MULTIPLE INTERNAL LAYER SOLUTIONS GENERATED BY SPATIALLY OSCILLATORY PERTURBATIONS

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ABSTRACT. For a singularly perturbed system of reaction–diffusion equations, we study the bifurcation of internal layer solutions due to the addition of a spatially oscillatory term. In the singular limit, the existence and stability of internal layer solutions are determined by the intersection of a **fast jump surface** Γ_1 and a **slow switching curve** $\mathcal{C}$. The case when the intersection is transverse was studied in [26]. In this paper, we show that when Γ_1 intersects with $\mathcal{C}$ tangentially, saddle-node or cusp type bifurcation may occur. Higher order expansions of internal layer solutions and eigenvalue–eigenfunctions are also presented. To find a true internal layer solution and true eigenvalue–eigenfunctions, we use a Newton's method in functions spaces that is suitable for numerical computations.

1. INTRODUCTION

There have been many studies of internal layer solutions in reaction diffusion systems for which the reaction term is independent of the spatial variables. Our objective here is to continue the work of [26] on the effect of spatial dependence in the reaction term by considering single layer solutions for a system of two equations modeling an activator inhibitor. To put the results in the context of existing literature, it is worthwhile to recall known results for a single equation

$$
\begin{aligned}
&u_t = \epsilon^2 u_{xx} + (1-u^2)(u-a(x)), \quad x \in (0,1),\\
&u_x = 0 \quad x = 0, 1,
\end{aligned}
\tag{1.1}
$$

where $0 < a(x) < 1$ is a C^1-function.

If $a = 0$ for all x, it is known that the only stable solutions of (1.1) are the constant functions ± 1. In [1], the authors have considered the case in which a is a function of x which assumes the value 0 at points $x_j \neq 0, 1$, $a'(x_j) \neq 0$, $1 \leq j \leq M$, $a'(0) \neq 0, a'(1) \neq 0$. They proved that there is an $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, (1.1) has the M^{th} Fibonacci number of exponentially stable solutions with sharp transition layer only at points from the set $\{x_1, \dots, x_M\}$. Furthermore, the dominant eigenvalue $\lambda(\epsilon)$ of a solution is $\lambda(\epsilon) = \epsilon\lambda_1 + O(\epsilon^2)$ with $\lambda_1 < 0$. The existence of stable solutions was obtained by constructing upper and lower solutions to obtain an invariant region and then invoke a result from Matano [27] on the existence of a stable solution. The fact that there were exactly the M^{th} Fibonacci number of stable solutions required some asymptotics and explicit calculation of the dominant

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eigenvalue. In [13], the authors used asymptotic methods to prove the existence of unstable solutions of the above type and calculated the asymptotic form of the positive eigenvalues.

In particular, the function $a(x) = \delta \sin \omega x$, $\delta \neq 0$, $\omega \neq (2m+1)\pi/2$, $m = 0, 1, 2 \dots$, satisfies the above properties. The number δ can be chosen to be as small as desired so that $(1-u^2)(u-a(x))$ is a small perturbation of $(1-u^2)u$ on any compact set. On the other hand, the number $\epsilon_0 = \epsilon_0(\delta)$ may have to go to zero as $\delta \to 0$. In this sense, δ is not a small perturbation uniformly in ϵ.

It is not known how these stable layered solutions occur through bifurcation from the zero function. However, if one is interested in one-layer solutions, the situation is much simpler. Let $a(x) = \delta \sin \omega x + \mu$ and let μ pass through δ (or $-\delta$), the zeroes of $a(x)$ are created pairwise through generic saddle-node type bifurcations. It is natural to expect that the monotonically increasing (or decreasing) one-layer solutions are created through saddle-node type bifurcations. This is indeed true and can be verified by the method employed in this paper.

A special case of an activator inhibitor system is

$$
\begin{aligned}
u_t &= \epsilon^2 u_{xx} + (1-u^2)(u-a) - y \\
y_t &= \frac{1}{\sigma} y_{xx} + (\delta u - y), \quad x \in (0,1)
\end{aligned}
\tag{1.2}
$$

with homogeneous Neumann boundary conditions, where $a \in (-1, 0)$, $\epsilon > 0, \sigma > 0, \delta > 0$ are constants. Suppose that δ is so that there is only one spatially independent solution of (1.2) (see Figure 3.1). For ϵ sufficiently large and σ sufficiently small, every solution of (1.2) approaches this constant solution. On the other hand, there are ϵ_0, σ_0 at which this solution bifurcates to a stable spatially dependent solution. This was observed by Turing in the seminal paper [35]. This solution also has a steep transition layer in u as ϵ becomes small (see [30]). The existence of these internal layer solutions can be traced to earlier papers [8, 10, 14, 28].

As in [1], it is reasonable to study the sensitivity of these single layer solutions when the vector field is subjected to a spatially dependent perturbation. As in [26], we choose the perturbation in the following way:

$$
\begin{aligned}
u_t &= \epsilon^2 u_{xx} + (1-u^2)(u-a) - y - \frac{k}{\omega}\sin(\omega x + b) \\
y_t &= \frac{1}{\sigma} y_{xx} + (\delta u - y), \quad x \in (0,1)
\end{aligned}
\tag{1.3}
$$

with the homogeneous Neumann boundary conditions. The parameters are $k > 0$, $\omega > 0$ and $b \in \mathbb{R}$. If k is in a compact set and ω is large, this can be considered as a small perturbation of the vector field in (1.2). However, as we will note below, this may not correspond to a small change in the dynamics of (1.3) uniformly in ϵ.

Under generic assumptions on the parameters k and b, and conditions on the largeness of ω and smallness of ϵ, it was shown in [26] that there can be many one layer solutions with some being exponentially stable and some unstable with index one. Asymptotic methods were used and the asymptotic form of the dominant eigenvalue was given as $\lambda(\epsilon) = \epsilon\lambda_1 + O(\epsilon^2)$ with $\lambda_1 \neq 0$.

The purpose of this paper is to complete the study of [26] by showing how these solutions appear through saddle node bifurcations and cusp bifurcations. In Section 2, we give an intuitive explanation of the results for the special case (1.3). The precise statement of the results is given in Section 3 for a system more general than (1.3) which satisfies the same hypotheses as in [30]. The remaining sections contain the detailed proof and are based on geometric methods in asymptotics.

Throughout the present paper, we use the following notations. $M_1 \pitchfork M_2$ means a nonempty and transverse intersection of two manifolds M_1 and M_2. The tangent space of a C^1 manifold $\mathcal{M}$ at a point $\wp \in \mathcal{M}$ is denoted $T_\wp\mathcal{M}$. For a piecewise continuous function $f(x)$ defined in a neighborhood of x_0, let $f(x_0+) = \lim_{x\to x_0+0}$, $f(x_0-) = \lim_{x\to x_0-0}$ and $[f](x_0) = f(x_0+) - f(x_0-)$. Let $\mathcal{P}_j(\mathbb{R})$, $\mathcal{P}_j(\mathbb{R}^-)$ and $\mathcal{P}_j(\mathbb{R}^+)$ be Banach spaces of continuous functions f defined on $\mathbb{R}$, $\mathbb{R}^-$ and $\mathbb{R}^+$ respectively and satisfy the following growth condition,

$$|f(\xi)| \le C(1 + |\xi|^j).$$

The norms are the weighted norms $|f| = \sup\{|f(\xi)(1 + |\xi|^j)^{-1}|,\ x \in \mathbb{R},\text{ or } \mathbb{R}^\pm\}$. When a system of linear equations has an exponential dichotomy on an interval I, the projection to stable and unstable spaces are denoted respectively by $P_s(t)$ and $P_u(t)$, $t \in I$. The definition of exponential dichotomies and some basic lemmas are presented in §5. An introduction to exponential dichotomies and their role in homoclinic bifurcation theory can be found in [6, 32].

2. Intuition for a special case

To understand how the spatial dependence of the vector field influences one layer solutions, it is worthwhile to briefly review the construction of such solutions for the spatially independent case (1.2). If we let

$$f(u) = (1 - u^2)(u - a)$$

and if y is a real number, then the cubic equation $f(u) - y = 0$ defines three curves $u = h_-(y), h_0(y), h_+(y)$ where respectively the derivative is negative, positive, negative (see Figure 3.1). The curves $u = h_\pm(y)$ are stable as solutions of the equation

$$u_t = \epsilon^2 u_{xx} + f(u) - y \tag{2.1}$$

with $u_x = 0$ at $x = 0, 1$ and the curve $u = h_0(y)$ is unstable.

For ϵ small and a special $\tilde{y}$, there is a stationary solution $u^\epsilon(x)$ which has a transition layer near x_0 where $y(x_0) = \tilde{y}$ and goes approximately from $h_-(\tilde{y})$ to $h_+(\tilde{y})$. It is unstable with index 1 and there is a positive constant $c(\tilde{y})$ such that the positive eigenvalue $\lambda(\tilde{y}) = O(e^{-c(\tilde{y})/\epsilon})$ as $\epsilon \to 0$. It is unstable but the degree of instability is exponentially small.

The construction of such an approximation of $u^\epsilon(x)$ and the determination of $\tilde{y}$ proceeds as follows. If $\xi = (x - x_0)/\epsilon$, $v(\xi) = u(x_0 + \epsilon\xi)$, then

$$v_{\xi\xi} + f(v) - y = 0, \quad \xi \in \left(-\frac{x_0}{\epsilon}, \frac{1 - x_0}{\epsilon}\right), \tag{2.2}$$

with $v_\xi = 0$ at the boundaries. For the ODE (2.2), with $\xi \in (-\infty, \infty)$, there is a unique $\tilde{y}$ such that there is a heteroclinic orbit v^0 of (2.2), with $y = \tilde{y}$, going from

$h_-(\tilde{y})$ to $h_+(\tilde{y})$. The constant $\tilde{y}$ is chosen so that the cubic function $f(v) - \tilde{y}$ is symmetric about the middle zero $h_0(\tilde{y})$; that is, the area under the curve described by the graph of $f(u)$ from $h_-(\tilde{y})$ to $h_0(\tilde{y})$ is equal to the area from $h_0(\tilde{y})$ to $h_+(\tilde{y})$ (*the equal area rule*).

From symmetry, the solution should be symmetric about zero and so we take $x_0 = 1/2$. The function $u_0^\epsilon(x) = v^0(\frac{x-0.5}{\epsilon})$, $x \in (0,1)$, should be an approximate solution to (2.1). One can now show that there is an exact solution $u^\epsilon(x)$ of (1.2) with

$$u^\epsilon(x) - u_0^\epsilon(x) = O(\epsilon), \quad u_0^\epsilon(0.5) = h_0(\tilde{y}).$$

A more difficult analysis shows that $u^\epsilon(x)$ is unstable with index 1 and the positive eigenvalue is $O(e^{-\tilde{c}(\tilde{y})/\epsilon})$ as $\epsilon \to 0$ (see [4, 11]).

If we now look at the coupled system of equations (1.2), then we do not expect that an equilibrium solution will have y remain at the constant value $\tilde{y}$ since y should satisfy approximately

$$y_{xx} + \sigma(\delta h_\pm(y) - y) = 0, \quad x \in (0,1) \tag{2.3}$$

$y_x(0) = y_x(1) = 0$, with the $-$ sign being used on the interval $(0, x_0)$ and the $+$ sign on $(x_0, 1)$. As we note below, the constant x_0 must be determined and is related to the equal area rule. It can not be specified a priori since we do not expect to preserve the symmetry as for the scalar equation.

The solution y of (2.3) is a C^2 function except at the point x_0, where there is a jump in the second derivative. The sign of $y_{xx}(x)$ assures that $y(x)$ is concave up for $x < x_0$ and concave down for $x > x_0$. The shape of the solution is plotted in Figure 3.2. We need to determine the point x_0 and the value $(y(x_0), y_x(x_0))$. Once $(y(x_0), y_x(x_0), x_0)$ is known, the solution y is determined by integrating (2.3) backward in $[0, x_0]$ and forward in $[x_0, 1]$.

From the equal area rule it is clear that

$$y(x_0) = \tilde{y} \tag{2.4}$$

in order to obtain a fast jump in the u equation. Define

$$\Gamma_1 = \{(y, z, x) \in \mathbb{R}^3 : y = \tilde{y}\}. \tag{2.5}$$

We must have $(y(x_0), y_x(x_0), x_0) \in \Gamma_1$ in order to obtain an approximate one layer solution of system (1.2).

The solution of (2.3) has to satisfy the boundary conditions at $x = 0, 1$. This also imposes some restriction on the point $(y(x_0), y_x(x_0), x_0)$. To give a geometric description of this restriction, let us rewrite (2.3) as

$$\begin{aligned} \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= \sigma(\delta h_\pm(y) - y) \\ \frac{dx}{dt} &= 1 \end{aligned} \tag{2.6}$$

and consider the solutions of (2.6) in $\mathbb{R}^3$. Notice that there are two vector fields in (2.6), one is defined with $h_-(y)$ and the other with $h_+(y)$. To have a solution that satisfies $z(0) = z(1) = 0$, it is natural to integrate the equation with $h_-(y)$ with initial value at $t = 0$ given by $(\alpha, 0, 0)$, $\alpha \in \mathbb{R}$ to obtain a two dimensional surface $\mathcal{M}_- \subset \mathbb{R}^3$ and to integrate the equation with $h_+(y)$ with initial value at 1 given by $(\beta, 0, 1)$, $\beta \in \mathbb{R}$, to obtain a two dimensional surface $\mathcal{M}_+ \subset \mathbb{R}^3$. It is shown [26] that $\mathcal{M}_-$ is transversal to $\mathcal{M}_+$. The curve $\mathcal{C} = \mathcal{M}_- \cap \mathcal{M}_+$ gives all of the possible switching points between the two vector fields of (2.6) if the boundary conditions $z(0) = z(1) = 0$ are to be satisfied. It is shown [26] that $\mathcal{C}$ has the form

$$\mathcal{C} = \{(y, z, x) : x = x^*(y), z = z^*(y), |y - \tilde{y}| \leq \eta\}$$

where $\eta > 0$ is a small constant and $\frac{\partial x^*}{\partial y} < 0$.

If we can find a point $(y(x_0), z(x_0), x_0) \in \mathcal{C} \cap \Gamma_1$, then we obtain an approximate equilibrium solution $(u^\epsilon(x), y^\epsilon(x))$ of (1.2). To obtain an exact solution, one uses the fact that $\mathcal{C}$ is transversal to the set Γ_1 (see [30, 26]).

The final part of the analysis involves showing that the exact solution is stable by showing that the dominant eigenvalue of this solution is $\lambda(\epsilon) = \epsilon\lambda_1 + O(\epsilon^2)$ as $\epsilon \to 0$ and $\lambda_1 < 0$.

Now let us study the existence of one layer solutions when the vector field is subjected to a small oscillatory perturbation; more precisely,

$$\begin{aligned} u_t &= \epsilon^2 u_{xx} + (1 - u^2)(u - a) - y - \frac{k}{\omega}\sin(\omega x + b) \\ y_t &= \frac{1}{\sigma} y_{xx} + (\delta u - y), \quad x \in (0, 1) \end{aligned} \tag{2.7}$$

with homogeneous Neumann boundary conditions. The parameters k, ω, b are constant with $k \geq 0$, $\omega > 0$.

We seek solutions with a transition layer in u near some point x_0 to be determined. For k in a compact set and ω large, this represents a small perturbation which is rapidly oscillating. However, as ϵ becomes small, this does not correspond to a small perturbation in the dynamics in the same way as we have noted in the discussion of (1.1) and the results in [1].

We proceed in the same way as remarked above for (2.1). We consider the equation

$$\begin{aligned} \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= \sigma(\delta h_\pm(y + \frac{k}{\omega}\sin(\omega x + b)) - y) \\ \frac{dx}{dt} &= 1 \end{aligned} \tag{2.8}$$

which defines two vector fields in $\mathbb{R}^3$. To obtain solutions that satisfy boundary conditions at $x = 0, 1$, the two dimensional surfaces $\mathcal{M}_\pm^{(k,\omega,b)}$ and the curve

$$\mathcal{C}^{(k,\omega,b)} = \mathcal{M}_-^{(k,\omega,b)} \cap \mathcal{M}_+^{(k,\omega,b)}$$

are used as before. The curve $\mathcal{C}^{(k,\omega,b)}$ is close to the curve $\mathcal{C}^{(0,\omega,b)} = \mathcal{C}$ to order $O(1/\omega)$ in the C^1 topology. Thus, the tangent vector to $\mathcal{C}^{(k,\omega,b)}$ is almost the same as the tangent vector of $\mathcal{C}$. The switching point $(y(x_0), z(x_0), x_0)$ between the two vector fields must lie on $\mathcal{C}^{(k,\omega,b)}$ so that the boundary values can be satisfied.

To obtain the approximate value of x_0 for which the u equation of (2.7) can have a fast jump from $h_-(\tilde{y})$ to $h_+(\tilde{y})$, we need to have

$$(y(x_0), z(x_0), x_0) \in \Gamma_1^{(k,\omega,b)} = \{(y, z, x) : y + \frac{k}{\omega}\sin(\omega x + b) = \tilde{y}\}, \tag{2.9}$$

due to the equal area rule again. We see that the switching point is on $\mathcal{C}^{(k,\omega,b)} \cap \Gamma_1^{(k,\omega,b)}$.

By a perturbation argument, we prove that the curve $\mathcal{C}^{(k,\omega,b)}$ can be represented as

$$\mathcal{C}^{(k,\omega,b)} = \{(y, z, x) : x = x^*(y, k, \omega, b), z = z^*(y, k, \omega, b), |y - \tilde{y}| \le \eta\}$$

where $\partial x^*(y, k, \omega, b)/\partial y < 0$. In fact, if ω is large, $\mathcal{C}^{(k,\omega,b)}$ is C^1 close to $\mathcal{C}$ and is monotone. However, the surface $\Gamma_1^{(k,\omega,b)}$ is oscillatory and therefore can have many intersections with $\mathcal{C}^{(k,\omega,b)}$ if k is sufficiently large. Most of these intersections are transversal intersections for which the existence of exact solutions near these was given in [26].

Our objective here is to prove that the nontransversal intersections of $\Gamma_1^{(k,\omega,b)}$ and $\mathcal{C}^{(k,\omega,b)}$ either occur with a quadratic tangency (corresponding to an exact saddle-node bifurcation of one layer solutions) or a cubic tangency (corresponding to a cusp bifurcation of one layer solutions). Moreover, each of these tangential intersections also gives rise to an exact solution near it.

The exact statement of the result for a more general system is given in the next section.

3. Main results

We study the following general system of fast-slow equations.

$$\begin{aligned} u_t &= \epsilon^2 u_{xx} + F(u, y + \frac{k}{\omega}\sin(\omega x + b)), && 0 < x < 1, \\ y_t &= y_{xx} + \sigma G(u, y), && u,\ y \in \mathbb{R}, \\ u_x &= y_x = 0, && x = 0,\ 1. \end{aligned} \tag{3.1}$$

The prototype of F and G are given in (1.3). Stationary solutions satisfy the following equations:

$$\begin{aligned} 0 &= \epsilon^2 u_{xx} + F(u, y + \frac{k}{\omega}\sin(\omega x + b)), && 0 < x < 1, \\ 0 &= y_{xx} + \sigma G(u, y), && u,\ y \in \mathbb{R}, \\ u_x &= y_x = 0, && x = 0,\ 1. \end{aligned} \tag{3.2}$$

It is possible to consider more general types of perturbation in the fast equation, but the special type in (3.1) makes the illustration simpler.

For $k = 0$, Nishiura and Fujii in [30, 31] used the SLEP method to show that the one layer solution u that jumps from near $h_-(y)$ to near $h_+(y)$ is unique and stable by proving that the unique critical eigenvalue $\lambda(\epsilon) = \sum_0^\infty \epsilon^j \lambda_j$ has $\lambda_0 = 0$, $\lambda_1 < 0$.

