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Using Melnikov's method to solve Silnikov's problems*

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Dedicated to Professor Jack K. Hale on the occasion of his 60th birthday

Synopsis

A function space approach is employed to obtain bifurcation functions for which the zeros correspond to the occurrence of periodic or aperiodic solutions near heteroclinic or homoclinic cycles. The bifurcation function for the existence of homoclinic solutions is the limiting case where the period is infinite. Examples include generalisations of Silnikov's main theorems and a retreatment of a singularly perturbed delay differential equation.

1. Introduction

Let $q(t)$ be a solution of an autonomous differential equation

$$\dot{x}(t)=f(x(t),\mu) \tag{1.1}$$

that is homoclinic to a hyperbolic equilibrium p when the parameter $\mu=0$. In an influential series of papers, Silnikov **[21–24]** studied periodic and aperiodic solutions near the orbit of $q(t)$, both for $\mu=0$ and for μ near 0. Silnikov's work has been carried on by Blazguez **[1]**, Chow and Deng [3], and Walther **[29]** to infinite dimensional spaces.

All these authors obtain their results by studying Poincaré maps both inside a neighbourhood of the equilibrium p and outside that neighbourhood but near the orbit of $q(t)$ and then matching the two maps. Our main goal in this paper is to provide an alternate approach that uses function spaces to analyse a linear variational equation around the solution $q(t)$ and then uses the Liapunov–Schmidt method to obtain bifurcation functions to the problem. The function space approach to homoclinic and heteroclinic bifurcation problems has been extensively developed following the original works of Chow, Hale and Mallet-Paret **[7]**, Kirchgassner **[26]**, Mielke **[27]**, Renardy **[28]**, and Palmer **[18]**. Our contribution is to extend this approach so that it also deals with nearby periodic and aperiodic solutions.

Let us briefly indicate how our approach works. Let Σ be an $(n-1)$-plane through $q(0)$ that is transverse to $\dot{q}(0)$. We fix a large number $\omega>0$ and try to find $\mu\cong 0$ and $x(t)$, $-\omega\leqq t\leqq\omega$, such that $x(t)$ satisfies (1.1), $x(0)\in\Sigma$, and $x(-\omega)=x(\omega)$. If we write $x(t)=q(t)+z(t)$, $-\omega\leqq t\leqq\omega$, then $z(t)$ satisfies a

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weakly nonlinear boundary value problem of the form

$$\begin{aligned} \dot{z}(t) - D_x f(q(t), 0) z(t) &= g(z(t), \mu, t), \\ z(\omega) - z(-\omega) &= b, \end{aligned} \tag{1.2}$$

where $b = q(-\omega) - q(\omega)$ is known. We now assume that the tangent spaces to the stable and unstable manifolds of p have only a one-dimensional intersection along $q(t)$. Then we can construct a real valued bifurcation function $G(\omega, \mu)$, defined for large ω and small μ, such that the boundary value problem (1.2) has a small solution $z(t)$ if and only if $G(\omega, \mu) = 0$. As $\omega \to \infty$, $G(\omega, \mu)$ approaches a function $G(\infty, \mu)$ that is zero if and only if (1.1) has a homoclinic orbit near $q(t)$. Thus $G(\infty, \mu)$ is the well-known bifurcation function for homoclinic orbits. It is known that

$$G(\infty, \mu) = \mu \int_{-\infty}^{\infty} \psi(t) D_\mu f(q(t), 0)\, dt + O(|\mu|^2),$$

where $\psi(t)$ is a bounded solution of the adjoint equation

$$\dot{z}(t) + \{D_x f(q(t), 0)\}^* z(t) = 0.$$

$\psi(t)$ is unique up to scalar multiple. A careful study shows that

$$G(\omega, \mu) = \psi(\omega) q(-\omega) - \psi(-\omega) q(\omega) + \mu \int_{-\infty}^{\infty} \psi(t) D_\mu f(q(t), 0)\, dt + \text{small terms}.$$

The integral term shows how the perturbation of the distance of the stable and unstable manifolds enters the problem. The boundary terms show how the unstable and stable eigenvalues enter the problem. The sign change of $\psi(\omega)q(-\omega)$ and $\psi(-\omega)q(\omega)$ also shows how the stable and unstable manifold are twisted along their common intersection $q(t)$, an idea that played important roles in **[25]**, **[14]**, and **[4]**. All the notions mentioned above are known to many investigators, but here we bring them together in a compact form.

Another merit of our approach is that it has a clear geometric interpretation. It is well-known that the bifurcation function $G(\infty, \mu)$ measures the distance between the unstable manifold W^u and the stable manifold W^s of $p(\mu)$ along a transverse direction, cf. **[9]**. Our bifurcation function $G(\omega, \mu)$ also measures the jump of a piecewise continuous solution $x(t)$ for (1.1) along the same direction. In this sense $G(\omega, \mu)$ is a natural generalisation of $G(\infty, \mu)$.

Our approach is sufficiently general so that it also applies to periodic and aperiodic solutions near heteroclinic cycles, and to delay differential equations. The key examples treated in this paper include generalisations of Silnikov's three theorems – the existence of a unique periodic solution **[23]**, infinitely many periodic solutions **[20, 21]**, and uncountably many aperiodic solutions **[24]**, as well as a complete reproof of the result of **[16, 17]** concerning a singularly perturbed differential-difference equation.

We shall now make some remarks on the organisation of this paper. In order to obtain a theory that applies to both homoclinic solutions and heteroclinic cycles, and to both periodic and aperiodic solutions, we shall develop our theory for *heteroclinic chains*. A sequence of solutions $\{q_i(t)\}_{i \in \mathbf{Z}}$ of an autonomous differential equation is called a *heteroclinic chain* if there is a sequence of equilibria $\{p_i\}_{i\in\mathbf{Z}}$ such that for every i, $\lim_{t\to-\infty} q_i(t) = p_i$ and $\lim_{t\to\infty} q_i(t) = p_{i+1}$. In Section 2 we shall develop the linear theory that we need. The theory is designed to be applicable to both ordinary and delay differential equations, so the reader interested only in ordinary differential equations may feel the presentation is somewhat unnatural. We have tried to make the presentation intelligible to such a reader while avoiding the necessity of redoing the theory for delay equations. In Section 3 we present results concerning solutions near a heteroclinic chain. In Section 4 we study periodic and aperiodic solutions near a homoclinic solution. (The results in this section are new since our assumptions are weaker than Silnikov's.) In Section 5 we study a singularly perturbed differential-difference equation $\varepsilon\dot{x} = -x(t) + f(x(t-1))$ that admits a heteroclinic cycle after some proper rescaling. We show that there is a periodic solution near the heteroclinic cycle. This implies the existence of a square wave-like solution for the singularly perturbed equation when $\varepsilon > 0$ is small.

The method described in this paper also extends to other types of heteroclinic or homoclinic bifurcations where the heteroclinic chain is not doubly infinite, for example, the bifurcation phenomena indicated in [**14**, Figure 0.1–0.4]. See also [**5**, **6**]. To treat those problems, a finite chain of length 2 should be used and we should look for solutions $x_1(t)$, $-\infty \leqq t \leqq \omega$, and $x_2(t)$, $-\omega \leqq t \leqq \infty$, for (1.1) with the matching condition $x_1(\omega) = x_2(-\omega)$. A system of two bifurcation functions $G_i(\omega, \mu)$, $i = 1, 2$ can be obtained. In particular, for the case indicated in [**14**, Figure 0.4], we can easily find that resonance of principal eigenvalues and a twist of the intersections of W^u and W^s are necessary for the homoclinic doubling to occur. See [**25**], [**14**], and [**4**] for some details as well as more complete references. Systems with periodic or almost periodic perturbations can also be handled by our method. However, we have decided to restrict this paper to bifurcations of infinite chains and to autonomous differential equations for clarity.

2. System of linear variational equations with discontinuity at boundaries

Throughout this paper $\mathcal{R}$ and $\mathcal{K}$ are used to denote the range and kernel of an operator. A two parameter family of linear bounded operators $T(t, s)$, $t \geqq s$ in a Banach space X is called an evolution operator if it is strongly continuous in t and s, and satisfies the semigroup properties: $T(t, t) = I$; $T(t, \sigma) = T(t, s)T(s, \sigma)$ for $t \geqq s \geqq \sigma$. $T(t, s)$ is said to have an exponential dichotomy on an interval I with a constant $K \geqq 0$ and an exponent $\alpha > 0$ if there are projections $P_s(s)$ and $P_u(s) = I - P_s(s)$, $s \in I$, strongly continuous in s and

(i) $T(t, s)P_s(s) = P_s(t)T(t, s)$, $t \geqq s$ in I;
(ii) $T(t, s)\colon \mathcal{R}P_u(s) \to \mathcal{R}P_u(t)$ is an isomorphism and $T(s, t)\colon \mathcal{R}P_u(t) \to \mathcal{R}P_u(s)$ is defined as the inverse of $T(t, s)|_{\mathcal{R}P_u(s)}$;
(iii) $|T(t, s)P_s(s)| \leqq Ke^{-\alpha(t-s)}$, $t \geqq s$, in I;
(iv) $|T(s, t)P_u(t)| \leqq Ke^{-\alpha(t-s)}$, $t \geqq s$, in I.

$\mathcal{R}P_s(t)$ and $\mathcal{R}P_u(t)$ are called the stable and unstable invariant subspaces of $T(t, s)$. We assume that $\mathcal{R}P_u(t)$ is finite dimensional. This covers many important cases like ODEs, parabolic PDEs and delay differential equations.

Let $T^*(s, t)$ be the adjoint of $T(t, s)$, i.e.

$$\langle x^*, T(t, s)x \rangle = \langle T^*(s, t)x^*, x \rangle, \quad x \in X, \quad x^* \in X^*.$$

$T^*(s, t)$ is a linear bounded operator, satisfies the semigroup properties for $s \leqq t$ in J, and is weak* continuous in s and t. $T^*(s, t)$ has an exponential dichotomy on I, with the projections $P_s^*(s)$ and $P_u^*(s)$ being the adjoint operators of $P_s(s)$ and $P_u(s)$.

(i)* $T^*(s, t)P_s^*(t) = P_s^*(s)T^*(s, t)$, $s \leqq t$, in I;

(ii)* $T^*(s, t)$: $\mathcal{R}P_u^*(t) \to \mathcal{R}P_u^*(s)$ is an isomorphism and $T^*(t, s)$: $\mathcal{R}P_u^*(s) \to \mathcal{R}P_u^*(t)$ is defined as the inverse of $T^*(s, t)|_{\mathcal{R}P_u^*(t)}$;

(iii)* $|T^*(s, t)P_s^*(t)| \leqq Ke^{-\alpha(t-s)}$, $s \leqq t$, in I;

(iv)* $|T^*(t, s)P_u^*(s)| \leqq Ke^{-\alpha(t-s)}$, $s \leqq t$, in I.

In this paper $T(t, s)$ comes from the linearisation of homoclinic or heteroclinic solutions of ODEs or delay differential equations. Let $[T(\tau, -\tau)]^{-1}\mathcal{R}P_s(\tau)$ be the pre-image of $\mathcal{R}P_s(\tau)$ under $T(\tau, -\tau)$. We assume throughout the paper that

(A) $T(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$ where $\tau > 0$ is a constant. Also, $\dim \mathcal{R}P_u(-\tau) = \dim \mathcal{R}P_u(\tau) = d^+$. $\mathcal{R}P_u(-\tau) \cap [T(\tau, -\tau)]^{-1}\mathcal{R}P_s(\tau)$ is one-dimensional, spanned by $\varphi_0 \neq 0$.

Remark 2.1. The condition in Hypothesis (A) cannot be replaced by a condition that $T(\tau, -\tau)\mathcal{R}P_u(-\tau) \cap \mathcal{R}P_s(\tau)$ is one-dimensional. In the case of a delay equation, it is possible that $\varphi_0 \neq 0$ but $T(\tau, -\tau)\varphi_0 = 0$.

Consider a linear bounded operator $\mathcal{F}$: $\mathcal{R}P_u(-\tau) \times \mathcal{R}P_s(\tau) \to X$, $\mathcal{F}$: $(u, v) \to x$,

$$x = j_1 v - T(\tau, -\tau)j_2 u. \tag{2.0}$$

Here $\mathcal{R}P_u(-\tau)$ and $\mathcal{R}P_s(\tau)$ are Banach spaces with norms induced from X; j_1 and j_2 are the natural embeddings from $\mathcal{R}P_s(\tau)$ and $\mathcal{R}P_u(-\tau)$ into X. Let $x^* \in X^*$. Now,

$$\langle x^*, x \rangle = \langle j_1^* x^*, v \rangle - \langle j_2^* T^*(-\tau, \tau)x^*, u \rangle.$$

Thus, $\mathcal{F}^*$: $X^* \to \mathcal{R}P_u^*(-\tau) \times \mathcal{R}P_s^*(\tau)$, $\mathcal{F}^* x^* = (-j_2^* T^*(-\tau, \tau)x^*, j_1^* x^*)$.

Lemma 2.2. *$\mathcal{F}$ is Fredholm with index $\mathcal{F} = 0$. $\mathcal{K}\mathcal{F}$ and $\mathcal{K}\mathcal{F}^*$ are both one-dimensional. $\mathcal{R}\mathcal{F}$ is of codimension one.*

$$\mathcal{K}\mathcal{F} = \{(u, v) \mid u = \xi\varphi_0, v = j_1^{-1}T(\tau, -\tau)j_2 u, \xi \in \mathbb{R}\};$$

$$\mathcal{K}\mathcal{F}^* = \{\xi\psi_0 \mid \psi_0 \in \mathcal{R}P_u^*(\tau), T^*(-\tau, \tau)\psi_0 \in \mathcal{R}P_s^*(-\tau), \xi \in \mathbb{R}\};$$

$$\mathcal{R}\mathcal{F} = \{x \mid \langle \psi_0, x \rangle = 0\}.$$

Proof. $\mathcal{F}$ is a compact perturbation of the Fredholm operator

$$x = j_1 v + 0 \,.\, u.$$

From (A), the index of the latter is $d^+ - d^+ = 0$. Therefore $\mathcal{F}$ is Fredholm with the same index (see [18]). From (A) again, $\mathcal{K}\mathcal{F}$ is one-dimensional. All the other assertions follow easily.

Let $\Delta \in X$ with $\langle \psi_0, \Delta \rangle = 1$. Then $\Delta \oplus \mathcal{R}\mathcal{F} = X$. Let $\omega_1 > \tau$, $\omega_2 > \tau$, and $J = [-\omega_1, \omega_2]$. Let $E(J)$ be the Banach space of piecewise continuous functions in J, with a possible jump at $t = \tau$. $E(J)$ is equipped with the sup norm. Let $E(J, \Delta)$ be the subspace of $E(J)$ which consists of piecewise continuous functions with a possible jump along the direction of Δ at $t=\tau$, i.e. $x(t)\in E(J,\Delta)$ if $x(\tau^-)-x(\tau^+)=\xi\Delta$, $\xi\in\mathbb{R}$. Here $x(\tau^-)=\lim_{t<\tau, t\to\tau} x(t)$ and $x(\tau^+)=\lim_{t>\tau, t\to\tau} x(t)$. For $h\in E(J)$ and $0\leqq\eta$, define the weighted norm as

$$\|h\|_\eta=\sup_{t\in J}\{2\,|h(t)|(e^{-\eta(\omega_1+t)}+e^{-\eta(\omega_2-t)})^{-1}\}.$$

Obviously $\|h\|_\eta\geqq\|h\|_0$. Since $\psi_0\in\mathscr{R}P_u^*(\tau)$ and $T^*(-\tau,\tau)\psi_0\in\mathscr{R}P_s^*(-\tau)$, $\psi(s)\overset{\text{def}}{=}T^*(s,\tau)\psi_0$ decays exponentially as $s\to\pm\infty$. Let $\bar\psi\in X^*$ with $\langle\bar\psi,\varphi_0\rangle\neq 0$.

We now consider the integral equation

$$x(t)=T(t,\sigma)x(\sigma)+\int_\sigma^t T(t,s)h(s)\,ds,\quad \sigma\leqq t,\ \text{in } J. \tag{2.1}$$

$x(t)$ is said to be a solution of (2.1) if $x: J\to X$ is continuous and satisfies (2.1) for all $\sigma\leqq t$ in J. Also, $x(t)$ is said to be a *piecewise continuous solution* of (2.1) with a possible jump at $t=\tau$ if $x(t)\in E(J)$ and satisfies (2.1) in $[-\omega_1,\tau)$ and $(\tau,\omega_2]$.

Lemma 2.3. *For $\varphi_1\in\mathscr{R}P_s(-\omega_1)$, $\varphi_2\in\mathscr{R}P_u(\omega_2)$ and $h\in E(J)$, equation (2.1) has a unique piecewise continuous solution $x\in E(J,\Delta)$ with $P_s(-\omega_1)x(-\omega_1)=\varphi_1$, $P_u(\omega_2)x(\omega_2)=\varphi_2$ and $\langle\bar\psi, x(-\tau)\rangle=0$. Let $x(\tau^-)-x(\tau^+)=\xi\Delta$, then*

$$\xi=\int_{-\omega_1}^{\omega_2}\langle\psi(s),h(s)\rangle\,ds+\langle\psi(-\omega_1),\varphi_1\rangle-\langle\psi(\omega_2),\varphi_2\rangle. \tag{2.2}$$

Moreover, we have the following estimates:

$$\|x\|_\eta\leqq C(|\varphi_1|+|\varphi_2|+\|h\|_\eta),\quad 0\leqq\eta<\alpha; \tag{2.3}$$

$$|\xi|+|P_u(-\tau)x(-\tau)|+|P_s(\tau)x(\tau)|$$
$$\leqq C(|\varphi_1|e^{-\alpha\omega_1}+|\varphi_2|e^{-\alpha\omega_2}+\|h\|_\eta(e^{-\eta\omega_1}+e^{-\eta\omega_2})); \tag{2.4}$$

$$|P_s(\omega_2)x(\omega_2)|\leqq C(\|h\|_0+|\varphi_1|e^{-\alpha(\omega_1+\omega_2)}+|\varphi_2|e^{-2\alpha\omega_2}); \tag{2.5}$$

$$|P_u(-\omega_1)x(-\omega_1)|\leqq C(\|h\|_0+|\varphi_1|e^{-2\alpha\omega_1}+|\varphi_2|e^{-\alpha(\omega_1+\omega_2)}). \tag{2.6}$$

Here the constants C, not necessarily equal, do not depend on ω_1 or ω_2.

