)$, $(\boldsymbol{\beta}, \mu) \in \bar{B}_1(\delta_2) \times \bar{B}_2(\delta_2)$, define a map $\varphi_3 : [0, 1] \to \mathcal{Z}$ by $\varphi_3(s) = F(sz_1 + (1 - s)z_2, \boldsymbol{\beta}, \mu)$. Then $\varphi_3 \in C^1$ and $\varphi_3(0) = 0$, then

$$\begin{aligned}
\|F(z_1, \boldsymbol{\beta}, \mu) - F(z_2, \boldsymbol{\beta}, \mu)\| = \|\varphi_3(1) - \varphi_3(0)\| = \|\int_0^1 \varphi_3'(p)dp\| \\
&\leqslant \|D_1 F(pz_1 + (1 - p)z_2^{(k)}, \boldsymbol{\beta}, \mu)\| \cdot \|z_1 - z_2\| \\
&\leqslant \frac{3}{4}\|z_1 - z_2\|.
\end{aligned}$$

Therefore F is a uniform contraction in $\bar{B}(\delta_1)$. By the contraction mapping principle, there are $\delta_{21}, \delta_{22} > 0$ and a C^1 map $\phi : B_1(\delta_{21}) \times B_2(\delta_{22}) \to B(\delta_1)$ such that $\phi(0,0) = 0$ and

$$\phi(\boldsymbol{\beta}, \mu) = F(\phi(\boldsymbol{\beta}, \mu), \boldsymbol{\beta}, \mu).$$

Let $\delta_0 = \min\{\delta_2, \delta_{21}, \delta_{22}\}$. From (4.7), we have

$$\phi(\boldsymbol{\beta}, \mu) = \sum_{p=1}^{2} \beta_p u_p + K(I - P)\widetilde{g}(\phi(\boldsymbol{\beta}, \mu), \mu). \tag{4.10}$$

Differentiating (4.10) with respect to $\boldsymbol{\beta}$, we have

$$D_1\phi(\boldsymbol{\beta}, \mu) = D_1F(\phi(\boldsymbol{\beta}, \mu), \boldsymbol{\beta}, \mu)D_1\phi(\boldsymbol{\beta}, \mu) + D_2F(\phi(\boldsymbol{\beta}, \mu), \boldsymbol{\beta}, \mu).$$

This, together with (4.9), implies that

$$D_1\phi = (I - D_1F(\phi, \boldsymbol{\beta}, \mu))^{-1} D_2F(\phi, \boldsymbol{\beta}, \mu).$$

By the smoothness of F, $D_1\phi$ is a C^1 function. Hence ϕ is C^2 in $\boldsymbol{\beta}$. Similarly, we can prove ϕ is C^2 in μ.

Differentiating (4.10) with respect to β_p and evaluating at $(0, 0)$, we get

$$\left.\frac{\partial\phi}{\beta_p}\right|_{(0,0)}(t) = u_p(t), \;\; p = 1, 2.$$

The proof has been completed. □

By Lemma 4.3, (4.5) has a solution $\phi(\boldsymbol{\beta}, \mu)$. Substituting $\phi(\boldsymbol{\beta}, \mu)$ into (4.6), we have the bifurcation equation

$$\begin{aligned} 0 &= P\widetilde{g}(\phi(\boldsymbol{\beta}, \mu), \mu) \\ &= \sum_{i=1}^{2} \psi_i(t) \int_{-\infty}^{+\infty} \langle \psi_i(s), \widetilde{g}(\phi(\boldsymbol{\beta}, \mu), \mu)(s) \rangle ds, \end{aligned} \tag{4.11}$$

where the definition of projection P is used. By the linear independence of ψ_1, ψ_2, we see that

$$H_i(\boldsymbol{\beta}, \mu) := \int_{-\infty}^{+\infty} \langle \psi_i(s), \widetilde{g}(\phi(\boldsymbol{\beta}, \mu), \mu)(s) \rangle ds = 0, \;\; i = 1, 2.$$

If there are some parameter values $(\boldsymbol{\beta}, \mu) \in \mathbb{R}^2 \times \mathbb{R}$ such that

$$H_i(\boldsymbol{\beta}, \mu) = 0, \;\; i = 1, 2,$$

then $z = \phi$ is a solution of (4.1) and hence the perturbed system (1.4) has a homoclinic orbit $x = \gamma + \phi$, where ϕ is given in (4.10). Let

$$H(\boldsymbol{\beta}, \mu) = (H_1(\boldsymbol{\beta}, \mu), H_2(\boldsymbol{\beta}, \mu)).$$

Through direct calculations, we can prove the following lemma.