When $k \ne 0$ and ω is large, system (3.2) is close to the one for $k = 0$ uniformly for $0 < \epsilon_0 \le \epsilon \le \epsilon_1$. However, it is not close to the one for $k = 0$ uniformly for $\epsilon > 0$. In fact, under some generic assumptions on the parameters k and b, Lin [26] has shown that system (3.2) can have several one layer solutions with each having a unique critical eigenvalue with $\lambda_0 = 0$. These solutions can be stable ($\lambda_1 < 0$) or unstable with index 1 ($\lambda_1 > 0$).

The analysis in [26] is valid only when the one layer solutions are hyperbolic with $\lambda_1 \neq 0$ and does not help in understanding how these solutions occur through bifurcation. The purpose of this paper is to complete the study in [26] by discussing the bifurcations in (k, b). As will be shown, they can be either saddle-node bifurcation (fold) or a cusp bifurcation.

We remark that system (3.2) can have solutions with several internal layers. In fact, for $k = 0$, such solutions can be obtained by an even extension of a one layer solution and concatenate the results [33, 29]. A new proof of the stability of such multiple layered solution is given is [26]. The analysis of multiple layer solutions will not be discussed in this paper. Also, to avoid confusion, in the sequel, by a single layer solution, we mean the solution that jumps up from near the slow manifold $u = h_-(y)$ to near $u = h_+(y)$. The other single layer solution which jumps downward will not be discussed in this paper.

Our assumptions on (3.1) are precisely the same as those in [30].

A1. The nullcline of F is sigmoidal and consists of three curves

$$\begin{aligned} R_- &= \{(u, y) : u = h_-(y),\ y \in (y_-, \infty)\}, \\ R_0 &= \{(u, y) : u = h_0(y),\ y \in (y_-, y_+)\}, \\ R_+ &= \{(u, y) : u = h_+(y),\ y \in (-\infty, y_+)\}. \end{aligned}$$

A2. If $J(y) = \int_{h_-(y)}^{h_+(y)} F(s, y)ds$, then there is a $\tilde{y}$ such that

$$J(\tilde{y}) = 0, \quad dJ(\tilde{y})/dy < 0.$$

A2 implies that

$$u_{\xi\xi} + F(u, y) = 0,$$

has a heteroclinic solution $q(\xi)$ connecting $h_-(\tilde{y})$ to $h_+(\tilde{y})$ if $y = \tilde{y}$. Define the Melnikov integral

$$\mathbf{n} = \int_{-\infty}^{\infty} \dot{q}(\xi) F_y(q(\xi), \tilde{y}) d\xi.$$

Elementary calculation shows that $\mathbf{n} = \frac{d}{dy} J(\tilde{y}) < 0$. This shows that the heteroclinic connection breaks with nonzero speed if y moves away from $\tilde{y}$.

The linear equation

$$U_{\xi\xi} + F_u(q(\xi), \tilde{y})U = 0,$$

has a bounded solution $\dot{q}$ that approaches zero exponentially as $\xi \to \pm\infty$, and any other such solution is a multiple of $\dot{q}$.

A3. $F_u < 0$ on R_- and R_+.

A4. $G < 0$ on R_- and $G > 0$ on R_+. $\frac{d}{dy} G(h_\pm(y), y) < 0$ for $y \in (y_-, \infty)$ or $(-\infty, y_+)$.

A5. $G_y|_{R_\pm} \leq 0$.

To ensure that the reduced boundary value problem on the slow manifold

$$y_{xx} + \sigma G(h(y), y) = 0, \quad y_x = 0 \text{ for } x = 0, 1.$$

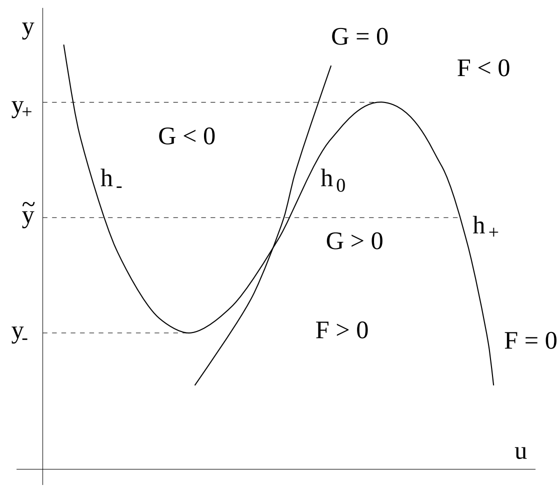

FIGURE 3.1. The nullclines of F and G

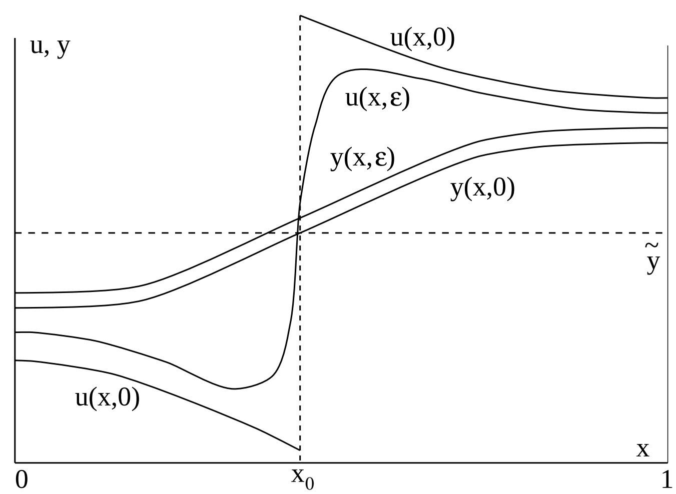

FIGURE 3.2. Internal layer solutions and their singular limits, $k = 0$

has a solution, Nishiura and Fujii assumed that $0 < \sigma \leq \sigma_0$ for some $\sigma_0 > 0$. Similarly, in Theorems 3.1 and 3.2, we assume that σ is a fixed constant, satisfying $0 < \sigma < \sigma_0$ with the same σ_0 as in their paper.

Our first result is essentially due to Lin [26].

Theorem 3.1. *There are positive constants ω_0, c_0 and positive continuous functions $\omega^*(k), \epsilon^*(k,\omega), k^*(\omega)$, $0 < k < \infty$, $0 < \omega < \infty$ with $k^*(\infty) = \lim_{\omega\to\infty} k^*(\omega)$ existing such that the following conclusions hold:*

(1) For any $k \geq 0$, if $\omega \geq \omega^*(k)$ and $0 < \epsilon \leq \epsilon^*(k,\omega)$, then (3.2) has at least one single internal layer solution.

(2) For any $\omega > \omega_0$, $0 \leq k \leq k^*(\omega)$, $0 < \epsilon \leq \epsilon^*(k,\omega)$ and $b \in \mathbb{R}$, there is a unique internal layer solution and it is stable ($\lambda_1 < 0$). If $k^*(\omega) < k \leq c_0\omega$, then there exist b and $\epsilon > 0$ such that (3.2) has more than one internal layer solution.

(3) For any integer N, there exist values of k, ω, b, ϵ such that system (3.2) has at least N stable single layer solutions and at least N unstable single layer solutions of index 1.

To state the next result on the manner in which the solutions in Theorem 3.1 occur through bifurcation, we introduce some additional notation. Throughout the paper, we let x_0 denote the position of the internal layer of a single layer solution. Let $[0, \mathcal{K}]$ be a compact interval in $\mathbb{R}^+$. Let $\omega > \omega_0$ and $\bar{\epsilon}(\mathcal{K}, \omega) = \sup_{0 \le k \le \mathcal{K}} \epsilon^*(k, \omega)$. Let

$$\begin{aligned} \mathcal{B}_{\mathcal{K},\omega,\epsilon} =& \{(k, b, x_0), 0 \le k \le \mathcal{K},\ b \in \mathbb{R},\ x_0 \in (0,1),\ \epsilon < \bar{\epsilon}(\mathcal{K}, \omega) : \\ & \text{there exists a one layer solution}\}, \\ \mathcal{B}_{\mathcal{K},\omega} =& \{(k, b, x_0), 0 \le k \le \mathcal{K},\ b \in \mathbb{R},\ x_0 \in (0,1) : \\ & \text{there exists a singular one layer solution}\}. \end{aligned}$$

Theorem 3.2. *For any $\mathcal{K} > 0$, there exists $\bar{\omega}(\mathcal{K}) > 0$ such that if $\omega > \bar{\omega}(\mathcal{K})$ then $\mathcal{B}_{\mathcal{K},\omega} = \lim_{\epsilon \to 0} \mathcal{B}_{\mathcal{K},\omega,\epsilon}$, in the topology of the distance of sets. The set $\mathcal{B}_{\mathcal{K},\omega}$ is a two-dimensional smooth manifold modeled on (k, b) coordinates except at points that form lower dimensional sets at which there is either a quadratic fold which occurs at points (k, b) with $k > k^*(\omega)$ or a cusp which occurs at $(k^*(\omega), b)$ for some b.*

At the fold points on $\mathcal{B}_{\mathcal{K},\omega}$, by moving b, internal layer solutions are created or eliminated through saddle-node type bifurcations. In a neighborhood of the cusp points of $\mathcal{B}_{\mathcal{K},\omega}$, the number of internal layer solutions locally ranges from one to three.

The method of proof of these results is to give a recursive procedure for obtaining formal matched asymptotic expansions, to any desired power of ϵ, of one layer solutions as well as expansions of the critical eigenvalue $\lambda(\epsilon) = \sum_0^\infty \epsilon^j \lambda_j$ and a corresponding eigenfunction. Some higher order expansions are needed near the fold and cusp since $\lambda_0 = \lambda_1 = 0$.

For the m^{th} order matched expansion of the one layer solution, there is an exact one layer solution $(u(\epsilon), y(\epsilon))$ near the formal expansion to within order $O(\epsilon^{\beta(m+1)})$ for some $0 < \beta < 1$. The same is true for $\lambda(\epsilon)$ and a corresponding eigenfunction.

The location of cusp points is 2π periodic in the b axis. The folds, projected to the (b, k) plane, are smooth curves issuing from the cusps. As k and ω increase, the folds of cusps get wider and overlap with each other. This creates an unbounded number of solutions through saddle–node type bifurcations. See Figure 3.3 for an illustration of a cusp in (k, b, x_0) space.

In the sequel, we will show that the surface $\mathcal{B}_{\mathcal{K},\omega}$ is approximately determined by the equation

$$\frac{k}{\omega} \sin(\omega x + b) = C(x - x_0),$$

where $-C$ is the slope of $\mathcal{C}$ at the switching point. After rescaling k we assume that $C = 1$. This allows us to plot an approximation of $\mathcal{B}$ numerically for a large, fixed ω. In Figure 3.4, we show that the trajectories of folds in the (b, k) plane, issuing from cusps and intersect with each other as k increases. In Figures of 3.5 and 3.6, we show cross sections of $\mathcal{B}_{\mathcal{K},\omega}$ for $k = 2$ and $k = 6$ respectively. The maxim number of mono layer solutions is three right after the forming of cusps (e.g. $k = 2$), and is five right after the first intersection of fold lines (e.g. $k = 6$).

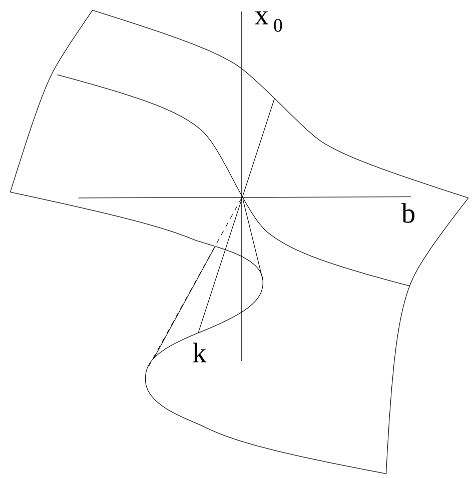

FIGURE 3.3. Cusp in the (k, b, x_0)-space. The k axis is perpendicular to the paper, while the x_0-axis is upward.

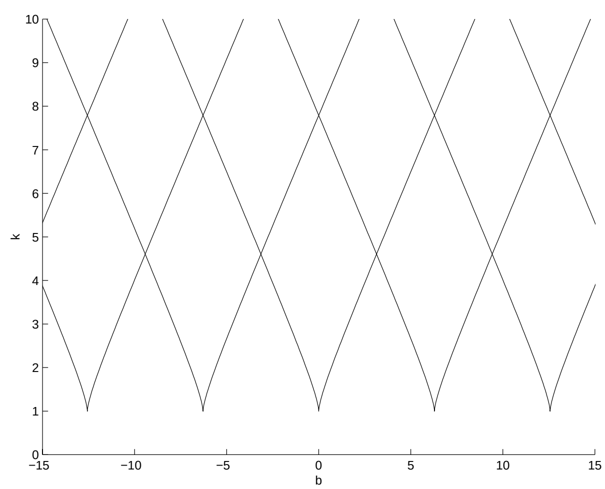

FIGURE 3.4. Trajectories of folds issuing from the cusps, located at $k = 1$. The number of mono layer solutions increases after the fold lines intersect.

Let Γ_1 be the codimension one surface in the space of slow variables defined as in (2.9). When $\epsilon = 0$ and $(y, z, x) \in \Gamma_1$, our assumptions imply that the unstable fibers of the slow manifold R_- intersect the stable fibers of the slow manifold R_+. The intersection is generic in the sense that the connection breaks with nonzero speed if moving along the normal direction of Γ_1. As in Fenichel [7], it follows that a heteroclinic solution connecting two center manifolds persists even when ϵ is positive and small. The surface Γ_1 will be called the **fast jump surface** on which a slow variable acting as a parameter guarantees a fast jump of solutions of the u equation to occur. However, additional knowledge about the relation of Γ_1 with the flow on the

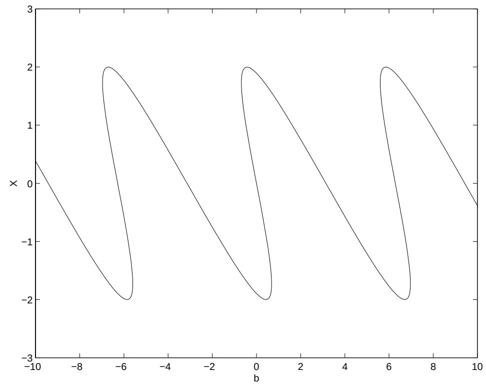

FIGURE 3.5. The ... $k = 2$, after cusps have formed.

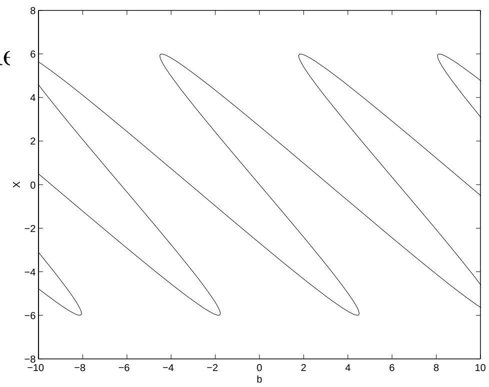

FIGURE 3.6. The cross section of the surface $\mathcal{B}$ at $k = 6$, after the first intersection of fold lines.

slow manifold also plays an important role. The naive guess that the flow on the slow manifold should intersect Γ_1 transversely turns out to be irrelevant. In §4, we will construct a **slow switching curve** $\mathcal{C}$ in the space of slow variables. All of the slow solutions have to switch from one vector field defined by $u = h_-(y + \frac{k}{\omega}\sin(\omega x + b))$ to another defined by $u = h_+(y + \frac{k}{\omega}\sin(\omega x + b))$ at $\mathcal{C}$ in order to satisfy the boundary conditions at $x = 0, 1$. As shown by the case of this paper, $\mathcal{C}$ is not a solution curve in general. See Figure 4.1 for Γ_1 and $\mathcal{C}$. The construction of higher order expansions, and the proof of the existence of a true solution near the formal solution heavily depends on whether the intersection of $\mathcal{C}$ and Γ_1 is transverse or not.

If $\mathcal{C}$ and Γ_1 intersect transversely, then the heteroclinic solution q breaks transversely when moving along $\mathcal{C}$. This case was treated in [26] where higher order expansions of $(u, y) = (\sum \epsilon^j u_j, \sum \epsilon^j y_j)$ were obtained together with the location of the internal layer $\sum_0^\infty \epsilon^j x_j$. The free parameter $x_j, j \geq 1$, was needed when computing u_j in the internal layer, denoted by u_j^S, since the the equation for u_j^S turns out to be not invertible. Physically $\{x_j\}_{j=1}^\infty$ acts as a phase perturbation since the stretched variable used in [26] was $\xi = (x - \sum \epsilon^j x_j)/\epsilon$. Therefore, the small perturbation in the switching time corresponds to a phase shift from $u_0^S(\xi) = q(\xi + x_1 + \epsilon x_2 + \dots)$ to $q(\xi)$. Proof of the existence of a true solution follows from the geometric method in [34, 16] or the analytic method in [8, 21].

If $\mathcal{C}$ intersects Γ_1 tangentially, the case considered in this paper, moving along $\mathcal{C}$ does not break the heteroclinic solution transversely. Some fundamental argument in [26] does not work here. This implies that the methods in [21] and [34] do not apply directly to this case. In particular, the linear system for (u_j, y_j) is not invertible even with the help of a free parameter x_j. We will employ a common trick in bifurcation theory, namely, assuming that the switching time x_0 is given and letting the system parameter $b = \sum \epsilon^j b_j$ depend on x_0. Each b_j is determined in the expansion of (or helps to determine) (u_j, y_j). We no longer need the terms $x_1, x_2, \cdots$ (They can be arbitrarily given and set to be zero). In the rest of the paper $\xi = (x - x_0)/\epsilon$ in the internal layer.

This paper is organized as follows. In §4, we construct leading terms for the asymptotic expansions of the internal layer solutions. We show that for some values of k and b, multiple internal layer solutions can be created through cusp and fold type bifurcations. The fold or cusp occurs when the slow switching curve $\mathcal{C}$ intersects with the fast jump surface Γ_1 at quadratic or cubic tangency points. In §5, we construct higher order expansions of the internal layer solutions. These expansions are determined by systems of linear differential-algebraic equations obtained by expanding (3.2) in powers of ϵ and matching inner and outer layers. Although the leading order expansion is sufficient to determine a true one-layer solution near it, the stability of this solution must be determined by the expansions up to the order ϵ^1. In §6, we discuss the stability of the internal layer solutions when the intersection of $\mathcal{C}$ and Γ_1 is tangential. We obtain expansions of critical eigenvalues and corresponding eigenfunctions to any order of ϵ. The stability of the internal layer solution is determined by the first nonzero coefficient of the expansion of the critical eigenvalue. Since $\lambda_0 = 0$, we need to compute the expansions up to at least $\epsilon^1 \lambda_1$. In §7, We justify that our formal series for internal layer solutions and eigenvalue–eigenfunctions are correct. We find correction terms to asymptotic series so that the result is an exact solution. Our main tool is Theorem 7.1 which uses a Newton's method in function spaces with an undetermined parameter. When applied to the internal layer solution, the parameter is b. When applied to the eigenvalue-eigenfunction problem, the parameter is the eigenvalue λ.

We remark that the method used in [26] also can be used to obtain the results in [1, 13] for the existence and stability of the one layer (or multiple layer) solutions of (1.1). In fact, it is only necessary to replace the function $y + \frac{k}{\omega}\sin(\omega x + b)$ by a function $\alpha y + \beta a(x)$ where α, β are constants with $a(x)$ satisfying the transversality conditions in the introduction. For $\alpha = 0$, we obtain the scalar equation of [1, 13] and for $\alpha \neq 0, \beta \neq 0$, we have a system. The results on bifurcation in the present paper should permit the understanding of the flow for nongeneric functions $a(x)$; that is, for functions $a(x)$ which have a quadratic tangency at zero at some point x_j.