The solution $x(t)$ will be denoted by $x(t)=\mathscr{X}(t;\varphi_1,\varphi_2,h,\omega_1,\omega_2)$, while ξ is denoted by $\xi=\Xi(\varphi_1,\varphi_2,h,\omega_1,\omega_2)$.

Proof. Projecting (2.1) into the stable and the unstable spaces, we have

$$P_u(\tau)x(\tau)=\int_{\omega_2}^{\tau}T(\tau,s)P_u(s)h(s)\,ds+T(\tau,\omega_2)\varphi_2,$$

$$P_s(-\tau)x(-\tau)=\int_{-\omega_1}^{-\tau}T(-\tau,s)P_s(s)h(s)\,ds+T(-\tau,-\omega_1)\varphi_1.$$

We try to find $\varphi_3\in\mathscr{R}P_u(-\tau)$, $\varphi_4\in\mathscr{R}P_s(\tau)$ and $\xi\in\mathbb{R}$ such that

$$\begin{aligned}\varphi_4-T(\tau,-\tau)\varphi_3=&\int_{-\omega_1}^{-\tau}T(\tau,s)P_s(s)h(s)\,ds+\int_{-\tau}^{\tau}T(\tau,s)h(s)\,ds\\&+\int_{\tau}^{\omega_2}T(\tau,s)P_u(s)h(s)\,ds\\&+T(\tau,-\omega_1)\varphi_1-T(\tau,\omega_2)\varphi_2-\xi\Delta.\end{aligned} \tag{2.7}$$

The left-hand side of (2.7) is the Fredholm operator $\mathcal{F}$: $\varphi_3 \times \varphi_4 \to X$ studied in Lemma 2.2. The right-hand side is in $\mathcal{R}\mathcal{F}$ if

$$\xi = \left\langle \psi_0, \int_{-\omega_1}^{-\tau} T(\tau, s)P_s(s)h(s)\,ds \right\rangle + \left\langle \psi_0, \int_{-\tau}^{\tau} T(\tau, s)h(s)\,ds \right\rangle$$
$$+ \left\langle \psi_0, \int_{\tau}^{\omega_2} T(\tau, s)P_u(s)h(s)\,ds \right\rangle + \langle \psi_0, T(\tau, -\omega_1)\varphi_1 \rangle - \langle \psi_0, T(\tau, \omega_2)\varphi_2 \rangle.$$

This is equivalent to (2.2) after some simplification, using the definition of $\psi(s)$ and the property that $\psi_0 \in \mathcal{R}P_u^*(\tau)$ and $T^*(-\tau, \tau)\psi_0 \in \mathcal{R}P_s^*(-\tau)$.

Let $\bar{\psi} = \bar{\psi}_u + \bar{\psi}_s$, where $\bar{\psi}_u \in \mathcal{R}P_u^*(-\tau)$ and $\bar{\psi}_s \in \mathcal{R}P_s^*(-\tau)$. Also, $\langle \bar{\psi}_u, \varphi_0 \rangle = \langle \bar{\psi}, \varphi_0 \rangle \neq 0$. From the general property of Fredholm operators, (2.7) has a unique solution $(\varphi_3^1, \varphi_4^1)$ such that $\langle \bar{\psi}_u, \varphi_3^1 \rangle = 0$, if (2.2) holds. Let the solution of (2.7) be $\varphi_3 = \varphi_3^1 + \zeta\varphi_0$ and $\varphi_4 = \varphi_4^1 + \zeta T(\tau, -\tau)\varphi_0$, $\zeta \in \mathbb{R}$. The parameter ζ is determined by the condition

$$\langle \bar{\psi}_u, \varphi_3 \rangle + \langle \bar{\psi}_s, P_s(-\tau)x(-\tau) \rangle = 0, \tag{2.8}$$

using the fact that $\langle \bar{\psi}_u, \varphi_0 \rangle \neq 0$. Define $x(t)$, $t \in [-\omega_1, \omega_2]$ by

$$x(t) = T(t, -\omega_1)\varphi_1 + T(t, \tau)\varphi_3 + \int_{-\omega_1}^{t} T(t, s)P_s(s)h(s)\,ds$$
$$+ \int_{-\tau}^{t} T(t, s)P_u(s)h(s)\,ds, \quad -\omega_1 \leqq t \leqq -\tau;$$

$$x(t) = T(t, -\tau)x(-\tau) + \int_{-\tau}^{t} T(t, s)h(s)\,ds, \quad -\tau \leqq t < \tau;$$

$$x(t) = T(t, \omega_2)\varphi_2 + T(t, \tau)\varphi_4 + \int_{\tau}^{t} T(t, s)P_s(s)h(s)\,ds$$
$$+ \int_{\omega_2}^{t} T(t, s)P_u(s)h(s)\,ds, \quad \tau \leqq t \leqq \omega_2.$$

It follows that the restriction of $x(t)$ on $[-\omega_1, \tau)$ and $(\tau, \omega_2]$ satisfies (2.1) on the indicated intervals. From (2.7) the jump at $t = \tau$ is $\xi\Delta$. Moreover, it can be verified that $P_u(-\tau)x(-\tau) = \varphi_3$ and $P_s(\tau)x(\tau) = \varphi_4$. Therefore $\langle \bar{\psi}, x(-\tau) \rangle = 0$ by virtue of (2.8). We conclude that $x(t) = \mathcal{X}(t; \varphi_1, \varphi_2, h, \omega_1, \omega_2)$ is the desired solution asserted by the lemma with the jump $\xi\Delta$, where $\xi = \Xi(\varphi_1, \varphi_2, h, \omega_1, \omega_2)$ is given by (2.2).

Observe that $|h(t)| \leqq \frac{1}{2} \|h\|_\eta (e^{-\eta(\omega_1+t)} + e^{-\eta(\omega_2-t)})$ and

$$\int_{-\omega_1}^{\omega_2} (e^{-\eta(\omega_1+t)} + e^{-\eta(\omega_2-t)})e^{-\alpha|t|}\,dt \leqq C(e^{-\eta\omega_1} + e^{-\eta\omega_2}), \quad 0 \leqq \eta < \alpha.$$

Thus from (2.2) and $|\psi(s)| \leqq Ce^{-\alpha|s|}$, we have that $|\xi|$ is bounded by the right-hand side of (2.4). Similarly, the right-hand side of (2.7) is bounded by the right-hand side of (2.4) with possibly a different $C > 0$. Since $\mathcal{F}^{-1}$ is bounded from $\mathcal{R}\mathcal{F}$ to $\mathcal{K}\mathcal{F}^\perp = \{u, v) \mid u \in \mathcal{R}P_u(-\tau),\ v \in \mathcal{R}P_s(\tau),\ \langle \bar{\psi}, u \rangle = 0\}$, we have that $|\varphi_3^1| + |\varphi_4^1|$ is bounded by the right-hand side of (2.4). From (2.8) it is clear that

$|\zeta|$ is bounded by $C\,|P_s(-\tau)x(-\tau)|$, and the latter is also bounded by the right-hand side of (2.4). Estimate (2.4) now follows easily. Estimate (2.3) follows from (2.4) and the definition of $x(t)$. Observe that

$$P_u(-\omega_1)x(-\omega_1) = T(-\omega_1, -\tau)\varphi_3 + \int_{-\tau}^{-\omega_1} T(-\omega_1, s)P_u(s)h(s)\,ds.$$

Estimate (2.6) then follows from (2.4). The proof of (2.5) is similar to (2.6). □

The following corollary is a useful tool in deriving bifuration functions of homoclinic solutions, cf. **[18]** and **[15]** where $T(t, s)$ has exponential dichotomies on $(-\infty, 0]$ and $[0, +\infty)$. A proof for the case that $T(t, s)$ only has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$, $\tau > 0$, was given in **[11]** using a much involved argument.

COROLLARY 2.4. *Let $h \in E(\mathbb{R})$. Then the integral equation* (2.1) *has a unique piecewise continuous solution x in $E(\mathbb{R}, \Delta)$, with $\langle \bar{\psi}, x(-\tau)\rangle = 0$. Let $x(\tau^-) - x(\tau^+) = \xi\Delta$; then*

$$\xi = \int_{-\infty}^{\infty} \langle \psi(s), h(s)\rangle\,ds. \tag{2.9}$$

Proof. Let $\varphi_1(-\omega)$ and $\varphi_2(\omega)$ be any bounded functions for $\omega \geqq \tau$. Let $\omega \to \infty$, clearly $\Xi(\varphi_1, \varphi_2, h, \omega, \omega) \to \int_{-\infty}^{\infty} \langle \psi(s), h(s)\rangle\,ds$. Let $J \subset \mathbb{R}$ be any bounded interval, and let $0 < \eta < \alpha$. From (2.3) $\mathscr{X}(t; \varphi_1, \varphi_2, h, \omega, \omega)$ approaches a limit $x(t)$ uniformly on J as $\omega \to \infty$. Details will not be given here but can be supplied by an argument similar to that in the proof of Theorem 4.2. Note that $x(t)$ is defined for all $t \in \mathbb{R}$ and satisfies (2.1) with the desired supplementary conditions; $x(t)$ does not depend on the choice of $\varphi_1(-\omega)$ or $\varphi_2(\omega)$, and this implies the uniqueness of $x(t)$.

We now state the main result of this section. Consider the system of integral equations, each defined on $[-\omega_i, \omega_{i+1}]$, $i \in \mathbb{Z}$, with some jump boundary conditions,

$$x_i(t) = T^i(t, \sigma)x_i(\sigma) + \int_\sigma^t T^i(t, s)h_i(s)\,ds, \quad -\omega_i \leqq \sigma \leqq t \leqq \omega_{i+1}, \tag{2.10}$$

$$x_{i-1}(\omega_i) - x_i(-\omega_i) = b_i. \tag{2.11}$$

We assume that the following hypotheses are valid, where all the parameters without index $i \in \mathbb{Z}$ do not depend on i.

(A_1) $T^i(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$ for some $\tau > 0$, with projections $P_s^i(t)$ and $P_u^i(t)$, exponent α and constant K. The dimension of $\mathscr{R}P_u^i(t)$ is d^+.

(A_2) $\mathscr{R}P_u^i(-\tau) \cap [T(\tau, -\tau)]^{-1}\mathscr{R}P_s^i(\tau)$ is one-dimensional, spanned by $\varphi_i \neq 0$. Here again $[T(\tau, -\tau)]^{-1}\mathscr{R}P_s^i(\tau)$ denotes the pre-image of $\mathscr{R}P_s^i(\tau)$.

Let $\bar{\psi}_i \in X^*$ and $\beta > 0$, with the property that $|\langle \bar{\psi}_i, \varphi_i\rangle| \geqq \beta\,|\bar{\psi}_i|\,|\varphi_i| > 0$ for all $i \in \mathbb{Z}$. Define $\mathscr{F}_i$ as we did for $\mathscr{F}$ in Lemma 2.2. $\mathscr{K}\mathscr{F}_i$ is one-dimensional. Define $(\mathscr{K}\mathscr{F}_i)^\perp = \{(u, v) \mid u \in \mathscr{R}P_u^i(-\tau)$ and $v \in \mathscr{R}P_s^i(\tau)$, $\langle \bar{\psi}_i, u\rangle = 0\}$. From ($A_2$) and

Lemma 2.2, $\mathcal{F}_i$ is Fredholm with index $\mathcal{F}_i = 0$. $\mathcal{K}\mathcal{F}_i^*$ is one-dimensional, spanned by ψ_i, with $|\psi_i| = 1$. Let $\psi_i(s) = T^*(s, \tau)\psi_i$. Let $\Delta_i \in X$ be complementary to $\mathcal{R}\mathcal{F}_i$. $\langle \psi_i, \Delta_i \rangle = 1$ and $|\Delta_i| \leqq K_1$, where $K_1 \geqq 1$ is a constant. For any subspaces $X_1, X_2 \subset X$ with $X_1 \oplus X_2 = X$, let $P(X_1, X_2)$ be the projection operator with the range being X_1, and the kernel X_2.

(A_3) There is a constant $\bar{\omega} > \tau$ and a constant $M \geqq 1$ such that for $\omega_i \geqq \bar{\omega}$ and $i \in \mathbb{Z}$,

$$\mathcal{R}P_s^i(-\omega_i) \oplus \mathcal{R}P_u^{i-1}(\omega_i) = X,$$

and $|P(\mathcal{R}P_u^{i-1}(\omega_i), \mathcal{R}P_s^i(-\omega_i))| \leqq M$.

(A_4) (Uniformity assumptions with respect to $i \in \mathbb{Z}$) The constants α, K, β, K_1 and M do not depend on $i \in \mathbb{Z}$. $\mathcal{F}_i^{-1}$: $\mathcal{R}\mathcal{F}_i \to (\mathcal{K}\mathcal{F}_i)^\perp$ is uniformly bounded with respect to $i \in \mathbb{Z}$.

It can be shown that $|\psi_i(s)| \leqq Ce^{-\alpha|s|}$ for all $s \in \mathbb{R}$, $i \in \mathbb{Z}$. The constant C does not depend on i. In the sequel any sequence indexed by $i \in \mathbb{Z}$ will be denoted by a bold faced letter for brevity, e.g. $\boldsymbol{\omega} = \{\omega_i\}_{i \in \mathbb{Z}}$.

THEOREM 2.5. *There is a constant $\bar{\bar{\omega}} \geqq \bar{\omega}$ with the following property. If $\omega_i \geqq \bar{\bar{\omega}}$ for all $i \in \mathbb{Z}$, then there exists a unique piecewise continuous solution $\mathbf{x}(t)$ for system* (2.10) *and* (2.11), *with $x_i \in E([-\omega_i, \omega_{i+1}], \Delta_i)$ and $\langle \bar{\psi}_i, x_i(-\tau) \rangle = 0$. Let $x_i(\tau^-) - x_i(\tau^+) = \xi_i \Delta_i$; then*

$$\xi_i = \int_{-\omega_i}^{\omega_{i+1}} \langle \psi_i(s), h_i(s) \rangle \, ds + \langle \psi_i(-\omega_i), x_i(-\omega_i) \rangle - \langle \psi_i(\omega_{i+1}), x_i(\omega_{i+1}) \rangle. \quad (2.12)$$

Let $\varphi_1^i = P_s^i(-\omega_i)x_i(-\omega_i)$ and $\varphi_2^i = P_u^i(\omega_{i+1})x_i(\omega_{i+1})$. Then

$$|\boldsymbol{\varphi}_1| + |\boldsymbol{\varphi}_2| \leqq C_1(|\mathbf{b}| + \|\mathbf{h}\|_0); \quad (2.13)$$

$$\|x_i\|_\eta \leqq C_2(|\mathbf{b}| + \|\mathbf{h}\|_0 + \|h_i\|_\eta), \quad 0 \leqq \eta < \alpha. \quad (2.14)$$

Here $|\boldsymbol{\varphi}_1| = \sup_{i \in \mathbb{Z}} \{|\varphi_1^i|\}$, similarly for $|\boldsymbol{\varphi}_2|$ and $|\mathbf{b}|$. $\|\mathbf{h}\|_\eta = \sup_{i \in \mathbb{Z}} \{\|h_i\|_\eta\}$, similarly for $\|\mathbf{x}\|_\eta$. $\|h_i\|_\eta$ is defined after Lemma 2.2, but over $J_i = [-\omega_i, \omega_{i+1}]$ instead of $[-\omega_1, \omega_2]$.

The solution $\mathbf{x}(t) = \{x_i(t)\}_{i \in \mathbb{Z}}$ will be denoted by $x_i(t) = X_i(t; \mathbf{b}, \mathbf{h}, \boldsymbol{\omega})$ while ξ_i is denoted by $\xi_i = \Gamma_i(\mathbf{b}, \mathbf{h}, \boldsymbol{\omega})$.

Proof. We try to find $\varphi_1^i \in \mathcal{R}P_s^i(-\omega_i)$ and $\varphi_2^i \in \mathcal{R}P_u^i(\omega_{i+1})$, $i \in \mathbb{Z}$ such that $\mathcal{X}_{i-1}(\omega_i; \varphi_1^{i-1}, \varphi_2^{i-1}, h_{i-1}, \omega_{i-1}, \omega_i) - \mathcal{X}_i(-\omega_i; \varphi_1^i, \varphi_2^i, h_i, \omega_i, \omega_{i+1}) = b_i$, $i \in \mathbb{Z}$, or equivalently

$$\varphi_2^{i-1} + P_s^{i-1}(\omega_i)\mathcal{X}_{i-1}(\omega_i) - \varphi_1^i - P_u^i(-\omega_i)\mathcal{X}_i(-\omega_i) = b_i, \quad i \in \mathbb{Z}.$$

Here, $\mathcal{X}_i(t)$ is abbreviated from $\mathcal{X}_i(t; \varphi_1^i, \varphi_2^i, h_i, \omega_i, \omega_{i+1})$. Using projections, we have

$$\begin{cases} \varphi_1^i = P(\mathcal{R}P_s^i(-\omega_i), \mathcal{R}P_u^{i-1}(\omega_i))\{P_s^{i-1}(\omega_i)\mathcal{X}_{i-1}(\omega_i) - P_u^i(-\omega_i)\mathcal{X}_i(-\omega_i) - b_i\}, \\ \varphi_2^i = P(\mathcal{R}P_u^i(\omega_{i+1}), \mathcal{R}P_s^{i+1}(-\omega_{i+1}))\{b_{i+1} - \mathcal{R}P_s^i(\omega_{i+1})\mathcal{X}_i(\omega_{i+1}) \\ \qquad + P_u^{i+1}(-\omega_{i+1})\mathcal{X}_{i+1}(-\omega_{i+1})\}. \end{cases} \quad (2.15)$$

From (2.5) and (2.6), if $\bar{\omega}$ is large and $\omega_i \geqq \bar{\omega}$ for all $i \in \mathbb{Z}$, then (2.15) is a contraction map from $\prod_{i\in\mathbf{Z}} \mathcal{R}P_s^i(-\omega_i) \times \prod_{i\in\mathbf{Z}} \mathcal{R}P_u^i(\omega_i)$ into itself. Therefore (2.15) has a unique solution $\varphi_1^i = \Phi_1^i(\mathbf{b}, \mathbf{h}, \boldsymbol{\omega})$ and $\varphi_2^i = \Phi_2^i(\mathbf{b}, \mathbf{h}, \boldsymbol{\omega})$, $i \in \mathbb{Z}$. From (2.5) and (2.6) again we can show that (2.13) is valid. Define $x_i(t) = X_i(t; \mathbf{b}, \mathbf{h}, \boldsymbol{\omega}) = \mathcal{X}_i(t; \Phi_1^i, \Phi_2^i, h_i, \omega_i, \omega_{i+1})$. $x(t)$ is the desired solution for system (2.10) and (2.11). (2.14) follows from (2.3) and (2.13). Define $\xi_i = \Gamma_i(\mathbf{b}, \mathbf{h}, \boldsymbol{\omega}) = \Xi_i(\Phi_1^i, \Phi_2^i, h_i, \omega_i, \omega_{i+1})$. Obviously ξ_i satisfies (2.12).

