Homoclinic Bifurcations with Weakly Expanding Center Manifolds

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Abstract: Interaction of homoclinic bifurcation and bifurcation on the center manifold is studied. We show that the occurrence of different types of solutions near the homoclinic orbit is determined asymptotically by a reduced system on the center manifold. The method is applied to cases where the center manifold is one- or two-dimensional. When the center manifold is one-dimensional, we can obtain all the solutions near the homoclinic orbit. When a Hopf bifurcation occurs on a two-dimensional center manifold, the system can have infinitely many periodic and aperiodic solutions. These solutions disappear in a manner predicted by the reduced system when the perturbation term is increased. We prove that certain periodic and aperiodic solutions disappear through inverse period doubling or saddle-node bifurcation.

1. Introduction

Nonlinear bifurcation phenomena near a homoclinic solution have drawn much attention following the early work of Silnikov [27, 28], who discovered various periodic and aperiodic solutions near a homoclinic solution that is asymptotic to a hyperbolic equilibrium. Silnikov's work shows that a solution homoclinic to an equilibrium can produce chaos; this mechanism is distinct from the better known one studied by Smale, who showed that a solution homoclinic to a periodic solution can produce chaos. See the books [13] and [30] for additional references.

Homoclinic bifurcation at a nonhyperbolic equilibrium with center manifold has been studied by [20], [24, 25], [4], [9], and [18]. As in these papers we shall study a homoclinic solution $q(t)$ that approaches an equilibrium exponentially in one direction and is tangent to its center manifold in the other direction. But we shall develop methods that in principle apply to center manifolds of arbitrary dimension. The case in which the nonhyperbolic equilibrium is undergoing a Hopf bifurcation, which was recently studied by Deng and Sakamoto [10] is particularly interesting and has motivated our work. Perturbations of such a vector field can exhibit, in addition to Silnikov's phenomenon, a small periodic solution whose Poincaré map can have a Smale horseshoe. Moreover, this horseshoe can degenerate into a tangential intersection of the stable and unstable manifolds of the periodic solution, a very complicated situation that has been studied by [22], [23], and [31]. Thus, the unfolding of the Hopf/homoclinic bifurcation involves at least three of the most interesting phenomena studied in dynamical systems theory.

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We now preview in some detail the results of this paper. The equation we treat can be written as

$$\begin{aligned} \dot{y} &= A_0 y + g_0(y, u, v, \mu), \\ \dot{u} &= A_1 u + g_1(y, u, v, \mu), \\ \dot{v} &= A_2 v + g_2(y, u, v, \mu). \end{aligned} \tag{1.1}$$

Here $x = (y, u, v) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^m$, A_0 is an $n \times n$ matrix with $Re\,\sigma(A_0) = 0$, A_1 is an $\ell \times \ell$ matrix with $Re\,\sigma(A_1) \geq \alpha_0 > 0$ and A_2 is an $m \times m$ matrix with $Re\,\sigma(A_2) \leq -\alpha_0 < 0$, $\mu = (\mu_1, \mu_2)$ is a parameter where μ_1 determines the flow near $(y, u, v) = 0$ and μ_2 is related to the distance of $W^{cu}(\mu)$ and $W^s(\mu)$. g_0, g_1 and g_2 are higher order terms when $\mu_1 = 0$. The invariant manifolds near $(y, u, v, \mu) = 0$ are $W^{cu}_{loc}(\mu) = \{v = 0\}$, $W^{cs}_{loc}(\mu) = \{u = 0\}$ and $W^c_{loc}(\mu) = \{u = 0,\ v = 0\}$. When $\mu = 0$, system (1.1) has a homoclinic solution $x = q(t)$ that approaches $x = 0$ as $t \to \pm\infty$. Assume that $q(\tau) \in W^s_{loc}(0)$ and $q(-\tau) \in W^c_{loc}(0)$ for a large constant $\tau > 0$. Let the solution map for $\dot{y} = g_0(y, 0, 0, \mu)$ be $\Phi(t, \mu, y_0)$.