4. Leading terms of the internal layer solutions

We first construct the leading terms (u_0, y_0) of the asymptotic expansion

$$u(x, \epsilon) = \sum \epsilon^j u_j, \quad y(x, \epsilon) = \sum \epsilon^j y_j.$$

In regular layers, let $\epsilon = 0$ in (3.2). The u equation $0 = F(u, y + \frac{k}{\omega}\sin(\omega x + b))$ has two branches of solutions,

$$u = h_\pm(y + \frac{k}{\omega}\sin(\omega x + b)).$$

Inserting the expression into the y equation yields

$$y_{xx} + \sigma G(h_\pm(y + \frac{k}{\omega}\sin(\omega x + b)), y) = 0.$$

With x_0 remaining to be determined and using h_- for $x < x_0$ and h_+ for $x > x_0$, we obtain the equation for $y(x)$ on $(0, x_0) \cup (x_0, 1)$.

In the internal layer, letting $\xi = (x - x_0)/\epsilon$ in the first equation in (3.2), we have

$$u_{\xi\xi} + F(u, y(x_0 + \epsilon\xi) + \frac{k}{\omega}\sin(\omega(x_0 + \epsilon\xi) + b)) = 0.$$

For $\epsilon = 0$, we obtain the u equation with $y(x_0)$ and x_0 as parameters:

$$u_{\xi\xi} + F(u, y(x_0) + \frac{k}{\omega}\sin(\omega x_0 + b)) = 0.$$

At $x = x_0$, from **A2**, the u equation has a heteroclinic solution $q(\xi)$ if

$$y(x_0) + \frac{k}{\omega}\sin(\omega x_0 + b) = \tilde{y}.$$

We now return to the construction of y_0 in regular layers. Consider a first order system in (y, z, x) space:

$$\begin{aligned} dy/dt &= z, \\ dz/dt &= -\sigma G(h(y + \tfrac{k}{\omega}\sin(\omega x + b)), y), \\ dx/dt &= 1, \end{aligned} \tag{4.1}$$

where $h = h_-$ or h_+. Define the following subsets of $\mathbb{R}^3$:

$$\begin{aligned} \Gamma_0 &= \{(y, z, x) | x = 0\}, \\ \Gamma_1 &= \{(y, z, x) | y + \tfrac{k}{\omega}\sin(\omega x + b) = \tilde{y}\}, \\ \Gamma_2 &= \{(y, z, x) | x = 1\}, \\ \mathcal{S}_0 &= \{(y, z, x) | x = 0,\ z = 0\}, \\ \mathcal{S}_1 &= \{(y, z, x) | x = 1,\ z = 0\}. \end{aligned}$$

The **fast jump surface** Γ_1 will be denoted $\Gamma_1^{(k,\omega,b)}$ when we want to express its dependence on (k, ω, b) .

The solution (y, z, x) of (4.1) must start from $\mathcal{S}_0$ and end at $\mathcal{S}_1$. Notice that $-\sigma G > 0$ for $x < x_0$ and < 0 for $x > x_0$. This means that z is increasing if $x < x_0$ and decreasing if $x > x_0$ (or y is concave up if $x < x_0$ and concave down if $x > x_0$). In order to satisfy the boundary conditions at $x = 0, 1$, the solution must switch from one vector field of (4.1) related to h_- to another related to h_+ at a switching point $\wp = (y(x_0), z(x_0), x_0)$ where $dz/dt = d^2y/dt^2$ changes sign. We now describe the set of all the switching points $\wp$.

Let $\Phi_-(t)$ or $\Phi_+(t)$ be the solution map of (4.1) with $h = h_-$ or $h = h_+$ respectively. Let

$$\mathcal{M}_- = \{\Phi_-(t)\mathcal{S}_0,\ t \geq 0\},$$
$$\mathcal{M}_+ = \{\Phi_+(t)\mathcal{S}_1,\ t \leq 0\}.$$

Notations $\Phi_\pm^{(k,\omega,b)}$ and $\mathcal{M}_\pm^{(k,\omega,b)}$ will be used when we want to express their dependence on (k, ω, b).

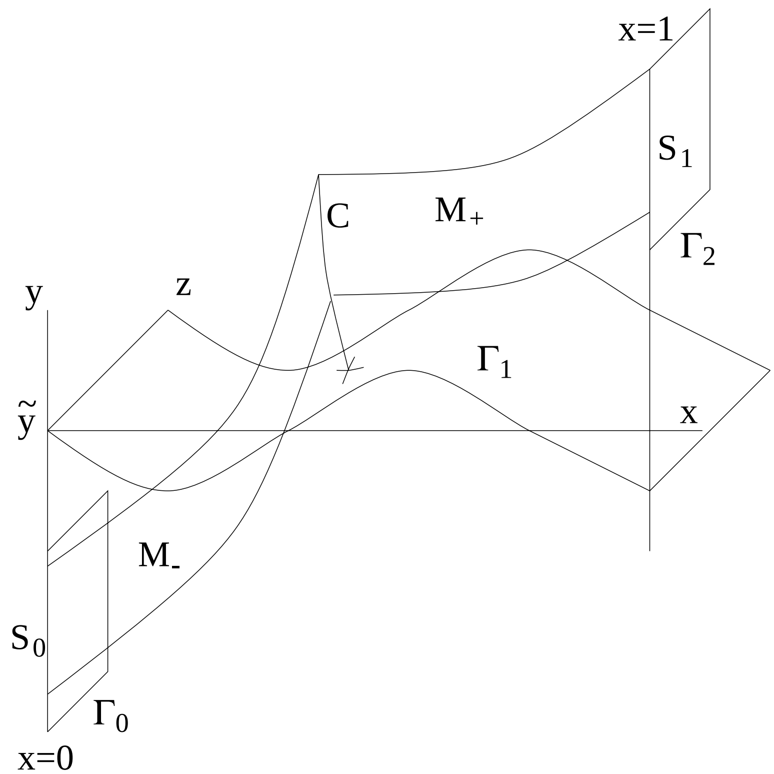

FIGURE 4.1. In the (x, y, z) space, the switching point is determined by the intersection of a slow switching curve $\mathcal{C}$ and a fast jump surface Γ_1.

Since Γ_0 and Γ_2 are transversal to the flow of (4.1), the sets $\mathcal{M}_\pm$ are smooth two-dimensional manifolds. Each switching point $\wp \in \mathcal{M}_- \cap \mathcal{M}_+ \cap \Gamma_1$.

Lemma 4.1. *Let $k = 0$ in (4.1). Let $\bar{\Pi} = \{(y, z, x)|x = x_0\}$ where x_0 is the switching time, and let $\mu_\pm = \mathcal{M}_\pm \cap \bar{\Pi}$. Then $\mu_- \pitchfork \mu_+$ in $\bar{\Pi}$. In particular, $\mathcal{M}_- \pitchfork \mathcal{M}_+$. Moreover, the curve $\mathcal{C} = \mathcal{M}_- \cap \mathcal{M}_+$ is a C^1 submanifold and can be written as $\mathcal{C} = \{(y, z, x)|x = x^*(y, b),\ z = z^*(y, b),\ |y - \tilde{y}| < \eta, \eta > 0\}$, with $\partial x^*(y, b)/\partial y < 0$.*

Proof. We will use results from [30] concerning the regular solution y_0 of (4.1) when $k = 0$. In regular layers, linearize around the solution y_0 and consider an initial value problem for $x \geq 0$:

$$
\begin{aligned}
&Y_x = Z,\\
&Z_x = -\sigma \tfrac{d}{dy} G(h_-(y_0(x)), y_0(x))Y,\\
&Y(0) = 1,\\
&Z(0) = 0,
\end{aligned}
$$

Since $\frac{d}{dy}G(h(y_0(x)), y_0(x)) < 0$, the solution (Y^-, Z^-) satisfies $Y^-(x_0) > 1$, $Z^-(x_0) > 0$. Consider a similar initial value problem for $x \leq 1$:

$$\begin{aligned}
&Y_x = Z, \\
&Z_x = -\sigma \tfrac{d}{dy} G(h_+(y_0(x)), y_0(x))Y, \\
&Y(1) = 1, \\
&Z(1) = 0,
\end{aligned}$$

The solution (Y^+, Z^+) satisfies $Y^+(x_0) > 1$, $Z^+(x_0) < 0$.

Recall that $T\wp\mathcal{M}$ denotes the tangent space of a manifold $\mathcal{M}$ at $\wp$. Observe that $(Y^-(x_0), Z^-(x_0), 0) \in T\wp\mathcal{M}_-$ and $(Y^+(x_0), Z^+(x_0), 0) \in T\wp\mathcal{M}_+$ and the two vectors are linearly independent. Thus $\mu_- \pitchfork \mu_+$ on $\bar{\Pi}$. Since $\bar{\Pi}$ is transversal to the flow of (4.1), it follows that the intersection of $\mathcal{M}_-$ and $\mathcal{M}_+$ is transverse.

Based on the properties of $(Y^\pm(x_0), Z^\pm(x_0))$, the following linear system has a unique solution (Y^c, Z^c).

$$\begin{aligned}
&dY/dx = Z, \\
&dZ/dx = -\sigma \tfrac{d}{dy} G(h(y_0(x)), y_0(x))Y, \\
&Z(0) = 0,\ Z(1) = 0, \\
&[Y](x_0) = 0, \\
&[Z](x_0) = \sigma(G(h_-(\tilde{y}), \tilde{y}) - G(h_+(\tilde{y}), \tilde{y})),
\end{aligned} \tag{4.2}$$

where $h = h_-$ or h_+ if $x < x_0$ or $> x_0$. We now have two tangent vectors on $\mathcal{M}_-$:

$$(Y^c(x_0), Z^c(x_0-), 0) \text{ and } (y_{0x}(x_0), -\sigma G(h_-(\tilde{y}), \tilde{y}), 1),$$

and two tangent vectors on $\mathcal{M}_+$:

$$(Y^c(x_0), Z^c(x_0+), 0) \text{ and } (y_{0x}(x_0), -\sigma G(h_+(\tilde{y}), \tilde{y}), 1).$$

Denote $y_{0x} = z_0$. A tangent vector of $\mathcal{C}$ at the switching point has the form

$$\begin{aligned}
&(Y^c(x_0) - z_0(x_0), Z^c(x_0-) + \sigma G(h_-(\tilde{y}), \tilde{y}), -1) \\
=&(Y^c(x_0) - z_0(x_0), Z^c(x_0+) + \sigma G(h_+(\tilde{y}), \tilde{y}), -1)
\end{aligned}$$

Nishiura and Fujii [30] have shown that $Y^c(x_0) - z_0(x_0) > 0$, which implies that locally we can express the x and z coordinates of $\mathcal{C}$ as functions of y, $|y - \tilde{y}| < \eta$. It is also obvious that if $\eta > 0$ is small, $\frac{\partial x^*(y,b)}{\partial y} < 0$ for all $|y - \tilde{y}| < \eta$.

□

Observe that the distances between $\mathcal{M}_\pm^{(k,\omega,b)}$ and $\mathcal{M}_\pm^{(0,\omega,b)}$ are $O(\frac{k}{\omega^2})$ in the C^0 metric and $O(\frac{k}{\omega})$ in the C^1 metric. For any $k \neq 0$, if ω is sufficiently large, $\mathcal{M}_-^{(k,\omega,b)}$ and $\mathcal{M}_+^{(k,\omega,b)}$ still intersect transversely along a smooth curve $\mathcal{C}^{(k,\omega,b)}$. The curve $\mathcal{C}^{(k,\omega,b)}$ is called the **slow switching curve** on which y_{xx} has to change sign in order to satisfy boundary conditions at $x = 0, 1$. The distance between $\mathcal{C}^{(k,\omega,b)}$ and $\mathcal{C}^{(0,\omega,b)}$ is $O(k/\omega^2)$ in the C^0 metric and is $O(k/\omega)$ in the C^1 metric. This implies that $\mathcal{C}^{(k,\omega,b)}$ also has the form

$$\mathcal{C}^{(k,\omega,b)} = \{(y, z, x) : x = x^*(y, k, \omega, b), z = z^*(y, k, \omega, b), |y - \tilde{y}| \leq \eta\}$$

where $\partial x^*(y, k, \omega, b)/\partial y < 0$.

The y coordinate of $\Gamma_1^{(k,\omega,b)}$ oscillates between $\tilde{y} \pm k/\omega$. When $\omega > k/\eta$, $\mathcal{C}^{(k,\omega,b)} \cap \Gamma_1^{(k,\omega,b)}$ is nonempty. Each point on $\mathcal{C}^{(k,\omega,b)} \cap \Gamma_1^{(k,\omega,b)}$ gives rise to an approximate internal layer solution. Observe that the slope of $\mathcal{C}^{(k,\omega,b)}$ is nonzero and depends very little on k if ω is sufficiently large. Therefore, if k is sufficiently large, $\mathcal{C}^{(k,\omega,b)} \cap \Gamma_1^{(k,\omega,b)}$ consists of multiple points. This gives rise to multiple existence of internal layer solutions.

If $(y, z, x) \in \mathcal{C} \cap \Gamma_1^{(k,\omega,b)}$, then $x = x^*(\tilde{y} - \frac{k}{\omega}\sin\phi, k, \omega, b)$ with $\omega x^*(\tilde{y} - \frac{k}{\omega}\sin\phi, k, \omega, b) + b - \phi = 0$. It was proved in [26] that, for each $(\phi, k, \omega) \in \mathbb{R}^3$ with sufficiently large ω, there exist a unique $b^*(\phi, k, \omega) \in \mathbb{R}$ satisfying this latter equation. Therefore, the x coordinate of $\mathcal{C} \cap \Gamma_1$ is $x_0 = (\phi - b^*(\phi, k, \omega))/\omega$. In the following figures, we depict some possible intersections of $\mathcal{C}^{(k,\omega,b)}$ and $\Gamma_1^{(k,\omega,b)}$. At $\phi = \phi_1$, $\mathcal{C}^{(k,\omega,b)}$ intersects $\Gamma_1^{(k,\omega,b)}$ transversely. At $\phi = \phi_2$, the intersection is of second degree. The curve $\mathcal{C}^{(k,\omega,b)}$ can intersect $\Gamma_1^{(k,\omega,b)}$ transversely at three points corresponding to $\phi = \phi_3$, ϕ_4, ϕ_5 respectively. At $\phi = \phi_6$, the intersection is of third degree.

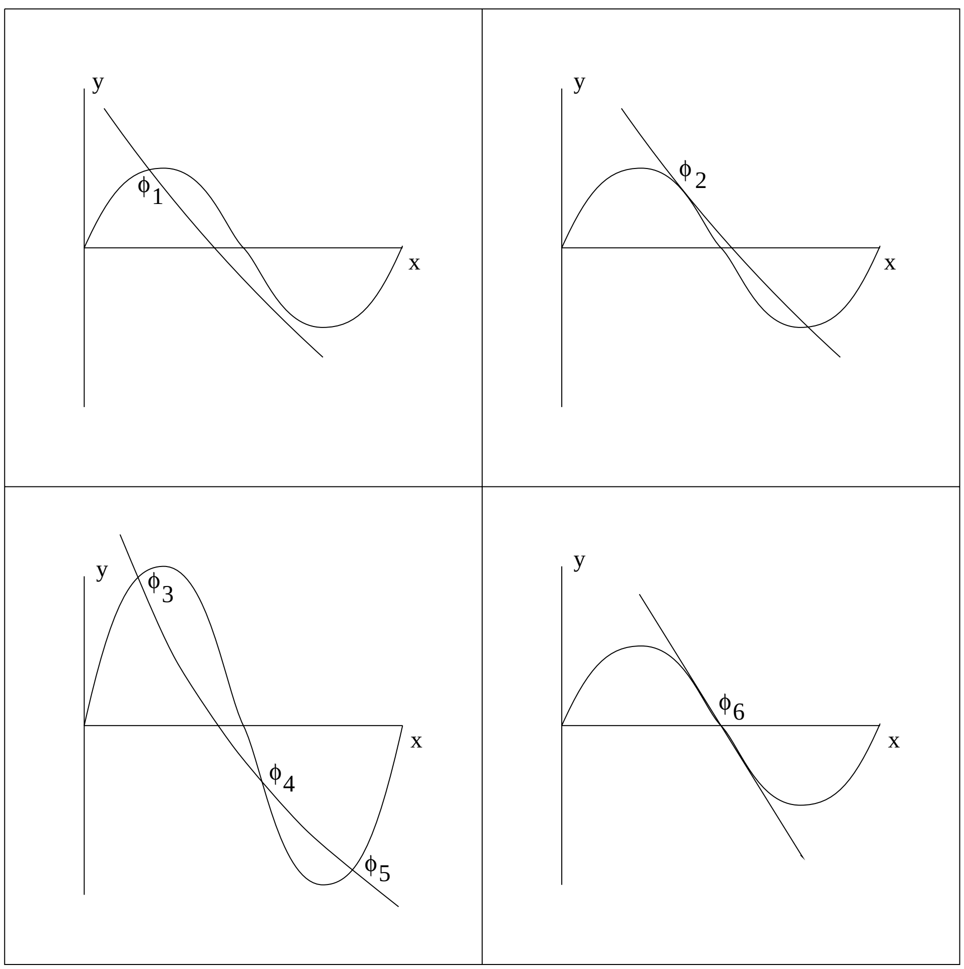

FIGURE 4.2. Several possible intersections of $\mathcal{C}^{(k,\omega,b)}$ and $\Gamma_1^{(k,\omega,b)}$ are depicted.

The existence of a true solution and its stability corresponding to a point at which $\mathcal{C}^{(k,\omega,b)}$ intersects $\Gamma_1^{(k,\omega,b)}$ transversally were discussed in [26]. To discuss solutions that correspond to points near the tangential intersection of $\mathcal{C}^{(k,\omega,b)}$ and $\Gamma_1^{(k,\omega,b)}$, we need the following lemma.

Lemma 4.2. *For a fixed $k = k^\dagger$, let $\wp^\dagger = (y^\dagger, z^\dagger, x^\dagger)$ be a tangential intersection of $\mathcal{C}$ and Γ_1. Then there exists a smooth function $\tilde{b}(x,k)$ defined in a neighborhood of $x^\dagger$ and $k^\dagger$, such that, if $b = \tilde{b}(x_0, k)$, then there is a point in $\Gamma_1 \cap \mathcal{C}$ whose x coordinate is x_0. Moreover, if $\wp^\dagger$ corresponds to a quadratic tangency, then*

$$\tilde{b}(x_0,k) = b_0(k) + b_1(k)(x_0 - x^\dagger) + b_2(k)(x_0 - x^\dagger)^2 + O((x_0 - x^\dagger)^3),$$

with $b_1(k^\dagger) = 0, b_2(k^\dagger) \neq 0$. If $\wp^\dagger$ corresponds to a cubic tangency, then

$$\tilde{b}(x_0,k) = c_0(k) + c_1(k)(x_0 - x^\dagger) + c_2(k)(x_0 - x^\dagger)^2 + c_3(k)(x_0 - x^\dagger)^3 + O((x_0 - x^\dagger)^4),$$

with $c_1(k^\dagger) = c_2(k^\dagger) = 0, c_3(k^\dagger) \neq 0$. Moreover,

$$dc_0(k^\dagger)/dk = O(\omega^{-1}), \quad dc_1(k^\dagger)/dk \neq 0, \quad dc_2(k^\dagger)/dk = O(\omega^{-1}). \tag{4.3}$$

Here, $\neq 0$ means that the quantity is bounded away from zero uniformly as $\omega \to \infty$.

Proof. The manifolds $\mathcal{M}_\pm^{(k,\omega,b)}$ can be expressed as graphs of solution maps of (4.1) with a parameter (k,ω,b). Using linear variational equations, we can prove the estimate

$$\left|\frac{\partial^{i+j+\ell} x^*(y,k,\omega,b)}{\partial y^i \partial b^j \partial k^\ell}\right| \leq \frac{C\omega^i}{\omega^2}, \quad \text{if } j + \ell \geq 1. \tag{4.4}$$

The ω^{-2} factor comes from the fact that the L^1 norm of $|\frac{k}{\omega}\sin(\omega x + b)|$ is $O(\omega^{-2})$ within one period. The method used to prove (4.4) is the same as used in [26], Lemma 5.3.