LEMMA 2.6. *If* $|h_i(t)| \leqq \varepsilon\, |x_i(t)|$, $t \in [-\omega_i, \omega_{i+1}]$, $i \in \mathbb{Z}$, *then for each* $0 < \eta < \alpha$,

$$\|\mathbf{x}\|_\eta \leqq C\, |\mathbf{b}|, \tag{2.16}$$

$$|\boldsymbol{\varphi}_1| + |\boldsymbol{\varphi}_2| \leqq C\, |\mathbf{b}|, \tag{2.17}$$

$$\|x_i\|_\eta \leqq C(|\varphi_1^i| + |\varphi_2^i|), \tag{2.16'}$$

and

$$|\varphi_1^i| + |\varphi_2^{i-1}| \leqq C\{|b_i| + (|\varphi_1^{i-1}| + |\varphi_2^i|)(e^{-\eta\omega_{i-1}} + e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})\} \tag{2.18}$$

provided that $0 < \varepsilon$ *is sufficiently small.*

Proof. From (2.14), and $\|\mathbf{h}\|_\eta \leqq \varepsilon\, \|\mathbf{x}\|_\eta$, (2.16) follows if ε is sufficiently small so that $C_2\varepsilon < \frac{1}{2}$. (2.17) follows from (2.13) and (2.16).

Note that (2.3) implies (2.16)'. (2.4) implies that

$$\begin{aligned} |\varphi_3^i| + |\varphi_4^i| &\leqq C\{|\varphi_1^i| e^{-\alpha\omega_i} + |\varphi_2^i| e^{-\alpha\omega_{i+1}} + (|\varphi_1^i| + |\varphi_2^i|)(e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})\} \\ &\leqq C(|\varphi_1^i| + |\varphi_2^i|)(e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}}), \end{aligned} \tag{2.19}$$

where $\varphi_3^i = P_u^i(-\tau)x_i(-\tau)$ and $\varphi_4^i = P_s^i(\tau)x_i(\tau)$. From the definition of $x_i(t)$, $t \in [-\omega_i, -\tau]$, we have

$$\begin{aligned} |x_i|_{C[-\omega_i, -\tau]} &\leqq C(|\varphi_1^i| + |\varphi_3^i| + |h_i|_{C[-\omega_i, -\tau]}) \\ &\leqq C(|\varphi_1^i| + |\varphi_2^i|(e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})). \end{aligned} \tag{2.20}$$

Similarly

$$|x_{i-1}|_{C[\tau, \omega_i]} \leqq C(|\varphi_2^{i-1}| + |\varphi_1^{i-1}|(e^{-\eta\omega_{i-1}} + e^{-\eta\omega_i})). \tag{2.21}$$

Here $C[a, b]$ is the space of continuous functions defined on $[a, b]$ with the sup norm $|\cdot|_{c[a,b]}$. By (2.15) and (2.19), we then have

$$\begin{aligned} |\varphi_1^i| + |\varphi_2^{i-1}| &\leqq C(|b_i| + |P_s^{i-1}(\omega_i)x_{i-1}(\omega_i)| + |P_u^i(-\omega_i)x_i(-\omega_i)|) \\ &\leqq C\bigg\{|b_i| + \bigg|T^{i-1}(\omega_i, \tau)\varphi_4^{i-1} + \int_\tau^{\omega_i} T^{i-1}(\omega_i, s)P_s^{i-1}(s)h_{i-1}(s)\, ds\bigg| \\ &\quad + \bigg|T^i(-\omega_i, -\tau)\varphi_3^i + \int_{-\tau}^{-\omega_i} T^i(-\omega_i, s)P_u^i(s)h_i(s)\, ds\bigg|\bigg\} \\ &\leqq C\{|b_i| + (|\varphi_1^{i-1}| + |\varphi_2^i|)(e^{-\eta\omega_{i-1}} + e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})e^{-\alpha\omega_i} \\ &\quad + \varepsilon(|x_{i-1}|_{C[\tau,\omega_i]} + |x_i|_{C[-\omega_i-\tau]})\} \\ &\leqq C\{|b_i| + (|\varphi_1^{i-1}| + |\varphi_2^i|)(e^{-\eta\omega_{i-1}} + e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})\}. \end{aligned}$$

3. A general result on the bifuraction of heteroclinic chains

Consider the ordinary differential equation (1.1) where $x \in \mathbb{R}^n$, $\mu \in M$, M is a Banach space of parameters, $f: \mathbb{R}^n \times M \to \mathbb{R}^n$ is C^k, $k \geqq 2$. $D^i f$, $|i| \leqq 2$ is uniformly bounded for $x \in \mathcal{O}$ and $|\mu| \leqq \mu_0$, $\mathcal{O} \subset \mathbb{R}^n$ is an open subset. When $\mu = 0$ there is a heteroclinic chain consisting of a sequence of equilibria $\{p_i\}_{i \in \mathbf{Z}}$, and a sequence of heteroclinic solutions $\{q_i(t)\}_{i \in \mathbf{Z}}$ with $q_i(t) \to p_i$ as $t \to -\infty$ and $q_i(t) \to p_{i+1}$ as $t \to +\infty$. Assume that the orbit of $q_i(t)$, $i \in \mathbb{Z}$, together with its ε-neighbourhood, is contained in $\mathcal{O}$.

Let Σ_i be a codimension 1 plane through $q_i(-\tau)$ and is transverse to $\dot{q}_i(-\tau)$, $\tau > 0$ is a constant. There is a $\bar{\psi}_i \in \mathbb{R}^{n^*}$ such that $\Sigma_i = \{x \mid \langle \bar{\psi}_i, x - q_i(-\tau) \rangle = 0\}$. Let $x(t)$ be a solution that orbitally lies near the heteroclinic chain. Denote the time spent by $x(t)$ between Σ_{i-1} and Σ_i by $2\omega_i$. We intend to derive conditions on $\omega = \{\omega_i\}_{i \in \mathbf{Z}}$ and μ so that the existence of such solution $x(t)$ will be possible. Let the orbit of $x(t)$ be the union of those of $x_i(t)$, each defined on $t \in [-\omega_i, \omega_{i+1}]$ and subject to the condition $x_i(-\tau) \in \Sigma_i$; that is, there exists $t_i \in \mathbb{R}$ such that $x(-\tau + t_i) \in \Sigma_i$ and $x_i(t) = x(t + t_i)$ for $t \in [-\omega_i, \omega_{i+1}]$.

The reader may have noticed that it is more natural to set $\tau = 0$ as in the Introduction. The reason we set $\tau > 0$ is to unify the notation with that to be used in Section 5, where τ is a large positive constant.

Let $x_i(t) = q_i(t) + z_i(t)$, $-\omega_i \leqq t \leqq \omega_{i+1}$, $i \in \mathbb{Z}$. $\{z_i\}_{i \in \mathbf{Z}}$ satisfies a variational system with boundary conditions at $\pm\omega_i$, $i \in \mathbb{Z}$.

$$\begin{aligned} \dot{z}_i(t) - A_i(t) z_i(t) &= g_i(z_i(t), \mu, t), \\ z_{i-1}(\omega_i) - z_i(-\omega_i) &= b_i, \\ \langle \bar{\psi}_i, z_i(-\tau) \rangle &= 0. \end{aligned} \tag{3.1}$$

Here $A_i(t) = D_x f(q_i(t), 0)$, $g_i(z, \mu, t) = f(q_i(t) + z, \mu) - f(q_i(t), 0) - A_i(t) z = 0(|\mu| + |z|^2)$, $b_i = q_i(-\omega_i) - q_{i-1}(\omega_i)$. The first equation of (3.1) is equivalent to an integral equation

$$z_i(t) = z_i(\sigma) + \int_\sigma^t T^i(t, s) g_i(z_i(s), \mu, s)\, ds, \quad -\omega_i \leqq \sigma \leqq t \leqq \omega_{i+1},$$

where $T^i(t, s)$ is the principal matrix solution of the homogeneous equation

$$\dot{z}(t) - A_i(t) z(t) = 0. \tag{3.2}$$

Denote $g_i(q_i(t), \mu, t)$ by $h_i(t)$. Equation (3.1) becomes a system studied in Section

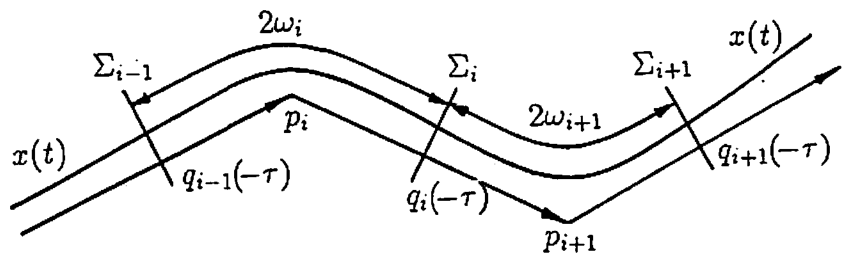

Figure 3.1

2. We state conditions on f so that the results in Section 2 can be used here. Symbols without the index i do not depend on $i \in \mathbb{Z}$.

(B_1) $f(p_i, 0) = 0$ with $|\mathrm{Re}\, \sigma\{Dxf(p_i, 0)\}| \geqq \alpha_0 > 0$.

(B_2) $\dot{q}_i(t)$ is the only bounded solution for the linear equation (3.2).

From (B_1), $T^i(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$. The exponent can be any $0 < \alpha < \alpha_0$. From (B_2), (A_2) is satisfied with $\varphi_i = \dot{q}_i(-\tau)$. Define $\mathcal{F}_i$, $(\mathcal{K}\mathcal{F}_i)^\perp$, ψ_i, $\psi_i(s)$ and Δ_i as in Section 2 for all $i \in \mathbb{Z}$. We now assume that (A_1), (A_3) and (A_4) are satisfied. In particular, the exponent $0 < \alpha < \alpha_0$ and the constant $K(\alpha)$ do not depend on $i \in \mathbb{Z}$, for example. Though most of these assumptions can be derived from assumptions on $f(x, 0)$, we shall not pursue this, since they are clearly valid in the problems to be discussed in Sections 4 and 5.

THEOREM 3.1. *Let* (B_1) *and* (B_2) *be satisfied. Let* (A_1), (A_3) *and* (A_4) *and all the assumptions on* $T^i(t, s)$, $\bar{\psi}_i$, φ_i, ψ_i, Δ_i *and* $\mathcal{F}_i$ *as made in Theorem* 2.5 *be satisfied. Then there are positive constants* $\hat{\omega}$, μ_1 *and* ε_1 *with the following property. For any sequence* $\boldsymbol{\omega} = \{\omega_i\}_{i \in \mathbf{Z}}$ *with* $\omega_i \geqq \hat{\omega}$ *and* $|\mu| \leqq \mu_1$, *there exists a unique piecewise continuous solution* $x(t)$ *for* (3.1) *that orbitally lies in a* ε_1*-neighbourhood of* $\mathbf{q}(t)$. *The time spent by* $x(t)$ *between* Σ_{i-1} *and* Σ_i *is* $2\omega_i$. $x(t)$ *has infinitely many jumps of the form* $\xi_i \Delta_i$, $\xi_i \in \mathbb{R}$, *each occurring at the time* 2τ *after it meets* Σ_i. *Moreover, if* $x(-\tau + t_i) \in \Sigma_i$ *and* $x_i(t) = q_i(t) + z_i(t) = x(t + t_i)$ *for* $t \in [-\omega_i, \omega_{i+1}]$, *then*

$$\xi_i = \int_{-\omega_i}^{\omega_{i+1}} \psi_i(s) g_i(z_i(s), \mu, s)\, ds + \psi_i(-\omega_i) z_i(-\omega_i) - \psi_i(\omega_{i+1}) z_i(\omega_{i+1}). \quad (3.3)$$

Denote the solution by $z_i = Z_i(t; \boldsymbol{\omega}, \mu)$ *and* $\xi_i = G_i(\boldsymbol{\omega}, \mu)$. $Z_i(.\,; \boldsymbol{\omega}, \mu) \in E([-\omega_i, \omega_{i+1}], \Delta_i)$ *and* $G_i(\boldsymbol{\omega}, \mu) \in \mathbb{R}$ *are* C^k *in* μ *for fixed* $\boldsymbol{\omega}$. *Moreover*

$$\|Z_i\|_{E([-\omega_i, \omega_{i+1}])} \leqq C(e^{-\alpha\omega} + |\mu|), \quad (\omega = \inf\{\omega_i\}_{i \in \mathbf{Z}}), \quad (3.4)$$

$$\left\| \frac{\partial Z_i}{\partial \mu} \right\|_{E([-\omega_i, \omega_{i+1}])} \leqq C \left| \frac{\partial f}{\partial \mu} \right|. \quad (3.5)$$

Here $\left| \dfrac{\partial f}{\partial \mu} \right| = \sup \left\{ \left| \dfrac{\partial f(x, \mu)}{\partial \mu} \right| x \in \mathcal{O}, |\mu| \leqq \mu_0 \right\}$.

Proof. The norms for functions will be sup norms in this proof. Since all the hypotheses of Theorem 2.5 are satisfied, there exists $\bar{\omega} > 0$ such that for each $\boldsymbol{\omega}$ with $\omega_i \geqq \bar{\omega}$ and each $\mathbf{h}$ with $h_i \in E([-\omega_i, \omega_{i+1}])$, $X_i(t; \mathbf{b}, \mathbf{h}, \boldsymbol{\omega})$ can be defined as in Theorem 2.5. It is clear that (3.1) has a solution $z_i \in E([-\omega_i, \omega_{i+1}], \Delta_i)$, $i \in \mathbb{Z}$ if and only if

$$z_i(t) = X_i(t; \mathbf{b}, \{g_i(z_i(\cdot), \mu, \cdot)\}_{i \in \mathbf{Z}}, \boldsymbol{\omega}), \quad i \in \mathbb{Z}. \quad (3.6)$$

Let $U_{\varepsilon_1, \mu_1} = \{(\mathbf{z}, \mu) \mid z_i \in E([-\omega_i, \omega_{i+1}), \Delta_i), |z_i| < \varepsilon_1, |\mu| < \mu_1\}$. The right-hand side of (3.6) together with $\mu = \mu$ is a C^k mapping from U_{ε_1, μ_1} to itself, and is a uniform contraction with respect to $\mathbf{z}$, for $|\mu| < \mu_1$. Therefore (3.6) has a unique fixed point $\{Z_i(t; \boldsymbol{\omega}, \mu)\}_{i \in \mathbf{Z}}$ which is C^k: $\mu \to \prod_{i \in \mathbf{Z}} E([-\omega_i, \omega_{i+1}], \Delta_i)$. Let $\omega =$

$\inf\{\omega_i\}$. From (3.6)

$$\begin{aligned}|Z_i| &\leqq C(|\mathbf{b}| + |\{g_i(z_i(\cdot), \mu_i, \cdot)\}|)\\ &\leqq C(e^{-\alpha\omega} + |Z_i(\cdot)|^2 + |\mu|)\\ &\leqq C(e^{-\alpha\omega} + |\mu|),\end{aligned}$$

$$\left|\frac{\partial \mathbf{Z}}{\partial \mu}\right| \leqq \left|\frac{\partial \mathbf{X}}{\partial \mathbf{g}} \cdot \frac{\partial \mathbf{g}}{\partial \mathbf{Z}} \cdot \frac{\partial \mathbf{Z}}{\partial \mu} + \frac{\partial \mathbf{X}}{\partial \mathbf{g}} \cdot \frac{\partial \mathbf{g}}{\partial \mu}\right|.$$

For an estimate of $\frac{\partial \mathbf{X}}{\partial \mathbf{g}}$, see (2.14) with $\eta = 0$. Since $\frac{\partial g_i(z_i(\cdot), \mu, \cdot)}{\partial \mathbf{z}_i} = O(|z_i| + |\mu|)$, if ω is sufficiently large and $|\mu|$ is small, $\left|\frac{\partial \mathbf{X}}{\partial \mathbf{g}} \cdot \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right| \leqq \frac{1}{2}$, then

$$\left|\frac{\partial \mathbf{Z}}{\partial \mu}\right| \leqq C\left|\frac{\partial \mathbf{g}}{\partial \mu}\right| \leqq C\left|\frac{\partial f}{\partial \mu}\right|.$$

Substituting $h_i(t) = g_i(Z_i(t; \boldsymbol{\omega}, \mu), \mu, t)$ into (2.12), we have (3.3). It is also clear that $G_i(\boldsymbol{\omega}, \mu)$ is C^k in μ for each fixed $\boldsymbol{\omega}$.