Lemma 4.4. *For $i, p, q \in \{1, 2\}$, the function $H(\boldsymbol{\beta}, \mu)$ has the following properties:*

(i) If there are some $(\boldsymbol{\beta}, \mu) \in \mathbb{R}^2 \times \mathbb{R}$ such that $H(\boldsymbol{\beta}, \mu) = 0$, then ϕ is a solution of (4.1);

(ii) $H_i(0, 0) = 0$, $\dfrac{\partial H_i}{\partial \beta_p}(0, 0) = 0$;

(iii) $b_{pq}^{(i)} = \dfrac{\partial^2 H_i}{\partial \beta_p \partial \beta_q}(0, 0) = \displaystyle\int_{-\infty}^{+\infty} \langle \psi_i(t), D^2 f(\gamma(t)) u_p(t) u_q(t) \rangle dt$;

(iv) $\tilde{a}_i = \dfrac{\partial H_i}{\partial \mu}(0, 0) = \displaystyle\int_{-\infty}^{+\infty} \langle \psi_i(t), g(\gamma(t), 0, t) \rangle dt$.

Let $M : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be given by

$$M(\boldsymbol{\beta}, \mu) = (M_1(\boldsymbol{\beta}, \mu), M_2(\boldsymbol{\beta}, \mu)),$$

where $M_i(\boldsymbol{\beta}, \mu) = \frac{1}{2}\boldsymbol{\beta}^T B^{(i)} \boldsymbol{\beta} + \tilde{a}_i \mu$ contains the lowest order terms of $H_i(\boldsymbol{\beta}, \mu)$. Compare this with (2.6). If we let $\tilde{a}_i = -a_i/2$, then the simple solutions of (2.6) are the simple solutions of $M(\boldsymbol{\beta}, \mu) = 0$. From the discussions in Theorems 3.4, 3.11 and 3.12 of Section 3, for some fixed μ, Eq. (2.6) have four simple solutions $\boldsymbol{\beta}_0^{(1)}, \dots, \boldsymbol{\beta}_0^{(4)}$. Hence $\boldsymbol{\beta}_0^{(1)}, \dots, \boldsymbol{\beta}_0^{(4)}$ are the simple zeros of $M(\boldsymbol{\beta}, \mu)$.

Lemma 4.5. *There are some fixed μ_0 such that $M(\boldsymbol{\beta}, \mu_0)$ has four simple zeros $\boldsymbol{\beta}_0^{(1)}, \dots, \boldsymbol{\beta}_0^{(4)}$. For each $\boldsymbol{\beta} = \boldsymbol{\beta}_0^{(j)}$, $1 \le j \le 4$, there exist an open region $I_j \subset \mathbb{R}$ containing zero and differentiable function, $\omega_j : I_j \to \mathbb{R}^2$ such that $\omega_j(0) = 0$, and $H(s(\boldsymbol{\beta}_0^{(j)} + \omega_j(s)), s^2\mu_0) = 0$ for $s \in I_j$ and $s \neq 0$.*

Proof. Since $\boldsymbol{\beta}_0^{(j)}$ are simple solutions, we have $M(\boldsymbol{\beta}_0^{(j)}, \mu_0) = 0$ and $D_{\boldsymbol{\beta}} M(\boldsymbol{\beta}_0^{(j)}, \mu_0)$ is a 2×2 nonsingular matrix. For each $\boldsymbol{\beta} = \boldsymbol{\beta}_0^{(j)}$, $j = 1, 2, 3, 4$, define a C^2 function $W : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2$ by

$$W(x, s) = \begin{cases} \frac{1}{s^2} H(s(\boldsymbol{\beta}_0^{(j)} + x), s^2\mu_0), & \text{for } s \neq 0, \\ M(\boldsymbol{\beta}_0^{(j)} + x, \mu_0), & \text{for } s = 0. \end{cases}$$

Clearly, $H = 0$ if and only if $W = 0$ for $s \neq 0$. Through direct calculations, we have $W(0, 0) = 0$ and $D_x W(0, 0) = D_{\boldsymbol{\beta}} M(\boldsymbol{\beta}_0^{(j)}, \mu_0)$ is a nonsingular matrix. By the implicit function theorem there exist an open region $I_j \subset \mathbb{R}$ containing zero and a differentiable functions, $\omega_j : I_j \to \mathbb{R}^2$ with $\omega_j(0) = 0$ such that $W(\omega_j(s), s) = 0$ for $s \in I_j$. Hence we have

$$H(s(\boldsymbol{\beta}_0^j + \omega_j(s)), s^2\mu_0) = 0 \ \text{ for } s \neq 0.$$