The function $b^*(\phi,k,\omega)$ satisfies,

$$\omega x^*(\tilde{y} - \frac{k}{\omega}\sin\phi, k, \omega, b) + b - \phi = 0. \tag{4.5}$$

Differentiating (4.5), we have

$$\frac{\partial b^*}{\partial k} = (1 + \omega\frac{\partial x^*}{\partial b})^{-1}\frac{\partial x^*}{\partial y}\sin\phi + O(\omega^{-1}), \tag{4.6}$$

$$\frac{\partial b^*}{\partial \phi} = (1 + \omega\frac{\partial x^*}{\partial b})^{-1}(1 + k\frac{\partial x^*}{\partial y}\cos\phi). \tag{4.7}$$

In particular, $\frac{\partial b^*}{\partial \phi} = 0$ if the intersection is tangential. Using (4.4) and (4.7), we find

$$\frac{\partial^2 b^*}{\partial \phi^2} = -k\frac{\partial x^*}{\partial y}(1 + \omega\frac{\partial x^*}{\partial b})^{-1}\sin\phi + O(\omega^{-1}), \quad \text{if } \frac{\partial b^*}{\partial \phi} = 0, \tag{4.8}$$

$$\frac{\partial^3 b^*}{\partial \phi^3} = -k\frac{\partial x^*}{\partial y}(1 + \omega\frac{\partial x^*}{\partial b})^{-1}\cos\phi + O(\omega^{-1}), \quad \text{if } \frac{\partial b^*}{\partial \phi} = \frac{\partial^2 b^*}{\partial \phi^2} = 0. \tag{4.9}$$

Assume now $\frac{\partial b^*}{\partial \phi} = \frac{\partial^2 b^*}{\partial \phi^2} = 0$. Differentiating (4.5) with respect to both ϕ and k, we have

$$\frac{\partial^2 b^*}{\partial \phi \partial k} = \frac{\partial x^*}{\partial y}(1 + \omega \frac{\partial x^*}{\partial b})^{-1} \cos\phi + O(\omega^{-1}), \tag{4.10}$$

$$\frac{\partial^3 b^*}{\partial \phi^2 \partial k} = -\frac{\partial x^*}{\partial y}(1 + \omega \frac{\partial x^*}{\partial b})^{-1} \sin\phi + O(\omega^{-1}). \tag{4.11}$$

Though the details are quite tedious to verify, the above can be obtained by formally partial differentiating (4.6) with respect to ϕ, since x^* depends weakly on k and b.

Consider the equation

$$x = \frac{\phi - b^*(\phi, k, \omega)}{\omega}.$$

Observe that $\frac{\partial x}{\partial \phi} = \frac{1}{\omega}$ if $\partial b^*/\partial \phi = 0$. We can solve for ϕ as a function $\phi^*(x, k, \omega)$ of (x, k, ω) in a neighborhood of $(x^\dagger, k^\dagger, \omega)$. It is easy to verify that, if $\frac{\partial b^*}{\partial \phi} = 0$, then $\partial \phi^*(x, k, \omega)/\partial x = \omega$ and $\partial \phi^*(x, k, \omega)/\partial k = \partial b^*/\partial k$. Other derivatives of $\phi^*(x, k)$ are also computable. Let $\tilde{b}(x, k, \omega) = b^*(\phi^*(x, k, \omega), k, \omega)$.

At any tangential intersection, we have $\frac{\partial \tilde{b}}{\partial x} = \omega \frac{\partial b^*}{\partial \phi} = 0$.

At a point of quadratic tangency, $\sin\phi \neq 0$. Therefore, from (4.8), $\frac{\partial^2 b^*}{\partial \phi^2} \neq 0$ and $\frac{d^2 \tilde{b}}{dx^2} \neq 0$. We have proved that $b_1(k^\dagger) = 0$, $b_2(k^\dagger) \neq 0$.

Similarly, at a point of cubic tangency, $\cos\phi \neq 0$ and $\frac{\partial^2 b^*}{\partial \phi^2} = 0$. From (4.9), $\frac{\partial^3 b^*}{\partial \phi^3} \neq 0$. We then conclude that $\frac{d^2 \tilde{b}}{dx^2} = 0$ and $\frac{d^3 \tilde{b}}{dx^3} \neq 0$. Therefore $c_1(k^\dagger) = c_2(k^\dagger) = 0$ but $c_3(k^\dagger) \neq 0$. Based on (4.8), $\sin\phi = O(\omega^{-1})$. Estimates (4.3) can be derived from (4.6), (4.10) and (4.11).

□

Let ω be large and fixed. The singular limit bifurcation surface $\mathcal{B}_{\mathcal{K},\omega} = \{(k, b, x_0) | b = \tilde{b}(x_0, k)\}$ has the desired fold and cusp structure as stated in Theorem 3.2. Since $b_1(k^\dagger) = 0, b_2(k^\dagger) \neq 0$, the structure of a fold is clear at points where $\mathcal{C}$ intersects Γ_1 quadratically. To see the cusp structure at a point of cubic tangency, consider finding zeros of the function

$$\begin{aligned} H(k, b, x_0) &= c_0(k) - b + c_1(k)(x_0 - x^\dagger) + c_2(k)(x_0 - x^\dagger)^2 \\ &\quad + c_3(k)(x_0 - x^\dagger)^3 + O((x_0 - x^\dagger)^4) \\ &\equiv \gamma_0 + \gamma_1(x_0 - x^\dagger) + \gamma_2(x_0 - x^\dagger)^2 + \gamma_3(x_0 - x^\dagger)^3 + h(x_0, k), \end{aligned}$$

where $h = O(|x_0 - x^\dagger|^4)$. The method in [5] allows us to eliminate γ_2 by a shifting in x_0 to obtain

$$H = \tilde{\gamma}_0 + \tilde{\gamma}_1(x_0 - x^\dagger - \tilde{x}(k)) + \tilde{\gamma}_3(x_0 - x^\dagger - \tilde{x}(k))^3 + \tilde{h}(x_0, k),$$

where $\tilde{x}(k) = \gamma_2/(3\gamma_3) = O(k/\omega)$ by (4.3) and $\tilde{h} = O(|(x_0 - x^\dagger|^4 + \omega^{-4}|k - k^\dagger|^4)$. The cusp in $(\tilde{\gamma}_0, \tilde{\gamma}_1)$ is approximately of the parameterized form

$$\tilde{\gamma}_0 = 2\tilde{\gamma}_3(x_0 - x^\dagger - \tilde{x}(k))^3, \quad \tilde{\gamma}_1 = -3\tilde{\gamma}_3(x_0 - x^\dagger - \tilde{x}(k))^2.$$

One can verify that the map $(k, b) \to (\tilde{\gamma}_0, \tilde{\gamma}_1)$ is a diffeomorphism. The cusp structure in (k, b) has thus been obtained.

In §7, we will show that a true internal layer solution exists if $\epsilon > 0$ is small and $b = \tilde{b}(x_0, k, \omega) + O(\epsilon^\beta)$ for some $0 < \beta < 1$. The surface $\mathcal{B}_{\mathcal{K},\omega,\epsilon}$, defined before Theorem 3.2, is within ϵ^β to the surface $\mathcal{B}_{\mathcal{K},\omega}$ which has been shown to bear a cusp structure. It is possible to have up to C^4 estimates of the correction term on $\tilde{b}(x_0, k, \omega)$. By doing so, we can prove that $\mathcal{B}_{\mathcal{K},\omega,\epsilon}$ also has a cusp structure. For simplicity, we will only give C^0 estimate of the correction term in this paper.

5. Higher order expansions of the internal layer solutions

For every switching point near a tangential intersection of $\mathcal{C}$ and Γ_1, there corresponds a zeroth order expansion (y_0, u_0) in both regular and singular layers. The purpose of this section is to find higher order expansions in both regular and singular layers. There are two regular layers, defined in $(0, x_0)$ and $(x_0, 1)$. There are three singular layers, two of them are boundary layers, at $x = 0$ and $x = 1$; the other is an internal layer, at $x = x_0$.

Regular layers are points $x \in (0,1)$ where $u(x, \epsilon) \to u(x, 0)$, $\epsilon^2 u_{xx}(x, \epsilon) \to 0$ as $\epsilon \to 0$ with the convergence being uniform in a compact interval surrounding x. These two properties fail at $x = x_0$. When ϵ is small, from Figure 3.2, we see that there is a narrow interval surrounding x_0 where $u_{xx}(x, \epsilon)$ is not small so that $\epsilon^2 u_{xx}(x, \epsilon) \nrightarrow 0$ and $u(x, \epsilon) \nrightarrow u(x, 0)$ uniformly no matter how small the interval is. In the singular limit, the internal layer is at a point $x = x_0$. An important observation is that using the scale $\xi = (x - x_0)/\epsilon$ to blow up the internal layer, the result $u^S(\xi, \epsilon)$ does have a limit as $\epsilon \to 0$. The limit is a heteroclinic solution connecting $u(x_0-, 0)$ to $u(x_0+, 0)$.

For the problem under consideration, the ϵ^0th order expansion of $(u(x, \epsilon), y(x, \epsilon))$ satisfies the Neumann boundary conditions at $x = 0, 1$ and converges uniformly near $x = 0, 1$. However, higher order expansions of $(u(x, \epsilon), y(x, \epsilon))$ do not satisfy boundary conditions at $x = 0, 1$. Therefore boundary layers near $x = 0, 1$ must be added. Again, the boundary layers are points in the x scale. Using $\xi = x/\epsilon$ and $(x-1)/\epsilon$ to blow up the neighborhood of $x = 0$ and $x = 1$, they become intervals $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively.

We use superscript S to denote singular layers and superscript R to denote regular layers. We will label the three singular layers by $i = 0, 1, 2$. The points $x^0 = 0, x^1 = x_0$ and $x^2 = 1$ denote the locations of singular layers. With $\xi = (x - x^i)/\epsilon, i = 0, 1, 2,$ we look for expansions of solutions and the parameter b in the form

$$\sum_0^\infty \epsilon^j u_j^{Ri}(x), \quad \sum_0^\infty \epsilon^j y_j^{Ri}(x), \begin{cases} x \in (0, x_0), \ i = 1, \\ x \in (x_0, 1), \ i = 2; \end{cases}$$

$$\sum_0^\infty \epsilon^j u_j^{Si}(\xi), \quad \sum_0^\infty \epsilon^j y_j^{Si}(\xi), \begin{cases} \xi \geq 0, \ i = 0, \\ \xi \in \mathbb{R}, \ i = 1, \\ \xi \leq 0, \ i = 2; \end{cases}$$

$$b = \sum_0^\infty \epsilon^j b_j.$$

If no confusion arises, superscripts are sometimes dropped for notational simplicity. Let $z = y_x$ in regular layers and $z = y_\xi/\epsilon$ in singular layers.

The governing equations for higher order expansions can be obtained by expanding (3.2) in powers of ϵ. Auxiliary conditions are imposed on the expansions through the matching of adjacent regular and singular layers, and the boundary conditions.

There are many publications discussing the principles of matching. The one used in this paper may not be the most general but is convenient for systems that possess hyperbolic slow manifolds. Let (u^R, y^R) be the outer solution in one of the regular layers adjacent to x^i. The expansion of (u^R, y^R, z^R) by the inner variable ξ is denoted $(\tilde{u}^R, \tilde{y}^R, \tilde{z}^R)$ with

$$\sum_0^\infty \epsilon^j \tilde{w}_j^R(\xi) = \sum_0^\infty \epsilon^j w_j^R(x^i + \epsilon\xi),$$

where $w = u, y$ or z. It is reasonable to expect that the inner expansions (u^S, y^S, z^S) should approach $(\tilde{u}^R, \tilde{y}^R, \tilde{z}^R)$ as $\xi \to \pm\infty$. In fact, we can be more precise about the rate of convergence:

The exponential matching principle

$$|\tilde{u}_j^R(\xi) - u_j^S(\xi)| + |\tilde{u}_{j\xi}^R(\xi) - u_{j\xi}^S(\xi)| \le C(1 + |\xi|^j)e^{-\gamma|\xi|}, \tag{5.1}$$

$$|\tilde{y}_j^R(\xi) - y_j^S(\xi)| + |\tilde{y}_{j\xi}^R(\xi) - y_{j\xi}^S(\xi)| \le C(1 + |\xi|^j)e^{-\gamma|\xi|}. \tag{5.2}$$

(5.2) is equivalent to

$$|\tilde{y}_j^R(\xi) - y_j^S(\xi)| + |\tilde{z}_j^R(\xi) - z_j^S(\xi)| \le C(1 + |\xi|^j)e^{-\gamma|\xi|}.$$

The above exponential-polynomial rate was first used in [20]. Since $(\tilde{u}_j^R, \tilde{y}_j^R, \tilde{z}_j^R)$ are polynomials of order j obtained through the Taylor expansions, therefore, as $\xi \to \pm\infty$, (u_j^S, y_j^S, z_j^S) are asymptotically polynomials of order j also. Consider the u equations. If $j = 0$, we are back to the statement that $q^i(\xi) \to u_0^R(x^i\pm)$ exponentially as $\xi \to \pm\infty$. The loss of the rate of convergence for $j > 0$ was also explained in [20]. Briefly, u_j^S and $\tilde{u}_j^R$ satisfy two linear nonhomogeneous systems

$$Lu_j^S = \mathcal{P}(u_1^S, u_2^S, \cdots, u_{j-1}^S), \quad \tilde{L}\tilde{u}_j^R = \tilde{\mathcal{P}}(\tilde{u}_1^R, \tilde{u}_2^R, \cdots, \tilde{u}_{j-1}^R).$$

Here, $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are two multi-linear forms with their coefficients approaching each other exponentially as $\xi \to \pm\infty$. Also, $\tilde{L}$ and L are differential operators with their coefficients approaching each other exponentially as $\xi \to \pm\infty$. These are all due to the fact that $q^i(\xi) \to u_0^R(x^i\pm)$ as $\xi \to \pm\infty$. Now the difference $\Delta u_j = u_j^S - \tilde{u}_j^R$ satisfies a nonhomogeneous linear equation

$$\tilde{L}\Delta u_j = (\tilde{L} - L)u_j^S + \mathcal{P}(u_1^S, u_2^S, \cdots, u_{j-1}^S) - \tilde{\mathcal{P}}(\tilde{u}_1^R, \tilde{u}_2^R, \cdots, \tilde{u}_{j-1}^R).$$

It is easy to prove, by induction in j, that the right hand side of the above equation is of $O((1+|\xi|^j)e^{-\gamma|\xi|})$. Thus, it is reasonable to require that Δu_j is of $O((1+|\xi|^j)e^{-\gamma|\xi|})$.

The Neumann boundary conditions at $x = 0, 1$ induce some initial/terminal conditions on the boundary layers: $u_\xi^{Si}(0, \epsilon) = 0$, $y_\xi^{Si}(0, \epsilon) = 0$, $i = 0, 2$. Therefore, we have

Boundary conditions in boundary layers

(5.3) $$u^{Si}_{j\xi}(0) = 0, \quad z^{Si}_j(0) = 0, \quad i = 0, 2.$$

In §4, we have shown that the points in $\Gamma_1 \cap \mathcal{C}$ near a tangential intersection can be exhibited through the expression $b_0 = \tilde{b}(x_0, k)$. With this x_0 and b_0, there is a unique point $(y^\dagger, z^\dagger, x_0) \in \mathcal{C} \cap \Gamma_1$. The ϵ^0th order expansion in regular layers can be obtained by solving (y^R_0, z^R_0) from (4.1) using $(y(x_0), z(x_0)) = (y^\dagger, z^\dagger)$ as an initial condition. To complete solutions in regular layers, let $u^R_0(x) = h_\pm(y^R_0(x) + \frac{k}{\omega}\sin(\omega x + b_0))$ for $x < x_0$ or $x > x_0$ respectively. In singular layers, let $u^{Si}_0(\xi) = q^i(\xi)$ where $q^1 = q$ is the heteroclinic solution if $i = 1$. $q^i = u^{Ri}_0(x^i)$ is a constant solution if $i = 0, 2$. In singular layers, $y^{Si}_0(\xi) = y^R_0(x^i)$ and $z^{Si}_0(\xi) = z^R_0(x^i)$ are constant solutions.

For the convenience of typing, let $f = F$ and $g = \sigma G$. In regular layers, let $f_u(x) = f_u(u^R_0(x), y^R_0(x) + \frac{k}{\omega}\sin(\omega x + b_0))$, $g_u(x) = g_u(u^R_0(x), y^R_0(x))$. The function $f_y(x)$ is defined similar to $f_u(x)$, $g_y(x)$ similar to $g_u(x)$. In singular layers, let $f_u(\xi) = f_u(q^i(\xi), y^S_0(\xi) + \frac{k}{\omega}\sin(\omega x^i + b_0))$, $g_u(\xi) = g_u(q^i(\xi), y^S_0(\xi))$. The function $f_y(\xi)$ is defined similar to $f_u(\xi)$, $g_y(\xi)$ similar to $g_u(\xi)$.

Some basic lemmas

Definition . Let I be a finite or infinite interval. Let $\Phi(t, s)$ be the principal matrix solution for a linear system $U' = \mathcal{A}(t)U$, $t \in I$. The system is said to have an exponential dichotomy on I if there exist positive constants K, α and projections $P_s(t) + P_u(t) = id$ such that for $t, s \in I$, we have
(i) $\Phi(t, s)P_s(s) = P_s(t)\Phi(t, s)$.
(ii) $|\Phi(t, s)P_s(s)| \le Ke^{-\alpha(t-s)}, \quad s \le t$.
(iii) $|\Phi(t, s)P_u(s)| \le K^{-\alpha(s-t)}, \quad t \le s$.

$P_s(t)$ and $P_u(t)$ are called respectively stable and unstable projections, and the ranges of $P_s(t)$ and $P_u(t)$ are called stable and unstable subspaces of the exponential dichotomy at the time $t \in I$.

If $I = [t_1, t_2]$ is a finite interval, then corresponding to any continuous t-dependent projections $P_s(t), P_u(t)$, there is always an exponential dichotomy on I. However, we are only interested in dichotomies where K is not too large and $t_2 - t_1$ is not too small so that $Ke^{-\alpha(t_2-t_1)} << 1$. In singular perturbation problems, the length of regular layers are $O(1/\epsilon)$, in the stretched variable ξ. If K and α are independent of ϵ, then if ϵ is small, the notion of the exponent dichotomy becomes very useful.

Definition . The second order equation $u_{\xi\xi} + c(\xi)u = 0$ is said to have an exponential dichotomy on an interval I if the associated first order system $u_\xi = v$, $v_\xi = -c(\xi)u$ has an exponential dichotomy on I.

We present some lemmas concerning linear variational equations around the 0th expansions in regular or in singular layers. The notation $f_u(\cdot)$ means $f_u(u^R_0(x), y^R_0(x))$ in regular layers and $f_u(u^S_0(\xi), y^S_0(\xi))$ in singular layers.