The following lemmas are useful in Section 4:

Lemma 3.2.

$$\frac{\partial G_i(\boldsymbol{\omega}, \mu)}{\partial \mu} = \int_{-\infty}^{\infty} \psi_i(s) D_\mu f(q_i(s), 0)\, ds + O(e^{-\alpha\omega} + |\mu|) \quad \textit{where} \quad \omega = \inf\{\omega_i\}_{i\in\mathbf{Z}}$$

Proof.

$$\begin{aligned}\frac{\partial G_i(\boldsymbol{\omega}, \mu)}{\partial \mu} &= \psi_i(-\omega_i)\frac{\partial Z_i(-\omega_i; \boldsymbol{\omega}, \mu)}{\partial \mu} - \psi_i(\omega_{i+1})\frac{\partial Z_i}{\partial \mu}(\omega_{i+1}; \boldsymbol{\omega}, \mu)\\ &\quad + \int_{-\omega_i}^{\omega_{i+1}} \psi_i(s)[D_z g_i(Z_i, \mu, s) D_\mu Z_i(s; \boldsymbol{\omega}, \mu) + D_\mu g_i(Z_i, \mu, s)]\, ds.\end{aligned}$$

The first two terms are clearly $O(e^{-\alpha\omega})$ since $D_\mu Z_i(\cdot\,; \boldsymbol{\omega}, \mu)$ is uniformly bounded for $\omega_i \geqq \hat{\omega}$ and $|\mu| \leqq \mu_1$ (see (3.5)). The integral term can be rewritten as

$$\int_{-\omega_i}^{\omega_{i+1}} \psi_i(s)\frac{\partial f}{\partial \mu}(q_i(s), 0)\, ds + O(e^{-\alpha\omega} + |\mu|),$$

due to $|D_z g_i(Z_i, \mu, s)| = O(|Z_i| + |\mu|) = O(e^{-\alpha\omega} + |\mu|)$, see (3.4), and

$$\begin{aligned}D_\mu g_i(Z_i, \mu, s) = D_\mu f(q_i(s) + Z_i, \mu) &= D_\mu f(q_i(s), 0) + O(|Z_i| + |\mu|)\\ &= D_\mu f(q_i(s), 0) + O(e^{-\alpha\omega} + |\mu|),\end{aligned}$$

and also that $\int_{-\omega_i}^{\omega_{i+1}}$ can be replaced by $\int_{-\infty}^{\infty}$ with an error of $O(e^{-\alpha\omega})$.

LEMMA 3.3. *Let* $0 < \eta < \alpha$. *Then*

$$
\begin{aligned}
G_i(\boldsymbol{\omega}, 0) = &-\psi_i(-\omega_i)P(\mathscr{R}P_s^i(-\omega_i),\ \mathscr{R}P_u^{i-1}(\omega_i))b_i - \psi_i(\omega_{i+1})P(\mathscr{R}P_u^i(\omega_{i+1}),\\
&\mathscr{R}P_s^{i+1}(-\omega_{i+1}))b_{i+1}\\
&+ O\{|\psi_i(-\omega_i)|\,(|b_i|^2 + |\mathbf{b}|\,(e^{-2\eta\omega_{i-1}} + e^{-2\eta\omega_i} + e^{-2\eta\omega_{i+1}}))\}\\
&+ O\{|\psi_i(\omega_{i+1})|\,(|b_{i+1}|^2 + |\mathbf{b}|\,(e^{-2\eta\omega_i} + e^{-2\eta\omega_{i+1}} + e^{-2\eta\omega_{i+2}}))\}\\
&+ O\{(|b_i|^2 + |b_{i+1}|^2 + |\mathbf{b}|^2\,(e^{-4\eta\omega_{i-1}} + e^{-4\eta\omega_i} + e^{-4\eta_{i+1}} + e^{-4\eta\omega_{i+2}}))\\
&\qquad\times (e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})\}.
\end{aligned}
$$

Proof. We shall use Lemma 2.6 with $h_i(t) = g_i(z_i(t), 0, t) = O(|z_i(t)|^2)$. From (3.4) the hypothesis in Lemma 2.6 is valid if $\omega = \inf\{\omega_i \mid i \in \mathbb{Z}\}$ is sufficiently large.

From (2.15), we have

$$
\begin{aligned}
|\varphi_1^i + P(\mathscr{R}P_s^i(-\omega_i),\ &\mathscr{R}P_u^{i-1}(\omega_i))b_i| + |\varphi_2^{i-1} - P(\mathscr{R}P_u^{i-1}(\omega_i),\ \mathscr{R}P_s^i(-\omega_i))b_i|\\
&\leqq C\{|P_s^{i-1}(\omega_i)z_{i-1}(\omega_i)| + |P_u^i(-\omega_i)z_i(-\omega_i)|\}\\
&\leqq C\left\{\left|T^{i-1}(\omega_i, \tau)\varphi_4^{i-4} + \int_\tau^{\omega_i} T^{i-1}(\omega_i, s)P_s^{i-1}(s)g_{i-1}(z_{i-1}(s), 0, s)\, ds\right|\right.\\
&\qquad\left. + \left|T^i(-\omega_i, -\tau)\varphi_3^i + \int_{-\tau}^{-\omega_i} T^i(-\omega_i, s)P_u^i(s)g_i(z_i(s), 0, s)\, ds\right|\right\}\\
&\leqq C\{|\mathbf{b}|\,(e^{-\eta\omega_{i-1}} + e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})e^{-\alpha\omega_i} + |z_{i-1}|^2_{C[\tau,\omega_i]} + |z_i|^2_{C[-\omega_i,-\tau]}\}\\
&\leqq C\{|\mathbf{b}|\,(e^{-2\eta\omega_{i-1}} + e^{-2\eta\omega_i} + e^{-2\eta\omega_{i+1}}) + |\varphi_1^i|^2 + |\varphi_2^{i-1}|^2\}\\
&\leqq C\{|b_i|^2 + |\mathbf{b}|\,(e^{-2\eta\omega_{i-1}} + e^{-2\eta\omega_i} + e^{-2\eta\omega_{i+1}})\}. \qquad (3.7)
\end{aligned}
$$

Here we have used (2.18), (2.19), (2.20) and (2.21) and the fact $|\mathbf{b}|^2 \leqq |\mathbf{b}|$ if ω is large.

It can also be verified that

$$
\begin{aligned}
&\left|\int_{-\omega_i}^{\omega_{i+1}} \psi_i(s)g_i(z_i(s), 0, s)\, ds\right|\\
&\quad\leqq C\int_{-\omega_i}^{\omega_{i+1}} e^{-\alpha|s|}\,\|z_i\|_\eta^2\,(e^{-2\eta(\omega_i+s)} + e^{-2\eta(\omega_{i+1}-s)})\, ds\\
&\quad\leqq C\,\|z_i\|_\eta^2\,(e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})\\
&\quad\leqq C(|\varphi_1^i|^2 + |\varphi_2^i|^2)(e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}})\\
&\quad\leqq C\{|b_i|^2 + |b_{i+1}|^2 + |\mathbf{b}|^2\,(e^{-4\eta\omega_{i-1}} + e^{-4\eta\omega_i}\\
&\qquad + e^{-4\eta\omega_{i+1}} + e^{-4\eta\omega_{i+2}})\}(e^{-\eta\omega_i} + e^{-\eta\omega_{i+1}}) \qquad (3.8)
\end{aligned}
$$

Here we have employed (2.16)′ and (3.7). The desired result then follows from (3.7) and (3.8).

LEMMA 3.4. *Let* $\boldsymbol{\omega}^{(j)}=\{\omega_i^{(j)}\}_{i\in\mathbb{Z}}$, $j=1,2$, *be such that* $\omega_i^{(j)}\geqq\hat{\omega}$ *for* $i\in\mathbb{Z}$, $j=1,2$, *and* $|\omega_i^{(1)}-\omega_i^{(2)}|\leqq\frac{1}{N}$ *for* $l-N\leqq i\leqq l+N$, *where* l *is an integer. Then*

$$G_l(\boldsymbol{\omega}^{(1)},\mu)-G_l(\boldsymbol{\omega}^{(2)},\mu)\to 0 \quad \text{as } N\to\infty,$$

provided that $\hat{\omega}>0$ *is sufficiently large and* $|\mu|\leqq\mu_1$ *is sufficiently small.*

Proof. It suffices to set $l=0$. Let $\boldsymbol{\omega}^{(3)}=\{\omega_i^{(3)}\}_{i\in\mathbb{Z}}$ be defined as $\omega_i^{(3)}=\inf\{\omega_i^{(1)},\omega_i^{(2)}\}$. Then $|\omega_i^{(3)}-\omega_i^{(j)}|\leqq\frac{1}{N}$ for $-N\leqq i\leqq N$, $j=1,2$. We shall give an estimate on $G_0(\boldsymbol{\omega}^{(1)},\mu)-G_0(\boldsymbol{\omega}^{(3)},\mu)$. Let $z_i^{(1)}(t)=Z_i(t;\boldsymbol{\omega}^{(1)},\mu)$ be as in Theorem 3.1. Let $z_i^{(4)}(t)$ be the restriction of $z_i^{(1)}(t)$ on $[-\omega_i^{(3)},\omega_{i+1}^{(3)}]$. Then

$$z_i^{(4)}(t)=X_i(t;\mathbf{b}^{(4)},\{g_i(z_i^{(4)}(\cdot),\mu,\cdot)\}_{i\in\mathbb{Z}},\boldsymbol{\omega}^{(3)}),\quad i\in\mathbb{Z},$$

where $\mathbf{b}^{(4)}=\{b_i^{(4)}\}_{i\in\mathbb{Z}}$ is defined by $b_i^{(4)}=z_{i-1}^{(4)}(\omega_i^{(3)})-z_i^{(4)}(-\omega_i^{(3)})$. Let $z_i^{(3)}(t)=Z_i(t;\boldsymbol{\omega}^{(3)},\mu)$. Then

$$z_i^{(3)}(t)=X_i(t;\mathbf{b}^{(3)},\{g_i(z_i^{(3)}(\cdot),\mu,\cdot)\}_{i\in\mathbb{Z}},\boldsymbol{\omega}^{(3)}),\quad i\in\mathbb{Z},$$

where $\mathbf{b}^{(3)}=\{b_i^{(3)}\}_{i\in\mathbb{Z}}$ and $b_i^{(3)}=q_i(-\omega_i^{(3)})-q_{i-1}(\omega_i^{(3)})$. Let $\Delta z_i(t)=z_i^{(3)}(t)-z_i^{(4)}(t)$. Then

$$\Delta z_i(t)=X_i(t;\mathbf{b}^{(3)}-\mathbf{b}^{(4)},\Delta\mathbf{g},\boldsymbol{\omega}^{(3)}),$$

where $\Delta\mathbf{g}=\{\Delta g_i(t)\}_{i\in\mathbb{Z}}$ with $\Delta g_i(t)=g_i(z_i^{(3)}(t),\mu,t)-g_i(z_i^{(4)}(t),\mu,t)$. If $\hat{\omega}$ is large and μ_1 is small, we have $|\Delta g_i(t)|\leqq\varepsilon\,|\Delta z_i(t)|$ where $\varepsilon>0$ can be arbitrarily small. Therefore Lemma 2.6 applies. Let $\Delta b_i=b_i^{(3)}-b_i^{(4)}$. Observe that

$$\begin{aligned}|b_i^{(3)}-b_i^{(4)}|&\leqq|q_i(-\omega_i^{(3)})-q_i(-\omega_i^{(1)})|+|q_{i-1}(\omega_i^{(3)})-q_{i-1}(\omega_i^{(1)})|\\&\quad+|z_i^{(1)}(-\omega_i^{(3)})-z_i^{(1)}(-\omega_i^{(1)})|+|z_{i-1}(\omega_i^{(3)})-z_{i-1}(\omega_i^{(1)})|\\&=O\left(\frac{1}{N}\right),\quad -N\leqq i\leqq N.\end{aligned}$$

From (2.18) we then have

$$\begin{aligned}|\Delta\varphi_1^i|+|\Delta\varphi_2^{i-1}|&\leqq C\{|\Delta b_i|+(|\Delta\varphi_1^{i-1}|+|\Delta\varphi_2^i|)\,\delta\},\quad -N\leqq i\leqq N\\&\leqq C_1\left\{\frac{1}{N}+(|\Delta\varphi_1^{i-1}|+|\Delta\varphi_2^i|)\,\delta\right\}.\end{aligned}\tag{3.9}$$

Here $\Delta\varphi_1^i=P_s^i(-\omega_i^{(3)})\Delta z_i(-\omega_i^{(3)})$ and $\Delta\varphi_2^i=P_u^i(\omega_{i+1})\Delta z_i(\omega_{i+1})$. $\delta=3e^{-\eta\hat{\omega}}$ can be arbitrarily small if $\hat{\omega}$ is large. From (2.17), $|\Delta\boldsymbol{\varphi}_1|+|\Delta\boldsymbol{\varphi}_2|\leqq C_2\,|\Delta\mathbf{b}|$. Therefore from (3.19) and by induction

$$|\varphi_1^i|+|\varphi_2^{i-1}|\leqq C_1N^{-1}(1+C_1\delta+\ldots+(C_1\delta)^j)+(C_1\delta)^{j+1}C_2\,|\Delta\mathbf{b}|$$
$$\text{for } -N+j\leqq i\leqq N-j,\quad 0\leqq j\leqq N.$$

Thus

$$\begin{aligned}|\varphi_1^i|+|\varphi_2^{i-1}|&\leqq C_1(1-C_1\delta)^{-1}N^{-1}+(C_1\delta)^NC_2\,|\Delta\mathbf{b}|\\&=O\left(\frac{1}{N}\right),\quad -1\leqq i\leqq 1.\end{aligned}$$

From (2.16)′, $\|\Delta z_i\|_\eta = O\left(\frac{1}{N}\right)$ for $i = -1, 0$. Since $|\Delta g_0| \leqq \varepsilon |\Delta z_0|$, it follows that

$$G_0(\omega^{(1)}, \mu) - G_0(\omega^{(3)}, \mu) = \int_{-\omega_0^{(3)}}^{\omega_1^{(3)}} \psi_0(s)\Delta g_0(s)\, ds + O\left(\frac{1}{N}\right)$$
$$= O\left(\frac{1}{N}\right) \to 0, \quad \text{as } N \to \infty.$$

Similarly we can show that $G_0(\omega^{(2)}, \mu) - G_0(\omega^{(3)}, \mu) \to 0$ as $N \to \infty$. The desired result then follows.

4. Periodic and aperiodic solutions generated from a homoclinic orbit

Assume that equation (1.1) has a homoclinic solution $x = q(t)$ asymptotic to an equilibrium $p = 0$ when $\mu = 0$. Here μ is real, $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is C^k with $k \geqq 2$. Throughout this section we assume

(H_1) The equilibrium $p = 0$ is hyperbolic with $|\text{Re}\, \sigma\{D_x f(p, 0)\}| \geqq \alpha_0 > 0$.

(H_2) $x = \dot{q}(t)$ is the only bounded solution of the linear equation

$$\dot{x}(t) - D_x f(q(t), 0)x(t) = 0. \tag{4.1}$$

Let Σ be a codimension-1 plane, passing through $q(-\tau)$ and transverse to $\dot{q}(-\tau)$, where $\tau > 0$ is a constant. From (H_1), there exist stable and unstable manifolds W^s and W^u for the equilibrium $x = p$. The tangent spaces at $x = q(\tau)$ then span a codimension-1 subspace, $TW^u q(\tau) + TW^s q(\tau)$, of $\mathbb{R}^n$ by virtue of (H_2). Let $\Delta \in \mathbb{R}^n$ be complementary to $TW^u q(\tau) + TW^s q(\tau)$. Let $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ with $\omega_i > 0$, $i \in \mathbb{Z}$.

From (H_1), (4.1) has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$. From (H_2), the adjoint equation

$$\dot{x}(t) + D_x f(q(t), 0)^* x(t) = 0$$

has a unique bounded solution $x = \psi(t)$ up to scalar multiples. Without loss of generality, let $\langle \psi(\tau), \Delta \rangle = 1$. We can prove the following general result:

THEOREM 4.1. *Assume that* (H_1) *and* (H_2) *are satisfied. Then there are positive constants* $\hat{\omega}$, μ_1 *and* ε_1 *with the following property. For any sequence* ω *with* $\omega_i > \hat{\omega}$ *and* $|\mu| \leqq \mu_1$, *there exists a unique piecewise continuous solution* $x(t)$ *of* (1.1) *that orbitally lies in a* ε_1*-neighbourhood of* $q(t)$. *The time spent by* $x(t)$ *between two consecutive intersections with* Σ *is* $2\omega_i$, $i \in \mathbb{Z}$. $x(t)$ *has infinitely many jumps of the form* $\xi_i \Delta$, $\xi_i \in \mathbb{R}$, *each occurring at the time* 2τ *after it meets* Σ. *Moreover, if* $x(-\tau + t_i) \in \Sigma$ *and* $x_i(t) = q_i(t) + z_i(t) = x(t + t_i)$ *for* $t \in [-\omega_i, \omega_{i+1}]$, *then*

$$\xi_i = \int_{-\omega_i}^{\omega_{i+1}} \psi(s) g(z_i(s), \mu, s)\, ds + \psi(-\omega_i) z_i(-\omega_i) - \psi(\omega_{i+1}) z_i(\omega_{i+1}).$$

We shall denote the solution by $z_i(t) = Z_i(t; \omega, \mu)$ *and* $\xi_i = G_i(\omega, \mu)$, $i \in \mathbb{Z}$. *The existence of a* C^1 *solution* $x(t)$ *near the homoclinic orbit* $q(t)$ *is characterised by the condition* $G_i(\omega, \mu) = 0$ *for all* $i \in \mathbb{Z}$.