The proof has been completed. $\square$

By Lemma 4.5, the bifurcation function H vanishes at $\boldsymbol{\beta} = s(\boldsymbol{\beta}_0^{(j)} + \omega_j(s))$ and $\mu = s^2\mu_0$. Then system (4.1) has the solution $\phi(\boldsymbol{\beta}, \mu)$. Hence system (1.4) has four homoclinic solutions given by

$$\begin{aligned}\gamma_s^{(j)}(t) = \gamma(t) &+ \sum_{p=1}^{2} s(\beta_{0p}^{(j)} + \omega_{jp}(s))u_p(t) \\ &+ K(I-P)\widetilde{g}(\phi(s(\boldsymbol{\beta}_0^{(j)} + \omega_j(s)), s^2\mu_0), s^2\mu_0)(t),\end{aligned}$$

for $0 \neq s \in I_j$, $j = 1, \ldots, 4$. Clearly, $\lim_{s\to 0} \gamma_s^{(j)}(t) = \gamma(t)$.

5. The transversalities

If all the four homoclinic solutions $\gamma_s^{(j)}$ are transverse, then the periodic system (1.4) has four transverse homoclinic solution. Hence the periodic map of system (1.4) has four horseshoe chaotic motions. By Shadowing Lemma in [16], to prove the transversality of $\gamma_s^{(j)}$ suffices to prove Eq. (5.3) has no nonzero bounded solution.

Through calculations, we have

$$\frac{\partial \gamma_s^{(j)}}{\partial s}\Big|_{s=0} = \sum_{p=1}^{2} \beta_{0p}^{(j)} u_p. \tag{5.1}$$

Since $\gamma_s^{(j)}$ is a solution of (1.4) with $\mu = s^2\mu_0$, we get by substituting $\gamma_s^{(j)}$ into (1.4) that

$$\dot{\gamma}_s^{(j)} = f(\gamma_s^{(j)}) + s^2\mu_0 g(\gamma_s^{(j)}, s^2\mu_0, t).$$

Differentiating with respect to t, we have

$$\ddot{\gamma}_s^{(j)} = [Df(\gamma_s^{(j)}) + s^2\mu_0 D_1 g(\gamma_s^{(j)}, s^2\mu_0, t)]\dot{\gamma}_s^{(j)} + s^2\mu_0 D_3 g(\gamma_s^{(j)}, s^2\mu_0, t). \tag{5.2}$$

The variational equation of (1.4) along $\gamma_s^{(j)}$ is

$$\dot{u} = [Df(\gamma) + G(s)]u, \tag{5.3}$$

where

$$G(s) = Df(\gamma_s^{(j)}) - Df(\gamma) + s^2\mu_0 D_1 g(\gamma_s^{(j)}, s^2\mu_0, t).$$

We now prove that the variational equation (5.3) has no nonzero bounded solutions. It is easy to check that

$$\begin{aligned} G(0) &= 0 \\ \frac{\partial G}{\partial s}|_{s=0} &= \sum_{p=1}^{2} \beta_{0p}^{(j)} D^2 f(\gamma) u_p. \end{aligned} \tag{5.4}$$

Applying the projections P and $(I - P)$ on Eq. (5.3), we have

$$\dot{u} = Df(\gamma)u + (I - P)G(s)u, \tag{5.5}$$

$$0 = PG(s)u. \tag{5.6}$$

The general bounded solution u^* of (5.5) has the following form

$$u^* = \sum_{q=1}^{2} \eta_q u_q + K(I - P)G(s)u^*,$$

where $\eta_q \in \mathbb{R}$. Since $G(0) = 0$, there exists a small region $\tilde{I}$ around zero such that $(I - K(I - P)G(s))$ is invertible for $s \in \tilde{I}$. We get

$$u^* = [I - K(I - P)G(s)]^{-1} \sum_{q=1}^{2} \eta_q u_q \quad \text{for } s \in \tilde{I}.$$

Substituting $u = u^*$ into Eq. (5.6), we have

$$\begin{aligned} 0 &= PG(s)[I - K(I - P)G(s)]^{-1} \sum_{q=1}^{2} \eta_q u_q \\ &= \sum_{i=1}^{2} \psi_i \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I - P)G(s)]^{-1} \sum_{q=1}^{2} \eta_q u_q \rangle ds \\ &= \sum_{i,q=1}^{2} \psi_i \eta_q \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I - P)G(s)]^{-1} u_q \rangle ds \\ &= (\psi_1, \psi_2) V(G(s))(\eta_1, \eta_2), \end{aligned}$$

where matrix $V(G(s))$ is given by $V(G(s)) = [v_{iq}(s)]_{2\times 2}$ and

$$v_{iq}(s) = \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I - P)G(s)]^{-1} u_q \rangle dt. \tag{5.7}$$

Note that ψ_1, ψ_2 are linearly independent. If we can prove that $V(G(s))$ is a nonsingular matrix, then $\eta_1 = \eta_2 = 0$. Thus the only bounded solution for the linear variational equation along $\gamma_s^{(i)}$ is $u^* = 0$. The Shadowing Lemma implies that $\gamma_s^{(i)}$ is a transverse homoclinic solution of (1.4) and its periodic map exhibits chaotic motion.