Using the stretched variable $\xi = x/\epsilon$ in the regular layers, $x \in [0, x_0]$ and $[x_0, 1]$ corresponds to $\xi \in [0, x_0/\epsilon]$ and $[x_0/\epsilon, 1/\epsilon]$. Rewrite $\epsilon^2 u_{xx} + f_u(x)u = 0$ by the ξ

variable, and convert it into a first order system,

$$u_\xi = v, \quad v_\xi = -f_u(\epsilon\xi)u. \tag{5.4}$$

Lemma 5.1. *There exists $\epsilon > 0$ such that for $0 < \epsilon < \epsilon_0$, (5.4) has an exponential dichotomy in $[0, x_0/\epsilon]$ and $[x_0/\epsilon, 1/\epsilon]$. The constants K, α are independent of ϵ. As $\epsilon \to 0$, the projection $P_s(\bar{x}/\epsilon)$ approaches the spectral projection of an autonomous system $u' = v$, $v' = -f_u(\bar{x})u$ uniformly for $\bar{x} \in [0, x_0]$ or $[x_0, 1]$.*

Proof. For any fixed $\bar{x} \in [0, x_0]$ or $[x_0, 1]$, the above autonomous system is hyperbolic, with n-dimensional stable and unstable subspaces, since $f_u(\bar{x}) < 0$. If we observing that (5.4) is a slow varying system, then we can use Proposition 1, pp 50 of [6] to conclude that the nonautomous system also has an exponential dichotomy. The proof of the rest of the assertions is also in [6] □

To compute the expansions in boundary layers, we need the stable subspace (or unstable subspace) of the boundary layer at $x = 0$ (respectively at $x = 1$) to be transversal to the subspace defined by the Neumann boundary condition.

Lemma 5.2. *In each of the two boundary layers, $f_u(\xi)$ is a negative constant. The system*

$$u_\xi = v, \quad v_\xi = -f_u(\xi)u, \quad \xi \geq 0, \ \text{or}\ \xi \leq 0,$$

has an exponential dichotomy on $\mathbb{R}^-$ or $\mathbb{R}^+$. Moreover,

$$\mathcal{R}P_s \oplus \{(u, v)|u \in \mathbb{R}, v = 0\} = \mathbb{R}^2,$$
$$\mathcal{R}P_u \oplus \{(u, v)|u \in \mathbb{R}, v = 0\} = \mathbb{R}^2.$$

Proof. Since $(u_0^S(\xi), y_0^S(\xi))$ are constant functions on boundary layers, $f_u(\xi)$ is a constant function there, and is negative by A3. The spectral projections for the autonomous system are easy to compute. We will leave the verification of the transversality conditions in the lemma to the readers. □

Lemma 5.3. *In the internal layer, the homogeneous part of the system*

$$u_\xi = v, \quad v_\xi = -f_u(\xi)u + \mathcal{F}, \tag{5.5}$$

has an exponential dichotomy on $\mathbb{R}^-$ and $\mathbb{R}^+$. As $\xi \to \pm\infty$, the projection $P_s(\xi)$ approaches the spectral projection of an autonomous system

$$u_\xi = v, \quad v_\xi = -f_u(u_0^R(x_0^\pm), \tilde{y})u.$$

Up to constant multipliers, $(\dot{q}, \ddot{q})$ is the only bounded solution on $\mathbb{R}$ to (5.5) if $\mathcal{F} = 0$. The adjoint system

$$u_\xi = f_u(\xi)v, \quad v_\xi = -u,$$

has a unique bounded solution $\psi = (-\ddot{q}, \dot{q})$ *on* $\mathbb{R}$ *up to constant multipliers. Let* $t_1 < 0 < t_2$ *and consider (5.5) on* $[t_1, t_2]$*. Let* $\phi_s \in \mathcal{R}P_s(t_1)$*,* $\phi_u \in \mathcal{R}P_u(t_2)$ *be two given vectors. Let* $\mathcal{F}$ *be continuous on* $[t_1, t_2]$*. With the boundary conditions*

$$P_s(t_1)\begin{pmatrix} u \\ v \end{pmatrix} = \phi_s, \quad P_u(t_2)\begin{pmatrix} u \\ v \end{pmatrix} = \phi_u,$$

system (5.5) has a solution in $[t_1, t_2]$ *if and only if*

$$\psi(t_1)\phi(t_1) - \psi(t_2)\phi(t_2) + \int_{t_1}^{t_2} \dot{q}(\xi)\mathcal{F}(\xi)d\xi = 0.$$

If also $< \dot{q}, u > + < \ddot{q}, v >= 0$*, then the solution is unique and satisfies*

$$|u| \le C(|\phi_s| + |\phi_u| + |\mathcal{F}|),$$

where C *does not depend on* t_1 *and* t_2*.*

Proof. The existence of exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ follows from [32], where a Fredholm type condition for the solvability of (5.5), if $\xi \in \mathbb{R}$, is also presented. Generalization to boundary value problems on a finite interval can be found in [22], from which (5.5) has a solution if and only if

$$\psi(t_1)\phi(t_1) - \psi(t_2)\phi(t_2) + \int_{t_1}^{t_2} < \psi(\xi), (0, \mathcal{F}(\xi)) > d\xi = 0,$$

where $(0, \mathcal{F})$ is the forcing term of the system (5.5). Since $\psi = (-\ddot{q}, \dot{q})$, we have $< \psi, (0, \mathcal{F}) >= \dot{q}\mathcal{F}$. □

ϵ^1th order expansion

We first look at the ϵ^1th order expansion. The formula obtained here will be used to compute λ_1.

In regular layers, we have

$$\begin{aligned} &f_u u_1^R + f_y(y_1^R + \frac{k}{\omega}\cos(\omega x + b_0)b_1) = 0, \\ &y_{1xx}^R + g_u u_1^R + g_y y_1^R = 0. \end{aligned}$$

Solving u_1^R from the first equation and substituting into the second, we have

$$y_{1xx}^R - (g_u f_u^{-1} f_y - g_u)y_1^R - g_u f_u^{-1} f_y \frac{k}{\omega}\cos(\omega x + b_0)b_1 = 0. \tag{5.6}$$

In singular layers, including $i = 0, 1, 2$, we have

$$\begin{aligned} &u_{1\xi\xi}^{Si} + f_u u_1^{Si} + f_y(y_1^{Si} + \frac{k}{\omega}\cos(\omega x^i + b_0)(\omega\xi + b_1)) = 0, \\ &y_{1\xi}^{Si} = z_0^{Si}, \\ &z_{1\xi}^{Si} = -g(q^i(\xi), y_0^R(x^i)), \end{aligned} \tag{5.7}$$

where $z^S = \frac{dy^S}{dx} = \frac{dy^S}{\epsilon d\xi}$. Solving for (y_1^{Si}, z_1^{Si}), we have

$$y_1^{Si}(\xi) = y_1^{Si}(0) + z_0^R(x^i)\xi,$$
$$z_1^{Si}(\xi) = z_1^{Si}(0) - \int_0^\xi g(q^i(\xi), y_0^R(x^i))d\xi.$$

At x^i, $i = 0, 1$, let the inner expansion of the regular layer to the right of x^i, $(y^{R,i+1}(x,\epsilon), z^{R,i+1}(x,\epsilon))$, be denoted $(\tilde{y}^+(\xi,\epsilon), \tilde{z}^+(\xi,\epsilon))$. We have

$$\tilde{y}_1^+(\xi) = y_1^R(x^i+) + y_{0x}^R(x^i+)\xi,$$
$$\tilde{z}_1^+(\xi) = z_1^R(x^i+) + z_{0x}^R(x^i+)\xi.$$

If we recall that $y_{0x}^R(x^i+) = z_0^R(x^i)$ and $z_{0x}^R(x^i+) = -g(q^i(\infty), y_0^R(x^i))$, then the matching of $(\tilde{y}_1^+, \tilde{z}_1^+)$ and (y_1^S, z_1^S) leads to

$$y_1^R(x^i+) = y_1^S(0),$$
$$z_1^R(x^i+) = z_1^S(0) - \int_0^\infty [g(q^i(\xi), y_0^R(x_0)) - g(q^i(\infty), y_0^R(x_0))]d\xi.$$

Inner expansions $(\tilde{y}^-, \tilde{z}^-)$ for outer solutions to the left of x^i, $i = 1, 2$, satisfy similar formulas.

At the boundary layers, the boundary conditions (5.3) lead to,

$$z_1^R(0) = z_1^R(1) = 0. \tag{5.8}$$

At the internal layer, we obtain the jumps across $x^1 = x_0$,

$$\begin{aligned} &[y_1^R](x_0) = 0, \\ &[z_1^R](x_0) = -\int_0^\infty (g(q(\xi), y_0^R(x_0)) - g(q(\infty), y_0^R(x_0)))d\xi \\ &\qquad + \int_0^{-\infty} (g(q(\xi), y_0^R(x_0)) - g(q(-\infty), y_0^R(x_0)))d\xi. \end{aligned} \tag{5.9}$$

In the internal layer, since zero is an eigenvalue for equation (5.7), in order to have a solution $u_1^{S1} \in \mathcal{P}_1(\mathbb{R})$, we need to impose a Fredholm condition

$$\int_{-\infty}^\infty \dot{q}(\xi) f_y(y_1^{S1}(\xi) + \frac{k}{\omega}\cos(\omega x_0 + b_0)(\omega\xi + b_1))d\xi = 0.$$

Observe that $y_1^{S1}(0) = y_1^R(x_0)$. If we recall that $\mathbf{n} = \int_{-\infty}^\infty \dot{q}(\xi) f_y(q(\xi), y_0^R(x_0))d\xi$, we are led to the following condition on $y_1^R(x_0)$ and b_1:

$$\mathbf{n} \cdot (y_1^R(x_0) + \frac{k}{\omega}\cos(\omega x_0 + b_0)b_1) = -\int_{-\infty}^\infty \dot{q}(\xi) f_y(z_0^R(x_0) + k\cos(\omega x_0 + b_0))\xi d\xi. \tag{5.10}$$

ϵ^jth order expansion

The ϵ^jth, order expansion, $j > 1$, is similar to that of the ϵ^1th. When working on the jth order expansion, we assume that all the terms $u_\ell, y_\ell, b_\ell, \ell < j$ have been obtained and are denoted $\ell \cdot o \cdot t$.

In regular layers, we have

$$\begin{aligned}
&u^R_{j-2,xx} + f_u u^R_j + f_y(y^R_j + \frac{k}{\omega}\cos(\omega x + b_0)b_j) = \ell\cdot o\cdot t,\\
&y^R_{jxx} + g_u u^R_j + g_y y^R_j = \ell\cdot o\cdot t.
\end{aligned}$$

Solving for u^R_j from the the first equation and substituting into the second, we have

$$y^R_{jxx} - (g_u f_u^{-1} f_y - g_y)y^R_j - g_u f_u^{-1} f_y \frac{k}{\omega}\cos(\omega x + b_0)b_j = \ell\cdot o\cdot t \overset{def}{=} E_{0j}. \tag{5.11}$$

In singular layers, for $i = 0, 1, 2$, we have

$$\begin{aligned}
&u^{Si}_{j\xi\xi} + f_u u^{Si}_j = f_y(y^{Si}_j + \frac{k}{\omega}\cos(\omega x^i + b_0)b_j) = \ell\cdot o\cdot t,\\
&y^{Si}_{j\xi} = z^{Si}_{j-1},\\
&z^{Si}_{j\xi} = \ell\cdot o\cdot t.
\end{aligned} \tag{5.12}$$

The solution (y^{Si}_j, z^{Si}_j) is determined by $(y^{Si}_j(0), z^{Si}_j(0))$.

$$\begin{aligned}
y^{Si}_j(\xi) &= y^{Si}_j(0) + \ell\cdot o\cdot t\cdot,\\
z^{Si}_j(\xi) &= z^{Si}_j(0) + \ell\cdot o\cdot t\cdot.
\end{aligned}$$

The inner expansion of the regular layers to the right or left of x^i has the form

$$\begin{aligned}
\tilde{y}^\pm_j(\xi) &= y^R_j(x^i\pm) + (\ell\cdot o\cdot t)^\pm,\\
\tilde{z}^\pm_j(\xi) &= z^R_j(x^i\pm) + (\ell\cdot o\cdot t)^\pm.
\end{aligned}$$

Here the $(\ell\cdot o\cdot t))^\pm$ is a polynomial of degree j. It was proved in [26] that, as $\xi\to\pm\infty$, $y^{Si}_j(\xi) - \tilde{y}^\pm_j(\xi)$ approaches a constant determined by $y^{Si}_j(0)$. The same can be said about $z^{Si}_j(\xi) - \tilde{z}^\pm_j(\xi)$. In particular, $y^{Si}_j(0) = y^R_j(x_0^+) + \ell\cdot o\cdot t$. As a consequence, we conclude that, in boundary layers, the boundary conditions (5.3) and the matching imply that

$$z^R_j(0) = \ell\cdot o\cdot t \overset{def}{=} B_{0j}, \quad z^R_j(1) = \ell\cdot o\cdot t \overset{def}{=} B_{1j}. \tag{5.13}$$

From the matching of outer and inner solutions, at the internal layer, we again obtain jumps across x_0:

$$[y^R_j](x_0) = \ell\cdot o\cdot t \overset{def}{=} E_{1j}, \tag{5.14}$$

$$[z^R_j](x_0) = \ell\cdot o\cdot t \overset{def}{=} E_{2j}. \tag{5.15}$$

We have found the ϵ^jth order equation and auxiliary conditions for outer layers. Nishiura and Fujii [30] pointed out that although the length of the internal layer approaches zero as $\epsilon\to 0$, the effect of the internal layer on outer solutions is preserved as a δ function acting at the layer position. This causes a jump of (y^R_j, z^R_j) at x_0. Our scheme of solving the jth order fast-slow system is close to their idea. We will solve the regular layers with proper jumps determined by the internal layer first and then use the information on singular layers. The following lemma affirms that the regular layers can be uniquely determined with jump and boundary conditions.

Lemma 5.4. *Equation (5.11) with jump conditions (5.14), (5.15) and boundary conditions (5.13) has a unique solution y_j^R.*

Proof. Consider the same equation with $E_{0j} = 0$ and $B_{0j} = B_{1j} = E_{1j} = E_{2j} = b_j = 0$. From the shooting method, it is easy to see that $(y_j^R(x_0), z_j^R(0)) \in \mathrm{T}_{\mu_-} \cap \mathrm{T}_{\mu_+}$. Due to Lemma 4.1, $\mu_- \pitchfork \mu_+$ in $\bar{\Pi}$. Thus $(y_j^R(x_0), z_j^R(0)) = 0$. This proves that the homogeneous system has only the zero solution. Therefore, the nonhomogeneous system has a unique solution determined by the input terms. □

If we denote the solution given by Lemma 5.4 as a linear functional

$$y_j^R = \mathcal{K}(B_{0j}, B_{1j}, E_{1j}, E_{2j}, E_{0j} + g_u f_u^{-1} f_v \frac{k}{\omega} \cos(\omega x + b_0) b_j),$$

we have the estimate

$$|y_j^R| \le C(|B_{0j}| + |B_{1j}| + |E_{1j}| + |E_{2j}| + |E_{0j}| + \frac{k}{\omega^2}|b_j|). \tag{5.16}$$

Here the $\frac{k}{\omega^2}$ term is due to the fast oscillation in (5.11).

Observe that the $\ell \cdot o \cdot t$ in (5.12) is in $\mathcal{P}_j(\mathbb{R})$. When $i = 1$, zero is an eigenvalue of (5.12) in $\mathcal{P}_j(\mathbb{R})$. Therefore, equation (5.12) has a unique solution $u_j^S \in \mathcal{P}_j(\mathbb{R})$ that satisfies $u_j^S(0) \perp \dot{q}(0)$ if the Fredholm condition is satisfied: $\int \dot{q}(\xi) f_y(y_j^{Si}(\xi) + \frac{k}{\omega} \cos(\omega x^i + b_0) b_j) d\xi = 0$. This simplifies to

$$\mathbf{n} \cdot (y_j^R(x_0+) + \frac{k}{\omega} \cos(\omega x_0 + b_0) b_j) = \ell \cdot o \cdot t \stackrel{def}{=} E_{3j}. \tag{5.17}$$

Due to (5.16), the left hand side of (5.17) can be written as $\mathbf{n}\frac{k}{\omega}\cos(\omega x_0 + b_0) b_j + O(\frac{k}{\omega^2}) b_j$ plus terms which are bounded above by linear functions of $B_{0j}, B_{1j}, E_{0j}, E_{1j}$ and E_{2j}. Since $\cos\phi \ne 0$ near a tangential intersection of $\mathcal{C}$ and Γ_1, Since $\mathbf{n} \ne 0$, from A2, we can solve b_j as a function of $B_{0j}, B_{1j}, E_{0j}, E_{1j}, E_{2j}$ and E_{3j} from (5.17).

It remains to determine u_j^{Si} in boundary layers. Using Lemma 5.2 of this paper and Lemma 2.2 of [26], there exists a unique solution $u_j^{Si} \in \mathcal{P}_j(\mathbb{R}^{\pm})$, $i = 0, 2$ respectively that satisfies the boundary condition $u_{j\xi}^{Si}(0) = 0$.

This completes the ϵ^jth order expansion of the formal series solution.

The matching of u_j in the singular and regular layers can be proved based on the growth condition $|u_j^S(\xi)| \le C(1 + |\xi|^j)$. Details can be found in [20]. The matching of (y_j, z_j) in regular and singular layers is implied in condition (5.14,5.15). For details, please see [26].

6. Stability of the internal layer solution

The stability of the internal layer solution is determined by the critical eigenvalue $\lambda(\epsilon) = \sum_0^\infty \epsilon^j \lambda_j$, $\lambda_0 = 0$. Nishiura and Fujii [30] justified the SLEP method by showing that the critical eigenvalue constructed by the SLEP method is the only eigenvalue in a region $\mathrm{Re}\lambda \ge \mu$ where $\mu < 0$ is a constant. Their argument should apply to the system considered in this paper with some small change, since our system is close to theirs when ω is large. For this reason, we will discuss critical eigenvalues but not the non critical ones.

We want to formally solve an eigenvalue–eigenfunction problem $\lambda(\epsilon)\Xi(\epsilon) = \mathfrak{A}(\epsilon)\Xi(\epsilon)$, where $\mathfrak{A}$ is an differential operator and $\Xi = (U, Y)$ is an eigenfunction corresponding to $\lambda(\epsilon)$.

In regular layers, the eigenvalue problem is

$$\lambda U = \epsilon^2 U_{xx} + f_u^\epsilon U + f_y^\epsilon Y,$$
$$\lambda Y = Y_{xx} + g_u^\epsilon U + g_y^\epsilon Y.$$

In singular layers, with $\xi = (x - x^i)/\epsilon,\ i = 0, 1, 2$, the eigenvalue problem is

$$\lambda U = U_{\xi\xi} + f_u^\epsilon U + f_y^\epsilon Y,$$
$$\epsilon^2 \lambda Y = Y_{\xi\xi} + \epsilon^2 (g_u^\epsilon U + g_y^\epsilon Y).$$

In the above, $f_u^\epsilon = f_u(\sum \epsilon^j u_j, \sum \epsilon^j y_j + \frac{k}{\omega}\sin(\omega x + \sum \epsilon^j b_j))$, and similarly for f_y^ϵ, g_u^ϵ, g_y^ϵ. The convention for the arguments of f_u, f_y, g_u, g_y follows form that of §5 in the rest of this section.

Introducing $Z = Y_x$ in regular layers and $Z = Y_\xi/\epsilon$ in singular layers, our goal is to obtain expansions for $\lambda(\epsilon) = \sum \epsilon^j \lambda_j$ and $W(\epsilon) = \sum \epsilon^j W_j$ where $W = U$, Y or Z.

Boundary conditions Formal expansion of the Neumann boundary conditions yields:

$$U_{j\xi}^{Si}(0) = 0, \quad i = 0, 2,$$
$$Y_{j\xi}^{Si}(0) = Z_j^{Si}(0) = 0, \quad i = 0, 2.$$

Matching of inner and outer eigenfunctions

Let the inner expansion of a regular layer adjacent to the ith singular layer be

$$\sum_0^\infty \tilde{W}_j(\xi) = \sum_0^\infty W_j^R(x^i + \epsilon\xi),$$

where $W = U, Y$ or Z. Similar to the expansion of the formal solution in §5, we impose the following matching condition:

$$|U_j^{Si}(\xi) - \tilde{U}_j(\xi)| + |U_{j\xi}^{Si}(\xi) - \tilde{U}_{j\xi}(\xi)| \le C(1 + |\xi|^j)e^{-\gamma|\xi|},$$
$$|Y_j^{Si}(\xi) - \tilde{Y}_j(\xi)| + |Z_j^{Si}(\xi) - \tilde{Z}_j(\xi)| \le C(1 + |\xi|^j)e^{-\gamma|\xi|}.$$

If we observe that $\tilde{W}_j(\xi),\ W = U, Y, Z$, is a jth order polynomial of ξ, then

$$|U_j^{Si}(\xi)| + |Y_j^{Si}(\xi)| + |Z_j^{Si}(\xi)| \le C(1 + |\xi|^j).$$

The governing equations can be obtained by expanding the eigenvalue–eigenfunction system in powers of ϵ.