Proof. We can define a heteroclinic chain by setting $q_i(t) = q(t)$, $p_i = p$ for all $i \in \mathbb{Z}$. There exists $\bar{\psi} \in \mathbb{R}^{n^*}$ such that $\Sigma = \{x \mid \langle \bar{\psi}, x - q(-\tau)\rangle = 0\}$. Let $\Sigma_i = \Sigma$, $\bar{\psi}_i = \psi$, $\psi_i(s) = \psi(s)$, $\Delta_i = \Delta$, $\varphi_i = \dot{q}(-\tau)$, and $A_i(t) = D_x f(q(t), 0)$. Let $T^i(t, s) = T(t, s)$ be the principal matrix solution of equation (4.1). $T^i(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$. Define $\mathcal{F}_i = \mathcal{F}$ and $(\mathcal{H}\mathcal{F})^\perp$ as in Section 2.

It is clear that all the hypotheses of Theorem 3.1 are satisfied. In particular, the uniformity assumptions are trivially valid. The results of Theorem 4.1 follow from the corresponding ones of Theorem 3.1 □

We now study the bifurcation of periodic solutions from the homoclinic solution $q(t)$. The following theorem describes a bifurcation function for the existence of 2ω periodic solutions near $q(t)$. It also shows that the bifurcation function for homoclinic solutions near $q(t)$ is the limiting case as $\omega \to \infty$.

THEOREM 4.2. *Suppose that* (H_1) *and* (H_2) *are satisfied and thus Theorem* 4.1 *applies. Let* $\omega_i \equiv \omega > \hat{\omega}$ *for all* $i \in \mathbb{Z}$. *Then* $Z_i(t; \omega, \mu)$ *and* $G_i(\omega, \mu)$, *now denoted by* $Z(t; \omega, \mu)$ *and* $G(\omega, \mu)$ *for simplicity, do not depend on i. Moreover* $G(\omega, \mu)$ *is* C^k *jointly in* (ω, μ). *The existence of a* 2ω *periodic solution near the orbit of* $q(t)$ *is equivalent to the condition* $G(\omega, \mu) = 0$.

Let $\omega \to \infty$. *Then* $Z(t; \omega, \mu) \to Z(t; \infty, \mu)$ *uniformly for t on every compact subset* $V \subset \mathbb{R}$, *and* $G(\omega, \mu) \to G(\infty, \mu)$. *For* $|\mu| \leqq \mu_1$, *there exists a unique piecewise continuous solution* $x(t) = q(t) + Z(t; \infty, \mu) \in E(\mathbb{R}, \Delta)$ *that lies in an* ε_1 *neighbourhood of the orbit of* $q(t)$, *with the property that* $x(t) \to p(\mu)$ *as* $t \to \pm\infty$. $x(t)$ *has a possible jump* $x(\tau^-) - x(\tau^+) = \xi\Delta$ *occurring at the time* 2τ *after it meets* Σ. $\xi = G(\infty, \mu) = \int_{-\infty}^{\infty} \psi(s) g(Z(s; \infty, \mu), \mu, s)\, ds$. *The existence of a homoclinic solution for* $|\mu| \leqq \mu_1$ *in an* ε_1 *neighbourhood of* $q(t)$ *is equivalent to* $G(\infty, \mu) = 0$.

Proof. If $\omega_i \equiv \omega$ then $Z_i(t; \omega, \mu)$ and $G_i(\omega, \mu)$ do not depend on i by the uniqueness of Z_i and G_i. Therefore, if $G(\omega, \mu) = 0$ then the solution $x(t)$ is 2ω periodic. We then show $G(\omega, \mu)$ is C^k in (ω, μ). Let $\nu(t) \in C^\infty(\mathbb{R})$ such that $\nu(t) = 0$ for $t \in [-\tau - 1, \tau + 1]$ and $\nu(t) = 1$ for $t \in (-\infty, -\tau - 2] \cup [\tau + 2, \infty)$. Let $x_1(t) = x((1 + \beta\nu(t))t)$. Clearly $x_1(t) = x(t)$ for $t \in [-\tau - 1, \tau + 1]$. Let $x(t) \in E([-\omega, \omega], \Delta]$ be a piecewise continuous solution for (1.1) with $x(-\omega) = x(\omega)$, then $x(\tau^-) - x(\tau^+) = G(\omega, \mu)\Delta$. This implies $x_1(t) \in E([-\omega_1, \omega_1], \Delta)$ is a piecewise continuous solution for equation

$$\dot{x}_1(t) = (1 + \beta(\nu(t) + t\dot{\nu}(t))) f(x_1(t), \mu), \quad -\omega_1 \leqq t \leqq \omega_1, \tag{4.2}$$

with $x_1(-\omega_1) = x_1(\omega_1)$ where $\omega = (1 + \beta)\omega_1$. $x_1(t)$ extends to $t \in \mathbb{R}$ by period $2\omega_1$. Let ω_1 be fixed and β be in a neighbourhood of zero. Then ω is in a neighbourhood of ω_1. For equation (4.2), applying the same argument as we did to (1.1), we find that for each $\omega \geqq \hat{\omega}$, $|\mu| \leqq \mu_1$ and $|\beta| \leqq \beta_1$, (4.2) has a unique $2\omega_1$ periodic, piecewise continuous solution $x_1(t) = q(t) + z_1(t)$, with $\langle \bar{\psi}, z_1(-\tau)\rangle = 0$ and $x_1(\tau^-) - x_1(\tau^+) = \bar{G}(\beta, \mu)\Delta$. Moreover, $\bar{G}(\beta, \mu)$ is C^k in (β, μ). (Recall that in theorem 3.1, $G(\omega, \mu)$ is C^k with respect to the parameter μ for each fixed ω.) However, $G(\omega, \mu) = \bar{G}(\beta, \mu) = \bar{G}\left(\frac{\omega}{\omega_1} - 1, \mu\right)$. Thus $G(\omega, \mu)$ is C^k jointly in (ω, μ). The first part of the theorem has been proved.

Fixed $\zeta > \hat{\omega} > 0$ and consider $V = [-\zeta, \zeta]$. Let $\omega_i > \zeta$, $i = 1, 2$ and $\omega = \inf\{\omega_1, \omega_2\}$. Applying Lemma 2.3 to the interval $[-\omega, \omega]$, we have for $t \in V$

$$Z(t; \omega_i, \mu) = \mathscr{X}(t; P_s(-\omega)Z(-\omega; \omega_i, \mu), P_u(\omega)Z(\omega; \omega_i, \mu), g(Z(\cdot\,; \omega_i, \mu), \mu, \cdot), -\omega, \omega), \quad i = 1, 2.$$

Denote $Z(t; \omega_1, \mu) - Z(t; \omega_2, \mu)$ by $\Delta z(t)$ and $g(Z(s; \omega_1, \mu), \mu, s) - g(Z(s; \omega_2, \mu), \mu, s)$ by $\Delta g(s)$. We have

$$\|\Delta g(s)\|_\eta \leqq \tfrac{1}{2}\|\Delta z(s)\|_\eta$$

if $\|\Delta z\|_0$ is small, where $\|\cdot\|_\eta$ is the weighted norm in $E([-\omega, \omega])$. Since $\mathscr{X}(t; \varphi_1, \varphi_2, h, -\omega, \omega)$ is linear in $(\varphi_1, \varphi_2, h)$, we have

$$\begin{aligned}
\|\Delta z\|_\eta &= \|\mathscr{X}(t; P_s(-\omega)\Delta z(-\omega), P_u(\omega)\Delta z(\omega), \Delta g(\cdot), -\omega, \omega)\|_\eta \\
&\leqq C(|\Delta z(-\omega)| + |\Delta z(\omega)| + \|\Delta g\|_\eta) \\
&\leqq 2C(|\Delta z(-\omega)| + |\Delta z(\omega)|); \\
|\Delta z(t)| &\leqq \tfrac{1}{2}\|\Delta z\|_\eta (e^{-\eta(\omega+t)} + e^{-\eta(\omega-t)}).
\end{aligned}$$

Since $|\Delta z(-\omega)| + |\Delta z(\omega)|$ is uniformly bounded, $\|\Delta z\|_\eta$ is bounded as $\omega \to +\infty$. For $|t| \leqq \zeta$, we have $|\Delta z(t)| \to 0$ as $\omega = \inf\{\omega_1, \omega_2\} \to +\infty$. Therefore $\lim_{\omega\to\infty} Z(t; \omega, \mu)$ exists for all $t \in \mathbb{R}$ since ζ is arbitrary. Denote the limit by $Z(t; \infty, \mu)$. $Z(t; \infty, \mu) + q(t)$ is clearly a solution of (2.1) with a possible jump along Δ at $t = \tau$. By the saddle point property of the hyperbolic equilibrium $p(\mu)$, $Z(t; \infty, \mu) + q(t) \to p(\mu)$ as $t \to \pm\infty$. It is then straightforward to verify that $G(\omega, \mu) \to G(\infty, \mu)$ as $\omega \to +\infty$. □

We now make a new hypothesis:

(H$_3$) $\int_{-\infty}^{\infty} \psi(t) D_\mu f(q(t), 0)\, dt \neq 0$.

THEOREM 4.3. *Suppose that* (H$_1$), (H$_2$) *and* (H$_3$) *are satisfied and thus Theorems* 4.1 *and* 4.2 *apply. Then there are positive constants* $\hat{\omega}$, μ_1, ε_1 *and a* C^k *function* $\hat{\mu}(\omega)$: $\{\omega > \hat{\omega}\} \to \{|\mu| < \mu_1\}$ *with* $G(\omega, \hat{\mu}(\omega)) = 0$. *The* 2ω *periodic solution* $x(t) = q(t) + Z(t; \omega, \hat{\mu}(\omega))$ *that orbitally lies in the* ε_1 *neighbourhood of* $q(t)$ *is unique up to time translations.*

Proof. From Lemma 3.2, there exists $\delta > 0$ such that if $|\mu| < \mu_1$ and $\omega > \hat{\omega}$, $|D_\mu G(\omega, \mu)| \geqq \delta > 0$. Therefore $|G(\omega, \pm\mu_1) - G(\omega, 0)| \geqq \mu_1\delta > 0$. From Lemma 3.3 $|G(\omega, 0)| \leqq Ce^{-2\alpha\omega}$, $0 < \alpha < \alpha_0$. Therefore $G(\omega, \mu)$ is strictly monotone in $\mu \in [-\mu_1, \mu_1]$ and changes sign in that interval if $\hat{\omega}$ is chosen even larger so that $|G(\omega, 0)| < \mu_1\delta$ for $\omega > \hat{\omega}$. From the Intermediate Value Theorem, for each $\omega > \hat{\omega}$ there is a unique $\mu = \hat{\mu}(\omega)$ such that $G(\omega, \hat{\mu}(\omega)) = 0$ and $|\hat{\mu}(\omega)| < \mu_1$. It remains to prove that $\hat{\mu}(\omega)$ is C^k. Let $\omega_0 > \hat{\omega}$, consider $G(\omega, \mu) = 0$ in a neighbourhood of $(\omega_0, \hat{\mu}(\omega_0))$ and use the Implicit Function Theorem. □

Define the gap in the real part of $\sigma\{D_x f(p, 0)\}$ as $\mathrm{Gap}\,\{\sigma\{D_x f(p, 0)\}\} = \min\{|\lambda_1 - \lambda_2| : \lambda_1, \lambda_2 \in \mathrm{Re}\,\sigma\{D_x f(p, 0)\}, \lambda_1 \neq \lambda_2\}$. Let $0 < \varepsilon < \min\{\delta, \alpha_0\}$ where $\delta = \mathrm{Gap}\,\{\sigma\{D_x f(p, 0)\}\}$.

LEMMA 4.4. *There exists a solution $\bar{q}_i(t)$, $i = 1, 2$ for the linear autonomous equation $\dot{x}(t) - D_x f(p, 0)x(t) = 0$ such that*

$$|q(t) - \bar{q}_i(t)| \leqq o(|\bar{q}_i(t)|), \tag{4.3}$$

as $t \to -\infty$ for $i = 1$ or $t \to +\infty$ for $i = 2$. There also exists a solution $\bar{\psi}_i(t)$, $i = 1, 2$ for the linear autonomous equation $\dot{x}(t) + D_x f(p, 0)^* x(t) = 0$ such that

$$|\psi(t) - \bar{\psi}_i(t)| \leqq C |\bar{\psi}_i(t)| e^{-\epsilon|t|}, \tag{4.4}$$

as $t \to -\infty$ for $i = 1$ or $t \to +\infty$ for $i = 2$.

Proof. The proof of (4.4) will be left to the reader. Note that [**13**, Theorem 10.13.2] contains a related result. However, both the assumption and the conclusion in that book are weaker than ours. We now show (4.3). Let $\bar{P}_u$ be the spectral projection of $D_x f(p, 0)$ on the unstable space $\mathcal{R}\bar{P}_u$. It is known that $\bar{P}_u$: $W^u_{\text{loc}}(p) \to \mathcal{R}\bar{P}_u$ is a diffeomorphism with the inverse denoted by $\bar{P}_u^{-1}$: $\mathcal{R}\bar{P}_u \to W^u_{\text{loc}}(p)$. Since $W^u_{\text{loc}}(p)$ is locally invariant under the flow of (1.1) for $\mu = 0$, equation (1.1) then induces a vector field on $W^u_{\text{loc}}(p)$. In fact, if $\bar{P}_u x(t) = y(t)$, $y(t)$ satisfies the equation

$$\dot{y}(t) = \bar{P}_u f(\bar{P}_u^{-1} y(t), 0). \tag{4.5}$$

Equation (4.5) is an autonomous ODE in $\mathcal{R}\bar{P}_u$ with 0 being an unstable equilibrium. From a theorem in [**12**], there is a C^1 diffeomorphism $I + h$: $\mathcal{R}\bar{P}_u \to \mathcal{R}\bar{P}_u$ with $h(0) = 0$, $Dh(0) = 0$ such that (4.5) is equivalent to

$$\dot{y}(t) = D\bar{P}_u \,.\, D_x f(p, 0) D\bar{P}_u^{-1} y(t).$$

Here $D\bar{P}_u$: $TW^u_{\text{loc}}(p) \to \mathcal{R}\bar{P}_u$ has an inverse $D\bar{P}_u^{-1}$: $\mathcal{R}\bar{P}_u \to TW^u_{\text{loc}}(p)$. However, we identify $TW^u_{\text{loc}}(p)$ with $\mathcal{R}\bar{P}_u$, thus

$$\dot{y}(t) = D_x f(p, 0) y(t), \quad y(t) \in \mathcal{R}\bar{P}_u. \tag{4.6}$$

We then let $\bar{q}_1(t) = (I + h) \circ \bar{P}_u q(t)$ for $-t > 0$ sufficiently large so that $q(t) \in W^u_{\text{loc}}(p)$. By the definition $\bar{q}_1(t)$ is a solution of (4.6).

$$\begin{aligned} |q(t) - \bar{q}_1(t)| &\leqq |(I - \bar{P}_u) q(t)| + |h \circ \bar{P}_u q(t)| \\ &= O(|q(t)|^2) + o(|q(t)|) \\ &= o(|\bar{q}_1(t)|). \end{aligned}$$

Here we have employed the fact that $|(I - \bar{P}_u) q(t)| = O(|\bar{P}_u q(t)|^2)$, for $W^u_{\text{loc}}(p)$ is tangent to $\mathcal{R}\bar{P}_u$, and $h(0) = 0$, $Dh(0) = 0$. The case $i = 2$ can be proved similarly.

We now give a generalisation of Silnikov's work on the bifurcation of homoclinic solutions to periodic solutions. Let

$$\begin{aligned} \rho &= \min\{\operatorname{Re} \lambda \mid \lambda \in \sigma D_x f(p, 0), \operatorname{Re} \lambda > 0\}, \\ \nu &= \min\{-\operatorname{Re} \lambda \mid \lambda \in \sigma D_x f(p, 0), \operatorname{Re} \lambda < 0\}. \end{aligned}$$

Assume $0 < \rho \leqq \nu$. This implies $\alpha_0 = \rho$. We need the following hypotheses:

(H_4) $|q(t)| \sim C_1 e^{\rho t} |t|^l$, $t \to -\infty$ and $|\psi(t)| \sim C_2 e^{-\rho t} |t|^h$, $t \to +\infty$, where C_1 and C_2 are positive constants, l and h are nonnegative integers.

(H_5) $|\psi(-\omega) q(\omega) - \psi(\omega) q(-\omega)| / (|\psi(-\omega)|\, |q(\omega)| + |\psi(\omega)|\, |q(-\omega)|)$ does not approach zero as $\omega \to +\infty$.

(H_4) and (H_5) are generic by virtue of Lemma 4.4. If $\rho < v$ one may drop the term $\psi(-\omega)q(\omega)$ in (H_5).

LEMMA 4.5. *If* (H_1), (H_2), (H_4) *and* (H_5) *are satisfied, then there is* $\delta < 0$ *such that*

$$G(\omega, \mu) = \psi(-\omega)q(\omega) - \psi(\omega)q(-\omega) + \mu \int_{-\infty}^{\infty} \psi(t)D_\mu f(q(t), 0)\, dt$$
$$+ o(|e^{-(2\rho+\delta)\omega}| + |\mu|), \quad \text{as } \mu \to 0,\ \omega \to +\infty. \tag{4.7}$$

If (H_1)–(H_5) *are all satisfied, then* $\mu = \hat{\mu}(\omega)$ *satisfies*

$$\mu = (\psi(\omega)q(-\omega) - \psi(-\omega)q(\omega))\left(\int_{-\infty}^{\infty} \psi(t)D_\mu f(q(t), 0)\, dt\right)^{-1} + o(e^{-(2\rho+\delta)\omega}), \tag{4.8}$$

as $\omega \to +\infty$. *If* $\rho < v$, *one can drop* $\psi(-\omega)q(\omega)$ *in* (4.7) *and* (4.8).