It remains to show $V(G(s))$ is nonsingular. By (5.4) and (5.7), we have $v_{iq}(0) = 0$ and

$$\begin{aligned}\frac{\partial v_{iq}}{\partial s}|_{s=0} &= \sum_{p=1}^{2} \beta_{0p}^{(j)} \int_{-\infty}^{+\infty} \langle \psi_i, D^2 f(\gamma) u_q u_p \rangle dt.\\ &= \sum_{p=1}^{2} b_{pq}^{(i)} \beta_{0p}^{(j)} = \frac{\partial M_i}{\partial \beta_q}(\boldsymbol{\beta}_0^{(j)}, \mu_0)).\end{aligned}$$

We have the following approximation of $v_{ip}(s)$:

$$v_{iq}(s) = s \sum_{p=1}^{2} b_{pq}^{(i)} \beta_{0p}^{(j)} + O(s^2), \tag{5.8}$$

where $i, q = 1, 2$. Therefore

$$\begin{aligned}\det(V(G(s))) &= s^2 \det\left(\frac{\partial(M_1, M_2)}{\partial(\beta_1, \beta_2)}(\boldsymbol{\beta}_0^{(j)}, \mu_0)\right) + O(s^3)\\ &= s^2 \det(D_{\boldsymbol{\beta}} M(\boldsymbol{\beta}_0^{(j)}, \mu_0)) + O(s^3).\end{aligned}$$

Note that $D_{\boldsymbol{\beta}} M(\boldsymbol{\beta}_0^{(j)}, \mu_0)$ is nonsingular. Then there exists a region $\hat{I}$, $\hat{I} \subset \tilde{I}$ such that $V(G(s))$ is nonsingular when $0 \neq s \in \hat{I}$. Then the variational equation along $\gamma_s^{(j)}$ has no nonzero bounded solutions. So $\gamma_s^{(j)}$ is a transverse homoclinic solution of (1.4) and its periodic map exhibits chaotic motion.

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References

[1] S.N. Chow, J.K. Hale, J. Mallet-Parret, An example of bifurcation to homoclinic orbits, J. Differential Equations 37 (1980) 551–573.

[2] X.P. Chen, W.S. Wang, Some conditions for co-diagonalizations of two matrixes, J. Zaozhuang University 22 (2005) 11–13 (in Chinese).

[3] M. Fečkan, Bifurcation from degenerate homoclinics in periodically forced systems, Discrete Contin. Dyn. Syst. 5 (1999) 359–374.

[4] J.R. Gruendler, Homoclinic solutions for autonomous systems in arbitrary dimension, SIAM J. Math. Anal. 23 (1992) 702–721.

[5] J.R. Gruendler, Homoclinic solutions for autonomous ordinary differential equations with nonautonomous perturbation, J. Differential Equations 122 (1995) 1–26.

[6] J.R. Gruendler, The existence of transverse homoclinic solutions for higher order equations, J. Differential Equations 130 (1996) 307–320.

[7] J.R. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.

[8] J.K. Hale, Introduction to dynamic bifurcation, in: Bifurcation Theory and Applications, in: Lecture Notes in Math., vol. 1057, Springer-Verlag, Berlin, 1984, pp. 106–151.

[9] J.K. Hale, X.-B. Lin, Heteroclinic orbits for retarded functional differential equations, J. Differential Equations 65 (1986) 175–202.

[10] R. Horn, Topics in Matrix Analysis, Cambridge University Press, 1994.

[11] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[12] L. Jaeger, H. Kantz, Homoclinic tangencies and non-normal Jacobians – effects of noise in nonhyperbolic chaotic systems, Phys. D 105 (1997) 79–96.

[13] J. Knobloch, Bifurcation of degenerate homoclinics in reversible and conservative systems, J. Dynam. Differential Equations 9 (1997) 427–444.

[14] J. Li, X.-B. Lin, Traveling wave solutions for the Painlevé-integrable coupled KDV equations, Electron. J. Differential Equations 2008 (86) (2008) 1–11.

[15] V.K. Melnikov, On the stability of the center for time periodic perturbation, Trans. Moscow Math. Soc. 12 (1963) 1–57.

[16] K.J. Palmer, Exponential dichotomies and transversal homoclinic points, J. Differential Equations 55 (1984) 225–256.

[17] K.J. Palmer, Transversal heteroclinic orbits and Cherry'y example of a non-integrable Hamiltonian system, J. Differential Equations 65 (1986) 321–360.