(1) ϵ^0th order expansion:

First, in regular layers,

$$f_u U_0 + f_y Y_0 = 0,$$
$$Y_{0xx} + g_u U_0 + g_y Y_0 = 0,$$
$$U_0 = -f_u^{-1} f_y Y_0,$$
$$Y_{0xx}^R - (g_u f_u^{-1} f_y - g_u) Y_0^R = 0. \tag{6.1}$$

In singular layers, using $\lambda_0 = 0$, we have

$$U^{Si}_{0\xi\xi} + f_u U^{Si}_0 + f_u Y^{Si}_0 = 0, \tag{6.2}$$

$$Y^{Si}_{0\xi} = Z^{Si}_{0\xi} = 0. \tag{6.3}$$

We see that Y^S_0 and Z^S_0 are constant functions. From the matching principle and $Z^{Si}_0(0) = 0$, $i = 0, 2$, we have the boundary conditions,

$$Z^R_0(0) = Z^R_0(1) = 0. \tag{6.4}$$

The matching at $x = x_0$ yields the jump conditions,

$$[Y^R_0](x_0) = [Z^R_0](x_0) = 0. \tag{6.5}$$

From Lemma 5.4, equation (6.1) with boundary conditions (6.4) and jump condition (6.5) admits the unique solution $Y^R_0 = 0$, which implies that $Y^{Si}_0 = 0$, $i = 0, 1, 2$ and $U^R_0 = 0$.

In (6.2), substitute $Y^{Si}_0 = 0$. In boundary layers, due to the boundary condition $U^S_{0\xi}(0) = 0$, we find that $U^{Si}_0 = 0$ for $i = 0, 2$. Here we have used Lemma 2.2 from [26] again.

In the internal layer, $i = 1$, (6.2) has a unique bounded solution $\dot{q}(\xi)$ up to constant multiples. Let $U^{S1}_0 = \dot{q}(\xi)$. Since $\dot{q}^i(\xi) = 0$, $i = 0, 2$, and $\dot{q}^1 = \dot{q}$, we have $U^{Si}_0 = \dot{q}^i(\xi)$, $\quad i = 0, 1, 2$. We normalize the eigenfunction so that $< \dot{q}, U^{S1}_j >= 0, \quad j \geq 1$.

(2) ϵ^1th order expansion:

In regular layers, since $\lambda_0 U^R_1 + \lambda_1 U^R_0 = 0 = \lambda^0 Y^R_1 + \lambda_1 Y^R_0$, we have

$$\begin{aligned}
&f_u U^R_1 + f_y Y^R_1 = 0,\\
&Y^R_{1xx} + g_u U^R_1 + g_y Y^R_1 = 0,\\
&U^R_1 = -f_u^{-1} f_y Y^R_1,\\
&Y^R_{1xx} - (g_u f_u^{-1} f_y - g_y) Y^R_1 = 0.
\end{aligned} \tag{6.6}$$

In singular layers, we have

$$\begin{aligned}
\lambda_1 \dot{q}^i(\xi) &= U^{Si}_{1\xi\xi} + f_u U^{Si}_1 + f_y Y^{Si}_1\\
&\quad + f_{uu} \dot{q}^i u^{Si}_1 + f_{uy} \dot{q}^i [y^{Si}_1 + \frac{k}{\omega} \cos(\omega x_0 + b_0)(\omega \xi + b_1)],\\
Y^{Si}_{1\xi} &= Z^{Si}_0 = 0,\\
Z^{Si}_{1\xi} &= -g_u U^{Si}_0 - g_y Y^{Si}_0 = -g_u \dot{q}^i.
\end{aligned} \tag{6.7}$$

Thus, $Y^{Si}_1(\xi) = Y^R_1(x^i)$ is a constant function, and

$$Z^{Si}(\xi) = z^{Si}(0) - \int_0^\xi g_u \dot{q}^i(\xi) d\xi.$$

In boundary layers, using $\dot{q}^i = 0$, $i = 0, 2$, we have $Z^{Si}_1(\xi) = Z^{Si}_1(0) = 0$, $i = 0, 2$. From the matching principle, we have

$$Z^R_1(0) = Z^R_1(1) = 0. \tag{6.8}$$

At $x = x_0$, we compute the jumps

$$
\begin{aligned}
&[Y_1^R](x_0) = 0, \\
&[Z_1^R](x_0) = -\int_{-\infty}^{\infty} g_u \dot{q}^i(\xi) d\xi = g(q^i(-\infty), \tilde{y}) - g(q^i(\infty), \tilde{y}).
\end{aligned} \tag{6.9}
$$

From Lemma 5.4, equation (6.6), with boundary conditions (6.8) and jumps (6.9), has a unique solution $(Y_1^R, Z_1^R) = (Y^c, Z^c)$, which satisfies (4.2).

In order that (6.7) has a solution $U_1^{S1} \in \mathcal{P}_1(\mathbb{R})$, we need a Fredholm condition,

$$\lambda_1 |\dot{q}|^2 = < \dot{q},\, f_y Y_1^{S1} + f_{uu} \dot{q} u_1^{S1} + f_{uy} \dot{q} [\cdots] >,$$

where $[\cdots]$ are terms in the brackets of (6.7). The above can be simplified using integration by parts. Observe that

$$
\begin{aligned}
& f_{uu} \dot{q} u_1^{S1} + f_{uy} \dot{q} [\cdots] \\
=& \frac{\partial}{\partial \xi} (f_u u_1^{S1} + f_y [\cdots]) - f_u u_{1\xi}^{S1} - f_y (y_{1\xi}^{S1}(\xi) + k \cos(\omega x_0 + b_0)) \\
=& - u_{1\xi\xi\xi}^{S1} - f_u u_{1\xi}^{S1} - f_y (y_{1\xi}^{S1}(\xi) + k \cos(\omega x_0 + b_0)).
\end{aligned}
$$

Therefore, $< \dot{q},\, f_{uu} \dot{q} u_1^{S1} + f_{uy} \dot{q} [\cdots] > = - < \dot{q},\, f_y (y_{1\xi}^{S1} + k \cos(\omega x_0 + b_0))$. If we recall that $Y_1^{S1} = Y_1^R(x_0) = Y^c(x_0)$, $y_{1\xi}^{S1} = z_0^R(x_0)$ and $\mathbf{n} = \int_{-\infty}^{\infty} \dot{q}(\xi) f_y(q(\xi), \tilde{y}) d\xi$, then we finally arrive at

$$\lambda_1 |\dot{q}|^2 = \mathbf{n}(Y^c(x_0) - z_0^R(x_0) - k \cos(\omega x_0 + b_0)). \tag{6.10}$$

We project out the z-component of Γ_1 and $\mathcal{C}$ and let the images on the xy-plane be $\tilde{\Gamma}_1$ and $\tilde{\mathcal{C}}$. The slope of $\tilde{\Gamma}_1$ is $-k \cos(\omega x_0 + b_0)$ and the slope of $\tilde{\mathcal{C}}$ is $Y^c(x_0) - z_0^R(x_0)$ at $\mathcal{C} \cap \Gamma_1$. For the latter, please refer to Lemma 4.1. If we recall that $\mathbf{n} < 0$, from **A2**, then we can summarize our result in the following

Theorem 6.1. *For internal layer solutions corresponding to points near the tangential intersections of $\mathcal{C}$ and Γ_1,*

$$\lambda_1 = |\dot{q}|^{-2} \mathbf{n}(Y^c(x_0) - z_0^R(x_0) - k \cos(\omega x_0 + b_0)),$$

where Y^c satisfies (4.2). Furthermore,

$$
\lambda_1 \begin{cases}
< 0, & \text{(slope of } \tilde{\Gamma}_1\text{)>(slope of } \tilde{\mathcal{C}}\text{)}, \\
= 0, & \text{(slope of } \tilde{\Gamma}_1\text{)=(slope of } \tilde{\mathcal{C}}\text{)}, \\
> 0, & \text{(slope of } \tilde{\Gamma}_1\text{)<(slope of } \tilde{\mathcal{C}}\text{)}.
\end{cases}
$$

The solution U_1^{S1} of (6.7) is unique if $< \dot{q},\, U_1^{S1} > = 0$.

Finally, in boundary layers, $\dot{q}^i = 0$, $i = 0, 2$. From Lemma 5.2, equation (6.7) has a unique solution in $\mathcal{P}_1(\mathbb{R}^{\pm})$ with $U_{1\xi}^{Si}(0) = 0, \quad i = 0, 2$.

(3) jth order expansion, $j \geq 2$:

Assuming that we have obtained $\lambda_\ell,\ U_\ell,\ Y_\ell,\ Z_\ell,\quad 0 \leq \ell \leq j-1$, we want to compute $\lambda_j,\ U_j,\ Y_j,\ Z_j$. Any term that involves indices $0 \leq \ell \leq j-1$ will be denoted $\ell \cdot o \cdot t$.

In regular layers, since $\lambda_0 = 0$ and $U_0^R = Y_0^R = 0$,

$$\lambda_0 U_j^R + \cdots + \lambda_j U_0^R = \ell \cdot o \cdot t,$$
$$\lambda_0 Y_j^R + \cdots + \lambda_j Y_0^R = \ell \cdot o \cdot t.$$

The governing equations are

$$\begin{aligned} &f_u U_j^R + f_y Y_j^R = \ell \cdot o \cdot t,\\ &Y_{jxx}^R + g_u U_j^R + g_y Y_j^R = \ell \cdot o \cdot t,\\ &U_j^R = -f_u^{-1} f_y Y_j^R + \ell \cdot o \cdot t,\\ &Y_{jxx}^R - (g_u f_u^{-1} f_y - g_y) Y_j^R = \ell \cdot o \cdot t. \end{aligned} \tag{6.11}$$

In singular layers, using $Y_0^{Si} = 0$, $U_0^{Si} = \dot{q}^i$, we have

$$\lambda_0 U_j^{Si} + \cdots + \lambda_j U_0^{Si} = \lambda_j \dot{q}^i + \ell \cdot o \cdot t,$$
$$\lambda_0 Y_j^{Si} + \cdots + \lambda_j Y_0^{Si} = \ell \cdot o \cdot t.$$

The governing equations are:

$$\begin{aligned} &\lambda_j \dot{q}^i = U_{j\xi\xi}^{Si} + f_u U_j^{Si} + f_y Y_j^{Si} + \ell \cdot o \cdot t,\\ &Y_{j\xi}^{Si} = Z_{j-1}^{Si} = \ell \cdot o \cdot t,\\ &Z_{j\xi}^{Si} = -g_u U_{j-1}^{Si} - g_y Y_{j-1}^{Si} + \ell \cdot o \cdot t = \ell \cdot o \cdot t. \end{aligned} \tag{6.12}$$

Therefore $Y_j^{Si}(\xi) = Y_j^{Si}(0) + \ell \cdot o \cdot t, \quad Z_j^{Si}(\xi) = Z_j^{Si}(0) + \ell \cdot o \cdot t.$

From Lemma 3.3, [26], the matching of outer and inner solutions only needs to be done on constant terms, because the higher order powers of ξ are already matched. Therefore, we can deduce that

$$Z_j^R(0) = \ell \cdot o \cdot t, \quad Z_j^R(1) = \ell \cdot o \cdot t. \tag{6.13}$$

We also obtain the jumps at $x = x_0$.

$$[Y_j^R](x_0) = \ell \cdot o \cdot t, \quad [Z_j^R](x_0) = \ell \cdot o \cdot t. \tag{6.14}$$

Equation (6.11), with boundary condition (6.13) and jumps (6.14), has a unique solution Y_j^R which is now computable.

In boundary layers, using Lemma 2.2, [26], and the boundary condition $U_{j\xi}^{Si}(0) = 0$, and Lemma 5.2 of this paper, we have a unique solution $U_j^{Si} \in \mathcal{P}_j(\mathbb{R}^\pm)$, $i = 0, 2$, for (6.12).

In the internal layer, to have a solution of (6.12) in $\mathcal{P}_j(\mathbb{R})$, we need a Fredholm condition:

$$\lambda_j |\dot{q}|^2 =< \dot{q},\, f_y Y_j^{S1} + \ell \cdot o \cdot t > . \tag{6.15}$$

If we recall that $Y_j^{Si}(\xi) = Y_j^{Si}(0) + \ell \cdot o \cdot t$, $Y_j^{Si}(0) = Y_j^R(x_0+) + \ell \cdot o \cdot t$ and that Y_j^R has already been computed, we can then determine λ_j from (6.15).

The matching of inner and outer solutions for (Y_j, Z_j) has already been considered when deriving the boundary and jump condition of (Y_j^R, Z_j^R). The matching of U in inner and outer layers can be proved based on the growth conditions. Details can be found in [20, 26].

7. Existence of exact solutions and eigenvalue-eigenfunctions

The purpose of this section is to verify that the formal expansions of internal layer solutions and eigenvalue-eigenfunctions are valid. Unless otherwise specified, the norms are the supremum norms of continuous functions.

We truncate the formal series to form approximations of exact solutions. Let $0 < \beta < 1$ be a constant. Then ϵ^β is so called an intermediate variable that satisfies $\epsilon << \epsilon^\beta << 1$. Let a^i, $0 \le i \le 5$ be a sequence of points that divides $[0,1]$ into subintervals $I^i = \{x | a^{i-1} < x < a^i\}$. Here $a^0 = 0$, $a^1 = \epsilon^\beta$, $a^2 = x_0 - \epsilon^\beta$, $a^3 = x_0 + \epsilon^\beta$, $a^4 = 1 - \epsilon^\beta$ and $a^5 = 1$, where x_0 is the switching time in the formal construction. In §5 and §6, the indices $\ell = 1, 2$ and $\ell = 0, 1, 2$ were used for the regular and singular layers. Both layers are now uniformly indexed by i with $i = 2\ell$ for regular and $i = 2\ell+1$ for singular layers. Thus I^1, I^5 are the boundary layers, I^2, I^4 are the regular layers, and I^3 is the internal layer. Let $\xi^i = a^i/\epsilon$. In the stretched variable, $I^i = \{\xi | \xi^{i-1} < \xi < \xi^i\}$. The length of I^i approaches infinity as $\epsilon \to 0$, This makes the use of exponential dichotomies relevant.

Let us suppose that the approximations are given by

$$\lambda_{ap} = \sum_0^m \epsilon^j \lambda_j, \quad b_{ap} = \sum_0^m \epsilon^j b_j,$$

$$W^i_{ap}(x,\epsilon) = \sum_0^m \epsilon^j W^{R\ell}_j(x), \quad x \in I^i,\ i = 2\ell,\ \ell = 1, 2,$$

$$W^i_{ap}(x,\epsilon) = \sum_0^m \epsilon^j W^{S\ell}_j((x - x^\ell)/\epsilon), \quad x \in I^i,\ i = 2\ell+1,\ \ell = 0, 1, 2,$$

where $x^0 = 0, x^1 = x_0$ and $x^2 = 1$, and $W = (u, y)$ if we are dealing with internal layer solutions, and $W = (U, Y)$ if we are dealing with eigenfunctions. Let the exact critical eigenvalue be $\lambda_{ap} + \lambda$, the exact parameter be $b_{ap} + b$ and the exact solution to internal layer solution and eigenfunction be $W_{ap} + W$. Our goal is to find the correction terms λ, b and W. The linear variational system satisfied by λ, b and W will be solved first. The nonlinear system will then be solved by contraction mapping principles. We often need to express W_{ap} and W in the stretched variable $\xi = x/\epsilon$. With some abuse of notations, we let $W(\xi) = W(x)$, $x = \epsilon\xi$.

In the following, the coefficients in regular layers are $f^i_u = f_u(u^{R\ell}_0(x), y^{R\ell}_0(x) + \frac{k}{\omega}\sin(\omega x + b_0))$, $f^i_y = f_y(u^{R\ell}_0(x), y^{R\ell}_0(x) + \frac{k}{\omega}\sin(\omega x + b_0))$, $i = 2\ell$. Similar definitions apply to $g^i_u(x)$ and $g^i_y(x)$. If the stretched variable ξ is used in regular layers, x should be replaced by $\epsilon\xi$. In singular layers, The coefficient $f^i_u = f^i_u(u^{S\ell}_0(\xi - x^\ell/\epsilon), y^{S\ell}_0(\xi - x^\ell/\epsilon) + \frac{k}{\omega}\sin(\omega x^\ell + b_0))$, $i = 2\ell+1$. Similar definitions apply to $f^i_y(\xi), g^i_u(\xi)$ and $g^i_y(\xi)$. If unstretched variable x is used in singular layers, ξ is replaced by x/ϵ. We will drop the superscript i if it is clear from the context.

Let $s^i(\xi)$, $1 \le i \le 5$ be a given continuous and bounded function defined on $\mathbb{R}$.

A6. Assume that

$$\int_{-\infty}^{\infty} \dot{q}(\xi + x_0/\epsilon) s^3(\xi) d\xi \neq 0.$$

Our main tool is the following theorem concerning a linear system with forcing terms and boundary/jump conditions.

Theorem 7.1. *Consider the following system of equations where μ is an undetermined parameter, F^i and G^i are continuous, bounded functions on I^i. In regular layers,*

$$U_{\xi\xi} + f_u^i U + f_y^i Y = F^i(\xi), \quad \xi = x/\epsilon, \tag{7.1}$$

$$Y_{xx} + g_u^i U + g_y^i Y = G^i(x), \quad i = 2, 4. \tag{7.2}$$

In singular layers,

$$U_{\xi\xi} + f_u^i U + f_y^i Y + \mu s^i(\xi) = F^i(\xi), \tag{7.3}$$

$$Y_{xx} = G^i(x), \quad i = 1, 3, 5. \tag{7.4}$$

The boundary conditions are

$$U_\xi^1(\xi^0) = U_\xi^5(\xi^5) = 0, \quad Y_x^1(a^0) = Y_x^5(a^5) = 0. \tag{7.5}$$

The jump conditions for $1 \le i \le 4$ are

$$\begin{aligned} U^{i+1}(\xi^i) - U^i(\xi^i) &= J_1^i, \quad U_\xi^{i+1}(\xi^i) - U_\xi^i(\xi^i) = J_2^i, \\ Y^{i+1}(a^i) - Y^i(a^i) &= J_3^i, \quad Y_x^{i+1}(a^i) - Y_x^i(a^i) = J_4^i. \end{aligned} \tag{7.6}$$

Then, there exists a unique solution that satisfies (7.1)-(7.6), and there is a positive constant C such that

$$|\mu| + \sum_{i=1}^{5}(|U^i| + |Y^i|) \le C\{\sum_{i=1}^{5}(|F^i| + |G^i|_{L^1}) + \sum_{i=1}^{4}\sum_{j=1}^{4}|J_j^i|\}.$$

Remark . If we exam the proof of Theorem 7.1, which uses Lemma 7.5, we find that $|G^i|_{L^1}$ can be replaced by $\sup_{c,d\in I^i} |\int_c^d G^i(x)dx|$. This form of estimate is useful in the proof of Theorem 7.2.

Theorem 7.1 can be directly used to prove the existence of the exact critical eigenvalue and eigenfunctions near the asymptotic expansions obtained in §6. With some change, see Theorem 7.2, it can be used to prove the existence of exact internal layer solutions near the asymptotic series. The use of Theorem 7.1 is not limited to problems in this paper. It can be adapted to internal layer solutions not near bifurcation points in parameter spaces [26], as well as associated eigenvalue problems. We comment here that asymptotic expansions near and not near bifurcation points are quite different. A similar version of the methods used here can be adapted for singularly perturbed first order equations of the type in [21] and should simplify the original proof.