Proof. $|q(t) - p| = |q(t)| \leqq C_\alpha e^{-\alpha|t|}$, $t \in \mathbb{R}$, for any $0 < \alpha < \rho$, from Lemma 4.4. Therefore, $|D_x f(q(t), 0) - D_x f(p, 0)| \leqq C_\alpha e^{-\alpha|t|}$ as $t \to \pm\infty$. From [**18**, Lemma 3.4] we have that $|P_s(t) - \bar{P}_s| + |P_u(t) - \bar{P}_u| \leqq C_\alpha e^{-\alpha|t|}$ as $t \to \pm\infty$, where $\bar{P}_s$ and $\bar{P}_u$ are the spectral projections of $D_x f(p, 0)$ to the stable and unstable spaces. Based on this, it is easy to prove that

$$|P(\mathcal{R}P_s(-\omega), \mathcal{R}P_u(\omega)) - \bar{P}_s| + |P(\mathcal{R}P_u(\omega), \mathcal{R}P_s(-\omega)) - \bar{P}_u| \leqq C_\alpha e^{-\alpha\omega},$$

for $\omega > \hat{\omega}$. Observe that

$$q(-\omega) = \bar{P}_u q(-\omega) + (I - \bar{P}_u)q(-\omega)$$
$$= \bar{P}_u q(-\omega) + O(|q(-\omega)|^2),$$

by virtue of the fact that $q(-\omega) \in W^u_{\text{loc}}(p)$ that is tangent to $\mathcal{R}\bar{P}_u$. Thus, $\bar{P}_s q(-\omega) = O(|q(-\omega)|^2)$ and $\bar{P}_u q(-\omega) = q(-\omega) + O(|q(-\omega)|^2)$. Similarly, $\bar{P}_u q(\omega) = O(|q(\omega)|^2$ and $\bar{P}_s q(\omega) = q(\omega) + O(|q(\omega)|^2)$. Choosing α sufficiently close to $\rho = \alpha_0$, we can use Lemma 3.3 to conclude that

$$G(\omega, 0) = -\psi(-\omega)\bar{P}_s(q(-\omega) - q(\omega)) - \psi(\omega)\bar{P}_u(q(-\omega) - q(\omega)) + o(e^{-(2\rho+\delta)\omega})$$
$$= \psi(-\omega)q(\omega) - \psi(\omega)q(-\omega) + o(e^{-(2\rho+\delta)\omega}), \tag{4.9}$$

where $\delta > 0$ is a small constant. From Lemma 3.2,

$$G(\omega, \mu) - G(\omega, 0) = D_\mu G(\omega, \theta\mu) \,.\, \mu \quad (0 < \theta < 1)$$
$$= \left[\int_{-\infty}^{\infty} \psi(t)D_\mu f(q(t), 0)\, dt\right]\mu + O(e^{-\alpha\omega}|\mu| + |\mu|^2). \tag{4.10}$$

Estimate (4.7) follows from (4.9) and (4.10). If (H_3) is also valid, from Theorem 4.3, $\mu = \hat{\mu}(\omega)$. Estimate (4.8) then follows easily from (4.7). □

From Lemma 4.4, we have

$$\psi(-\omega)q(\omega) - \psi(\omega)q(-\omega) = \tilde{\psi}_1(-\omega)\tilde{q}_2(\omega) - \tilde{\psi}_2(\omega)\tilde{q}_1(-\omega)$$
$$+ o(|\psi(-\omega)|\,|q(\omega)| + |\psi(\omega)|\,|q(-\omega)|) \quad \text{as } \omega \to \infty.$$

Since solutions of linear equations with constant coefficients are linear combinations of products of generalised eigenvectors and exponential-polynomial functions, from (H_4) and (H_5) it is easy to see that

$$\psi(-\omega)q(\omega)-\psi(\omega)q(-\omega)=e^{-2\rho\omega}\omega^m\Theta(\omega)+o(e^{-2\rho\omega}\omega^m),$$

where $m \geqq 0$ is an integer and $\Theta(\omega)$ is a quasi-periodic function. In fact, $\Theta(\omega)$ is a sum of trigonometrical functions of period π/β, where β is such that $\pm\rho+i\beta$ is an eigenvalue of $D_xf(p,0)$.

THEOREM 4.6. (i) *If* (H_1)–(H_5) *are satisfied and if* $\inf|\Theta(\omega)|>0$, *then the sign of* $\hat{\mu}(\omega)$, $\omega>\hat{\omega}$ *is determined by the sign of the product of* $\Theta(\omega)$ *and* $\int_{-\infty}^{\infty}\psi(t)D_\mu f(q(t),0)\,dt$. *That is, bifurcation to periodic solution occurs only at one side of* $\mu=0$.

(ii) *If* (H_1), (H_2), (H_4) *and* (H_5) *are satisfied and if* $\inf\Theta(\omega)<0<\sup\Theta(\omega)$, *then there are positive constants* L, $\hat{\omega}$ *and* μ_1 *with the following property. For each* $|\mu|\leqq\mu_1$, *there is a constant* $\bar{\omega}(\mu)>\hat{\omega}$, $\bar{\omega}(\mu)\to\infty$ *as* $\mu\to 0$, *such that there exists a solution* ω *for* $G(\omega,\mu)=0$ *on each interval* $[\hat{\omega}+jL,\hat{\omega}+(j+1)L]$, *if* $j\geqq 0$ *and* $\hat{\omega}+(j+1)L\leqq\bar{\omega}(\mu)$. *In particular, if* $\mu=0$ *then* $\bar{\omega}(\mu)=\infty$ *and there exist infinitely many periodic solutions for equation* (1.1).

Proof. The assertion (i) follows from (4.8). To show (ii), solve $\mu=e^{-(2\rho+\delta)\bar{\omega}}$ and define $\bar{\omega}(\mu)=-\ln\mu/(2\rho+\delta)$ and $\bar{\omega}(0)=+\infty$. Let $\hat{\omega}<\omega\leqq\bar{\omega}(\mu)$. Then from (4.7)

$$G(\omega,\mu)=e^{-2\rho\omega}\omega^m\Theta(\omega)+o(e^{-2\rho\omega}\omega^m).$$

Let $\inf\Theta(\omega)<-\varepsilon<0<2\varepsilon<\sup\Theta(\omega)$. Since $\Theta(\omega)$ is almost periodic, there exists a relatively dense set $E(\varepsilon_2)$, $0<\varepsilon_2<\varepsilon$, and a length $L>0$ such that

$$E(\varepsilon_2)\cap(\beta,\beta+L)\neq\emptyset,\quad\text{for all }\beta\in\mathbb{R},$$

and for each $t\in E(\varepsilon_2)$,

$$|\Theta(\omega+t)-\Theta(\omega)|<\varepsilon_2,\quad\text{for all }\omega\in\mathbb{R}.$$

See [2] for the above notions. There exist ω_1 and ω_2 such that $\Theta(\omega_1)>2\varepsilon$ and $\Theta(\omega_2)<-\varepsilon$. By the almost periodicity, we have ω_3 with $\Theta(\omega_3)>\varepsilon$ and $\omega_k\in[\hat{\omega}+iL,\hat{\omega}+(i+1)L]$, $k=2,3$, for some integer i. By the almost periodicity again, on each interval $[\hat{\omega}+jL,\hat{\omega}+(j+1)L]$, there exist $\omega^{(1)}$ and $\omega^{(2)}$ with $\Theta(\omega^{(1)})>\varepsilon-\varepsilon_2$ and $\Theta(\omega^{(2)})<-(\varepsilon-\varepsilon_2)$. If we set $\hat{\omega}$ large and μ sufficiently small with $\hat{\omega}<\bar{\omega}(\mu)$, then on each interval $[\hat{\omega}+jL,\hat{\omega}+(j+1)L]\subset[\hat{\omega},\bar{\omega}(\mu)]$, we have $G(\omega^{(1)},\mu)>0$ and $G(\omega^{(2)},\mu)<0$. Thus there is at least one solution for $G(\omega,\mu)=0$ on such interval. □

We now study the bifurcation of aperiodic solutions from the homoclinic solution $q(t)$. Let $\{\omega_i\}_{i\in Z}$ be an aperiodic sequence. We look for a solution $x(t)$ that orbitally lies near $q(t)$ and spends the time $2\omega_i$ between two consecutive intersections with Σ. For simplicity we shall assume $0<\rho<\nu$ in what follows. Hypothesis (H_5) can be restated as

$(H_5)'$ $|\psi(\omega)q(-\omega)|/|\psi(\omega)|\,|q(-\omega)|)$ does not approach zero as $\omega\to+\infty$.

Let (H_1) and (H_2) be valid and let $\omega_i\geqq\hat{\omega}$, $i\in\mathbb{Z}$ and $|\mu|\leqq\mu_1$. Then Theorem 4.1 applies and $G_i(\omega,\mu)$, $i\in\mathbb{Z}$ is the bifurcation function described there.

LEMMA 4.7. *If $0<\rho<\nu$ and if* (H_1) *and* (H_2) *are satisfied, then there exist $\hat{\omega}>0$, $\mu_1>0$ (possibly different from those in Theorem* 4.1), *$\gamma>1$ and $\delta>0$ with the following property. If ω is such that $\omega_i \geqq \hat{\omega}$, $\omega_{i+1}/\omega_i<\gamma$ and $\omega_i/\omega_{i+1}<\gamma$ for all $i \in \mathbb{Z}$ and $|\mu| \leqq \mu_1$, then*

$$G_i(\omega, \mu) = -\psi(\omega_{i+1})q(-\omega_{i+1}) + O(e^{-(2\rho+\delta)\omega_{i+1}} + |\mu|).$$

Proof. Let $\rho/2<\alpha<\rho$, $0<\varepsilon<\min\{\alpha, \nu-\rho\}$ and ε be also smaller than Gap $\{\alpha\{D_x f(p, 0)\}\}$. There clearly exists $\gamma>1$ satisfying the following conditions simultaneously:

$$\gamma<\frac{\rho+\nu+\varepsilon}{2\rho}, \tag{4.11}$$

$$\gamma<\frac{3}{2}, \tag{4.12}$$

$$\frac{2\alpha}{\gamma^2}+\frac{\nu}{\gamma}>2\rho, \tag{4.13}$$

$$\gamma<\frac{2\alpha}{\rho}, \tag{4.14}$$

$$\gamma<\frac{2\rho+\alpha}{2\rho}. \tag{4.15}$$

We shall show that such γ is the desired constant. From the assumptions of this lemma, we have $\omega_{i+j} \geqq \frac{\omega_{i+1}}{\gamma^{|j-1|}}$ for all $i, j \in \mathbb{Z}$. Such relations will be used freely without explanation in the sequel.

Observe that $G_i(\omega, \mu) = G_i(\omega, 0) + O(|\mu|)$. The proof is then based on Lemma 3.3. We remark that the constant η can be replaced by $0<\alpha<\alpha_0$.

Define

$$\begin{aligned} I_1 = -\psi(-\omega_i)P(\mathcal{R}P_s(-\omega_i), \mathcal{R}P_u(\omega_i))(q(-\omega_i)-q(\omega_i)) \\ -\psi(\omega_{i+1})P(\mathcal{R}P_u(\omega_{i+1}), \mathcal{R}P_s(-\omega_{i+1}))(q(-\omega_{i+1})-q(\omega_{i+1})). \end{aligned}$$

As in Lemma 4.5, we can show that

$$P(\mathcal{R}P_s(-\omega_i), \mathcal{R}P_u(\omega_i)) = \bar{P}_s + O(e^{-\alpha\omega_i}),$$
$$P(\mathcal{R}P_u(\omega_{i+1}), \mathcal{R}P_s(-\omega_{i+1})) = \bar{P}_u + O(e^{-\alpha\omega_{i+1}}).$$

Therefore

$$\begin{aligned} I_1 &= -\psi(\omega_{i+1})q(-\omega_{i+1}) + O(e^{-(\rho+\nu+\varepsilon)\omega_i} + e^{-(2\rho+\varepsilon)\omega_{i+1}}) \\ &= -\psi(\omega_{i+1})q(-\omega_{i+1}) + O(e^{-(2\rho+\delta_1)\omega_{i+1}}), \end{aligned} \tag{4.16}$$

where $0<\delta_1<\min\left\{\frac{\rho+\nu+\varepsilon}{\gamma}-2\rho, \varepsilon\right\}$, by virtue of (4.11). From (4.12), there exists $0<\delta_2<\frac{3\rho}{\gamma}-2\rho$, and

$$|\psi(-\omega_i)|\,|b_i|^2 \leqq C\,|\omega_i|^{3n}e^{-(2\rho+\nu)\omega_i} \leqq Ce^{-3\rho\omega_i} \leqq Ce^{-(2\rho+\delta_2)\omega_{i+1}}. \tag{4.17}$$

From (4.13), there exists $0<\delta_3<\frac{\nu}{\gamma}+\frac{2\alpha}{\gamma^2}-2\rho$, and

$$|\psi(-\omega_i)|\,\|\mathbf{b}\|\,(e^{-2\alpha\omega_{i-1}}+e^{-2\alpha\omega_i}+e^{-2\alpha\omega_{i+1}})\leqq C\,|\omega_i|^n e^{-(\nu/\gamma+2\alpha/\gamma^2)\omega_{i+1}} \leqq Ce^{-(2\rho+\delta_3)\omega_{i+1}}. \tag{4.18}$$

Let $0<\delta_4<\rho$. We have

$$|\psi(\omega_{i+1})|\,|b_{i+1}|^2\leqq C\,|\omega_{i+1}|^{3n}e^{-3\rho\omega_{i+1}}\leqq Ce^{-(2\rho+\delta_4)\omega_{i+1}}. \tag{4.19}$$

By (4.14), there exists $0<\delta_5<\frac{2\alpha}{\gamma}-\rho$, and

$$|\psi(\omega_{i+1})|\,\|\mathbf{b}\|\,(e^{-2\alpha\omega_i}+e^{-2\alpha\omega_{i+1}}+e^{-2\alpha\omega_{i+2}})\leqq C\,|\omega_{i+1}|^n e^{-(\rho+2\alpha/\gamma)\omega_{i+1}} \leqq Ce^{-(2\rho+\delta_5)\omega_{i+1}}. \tag{4.20}$$

By (4.15), there exists $0<\delta_6<\frac{2\rho+\alpha}{\gamma}-2\rho$, and

$$|b_i|^2(e^{-\alpha\omega_i}+e^{-\alpha\omega_{i+1}})\leqq Ce^{-((2\rho+\alpha)/\gamma)\omega_{i+1}}\leqq C\,|\omega_i|^{2n}e^{-(2\rho+\delta_6)\omega_{i+1}}. \tag{4.21}$$

It is also clear that $0<\delta_6<\frac{\alpha}{\gamma}$. We then have

$$|b_{i+1}|^2(e^{-\alpha\omega_i}+e^{-\alpha\omega_{i+1}})\leqq C\,|\omega_{i+1}|^{2n}e^{-(2\rho+\alpha/\gamma)\omega_{i+1}}\leqq Ce^{-(2\rho+\delta_6)\omega_{i+1}}. \tag{4.22}$$

By (4.14), $\frac{4\alpha}{\gamma}>2\rho$, $\frac{4\alpha}{\gamma^2}+\frac{\alpha}{\gamma}>\frac{2\rho+\alpha}{\gamma}$. Therefore $\delta_6<\frac{4\alpha}{\gamma^2}+\frac{\alpha}{\gamma}-2\rho$ and

$$\|\mathbf{b}\|^2(e^{-4\alpha\omega_{i-1}}+e^{-4\alpha\omega_i}+e^{-4\alpha\omega_{i+1}}+e^{-4\alpha\omega_{i+2}})(e^{-\alpha\omega_i}+e^{-\alpha\omega_{i+1}}) \leqq Ce^{-(4\alpha/\gamma^2+\alpha/\gamma)\omega_{i+1}}\leqq Ce^{-(2\rho+\delta_6)\omega_{i+1}}. \tag{4.23}$$

Based on (4.16)–(4.23), we have

$$G_i(\omega, 0)=-\psi(\omega_{i+1})q(-\omega_{i+1})+O(e^{-(2\rho+\delta)\omega_{i+1}}),$$

where $0<\delta\leqq\inf\{\delta_i \mid 1\leqq i\leqq 6\}$. The desired estimate then follows easily. □

We now present a generalisation of Silnikov's theorem on bifurcation to aperiodic solutions. Assume that (H_4) and $(H_5)'$ are valid. Then

$$-\psi(\omega)q(-\omega)=e^{-2\rho\omega}\omega^m\Theta(\omega)+o(e^{-2\rho\omega}\omega^m),$$

where $m\geqq 0$ is an integer and $\Theta(\omega)$ is a quasi periodic function – a sum of trigonometric functions of period π/β where $\rho+i\beta$ is an eigenvalue of $D_xf(p, 0)$.

THEOREM 4.8. *Let $0<\rho<\nu$ and let (H_1), (H_2), (H_4) and $(H_5)'$ be valid. If $\inf\Theta(\omega)<0<\sup\Theta(\omega)$, then there are positive constants L, $\hat{\omega}$ and μ_1 with the following property. For each $|\mu|\leqq\mu_1$, there is a constant $\bar{\omega}(\mu)>\hat{\omega}$, $\bar{\omega}(\mu)\to\bar{\omega}(0)=\infty$ as $\mu\to 0$. Let $\{j(i)\}_{i\in\mathbb{Z}}$ be a sequence of nonnegative integers and $|\mu|\leqq\mu_1$, such that*

$$\hat{\omega}+(j(i)+1)L\leqq\bar{\omega}(\mu),\quad i\in\mathbb{Z}, \tag{4.24}$$

$$[\hat{\omega}+(j(i+1)+1)L]/[\hat{\omega}+j(i)L]<\gamma,\quad i\in\mathbb{Z}, \tag{4.25}$$

and

$$[\hat{\omega} + (j(i) + 1)L]/[\hat{\omega} + j(i+1)L] < \gamma, \quad i \in \mathbb{Z}. \tag{4.26}$$

Then there exists a sequence $\boldsymbol{\omega}$ *with* $\omega_i \in [\hat{\omega} + j(i)L, \hat{\omega} + (j(i)+1)L]$, *that satisfies* $G_i(\boldsymbol{\omega}, \mu) = 0$, $i \in \mathbb{Z}$. *In particular, if* μ *is sufficiently small, there are uncountably many ways to choose the aperiodic sequence* $\{j(i)\}_{i \in \mathbb{Z}}$. *Thus there exist uncountably many aperiodic orbits near the homoclinic orbit* $x = q(t)$.

