NOTES

EDITED BY DAVID J. HALLENBECK, DENNIS DETURCK, AND ANITA E. SOLOW

Another Brief Proof of the Sylvester Theorem

X. B. LIN
Mathematics Department, Michigan State University, East Lansing, MI 48824

A finite set of points, S, in a projective or affine space such that no lines intersects S in exactly two points is known as a Sylvester-Gallai (SG) configuration. In real space it is well known that there are no nonlinear SG's. There are several simple proofs of this fact (see **[1]**–**[4]**) and we offer here still another, as far as we know somewhat different from the others. The result in E^n follows from that in E^2 by projection.

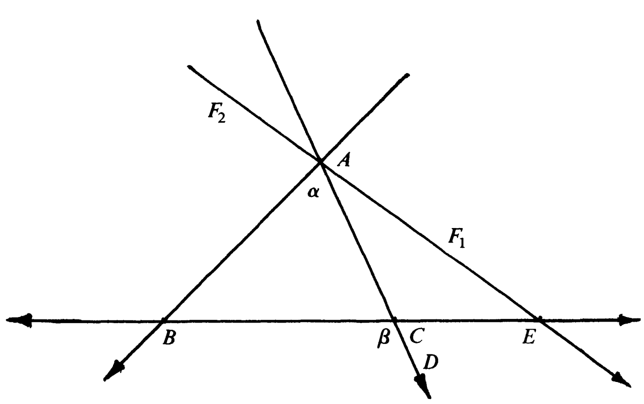
THEOREM. *If S is a finite set of points in E^2 such that no line intersects S in precisely two points, the S is a subset of a line.*

Proof. We use Hilbert's definition of angle as the union of two noncollinear rays with a common end point. If A, B, and C are in S, then the angle defined by rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted BAC or CAB. If $\overrightarrow{AB} \cup \overrightarrow{AC}$ contains a fourth point of S, the angle is called an admissible angle (relative to S). Suppose now that S is not linear. Then the angles at the vertices of the convex hull of S are certainly admissible so the set of such angles is not empty. Suppose BAC is the largest such angle with angle measure $\alpha < 180$ and that D is a fourth point of S on ray $\overrightarrow{AC}$. There is no loss of generality in assuming that C is between A and D.

Now BCD is larger than BAC so it is not an admissible angle. Thus ray $\overrightarrow{CB}$ cannot contain a third point of S. But the line $\overrightarrow{CB}$, by assumption, must contain another point E of S so E must be on the ray opposite to $\overrightarrow{CB}$.

But now the line AE presents us with a contradiction, since it must contain a third point F of S, and if F is on ray $\overrightarrow{AE}$, then BAE is admissible and larger than BAC; while if F is on the ray opposite to $\overrightarrow{AE}$, then FAC is admissible and greater than BAC.

This contradiction shows that S must be linear.



REFERENCES

1. H. S. M. Coxeter, Int. to Geom., 2nd ed., 1969, John Wiley and Sons, N.Y., pp. 65–66.
2. L. M. Kelly and W. O. S. Moser, On the number of ordinary lines determined by n points, *Canadian Jour. of Math.*, (10) 1958, 213.
3. Editorial Note, *Amer. Math. Monthly*, 51 (1944) 170–171.
4. G. D. Chakerian, Sylvester's Problem on Collinear Points and a Relative, *Amer. Math. Monthly*, 77 (1970) 164–167.

$A \geqq B \geqq 0$ Ensures $(BA^2B)^{1/2} \geqq B^2$— Solution to a Conjecture on Operator Inequalities

TAKAYUKI FURUTA
Department of Mathematics, Faculty of Science, Hirosaki University, Bunkyo-cho 3, Hirosaki 036, Aomori, Japan

DEDICATED TO PROFESSOR ZIRÔ TAKEDA WITH RESPECT AND AFFECTION

In an issue of this MONTHLY [**1**, p. 539], the following conjecture is stated.

CONJECTURE. *Let A and B be hermitian matrices on a finite dimensional Euclidean space. If* $A \geqq B \geqq 0$, *then*

$$(BA^2B)^{1/2} \geqq B^2$$

and

$$A^2 \geqq (AB^2A)^{1/2}.$$

In this short note, we shall prove this conjecture in a more general form.

THEOREM 1. *If A and B are positive bounded hermitian linear operators on a Hilbert space such that* $A \geqq B \geqq 0$, *then*

$$(BA^2B)^{1/2} \geqq B^2 \tag{1}$$

and

$$A^2 \geqq (AB^2A)^{1/2}. \tag{2}$$

We prove the following Lemma needed for Theorem 1.

LEMMA. *If A and B are positive bounded hermitian linear operators on a Hilbert space such that* $A \geqq B \geqq 0$, *then*

$$(B^{1/2}A^3B^{1/2})^{1/2} \geqq B^2 \tag{i}$$

and

$$(B^{1/2}A^2B^{1/2})^{1/3} \geqq B. \tag{ii}$$

We quote the following result to show the Lemma.

THEOREM A ([**2**]). *Let X and Y be bounded linear operators on a Hilbert space H. We suppose that* $X \geqq 0$ *and* $\|Y\| \leqq 1$. *If f is an operator monotone function defined on*