To make the theorem more useful, we summarize basic assumptions without direct reference to **A1-A5**. Assume that the homogeneous linear part of the U equation (7.1) or (7.3) has an exponential dichotomy in I^i, $i = 1, 2, 4, 5$, and that in I^3,(7.3) has an exponential dichotomy in each half of the interval $[\xi^2, x_0/\epsilon]$ or $[x_0/\epsilon, \xi^3]$. Also at each $\xi^i, 1 \le i \le 4$, $\mathcal{R}P_u^i(\xi^i) \oplus \mathcal{R}P_s^{i+1}(\xi^i) = \mathbb{R}^2$, at $\xi = x_0/\epsilon$, $\mathcal{R}P_u^3(\xi^-) = \mathcal{R}P_s^3(\xi^+)$, and at ξ^0 and ξ^5, the subspace defined by the boundary condition $U_\xi = 0$ intersects $\mathcal{R}P_s^1(\xi^0)$ or $\mathcal{R}P_u^5(\xi^5)$ transversely.

The conditions on the Y equation on two regular layers are the following. Consider the reduced flow on $[0, x_0]$ and $[x_0, 1]$,

$$Y_{xx} - (g_u(f_u)^{-1}f_y - g_y)Y = 0.$$

With Neumann boundary conditions at $x = 0, 1$ and jump conditions

$$Y(x_0^+) - Y(x_0^-) = 0, \quad Y_x(x_0^+) - Y_x(x_0^-) = 0,$$

the reduced Y equation has a unique solution on $[0, x_0]$ and $[x_0, 1]$.

With these conditions and **A6**, the conclusions of Theorem 7.1 are valid.

The proof of Theorem 7.1 is divided into three lemmas, Lemmas 7.5-7.7, and is deferred to the end of this section. We comment here that solving regular layers first while using information form singular layers as δ input function play a very important role in the analysis, see Lemma 7.6 and the proof of Lemma 7.7. The ideas also is used in [30].

To prove that the asymptotic expansions of layer solutions are valid, we need Theorem 7.2. Comparing to the systems in Theorem 7.1, the function $f_y\frac{k}{\omega}\cos(\omega x^\ell + b_0)b$ replaces $\mu s^i(\xi)$ in singular layers, but an extra term $f_y\frac{k}{\omega}\cos(\omega x + b_0)b$ is added to regular layers which has no counter part in Theorem 7.1.

Theorem 7.2. *Consider the following system of equations where F^i and G^i are continuous, bounded functions on I^i. In regular layers,*

$$U_{\xi\xi} + f_u^i U + f_y^i Y + f_y^i \frac{k}{\omega}\cos(\omega x + b_0)b = F^i(\xi), \tag{7.7}$$

$$Y_{xx} + g_u^i U + g_y^i Y = G^i(x), \quad i = 2, 4. \tag{7.8}$$

In singular layers,

$$U_{\xi\xi} + f_u^i U + f_y^i Y + f_y^i \frac{k}{\omega}\cos(\omega x^\ell + b_0)b = F^i(\xi), \tag{7.9}$$

$$Y_{xx} = G^i(x), \quad i = 1, 3, 5. \tag{7.10}$$

With the boundary and jump conditions (7.5) and (7.6), there exists a unique solution that satisfies (7.7)-(7.10). Moreover, there is a positive constant C, independent of ϵ and ω, such that

$$|b/\omega| + \sum_{i=1}^{5}(|U^i| + |Y^i|) \le C\{\sum_{i=1}^{5}(|F^i| + |G^i|_{L^1}) + \sum_{i=1}^{4}\sum_{j=1}^{4}|J_j^i|\}.$$

The proof of Theorem 7.2 uses Theorem 7.1 and is deferred to the end of this section.

Theorem 7.3. *For any integer $m \ge 0$, let (u_{ap}, y_{ap}) be the approximation of an internal layer solution and b_{ap} be an approximation of the parameter as constructed at the beginning of this section. In a small neighborhood of (u_{ap}, y_{ap}, b_{ap}), there exists a unique triplet $(u_{exact}, y_{exact}, b_{exact})$ such that (u_{exact}, y_{exact}) is an exact internal layer solution with the parameter b_{exact}. Moreover, if the approximation is obtained by the truncation to ϵ^mth terms, then*

$$|u_{exact} - u_{ap}| + |y_{exact} - y_{ap}| + |b_{exact} - b_{ap}| \le C\epsilon^{\beta(m+1)}, \quad 0 < \beta < 1.$$

Proof. Let $(-F^i, -G^i)$ be the residual error of the approximation in I^i.

$$\begin{aligned}
&\epsilon^2 u_{ap,xx} + f(u_{ap}, y_{ap} + \frac{k}{\omega}\sin(\omega x + b_{ap})) = -F^i, \quad i = 2,4,\\
&y_{ap,xx} + g(u_{ap}, y_{ap}) = -G^i, \quad i = 2,4,\\
&u_{ap,\xi\xi} + f(u_{ap}, y_{ap} + \frac{k}{\omega}\sin(\omega\epsilon\xi + b_{ap})) = -F^i, \quad i = 1,3,5,\\
&y_{ap,xx} + g(u_{ap}, y_{ap}) = -G^i, \quad i = 1,3,5.
\end{aligned}$$

It is easy to verify that $|F^i| + |G^i| = O(\epsilon^{m+1})$, $i = 2,4$. Since the width of singular layers are $O(\epsilon^{\beta-1})$ and the Taylor expansion of f and g involves polynomial growth terms of ξ, the residual error $|F^i| = O((\epsilon\xi)^{m+1}) = O(\epsilon^{\beta(m+1)})$. Recall that, in singular layers, we used $y_\xi = \epsilon z$, $z_\xi = -\epsilon g$ to do expansions. Only the ϵ^{m-1}th order expansion of g is used when computing the ϵ^mth expansion of y. Therefore, $G^i = O(\epsilon^{m\beta}), i = 1,3,5$. However, since the width of singular layers are $O(\epsilon^\beta)$ in the x scale, $|G^i|_{L^1} = O(\epsilon^{\beta(m+1)})$. In conclusion,

$$|F^i| + |G^i|_{L^1} \le C\epsilon^{\beta(m+1)}. \tag{7.11}$$

We need estimates on jump errors due to truncation. Consider a singular layer u^i_{ap}, $i = 1,3$ and the next regular layer $u^{i+1}_{ap}(x)$. Let $\tilde u^{i+1}(\xi)$ be the inner expansion of regular layers used in §5 for matching of inner and outer layers. Write

$$u^{i+1}_{ap}(a^i+) - u^i_{ap}(\xi^i-) = [u^{i+1}_{ap}(a^i+) - \tilde u^{i+1}(\epsilon^{\beta-1})]_1 + [\tilde u^{i+1}(\epsilon^{\beta-1}) - u^i_{ap}(\xi^i-)]_2.$$

The first term $[\cdots]_1 = O(\epsilon^{\beta(m+1)})$ due to the error of Taylor expansion when computing $\tilde u^{i+1}$. The second term $[\cdots]_2$ is $O(e^{-\alpha\epsilon^{\beta-1}}(1+\epsilon^{\beta-1})^m) = O(\epsilon^{\beta(m+1)})$ due to the exponential matching. The same can be said to jumps to the left of singular layers and to u_ξ, y, y_x. If we denote

$$\begin{aligned}
&u^{i+1}_{ap}(\xi^i) - u^i_{ap}(\xi^i) = -J^i_1, \quad u^{i+1}_{ap,\xi}(\xi^i) - u^i_{ap,\xi}(\xi^i) = -J^i_2,\\
&y^{i+1}_{ap}(a^i) - y^i_{ap}(a^i) = -J^i_3, \quad y^{i+1}_{ap,\xi}(a^i) - y^i_{ap,\xi}(a^i) = -J^i_4,
\end{aligned}$$

then we have

$$\sum_{i=1}^4 \sum_{j=1}^4 |J^i_j| \le C\epsilon^{\beta(m+1)}. \tag{7.12}$$

If we let $(u_{ap}+u, y_{ap}+y)$ be the exact solution with parameter $b_{ap}+b$, then (u,y,b) must cancel all the residual and jump errors. The functions (u,y) satisfy the following linear variational equations. In regular layers,

$$\begin{aligned}
&u_{\xi\xi} + f^i_u u + f^i_y y + f^i_y\frac{k}{\omega}\cos(\omega x + b_0)b = F^i(\xi) + M^i(u,y,b,\epsilon),\\
&y_{xx} + g^i_u u + g^i_y y = G^i(x) + N^i(u,y,\epsilon), \quad i = 2,4.
\end{aligned}$$

In singular layers,

$$\begin{aligned}
&u_{\xi\xi} + f^i_u u + f^i_y y + f^i_y\frac{k}{\omega}\cos(\omega x^\ell + b_0)b = F^i(\xi) + M^i(u,y,b,\epsilon),\\
&y_{xx} = G^i(x) + N^i(u,y,\epsilon), \quad i = 1,3,5.
\end{aligned}$$

We comment that a straight forward linearization of the y equation yields

$$y_{xx} + g_u^i u + g_y^i y = G^i + N^i,$$

in singular layers. But since the length of the domain is $O(\epsilon^\beta)$, the L^1 norm of $g_u^i u + g_y^i y$ is of $O(\epsilon^\beta(|u| + |y|))$ and is moved to N^i.

The nonlinear terms satisfy,

$$\begin{aligned}
&|M^i| \le C(|u^i|^2 + |y^i|^2 + |b|^2 + \epsilon^\beta(|u^i| + |y^i| + |b|)),\\
&|N^i|_{L^1} \le C(|u^i|^2 + |y^i|^2 + \epsilon^\beta(|u^i| + |y^i|)).
\end{aligned}$$

The boundary and jump conditions for (u^i, y^i) are

$$\begin{aligned}
u_\xi^1(\xi^0) = u_\xi^5(\xi^5) = 0, &\quad y_x^1(a^0) = y_x^5(a^5) = 0,\\
u^{i+1}(\xi^i) - u^i(\xi^i) = J_1^i, &\quad u_\xi^{i+1}(\xi^i) - u_\xi^i(\xi^i) = J_2^i,\\
y^{i+1}(a^i) - y^i(a^i) = J_3^i, &\quad y_x^{i+1}(a^i) - y_x^i(a^i) = J_4^i.
\end{aligned}$$

The system for (u, y, b) is exactly as in Theorem 7.2, except the presence of M^i, N^i terms. Let the solution of Theorem 7.2 be denoted

$$(\{U^i\}_1^5, \{Y^i\}_1^5, b) = \mathcal{F}(\{F^i\}_1^5, \{G^i\}_1^5, \{J_j^i\}_{i,j=1}^4).$$

We are led to the equation,

$$(\{u^i\}_1^5, \{y^i\}_1^5, b) = \mathcal{F}(\{F^i + M^i\}_1^5, \{G^i + N^i\}_1^5, \{J_j^i\}_{i,j=1}^4). \tag{7.13}$$

Let $\mathcal{O}_\delta = \{(\{u^i\}_1^5, \{y^i\}_1^5, b) : \sum(|u^i| + |y^i| + |b| \le \delta\}$. If $(\{u^i\}_1^5, \{y^i\}_1^5, b) \in \mathcal{O}_\delta$, then

$$|F^i + M^i| + |G^i + N^i|_{L^1} \le C(\epsilon^\beta \delta + \delta^2).$$

We first choose a small δ and then a sufficiently small ϵ, so that the right side of (7.13) is in $\mathcal{O}_\delta$ and $\mathcal{F}$ is a contraction mapping in $\mathcal{O}_\delta$. Therefore, there exists a unique fixed point for (7.13). The estimates of the solutions follows from (7.11), (7.12) and the estimates in Theorem 7.2. □

Theorem 7.4. *Let (U_{ap}, Y_{ap}) be the approximation of eigenfunctions and λ_{ap} be the approximation of the critical eigenvalue as constructed at the beginning of this section. Then, in a small neighborhood of $(U_{ap}, Y_{ap}, \lambda_{ap})$, there exists a unique triplet $(U_{exact}, Y_{exact}, \lambda_{exact})$ such that λ_{exact} is the exact critical eigenvalue corresponding to the eigenfunctions (U_{exact}, Y_{exact}). Moreover, if the approximation is obtained by the truncation to ϵ^mth terms, then*

$$|U_{exact} - U_{ap}| + |Y_{exact} - Y_{ap}| + |\lambda_{exact} - \lambda_{ap}| \le C\epsilon^{\beta(m+1)}.$$

Proof. Let $(-F^i, -G^i)$ be the residual error of the approximation of the eigenvalue problem in I^i.

$$\begin{aligned}
&-\lambda_{ap} U_{ap} + \epsilon^2 U_{ap,xx} + f_u^i(\text{exact})U + f_y^i(\text{exact})Y = -F^i,\\
&-\lambda_{ap} Y_{ap} + Y_{ap,xx} + g_u^i(\text{exact})U + g_y^i(\text{exact})Y = -G^i, \quad 1 \le i \le 5.
\end{aligned}$$

Here $f_u^i(\text{exact}) = f_u(u_{exact}, y_{exact} + \frac{k}{\omega}\sin(\omega x + b_{exact}))$ in regular layers, etc.. One can verify that $|F^i| + |G^i|_{L^1} \le C\epsilon^{\beta(m+1)}$. The jump errors

$$\begin{aligned} U_{ap}^{i+1}(\xi^i) - U_{ap}^i(\xi^i) &= -J_1^i, \quad U_{ap,\xi}^{i+1}(\xi^i) - U_{ap,\xi}^i(\xi^i) = -J_2^i, \\ Y_{ap}^{i+1}(a^i) - Y_{ap}^i(a^i) &= -J_3^i, \quad Y_{ap,x}^{i+1}(a^i) - Y_{ap,x}^i(a^i) = -J_4^i, \end{aligned}$$

satisfy $\sum_{i=1}^4 \sum_{j=1}^4 |J_j^i| \le C\epsilon^{\beta(m+1)}$ just as in the proof of Theorem 7.3.

Let $(U_{ap} + U, Y_{ap} + Y)$ be the exact eigenfunction corresponding to the eigenvalue $\lambda_{ap} + \lambda$. Then the variational equations for (U, Y, λ) are the following. In regular layers,

$$\begin{aligned} U_{\xi\xi} + f_u^i U + f_y^i Y &= F^i(\xi) + M^i(U, Y, \lambda, \epsilon), \\ Y_{xx} + g_u^i U + g_y^i Y &= G^i(x) + N^i(U, Y, \lambda, \epsilon), \quad i = 2, 4. \end{aligned}$$

In singular layers,

$$\begin{aligned} U_{\xi\xi} + f_u^i U + f_y^i Y - \lambda \dot{q}^\ell(x^\ell/\epsilon + \xi) &= F^i(\xi) + M^i(U, Y, \lambda, \epsilon), \\ Y_{xx} &= G^i(x) + N^i(U, Y, \lambda, \epsilon), \quad i = 1, 3, 5. \end{aligned}$$

The U equation in singular layers has an extra term $\dot{q}^\ell$ since this is the leading term of the expansion of eigenfunctions, while the leading term of the expansion of eigenfunctions is zero elsewhere. The nonlinear terms satisfy,

$$|M^i| + |N^i|_{L^1} \le C((|U^i|^2 + |Y^i|^2 + |\lambda|^2 + \epsilon^\beta(|U^i| + |Y^i| + |\lambda|)).$$

Recall also $\dot{q}^\ell = 0, \ell = 0, 2$ and $\dot{q}^\ell \ne 0, \ell = 1$.

The boundary and jump conditions for (U^i, Y^i) are

$$\begin{aligned} U_\xi^1(\xi^0) = U_\xi^5(\xi^5) &= 0, \quad Y_x^1(a^0) = Y_x^5(a^5) = 0, \\ U^{i+1}(\xi^i) - U^i(\xi^i) &= J_1^i, \quad U_\xi^{i+1}(\xi^i) - U_\xi^i(\xi^i) = J_2^i, \\ Y^{i+1}(a^i) - Y^i(a^i) &= J_3^i, \quad Y_x^{i+1}(a^i) - Y_x^i(a^i) = J_4^i. \end{aligned}$$

With $s^i = \dot{q}^\ell, i = 2\ell + 1$ and $\mu = \lambda$, all the hypotheses in Theorem 7.1 are satisfied. We have to solve a fixed point problem

$$(\{U^i\}_{i=1}^5, \{Y^i\}_{i=1}^5, \lambda) = \mathcal{F}(\{F^i + M^i\}_{i=1}^5, \{G^i + N^i\}_{i=1}^5, \{J_j^i\}_{i,j=1}^4),$$

where $\mathcal{F}$ is the solution map for the problem in Theorem 7.1. Details follow those in the proof of Theorem 7.3 and will be omitted. □

We now present the proof of Theorem 7.1. By the superposition principle, the proof can be divided into three parts, Lemma 7.5–Lemma 7.7. Since the linearized system is so close to the formal asymptotic expansions in section 5, we would like to follow closely the method used in that section. With $\epsilon \ne 0$, the u-equation in the regular layers is not an algebraic equation. Solving u algebraically and then substitute into the y equation, as we did in the formal expansions, is not possible. However, we still can approximately diagonalize the system by introducing the change of variable $U = V + (-f_u^{-1} f_y Y)$, where V is a correction term. Geometrically, this corresponds to decomposing U into two components: One at the tangential direction of the slow manifold, the other at the direction of stable and unstable fibers.

Lemma 7.5. *Consider the system (7.1)–(7.4) with no boundary or jump conditions imposed. Then there exists a (non unique) solution on each I^i and a positive constant C, independent of ϵ, such that*

$$|U^i| + |Y^i| \leq C(|F^i| + |G^i|_{L^1}), \tag{7.14}$$

$$|\mu| \leq C(|F^3| + |G^3|_{L^1}). \tag{7.15}$$

Proof. In singular layers, let $Y^i(x) = \int_{a^i}^x (x-t)G^i(t)dt$, $i = 1, 3, 5$. We have $|Y^i| \leq C|G^i|_{L^1}$, $i = 1, 3, 5$.

Next, extend the domains of F^i and Y^i to $\xi \in \mathbb{R}$ by constants outside I^i.

Observe that when $i = 3$, $\lambda = 0$ is a simple eigenvalue of $U_{\xi\xi} + f_u^i U = 0$ in L^∞. In order that (7.3) is solvable in L^∞, we need the Fredholm condition

$$\int_{-\infty}^{\infty} \dot{q}(\xi + x_0/\epsilon)\{F^i(\xi) - f_y^i Y^i(\xi) - \mu s^3(\xi)\}d\xi = 0.$$

From **A6**,this uniquely determines μ. If an additional condition $< \dot{q}(0), U^3(x_0/\epsilon) >= 0$ is imposed, then the solution U^3 is unique and satisfies (7.14) and (7.15). In boundary layers, using Lemma 5.2 of this paper and Lemma 2.3 of [24], there exists a unique solution of (7.3) that satisfies the boundary condition $U_\xi(0) = 0$ and (7.14).

The lemma has been proved for $i = 1, 3, 5$. Consider the regular layers, I^i, $i = 2, 4$. A severe difficulty occurs because the system is not decoupled. We use a change of variable that almost decouples the system and the final solution comes from an iteration scheme.