Proof. We shall use the result from Lemma 4.7. Solve $\mu = e^{-(2\rho+\delta)\bar{\omega}}$ to obtain $\bar{\omega}(\mu) = -\ln \mu/(2\rho + \delta)$, $\bar{\omega}(0) = \infty$. Let $\hat{\omega} < \omega_i < \bar{\omega}(\mu)$ for all $i \in \mathbb{Z}$. Then from Lemma 4.7

$$G_i(\boldsymbol{\omega}, \mu) = e^{-2\rho\omega_{i+1}}\omega_{i+1}^m \Theta(\omega_{i+1}) + o(e^{-2\rho\omega_{i+1}}\omega_{i+1}^m).$$

The lower case o is uniform with respect to $i \in \mathbb{Z}$. In a similar way to the proof of Theorem 4.6, we can show that there exist ω_i^M and ω_i^m in the interval $[\hat{\omega} + j(i)L, \hat{\omega} + (j(i)+1)L]$, for all $i \in \mathbb{Z}$, with $\Theta(\omega_i^M) > \varepsilon > -\varepsilon > \Theta(\omega_i^m)$. If $\hat{\omega}$ is sufficiently large and μ small, $\bar{\omega}(\mu) > \hat{\omega}$, then $G_i(\boldsymbol{\omega}^M, \mu) > 0 > G_i(\boldsymbol{\omega}^m, \mu)$. Here $\boldsymbol{\omega}^M$ (or $\boldsymbol{\omega}^m$) $= \{\omega_i\}_{i \in \mathbb{Z}}$ is a sequence with $\omega_{i+1} = \omega_{i+1}^M$ (or $\omega_{i+1} = \omega_{i+1}^m$). From Lemma 4.9 below, there exists at least one solution $\{\omega_i\}_{-k \leqq i \leqq k}$ for the system

$$G_i(\boldsymbol{\omega}, \mu) = 0, \quad -k \leqq i \leqq k.$$

Here ω_i, $|i| > k$, can be any constant in $[\hat{\omega} + j(i)L, \hat{\omega} + (j(i)+1)L]$. Denote the solution by $\boldsymbol{\omega}^{(k)}$. Finding a subsequence if necessary, we shall assume $\boldsymbol{\omega}^{(k)} \to \boldsymbol{\omega}$ coordinatewise. From the continuity of $G_i(\boldsymbol{\omega}^{(k)}, \mu)$ in $\boldsymbol{\omega}^{(k)}$, cf. Lemma 3.4, we have

$$G_i(\boldsymbol{\omega}, \mu) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

LEMMA 4.9. *Let* $Q = [-1, 1]^m$ *be a unit cube in* $\mathbb{R}^m$, $f: Q \to \mathbb{R}^m$ *is continuous with* $f_i(x) > 0$ (<0) *if* $x_i = 1$ *(or* $= -1$*), where* $f = (f_1, f_2, \ldots, f_m)$ *and* $x = (x_1, x_2, \ldots, x_m)$. *Then there exists at least one* $\bar{x} \in (-1, 1)^m$ *with* $f(\bar{x}) = 0$.

Proof. There is a homotopy F_t, $0 \leqq t \leqq 1$ on the boundary ∂Q of Z with $F_0 = f$, $F_1 = id$, and $0 \notin F_t(\partial Q)$. Therefore $\deg(0, Q, f) = \deg(0, Q, id) \neq 0$ (cf. [**20**]).

5. A singularly perturbed differential difference equation

Consider the singularly perturbed differential difference equation

$$\varepsilon \dot{z}(t) = -z(t) + f(z(t-1)) \tag{5.1}$$

Let $f \in C^k(\mathbb{R}, \mathbb{R})$, $k \geqq 1$, $f(a) = -b$ and $f(-b) = a$ for some $a > 0$, $b > 0$. For $\varepsilon = 0$, the difference equation $0 = -z(t) + f(z(t-1))$ admits a period 2 solution taking $z = a$ on intervals $(2n, 2n+1)$ and $x = -b$ on $(2n+1, 2n+2)$ for $n \in \mathbb{Z}$. Mallet-Paret and Nussbaum [**16, 17**] have shown that under some general conditions the slowly oscillating periodic solutions of (5.1) approach the period 2 step function as $\varepsilon \to 0$. Such a solution $z(t)$ consists of regular layers that stay near $z = a$ or $-b$ for nearly one unit of time and transition layers connecting a to $-b$ and $-b$ to a. Assuming that the period of $z(t)$ is $2 + 2\varepsilon r$, we introduce

$\eta(t)=z(t-1-\varepsilon r)$ and $\xi(t)=\eta(t-1-\varepsilon r)=z(t)$. Let $x(t)=\xi(-\varepsilon rt)$ and $y(t)=\eta(-\varepsilon rt)$. We have a system of two equations

$$\begin{aligned}\dot{x}(t)&=rx(t)-rf(y(t-1)),\\ \dot{y}(t)&=ry(t)-rf(x(t-1)).\end{aligned}\tag{5.2$_r$}$$

Observe that $(x, y)=(-b, a)$ and $(a, -b)$ are equilibria for (5.2). Although the existence of heteroclinic solutions between the equilibria is known from Mallet-Paret and Nussbaum's work, it can also be proved by a quite different method. Chow, Lin and Mallet-Paret [**8**], using perturbation and homotopy techniques, have shown that under the monotonicity on f and some general assumptions, there exists a unique $r_0>0$ such that $(5.2)_{r_0}$ possesses a unique heteroclinic solution $(p_1(t), q_1(t))$ connecting $(-b, a)$ to $(a, -b)$. By the symmetry of $(5.2)_r$, $(q_1(t), p_1(t))$ is a heteroclinic solution connecting $(a, -b)$ to $(-b, a)$. In this section, we shall use the method described in the previous sections to show the existence of a periodic solution of $(5.2)_r$ that orbitally lies near $(p_1(t), q_1(t))$ and $(q_1(t), p_1(t))$, with a large period 4ω. Such a periodic solution implies that (5.1) has a periodic solution of period $4\varepsilon r\omega=2+2\varepsilon r$. Here $\varepsilon=[(2\omega-1)r]^{-1}$. Therefore the method described here provides another approach to the problem studied by Mallet-Paret and Nussbaum.

We assume the following hypotheses in this section.

(D_1) The equilibria $(-b, a)$ and $(a, -b)$ of $(5.2)_r$ are hyperbolic.

(D_2) there exist $r_0>0$ and a heteroclinic solution $(p_1(t), q_1(t))$ for $(5.2)_{r_0}$ connecting $(-b, a)$ to $(a, -b)$.

(D_3) Bounded solutions of the linear variational system (5.3) form a one-dimensional linear space spanned by $(\dot{p}_1(t), \dot{q}_1(t))$.

$$\begin{aligned}\dot{x}(t)&=rx(t)-rDf(q_1(t-1))y(t-1),\\ \dot{y}(t)&=ry(t)-rDf(p_1(t-1))x(t-1).\end{aligned}\tag{5.3}$$

From a general theory of the Fredholm operator associated with (5.3), see [**15**], the formal adjoint system

$$\begin{aligned}\dot{x}(t)&=-rx(t)+rDf(p_1(t))y(t+1),\\ \dot{y}(t)&=-ry(t)+rDf(q_1(t))x(t+1),\end{aligned}\tag{5.4}$$

possesses a one-dimensional space of bounded solution, say spanned by $(\psi^{(1)}(t), \psi^{(2)}(t))$.

(D_4) $\int_{-\infty}^{\infty}\{\psi^{(1)}(t)\dot{p}_1(t)+\psi^{(2)}(t)\dot{q}_1(t)\}\,dt\neq 0$.

Remark 5.1. It has been shown in [**8**] that Hypotheses (B_1)–(B_5) in that paper imply (D_1)–(D_4). I also know some cases where (B_1)–(B_5) are not satisfied while (D_1)–(D_4) are.

THEOREM 5.2. *Suppose that the hypotheses* (D_1)–(D_4) *are valid. Then there are positive constants* $\hat{\omega}$, μ_1, ε_1 *and a* C^k *function* $r^*: (\hat{\omega}, +\infty)\to(r_0-\mu_1, r_0+\mu_1)$ *with the following property. If* $|r-r_0|<\mu$, $\omega>\hat{\omega}$ *and if* $(5.2)_r$ *admits a periodic solution* $(x(t), y(t))$ *of period* 4ω, *satisfying the estimate* $|x(t)-p_1(t)|+|y(t)-q_1(t)|<\varepsilon_1$ *for* $-\omega\leqq t\leqq\omega$ *and a symmetry condition* $x(t+2\omega)=y(t)$, *then it is sufficient and necessary to have* $r=r^*(\omega)$. *Moreover,* $(x(t), y(t))$ *is unique among such solutions up to time translations.*

THEOREM 5.3. *Suppose that the hypotheses* (D_1)–(D_4) *are valid. Then there exist* $\varepsilon_1, \varepsilon_2 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_2$, *there exist unique* ω *and* r, *satisfying* $\omega > \hat{\omega}$, $|r - r_0| < \mu_1$ *and* $4\varepsilon r\omega = 2 + 2\varepsilon r$ *with the following property. Equation* (5.1) *admits a unique periodic solution* $z(t)$ *of period* $2 + 2\varepsilon r$ *that satisfies the estimate*

$$|z(-\varepsilon rt) - p_1(t)| + |z(-\varepsilon rt - 1 - \varepsilon r) - q_1(t)| \leqq \varepsilon_1, \quad -\omega \leqq t \leqq \omega$$

We shall first introduce some notation. Let $Z = (x, y)'$, $F(Z) = (f(y), f(x))'$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Obviously $JF(Z) = F(JZ)$. We then rewrite $(5.2)_r$ as

$$\dot{Z}(t) = rZ(t) - rF(Z(t-1)) \tag{5.5$_r$}$$

If $Z(t)$ is a solution of $(5.5)_r$, so is $JZ(t)$. From (D_2), $W_1(t) \overset{\text{def}}{=} (p_1(t), q_1(t))'$ is a heteroclinic solution of $(5.5)_{r_0}$ connecting $(-b, a)'$ to $(a, -b)'$, thus $W_2(t) = JW_1(t)$ is a heteroclinic solution of $(5.5)_{r_0}$ connecting $(a, -b)'$ to $(-b, a)'$. Rewrite (5.3) as

$$\dot{Z}(t) = rZ(t) - rDF(W_1(t-1))Z(t-1). \tag{5.6}$$

Observe that $JDF(W_1(t-1))Z(t-1) = DF(JW_1(t-1))JZ(t-1)$. Thus if $Z(t)$ is a solution of (5.6), $JZ(t)$ is then a solution of

$$\dot{Z}(t) = rZ(t) - rDF(W_2(t-1))Z(t-1). \tag{5.7}$$

Let $u(t)$ be a continuous function in $\mathbb{R}^n$ with the domain $t \in [a, b]$. We define $u_t \in C([-1, 0], \mathbb{R}^n)$ and $u_t(\theta) = u(t + \theta)$, $-1 \leqq \theta \leqq 0$. u_t is a continuous function in $C([-1, 0], \mathbb{R}^n)$ with the domain $t \in [a + 1, b]$. Let $T^1(t, s)$, $t \geqq s$ be the solution map for (5.6) defined in the phase space $C([-1, 0], \mathbb{R}^2)$, (cf. [10]). Then $T^2(t, s) = JT^1(t, s)J$ is the solution map for (5.7). We also have $T^{2*}(s, t) = JT^{1*}(s, t)J$.

Recall that $(-b, a)'$ and $(a, -b)'$ are hyperbolic equilibria and $W_1(t) \to (-b, a)'$ or $(a, -b)'$ as $t \to -\infty$ or $+\infty$. From the roughness of exponential dichotomies, there exists $\tau > 0$ such that $T^1(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$ with projections $P_u^1(t)$ and $P_s^1(t)$. Similarly, $T^2(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$ with projections $P_u^2(t) = JP_u^1(t)J$ and $P_s^2(t) = JP_s^2(t)J$.

Let $C^*([-1, 0], \mathbb{R}^k)$ be the dual space for $C([-1, 0], \mathbb{R}^k)$. Let $\bar{\Psi}_1 \in C^*([-1, 0], \mathbb{R}^2)$ be such that $\langle \bar{\Psi}_1, \dot{W}_{1(-\tau)} \rangle \neq 0$. Here component-wise $\bar{\Psi}_1 = (\Psi_1^{(1)}, \Psi_1^{(2)})$, with $\Psi_1^{(j)} \in C^*([-1, 0], \mathbb{R})$, $j = 1, 2$. We then define $\bar{\Psi}_2 = \bar{\Psi}_1 J = (\Psi_1^{(2)}, \Psi_1^{(1)})$. Thus $\langle \bar{\Psi}_2, \dot{W}_{2(-\tau)} \rangle = \langle \bar{\Psi}_1 J, J\dot{W}_{1(-\tau)} \rangle = \langle \bar{\Psi}_1, \dot{W}_{1(-\tau)} \rangle \neq 0$.

Let $T^i(t, s) = T^{i \pm 2}(t, s)$, $i \in \mathbb{Z}$. $T^i(t, s)$ has exponential dichotomies on $(-\infty, -\tau]$ and $[\tau, +\infty)$ with projections $P_u^i(t) = P_u^{i \pm 2}(t)$ and $P_s^i(t) = P_s^{i \pm 2}(t)$, $i \in \mathbb{Z}$. We now define $\mathscr{F}_i$, $i \in \mathbb{Z}$ as in Section 2, replacing $T(t, s)$, $P_s(t)$ and $P_u(t)$ by $T^i(t, s)$, $P_s^i(t)$ and $P_u^i(t)$. From Lemma 2.2, $\mathscr{F}_i$ is Fredholm with index $\mathscr{F}_i = 0$. $\mathscr{K}\mathscr{F}_i$ and $\mathscr{K}\mathscr{F}_i^*$ is one-dimensional and $\mathscr{R}\mathscr{F}_i$ is of codimension 1. Let $\mathscr{K}\mathscr{F}^* = \{c\hat{\Psi}_1 \mid c \in \mathbb{R}\}$, $\Delta_1 \in C([-1, 0], \mathbb{R}^2)$ and $\langle \hat{\Psi}_1, \Delta_1 \rangle = 1$. Then

$$\Delta_1 \oplus \mathscr{R}\mathscr{F}_1 = C([-1, 0], \mathbb{R}^2).$$

Let $\hat{\Psi}_2 = \hat{\Psi}_1 J$ and $\Delta_2 = J\Delta_1$. It can be verified that $\hat{\Psi}_2 \in \mathcal{K}\mathcal{F}_2^*$ and $\langle \hat{\Psi}_2, \Delta_2 \rangle = 1$. Also $\Delta_2 \oplus \mathcal{R}\mathcal{F}_2 = C([-1, 0], \mathbb{R}^2)$.

Let us now define $W_i(t) = W_{i\pm 2}(t)$, $\bar{\Psi}_i = \bar{\Psi}_{i\pm 2}$, $\hat{\Psi}_i = \hat{\Psi}_{i\pm 2}$, $\Delta_i = \Delta_{i\pm 2}$ for $i \in \mathbb{Z}$. Let $\Psi_i(t) = T^*(t, \tau)\hat{\Psi}_i$ and $\psi_i(t) = \Psi_i(t)(\theta)\,|_{\theta=0}$. Here $\Psi_i(t)(\theta)$, $-1 \leqq \theta \leqq 0$ is an element in $C([-1, 0], \mathbb{R}^2)$ for each $t \in \mathbb{R}$, and $\psi_i(t)$ is a continuous function in $\mathbb{R}^2$ defined for $t \in \mathbb{R}$. $\psi_i(s) = (\psi_i^{(1)}(s), \psi_i^{(2)}(s))$ satisfies the formal adjoint equation like (5.4):

$$\dot{\psi}_i^{(1)}(t) = -r\psi_i^{(1)}(t) + rDf(p_i(t))\psi_i^{(2)}(t+1),$$
$$\dot{\psi}_i^{(2)}(t) = -r\psi_i^{(2)}(t) + rDf(q_i(t))\psi_i^{(1)}(t+1),$$

where $(p_i(t), q_i(t)) = W_i(t)$.