The homogeneous linear part of the equation $V_{\xi\xi} + f_u^i V = F^i$ has an exponential dichotomy on I^2, I^4. Using Lemma 5.1, the above has a solution V^i that satisfies $|V^i| \leq C|F^i|$. Let $U = V^i - f_u^{-1} f_y Y$. We now write (7.2) in the form,

$$Y_{xx} + g_u^i V^i - (g_u^i (f_u^i)^{-1} f_y^i - g_y^i)Y = G^i(x).$$

The above has a unique solution $\bar{Y}^i$ if an additional condition $(Y(a^i), Y_x(a^i)) = (0, 0)$ is imposed. The solution satisfies $|\bar{Y}^i| \leq C(|V^i| + |G^i|_{L^1}) \leq C(|F^i| + |G^i|_{L^1})$. The function $\bar{U}^i = V^i - f_u^{-1} f_y \bar{Y}^i$ satisfies

$$\bar{U}^i_{\xi\xi} + f_u \bar{U}^i + f_y \bar{Y}^i = V^i_{\xi\xi} - (f_u^{-1} f_y \bar{Y}^i)_{\xi\xi} + f_u V^i = F^i - (f_u^{-1} f_y \bar{Y}^i)_{\xi\xi}.$$

Equation (7.1) is not precisely satisfied by $\bar{U}^i$. The L^1 norm of the residual error $\bar{F}^i \equiv (f^{-1} f_y \bar{Y}^i)_{\xi\xi}$ satisfies

$$|\bar{F}^i|_{L^1(\xi^{i-1}, \xi^i)} = |\epsilon (f_u^{-1} f_y \bar{Y}^i)_{xx}|_{L^1(a^{i-1}, a^i)} \leq C\epsilon(|F^i| + |G^i|_{L^1(a^{i-1}, a^i)}).$$

We now solve (7.1), (7.2) with F^i replaced by $\bar{F}^i$ and G^i replaced by zero. Using the same process, we can find an approximation of the solution, denoted $(\tilde{U}^i, \tilde{Y}^i), i = 2, 4$, such that the residual error for the U equation in the $L^1(\xi^{i-1}, \xi^i)$ norm is $O(\epsilon |\bar{F}^i|_{L^1(\xi^{i-1}, \xi^i)})$. The proof is almost identical to the previous case, except that the L^1 norm of $\bar{F}^i$ is used instead the supremum norm of F^i. Since $|V^i| \leq C|\bar{F}^i|_{L^1(\xi^{i-1}, \xi^i)}$, using the L^1 norms will not affect the proof.

By the superposition principle, $(U^i, Y^i) = (\bar{U}^i + \tilde{U}^i, \bar{Y}^i + \tilde{Y}^i)$ is a better approximation than $(\bar{U}^i, \bar{Y}^i)$. The error in L^1 norm is reduced by a factor of $O(\epsilon)$. The process can be repeated so that the residual error in the L^1 norm is reduced by a factor of ϵ at each iteration. The iteration process converges to a true solution of the system. $\square$

Lemma 7.6. *In regular layers $I^i, i = 2, 4$, there exists a unique solution to the following system of equations with auxiliary boundary and jump conditions:*

$$(7.16) \qquad \begin{aligned} &Y^i_{xx} - (g^i_u(f^i_u)^{-1}f^i_y - g^i_y)Y^i = 0, \\ &Y^2_x(a^1) = B_1, \quad Y^4_x(a^4) = B_2, \\ &Y^4(a^3) - Y^2(a^2) = H_1, \\ &Y^4_x(a^3) - Y^2_x(a^2) = H_2. \end{aligned}$$

Moreover, we have

$$|Y^2|_{C^1} + |Y^4|_{C^1} \le C(|H_1| + |H_2| + |B_1| + |B_2|).$$

Proof. It suffices to show that if $H_1 = H_2 = B_1 = B_2 = 0$, then the system has only the unique zero solution. Convert (7.16) into a first order system

$$Y^i_x = Z^i, \quad Z^i_x = (g_u f_u^{-1} f_y - g_y)Y^i.$$

Let the solution map be Φ^i on each I^i. Let $\bar{S} = \{(Y, Z)|Z = 0\}$. If $a^1 = 0, a^2 = a^3 = x_0$ and $a^4 = 1$, then from Lemma 4.1,

$$\Phi^2(a^2, a^1)\bar{S} \oplus \Phi^4(a^3, a^4)\bar{S} = \mathbb{R}^2.$$

By the continuous dependence of $\Phi^i(t, s)$ on t and s, we conclude that the direct sum splitting is true if ϵ^β is sufficiently small and therefore a^2 and a^3 are sufficiently near x_0. Therefore, the solution is zero if $H_1 = H_2 = B_1 = B_2 = 0$. $\square$

Lemma 7.7. *Consider the system (7.1)–(7.4) with $F^i = G^i = 0$ for all $1 \le i \le 5$, and the boundary and jump conditions (7.5) and (7.6). Then, there exists a unique solution that satisfies (7.1)-(7.4) and the boundary and jump conditions. Moreover,*

$$|\mu| + \sum_{i=1}^{5}(|U^i| + |Y^i|) \le C\sum_{i=1}^{4}\sum_{j=1}^{4}|J^i_j|.$$

Proof. Following the idea of [30], we first solve the Y equation approximately and use the information to solve the U equation. If we write (7.4) as $Y_x = Z$, $Z_x = 0$, then we immediately see that $Z^1 = Z^5 = 0$ and Z^3 is a constant. Therefore, $Z^4(a^3) - Z^2(a^2) = J^2_4 + J^3_4$. It is also clear that Y^1 and Y^5 are constant solutions. Since $\epsilon^\beta \to 0$ as $\epsilon \to 0$, the change of $Y^3(x)$ across the interval I^3 is small.

We first approximate $Y^i, i = 2, 4$ in regular layers. If we let $H_1 = J^2_3 + J^3_3$, $H_2 = J^2_4 + J^3_4$ and impose the boundary conditions $Y^2_x(a^1) = J^1_4$, $Y^4_x(a^4) = -J^4_4$, then, due to Lemma 7.6, equation (7.16) has a unique solution $\bar{Y}^i$ on I^i, $i = 2, 4$. Let $\bar{Z}^i = \bar{Y}^i_x, i = 2, 4$. Let $\bar{Z}^1 = \bar{Z}^5 = 0$. Let $\bar{Z}^3(x) = \bar{Z}^2(a^2) + J^2_4$ or equivalently $\bar{Z}^4(a^3) - J^3_4$. It is easy to verify that all the boundary and jump conditions for $\bar{Z}^i, 1 \le i \le 5$ are satisfied.

Let $\bar{Y}^1(x) = \bar{Y}^2(a^1) - J^1_3$, $\bar{Y}^5(x) = \bar{Y}^4(a^4) + J^4_3$. From $Y_x = Z$, let $\bar{Y}^3(x) = \bar{Y}^2(a^2) + J^2_3 + (x - a^2)\bar{Z}^3$. It is easy to verify that the jumps of $\bar{Y}^i$ are satisfied at

a^1, a^2 and a^4. But at a^3, $\bar{Y}^4(a^3) - \bar{Y}^3(a^3) = J_3^3 + (a^3 - a^2)\bar{Z}^3$. The jump error at a^3 is of $O(\epsilon^\beta|\bar{Z}^3|)$.

From Lemmas 5.1 and 5.2, the homogeneous part of the systems (7.1) and (7.3) has an exponential dichotomy on $I^i, i = 1, 2, 4, 5$. From Lemma 5.3, the homogeneous part of system (7.3) has an exponential dichotomy on $[\xi^2, x_0/\epsilon]$ and $[x_0/\epsilon, \xi^3]$ respectively. Let the projections on I^i be denoted P_s^i, P_u^i. At each $\xi^i, 1 \le i \le 4$, $\mathcal{R}P_u^i(\xi^i) \oplus \mathcal{R}P_s^{i+1}(\xi^i) = \mathbb{R}^2$. Define $\phi_s^i \in \mathcal{R}P_s^{i+1}(\xi^i), 0 \le i \le 4$, and $\phi_u^i \in \mathcal{R}P_u^i(\xi^i), 1 \le i \le 5$ by

$$\phi_s^0 = \phi_u^5 = 0,$$
$$\phi_s^i - \phi_u^i = (J_1^i, J_2^i)^\tau, \quad 1 \le i \le 4.$$

Let $\bar{U}^3$ be a solution of (7.3) in I^3 satisfying

$$P_s^3(\xi^2)\begin{pmatrix} U \\ U_\xi \end{pmatrix} = \phi_s^2, \quad P_u^3(\xi^3)\begin{pmatrix} U \\ U_\xi \end{pmatrix} = \phi_u^3.$$

From Lemma 5.3, such $\bar{U}^3$ uniquely exists if $\dot{q}(0)U(0) + \ddot{q}(0)U_\xi(0) = 0$ and

$$\psi(\xi^2)\phi_s^2 - \psi(\xi^3)\phi_u^3 + \int_{\xi^2}^{\xi^3} \dot{q}(\xi + x_0/\epsilon)\{f_y^3\bar{Y}^3 + \mu s^3(\xi)\}d\xi = 0,$$

where $\psi(\xi - x_0/\epsilon) = (-\ddot{q}(\xi), \dot{q}(\xi))$. Since $\int \dot{q}(\xi + x_0/\epsilon)s^3(\xi)d\xi \ne 0$, μ is uniquely determined. Also

$$|\mu| + |\bar{U}^3| \le C(|\phi_s^2| + |\phi_u^3| + |\bar{Y}^3|) \le C\sum_{i=1}^4\sum_{j=1}^4(|J_j^i| + |J_j^i|).$$

Note that the homogeneous part of (7.1) or (7.3) has an exponential dichotomy on I^i, $i = 1, 2, 4, 5$. With $Y^i = \bar{Y}^i$ and $P_s^i(\xi^{i-1})U^i(\xi^{i-1}) = \phi_s^{i-1}$, $P_u^i(\xi^i)U^i(\xi^i) = \phi_u^i$, and μ, (7.1) or (7.3) has a unique solution for $i = 1, 2, 4, 5$, denoted by $\bar{U}^i$.

In regular layers, $i = 2, 4$, let $V^i = \bar{U}^i + (f_u^i)^{-1}f_y^i\bar{Y}^i$. With the x variable, we show that $|V^i|_{L^1(a^{i-1},a^i)} \le C\epsilon\sum_{i=1}^4\sum_{j=1}^4(|J_j^i| + |J_j^i|)$. In fact,

$$V_{\xi\xi} + f_uV - (f_u^{-1}f_y\bar{Y}^i)_{\xi\xi} = 0.$$

Let Φ be the principal matrix solution for the associated first order system

$$V_\xi = V_1,\ V_{1\xi} = -f_uV + (f_u^{-1}f_y\bar{Y}^i)_{\xi\xi}, \quad i = 2, 4,$$

which has an exponential dichotomy on I^i. See Lemma 5.1. The solution can be expressed by an integral equation

$$\begin{aligned}(V, V_\xi)(\xi) &= \Phi(\xi, \xi^{i-1})P_s(\xi^{i-1})(V, V_\xi)(\xi^{i-1}) + \Phi(\xi, \xi^i)P_u(\xi^i)(V, V_\xi)(\xi^i)\\ &\quad + \int_{\xi^{i-1}}^{\xi}\Phi(\xi, s)P_s(s)(0, (f_u^{-1}f_y\bar{Y}^i)_{\xi\xi}(s))ds + \int_{\xi^i}^{\xi}\Phi(\xi, s)P_u(s)(0, (f_u^{-1}f_y\bar{Y}^i)_{\xi\xi}(s))ds.\end{aligned}$$

Since $\bar{Y}_{xx}^i$ can be expressed by $\bar{Y}^i$ from the second order equation that defines $\bar{Y}^i$, we have $(f_u^{-1}f_y\bar{Y}^i)_{\xi\xi} = O(\epsilon^2|\bar{Y}^i|_{C^1})$ and

$$|V(\xi)| = O(\epsilon^2|\bar{Y}^i|_{C^1}) + |(V, V_\xi)(\xi^{i-1})|e^{-\gamma(\xi - \xi^{i-1})} + |(V, V_\xi)(\xi^i)|e^{-\gamma(\xi^i - \xi)}, \quad i = 2, 4.$$

We have $|V^i|_{L^1(\xi^{i-1},\xi^i)} \le C(\epsilon|\bar{Y}^i|_{C^1} + |(V,V_\xi)(\xi^{i-1})| + |(V,V_\xi)(\xi^i)|)$. However, from $V^i = \bar{U}^i + f_u^{-1}f_y\bar{Y}^i$, $|(V,V_\xi)| \le C(|\bar{U}^i|_{C^1} + |\bar{Y}^i|_{C^1})$ at ξ^i, $1 \le i \le 4$. If we observe that on the x scale, the L^1 norm of V^i gets a factor ϵ, the estimate for the L^1 norm of V^i on the interval (a^{i-1}, a^i) follows.

We rewrite (7.2) as

$$Y_{xx} - (g_u f_u^{-1} f_y - g_y)Y + g_u V = 0.$$

It is now clear that $\bar{Y}^i$, $i = 2,4$ does not satisfy (7.2). The residual error $|g_u V^i|$ is of $O(\epsilon)$ in L^1 norm. Using Lemma 7.5, there exists $(\tilde{U}^i, \tilde{Y}^i), i = 2,4$ that satisfies (7.1) and (7.2), and $|\tilde{U} - \bar{U}| + |\tilde{Y} - \bar{Y}| = O(\epsilon)$.

With the solutions $(\bar{U}^i, \bar{Y}^i), i = 1,3,5$, and $(\tilde{U}^i, \tilde{Y}^i), i = 2,4$, the residual errors of (7.1)–(7.4) are zero but the jump errors are of $O(\epsilon^\beta \sum |J_j^i|)$.

The procedure described above can be repeated indefinitely, each time reducing the jump error by a factor of ϵ^β. The iteration converges to a true solution of the system (7.1)-(7.4) that satisfies all of the boundary and jump conditions.

□

Proof of Theorem 7.2. Let $\{(U_1^i, Y_1^i)\}_{i=1}^5$ and b_1 be a solution to the following system. In regular layers,

$$\begin{aligned} &U_{\xi\xi} + f_u^i U + f_y^i Y = F^i(\xi), \\ &Y_{xx} + g_u^i U + g_y^i Y = G^i(x), \quad i = 2,4. \end{aligned}$$

In singular layers,

$$\begin{aligned} &U_{\xi\xi} + f_u^i U + f_y^i Y + f_y^i \frac{k}{\omega}\cos(\omega x^\ell + b_0)b_1 = F^i(\xi), \\ &Y_{xx} = G^i(x), \quad i = 1,3,5. \end{aligned}$$

The boundary and jump conditions are (7.5) and (7.6). Observe that $\int \dot{q}(\xi) f_y(q(\xi),\tilde{y})d\xi \ne 0$. If (x_0, b_0) is near to the tangential intersection of $\mathcal{C}$ and Γ_1, then $\cos(\omega x_0 + b_0) \ne 0$. With $s^3 = f_y^3 k\cos(\omega x_0 + b_0)$, **A6** is satisfied. Based on Theorem 7.1, the solution is unique and satisfies

$$|b_1/\omega| + \sum_{i=1}^5 (|U_1^i| + |Y_1^i|) \le C\{\sum_{i=1}^5 (|F^i| + |G^i|_{L^1}) + \sum_{i=1}^4 \sum_{j=1}^4 |J_j^i|\}.$$

Next let $\{(U_2^i, Y_2^i)\}_{i=1}^5$ and b_2 be a solution to the following system. In regular layers,

$$\begin{aligned} &U_{\xi\xi} + f_u^i U + f_y^i Y = 0, \\ &Y_{xx} + g_u^i U + g_y^i Y = g_u^i (f_u^i)^{-1} f_y^i \frac{k}{\omega}\cos(\omega x + b_0)b_1, \quad i = 2,4. \end{aligned}$$

In singular layers,

$$\begin{aligned} &U_{\xi\xi} + f_u^i U + f_y^i Y + f_y^i \frac{k}{\omega}\cos(\omega x^\ell + b_0)b_2 = 0, \\ &Y_{xx} = 0, \quad i = 1,3,5. \end{aligned}$$

The boundary conditions (7.5) and zero jump conditions, i.e. (7.6) with $J_j^i = 0$ for all i, j, are imposed. Again, from Theorem 7.1, the solution is unique. Using the Remark following Theorem 7.1, the solution satisfies

$$|b_2/\omega|+\sum_{i=1}^{5}(|U_2^i|+|Y_2^i|) \le C\sum_{i=2,4}\sup_{c,d\in I^i}\{|\int_c^d g_u^i(f_u^i)^{-1}f_y^i\frac{k}{\omega}\cos(\omega x+b_0)b_1dx|\} \le C\frac{k}{\omega^2}|b_1|.$$

Let $U = U_1 + U_2 - f_u^{-1}f_y\frac{k}{\omega}\cos(\omega x + b_0)b_1$ in regular layers, and let $U^i = U_1^i + U_2^i$ in singular layers. Let $Y^i = Y_1^i + Y_2^i$ and $b = b_1 + b_2$. One readily verifies that, with such (U, Y) and b, (7.8)-(7.10) are satisfied. The residual error for (7.7) is

$$E = -[f_u^{-1}f_y\frac{k}{\omega}\cos(\omega x + b_0)b_1]_{\xi\xi} + f_y\frac{k}{\omega}\cos(\omega x + b_0)b_2 = O((\epsilon^2\omega^2 + \omega^{-1})|b_1/\omega|).$$

If ω is sufficiently large and $\epsilon\omega$ is sufficiently small, $|E| << |b_1/\omega|$.

All of the boundary and jump conditions on Y and Y_x are satisfied. The jumps of the U variable are not satisfied due to the addition of the term $-f_u^{-1}f_y\frac{k}{\omega}\cos(\omega x+b_0)b_1$. Denote the jump in the U variable by

$$\tilde{J}_u^i = U^{i+1}(\xi^i) - U^i(\xi^i) = O(|b_1/\omega|), \quad 1 \le i \le 4,$$

which is not small. Let $U_3^i, 1 \le i \le 5$ and b_3 be a solution to the following system

$$\begin{aligned}
&U_{\xi\xi} + f_u^iU = 0, \quad i = 2, 4,\\
&U_{\xi\xi} + f_u^iU + f_y^i\frac{k}{\omega}\cos(\omega x^\ell + b_0)b_3 = 0,\\
&U_\xi^1(0) = U_\xi^5(\xi^5) = 0,\\
&U^{i+1}(\xi^i) - U^i(\xi^i) = -\tilde{J}_u^i.
\end{aligned}$$

Such U^3 and b_3 uniquely exist. In fact, it is a special case of Theorem 7.1 with $Y = 0$, $g_u^i = g_y^i = 0$ and $G^i = 0$. Moreover, one can show that

$$|U_3^i(\xi)| \le C(e^{-\alpha(\xi-\xi^{i-1})} + e^{-\alpha(\xi^i-\xi)})\sum_{i=1}^{4}|\tilde{J}_u^i|,$$

$$|b_3/\omega| \le Ce^{-\alpha\epsilon^{\beta-1}}\sum_{i=1}^{4}|\tilde{J}_u^i| \le C\epsilon^2|b_1/\omega|.$$

This is based the decay of the influence of jumps toward the interior of the intervals. Cf. [22].

Now let $U = U_1 + U_2 + U_3 - f_u^{-1}f_y\frac{k}{\omega}\cos(\omega x + b_0)b_1$ in regular layers, and let $U = U_1 + U_2 + U_3$ in singular layers. Let $Y = Y_1 + Y_2$ and $b = b_1 + b_2 + b_3$. The jump conditions in U and Y are all satisfied. The residual error in the U equations are increased by $O(|b_3/\omega|) = O(\epsilon^2|b_1/\omega|$. The residual error in the Y equations are increased by $O(|g_uU|_{L^1}) = O(\epsilon|b_1/\omega|)$. The residuals error with supremum norm in the U equation and L^1 norm in the Y equation are reduced by a small factor,

$O(\epsilon^2\omega^2 + \omega^{-1})$, of the same norms of the input

$$\{\sum_{i=1}^{5}(|F^i| + |G^i|_{L^1}) + \sum_{i=1}^{4}\sum_{j=1}^{4}|J_j^i|\}.$$

It is clear that the process can be repeated to further reduce the residual error and the iteration converges to a unique true solution (U, Y) and b. □

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