We now consider the following system of linear integral equations:

$$\zeta_{it} = T^i(t, \sigma)\zeta_{i\sigma} + \int_\sigma^t T^i(t, s)X_0 h_i(s)\, ds, \quad -\omega \leqq \sigma \leqq t \leqq \omega, \quad i \in \mathbb{Z},$$
$$\zeta_{(i-1)\omega} - \zeta_{i(-\omega)} = b_i, \quad i \in \mathbb{Z}, \tag{5.8}$$

where $h_i\colon [-\omega, \omega] \to \mathbb{R}^2$ is a piecewise continuous function, $\zeta_{it} \in C([-1, 0], \mathbb{R}^2)$ for each $t \in [-\omega, \omega]$, $b_i \in C([-1, 0], \mathbb{R}^2)$, X_0 is the matrix-valued jump function with $X_0(\theta) = 0$, $-1 \leqq \theta < 0$, and $X_0(0) = I$. Through $X_0 \notin C([-1, 0], \mathbb{R}^2)$, $T^i(t, s)X_0$ can still be defined through the initial value problem of the delay equation and $\int_\sigma^t T^i(t, s)X_0 h_i(s)\, ds$ is an element in $C([-1, 0], \mathbb{R}^2)$ (cf. [10]). It was shown in [15] that $P_u^i(t)X_0$ and $P_s^i(t)X_0$ can be defined, $T^i(t, s)P_s^i(s)X_0 = P_s^i(t)T^i(t, s)X_0$ holds and

$$|T^i(t, s)P_s^i(s)X_0| \leqq Ke^{-\alpha(t-s)}, \quad s \leqq t,$$
$$|T^i(s, t)P_u^i(t)X_0| \leqq Ke^{-\alpha(t-s)}, \quad s \leqq t.$$

In conclusion, all the assumptions (A_1)–(A_4) in Section 2 are valid for system (5.8) and all the uniformity assumptions with respect to $i \in \mathbb{Z}$ are trivially satisfied. Let $E([-\omega, \omega], \Delta_i)$ be the space of piecewise continuous functions: $[-\omega, \omega] \to C([-1, 0], \mathbb{R}^2)$ with jumps at $t = \tau$ along Δ_i. From Theorem 2.5 we have the following result for (5.8):

Lemma 5.4. *There is a constant $\hat{\omega} > 0$ with the following property. If $\omega > \hat{\omega}$, then there exists a unique sequence of piecewise continuous solutions $\zeta_i \in E([-\omega, \omega], \Delta_i)$, $i \in \mathbb{Z}$, for system (5.8) and*

$$\langle \bar{\Psi}_i, \zeta_{i(-\tau)} \rangle = 0, \quad i \in \mathbb{Z}. \tag{5.9}$$

Let $\zeta_{i\tau^-} - \zeta_{i\tau^+} = \xi_i \Delta_i$; then

$$\xi_i = \int_{-\omega}^{\omega} \psi_i(s)h_i(s)\, ds + \langle \Psi_i(-\omega), \zeta_{i(-\omega)} \rangle - \langle \Psi_i(\omega), \zeta_{i\omega} \rangle. \tag{5.10}$$

The piecewise continuous solution of (5.8) and (5.9) will be denoted by $\zeta_{it} = X_i(t; \mathbf{b}, \mathbf{h}, \omega)$. If $Jb_i = b_{i+1}$, then

$$\sigma^{-1}JX_i(t; \mathbf{b}, \mathbf{h}, \omega) = X_i(t; \mathbf{b}, \sigma^{-1}\{Jh_i\}_{i\in\mathbf{Z}}, \omega), \tag{5.11}$$

where $\sigma\{a_i\}_{i\in\mathbf{Z}} = \{a_{i+1}\}_{i\in\mathbf{Z}}$ is the shift mapping of sequences.

Proof. The existence of the unique solution $\{\zeta_i\}_{i\in\mathbf{Z}}$ follows from Theorem 2.5. It is also clear that

$$\xi_i = \int_{-\omega}^{\omega} \langle \Psi_i(s), X_0 h_i(s)\rangle\, ds + \langle \Psi_i(-\omega), \zeta_{i(-\omega)}\rangle - \langle \Psi_i(\omega), \zeta_{i\omega}\rangle.$$

Since $\langle \Psi_i(s), X_0\rangle = \Psi_i(s)(\theta)\,|_{\theta=0} = \psi_i(s)$, (5.10) follows. (5.11) can be proved by applying J to (5.8) and (5.9). Details will be left to the reader.

Proof of Theorem 5.2. We shall look for period 4ω solution $Z(t)$ for (5.5), with the property $Z(t+2\omega) = JZ(t)$, $-\omega \leqq t \leqq \omega$. Set $Z(t) = W_1(t) + \zeta_1(t)$ and $Z(t+2\omega) = W_2(t) + \zeta_2(t)$ for $-\omega \leqq t \leqq \omega$. Then $\zeta_2(t) = J\zeta_1(t)$. Define $\zeta_i(t) = \zeta_{i\pm 2}(t)$, $i \in \mathbb{Z}$. ζ satisfies a system of boundary value problems

$$\begin{cases} \dot{\zeta}_i(t) = r_0\zeta_i(t) - r_0 DF(W_i(t-1))\zeta_i(t-1) + N_i(\zeta_i(t-1), r, t-1), & -\omega \leqq t \leqq \omega, \\ \zeta_{(i-1)\omega} - \zeta_{i(-\omega)} = b_i, \quad i \in \mathbb{Z}, \end{cases} \tag{5.12}$$

where $b_i = W_{i(-\omega)} - W_{(i-1)\omega} \in C([-1, 0], \mathbb{R}^2)$, $N_i(\zeta, r, t) = -rF(W_i(t) + \zeta) + r_0F(W_i(t)) + r_0DF(W_i(t))\zeta + (r - r_0)(W_i(t) + \zeta) = O(|r - r_0| + |\zeta|^2)$. We also require (5.9) for each $i \in \mathbb{Z}$. We can rewrite (5.12) as a system of integral equations,

$$\begin{aligned} &\zeta_{it} = T^i(t, \sigma)\zeta_{i\sigma} + \int_\sigma^t T^i(t, s)X_0 N_i(\zeta_{is}(-1), r, s-1)\, ds, \quad -\omega \leqq t \leqq \omega, \\ &\zeta_{(i-1)\omega} - \zeta_{i(-\omega)} - b_i, \quad i \in \mathbb{Z}. \end{aligned} \tag{5.13}$$

From Lemma 5.4, system (5.13) and (5.9) is equivalent to

$$\zeta_{it} = X_i(t; \mathbf{b}, \{N_i(\zeta_{i\cdot}(-1), r, \cdot - 1)\}_{i\in\mathbf{Z}}, \omega), \quad i \in \mathbb{Z}. \tag{5.14}$$

On the other hand, for each given $\omega > \hat{\omega}$ and $|r - r_0| < \mu_1$, we shall try to find fixed points $|\zeta_i| < \varepsilon_1$, $i \in \mathbb{Z}$ for (5.14). Here $\hat{\omega}$ is sufficiently large, $\mu_1 > 0$ and $\varepsilon_1 > 0$ are sufficiently small. (5.14) can be solved by the uniform contraction principle in $\prod_{i\in\mathbf{Z}} E([-\omega, \omega], \Delta_i)$. The unique solution ζ_{it} denoted by $\mathscr{Z}_i(t; \omega, r)$ is C^k in r for each fixed ω. Let us now apply $\sigma^{-1}J$ to both sides of (5.14). From (5.11) and the fact that

$$\sigma^{-1}JN_i(\beta_i, r, t) = N_i(\sigma^{-1}J\beta_i, r, t),$$

we then have

$$\sigma^{-1}J\zeta_{it} = X_i(t; \mathbf{b}, \{N_i(\sigma^{-1}J\zeta_{i\cdot}(-1), r, \cdot - 1)\}_{i\in\mathbf{Z}}, \omega), \quad i \in \mathbb{Z}.$$

The uniqueness of the fixed point of the contraction map implies that

$$J\zeta_{it} = \zeta_{(i+1)t}, \quad i \in \mathbb{Z}.$$

Let $\zeta_{i\tau^-} - \zeta_{i\tau^+} = \xi_i\Delta_i$. It is not hard to verify that $\xi_i = \xi_{i+1}$ for all $i \in \mathbb{Z}$, based on (5.10) and $h_i(s) = N_i(\zeta_{is}(-1), r, s-1)$, with the property $Jh_i(s) = h_{i+1}(s)$. Denote ξ_i, $i \in \mathbb{Z}$ by $G(\omega, r)$:

$$G(\omega, r) = \int_{-\omega}^{\omega} \psi_1(s)N_1(\zeta_{1s}(-1), r, s-1)\, ds + \langle \Psi_1(-\omega), \zeta_{1(-\omega)}\rangle - \langle \Psi_1(\omega), \zeta_{1\omega}\rangle.$$

The proof of the existence of a unique solution $r = r^*(\omega)$ of $G(\omega, r) = 0$ for $\omega > \hat{\omega}$, $|r - r_0| < \mu_1$ follows closely the proof of Theorem 4.3. Here Lemmas 3.2 and 3.3, which are also valid for delay equations, play important roles. Note also from (D_4) that

$$\frac{\partial G(\infty, r_0)}{\partial r} = \int_{-\infty}^{\infty} \{\psi_1^{(1)}(t)(p_1(t) - f(q_1(t-1)) + \psi_1^{(2)}(t)(q_1(t) - f(p_1(t-1)))\}\,dt$$
$$= \frac{1}{r_0}\int_{-\infty}^{\infty} [\psi_1^{(1)}(t)\dot{p}_1(t) + \psi_1^{(2)}(t)\dot{p}_2(t)]\,dt \neq 0.$$

The function $r^*(\omega)$ so obtained is continuous for $\omega > \hat{\omega}$. Details will not be given here.

In order to show that $r^*(\omega)$ is C^k, it suffices to prove that $G(\omega, r)$ is jointly C^k for $\omega > \hat{\omega}$ and $|r - r_0| < \mu_1$. Here again we use the geometric interpretation of $G(\omega, r)$, and the scaling of time used in the proof of Theorem 4.2. First, we define a pair of functions in $\mathbb{R}^2$, $\zeta_1^-(t)$, $t \in [-\omega - 1, -\tau]$ and $\zeta_1^+(t)$, $t \in [\tau - 1, \omega]$.

$$\zeta_1^+(t+\theta) = \zeta_{1t}(\theta), \quad t \in (\tau, \omega], \quad \theta \in [-1, 0],$$
$$\zeta_1^-(t+\theta) = \zeta_{1t}(\theta), \quad t \in [-\omega, \tau), \quad \theta \in [-1, 0].$$

The semi-open intervals can be made closed by taking the right and left limits. Let $\zeta_1(t) = \zeta_1^-(t)$ for $t \in [-\omega - 1, \tau]$ and $\zeta_1^+(t)$ for $t \in [\tau - 1, \omega]$. $\zeta_1(t)$ is bivalued on $[\tau - 1, \tau]$ and has two branches, each of which can be reached by continuous extension from the right or left. Let $Z(t) = W_1(t) + \zeta_1(t)$ and $Z(t + 2\omega) = JZ(t)$, for $t \in [-\omega, \omega]$. We then extend $Z(t)$ as a 4ω periodic function for $t \in \mathbb{R}$. It is clear from the construction that $Z(t)$ is a piecewise continuous function bivalued on the intervals $[2j\omega + \tau - 1, 2j\omega + \tau]$, $j \in \mathbb{Z}$. The jumps on the two branches are

$$Z^-_{2j\omega+\tau} - Z^+_{2j\omega+\tau} = G(\omega, \mu)\Delta_i \in C([-1, 0], \mathbb{R}^2),$$

where $i = 1$ if j is even and $i = 2$ if j is odd. It is also clear that each continuous branch of $Z(t)$ satisfies equation $(5.5)_r$. Consequently we infer that $Z(t)$ is C^{k+1} on each interval $[2j\omega + \tau + k, 2j\omega + 2\omega + \tau - 1]$.

To rescale the time, let us define $\nu(t) \in C^\infty(\mathbb{R})$ as follows: $\nu(t) = 0$ for $t \in [0, \tau + k]$, $\nu(t) = 1$ for $t \in [\tau + k + 1, \infty)$ and $\nu(-t) = \nu(t)$ for $t \in \mathbb{R}$. Let $\bar{Z}(t) = Z((1 + \beta\nu(t))t)$ and $\bar{Z}(t + 2\omega_1) = J\bar{Z}(t)$ for $t \in [-\omega_1, \omega_1]$. Here $\omega = (1 + \beta)\omega_1$, ω_1 is fixed and β is in a neighbourhood of 0. We then extend $\bar{Z}(t)$ to be a piecewise continuous, $4\omega_1$ periodic function which is bivalued on intervals $[2j\omega_1 + \tau - 1, 2j\omega_1 + \tau]$, $j \in \mathbb{Z}$. Each continuous branch of $\bar{Z}(t)$ satisfies a delay equation. For example, if $t \in [-\omega_1, \omega_1]$, we have

$$\frac{d}{dt}\bar{Z}(t) = (1 + \beta(\nu(t) + t\dot{\nu}(t)))[r\bar{Z}(t) - rF(\bar{Z}(t - d(\beta, t)))], \qquad (5.15)$$

where $d(\beta, t) = (1 + \beta\nu(t))^{-1}$ is the time dependent delay. Here we also have to be careful in choosing the correct branches when $t \in [\tau - 1, \tau]$.

The advantage for the solution $\bar{Z}(t)$ is that the period $4\omega_1$ is fixed, and β appears to be a parameter. We can use the previous method to show that for each $|\beta| < \beta_1$ and $|r - r_0| < \mu_1$, there exists a unique piecewise continuous solution for (5.15) satisfying $\bar{Z}_{\omega_1} + \bar{W}_{1\omega_1} = J(\bar{Z}_{(-\omega_1)} + \bar{W}_{1(-\omega_1)})$, where $\bar{W}_1$ is rescaled from W_1, with a jump $\tilde{Z}_\tau^- - \tilde{Z}_\tau^+ = \tilde{G}(\beta, r)\Delta_1$ between two branches on the interval $[\tau-1, \tau]$, and satisfies a phase condition at $t = -\tau$. The function $\tilde{G}(\beta, r)$ is C^k jointly in (β, r); however, we shall not give the proof here. Interested readers may refer to [**10**, Chapter 10, Theorem 2.2] and [**11**] for a similar problem. Since $Z(t) = \tilde{Z}(t)$ for $t \in [\tau-1, \tau]$, we have

$$G(\omega, \mu) = \tilde{G}(\beta, \mu) = \tilde{G}\left(\frac{\omega}{\omega_1} - 1, \mu\right),$$

which is C^k jointly in (ω, μ).

Proof of Theorem 5.3. If $(x(t), y(t))$ is a 4ω periodic solution of $(5.2)_r$, then $r = r^*(\omega)$ and $z(t) = x(-t/\varepsilon r)$ is a $2 + 2\varepsilon r$ periodic solution of (5.1). From $4\varepsilon r\omega = 2 + 2\varepsilon r$, $\varepsilon = [(2\omega - 1)r^*(\omega)]^{-1}$. We want to show that ε approaches zero monotonically as $\omega \to \infty$.

Let $\omega_2 > \omega_1 > \hat{\omega}$, $\zeta_{it}^{(1)} = \mathscr{Z}_i(t; \omega_1, r)$ and $\zeta_{it}^{(2)} = \mathscr{Z}_i(t; \omega_2, r)$. Define $b_i^{(j)} = \zeta_{(i-1)\omega_1}^{(j)} - \zeta_{i(-\omega_1)}^{(j)}$ for $i \in \mathbb{Z}$, $j = 1, 2$. In a similar way to the proof of Theorem 3.4, we can show that $|b_i^{(1)} - b_i^{(2)}| = 0(\omega_2 - \omega_1)$. From Lemma 2.6, (2.16), it follows that $\|\zeta_i^{(1)} - \zeta_i^{(2)}\|_\eta = O(\omega_2 - \omega_1)$. Here $\| \ \|_\eta$ is the weighted norm on $E([-\omega_1, \omega_1], \Delta_i)$.

We now obtain an estimate on $G(\omega_2, r) - G(\omega_1, r)$. First,

$$\left|\int_{-\omega_2}^{\omega_2} \psi_1(s) N_1(\zeta_{1s}^{(2)}(-1), r, s-1)\, ds - \int_{-\omega_1}^{\omega_1} \psi_1(s) N_1(\zeta_{1s}^{(1)}(-1), r, s-1)\, ds\right|$$

$$\leqq Ce^{-\alpha\omega_1}(\omega_2 - \omega_1) + C\int_{-\omega_1}^{\omega_1} |\psi_1(s)|\, |\zeta_{1s}^{(2)} - \zeta_{1s}^{(1)}|\, ds$$

$$\leqq Ce^{-\alpha\omega_1}(\omega_2 - \omega_1) + C\int_{-\omega_1}^{\omega_1} e^{-\alpha|s|}\, \|\zeta_1^{(2)} - \zeta_1^{(1)}\|_\eta (e^{-\eta(\omega_1+s)} + e^{-\eta(\omega_1-s)})\, ds$$

$$\leqq Ce^{-\eta\omega_1}(\omega_2 - \omega_1).$$

It is also clear that

$$|\langle \Psi_1(-\omega_2), \zeta_{1(-\omega_2)}^{(2)}\rangle - \langle \Psi_1(\omega_2), \zeta_{1\omega_2}^{(2)}\rangle - \langle \Psi_1(-\omega_1), \zeta_{1(-\omega_1)}^{(1)}\rangle + \langle \Psi_1(\omega_1), \zeta_{1\omega_1}^{(1)}\rangle|$$

$$\leqq C(\omega_2 - \omega_1)\left(\sup_{|\omega| \geqq \omega_1} (|\dot{\Psi}_1(\omega)| + |\Psi_1(\omega)|)\right) \leqq Ce^{-\eta\omega_1}(\omega_2 - \omega_1).$$

Thus

$$|G(\omega_2, r) - G(\omega_1, r)| \leqq Ce^{-\eta\omega_1}(\omega_2 - \omega_1). \tag{5.16}$$

Let $\omega_1 + \frac{1}{2} > \omega_2 > \omega_1 > \hat{\omega}$, and $r_2 = r^*(\omega_2)$, $r_1 = r^*(\omega_1)$. If $\hat{\omega}$ is sufficiently large, we have $r_1 > \frac{r_0}{2}$. This is due to the fact that $r^*(\omega) \to r_0$ as $\omega \to \infty$, as can be seen from a FDE analogue of (4.8).

From $G(\omega_2, r_2) - G(\omega_1, r_1) = 0$, we have

$$\begin{aligned} Ce^{-\eta\omega_1}(\omega_2 - \omega_1) &\geqq |G(\omega_2, r_2) - G(\omega_1, r_2)| \\ &\geqq |G(\omega_1, r_2) - G(\omega_1, r_1)| \\ &\geqq C_1 |r_2 - r_1|. \end{aligned}$$

The last inequality is due to a FDE analogue of Lemma 3.2. Thus, $|r_2 - r_1| \leqq Ce^{-\eta\omega_1}(\omega_2 - \omega_1)$. We then have

$$(2\omega_2 - 1)r_2 - (2\omega_1 - 1)r_1 = 2(\omega_2 - \omega_1)r_1 + (2\omega_2 - 1)(r_2 - r_1)$$
$$\geqq (\omega_2 - \omega_1)r_0 - 2\omega_1 \,.\, Ce^{-\eta\omega_1}(\omega_2 - \omega_1) > 0,$$

provided $\hat{\omega}$ is sufficiently large. It is now clear that for every $\varepsilon > 0$, sufficiently small, there is a unique $\omega > \hat{\omega}$ such that $\varepsilon = [(2\omega - 1)r^*(\omega)]^{-1}$. □

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