This paper is divided into two parts. The first part consists of section 2 to section 5, and the second part consists of section 6 to section 9. In the first part of this paper, we present a general method to treat homoclinic bifurcation at a nonhyperbolic equilibrium whose center manifold is weakly expanding and has any finite dimension. Bifurcation equations for homoclinic, heteroclinic, periodic and aperiodic solutions will be derived by a method inspired by the Shadowing Lemma from dynamical systems theory ([22]). The bifurcation equations will then be asymptotically reduced onto the center manifold. Under some nondegeneracy conditions, that are posed on the reduced system, the bifurcation equations can be solved, and a one-to-one correspondence between solutions and their "symbols" can be proved.

We first construct a codimension-one submanifold Σ that is transverse to the orbit of $q(t)$ at $t = \tau$ and a vector $\vec{\Delta} = (\Delta_1, 0, 0)$ that is transverse to $TW^{cu}_{loc}(0) + TW^s_{loc}(0)$. We show that under some general conditions, for any positive infinite sequence $\{t_i\}^\infty_{-\infty}$, there exists a unique piecewise continuous solution $x(t)$ that is orbitally ϵ-near the homoclinic orbit Γ_o, and may have jump along $\vec{\Delta}$ direction each time it hits Σ. Moreover, let $\{\zeta_i\}^\infty_{-\infty}$ be the time sequence that $x(\zeta_i^-) \in \Sigma$, then $\zeta_{i+1} - \zeta_i = t_i$, and $x(\zeta_i^-) - x(\zeta_i^+) = \xi_i \vec{\Delta}$. The unknown sequence $\{\xi_i\}^\infty_{-\infty}$ together with $x(t)$ are continuous functions of $\{t_i\}^\infty_{-\infty}$ and μ_2 both in the uniform and product topologies and $\{\xi_i\}^\infty_{-\infty}$ is denoted by $\xi_i = G_i(\{t_j\}, \mu)$, $i \in \mathbb{Z}$. To have a genuine solution of (1.1) we need to solve bifurcation equations

$$G_i(\{t_j\}, \mu) = 0, \quad i \in \mathbb{Z}. \tag{1.2}$$

We then show that the bifurcation functions can be asymptotically reduced to the center manifold. Let $p(t, \mu) = \Phi(t, \mu, q_y(-\tau))$ where (q_y, q_u, q_v) are the (y, u, v) coordinates for $q(t)$. It is shown that $\tilde{\mathscr{C}}(\mu) = W^{cu}(\mu) \cap W^{cs}(\mu) \cap \Sigma$ is an (n-1)-dimensional smooth submanifold, so is $\mathscr{C}(\mu) = \pi\tilde{\mathscr{C}}(\mu)$ where π is the projection of $W^{cs}_{loc}(\mu)$ onto $W^c_{loc}(\mu)$ along the stable fibers. Let $d(y, \mu)$ be the distance between $y \in \mathbb{R}^n$ to $\mathscr{C}(\mu)$ along $\vec{\Delta}$. (It is shown that $\vec{\Delta} \pitchfork \mathscr{C}(\mu)$ on $\mathbb{R}^n$). Then we show that

$$G_i(\{t_j\}, \mu) - d(p(-t_i + 2\tau, \mu), \mu), \quad i \in \mathbb{Z} \tag{1.3}$$

is a small quantity if $|\mu| \to 0$ and $\epsilon \to 0$. The bifurcation equations have been solved under some nondegeneracy conditions of the function $d(p(-t_i+2\tau, \mu), \mu)$. See H_6) and H_7). The same nondegeneracy conditions also insure the hyperbolicity of the obtained solutions. Our method also extends to solutions $x(t)$ that are near Γ_0 for $\zeta_1 \le t \le \zeta_2$, where we need to specify $x_v(\zeta_1) = \tilde{v}$ and $x_w(\zeta_2) = \tilde{w}$. Here $\tilde{w} = (\tilde{y}, \tilde{u})$, x_v and x_w are the v and w coordinates for $x = (w, v) = (y, u, v)$.

The purpose of the second part of the paper is to treat examples with one and two dimensional center manifolds by the general method. Examples with one dimensional center manifolds have been extensively studied, [20], [24, 25], [4], [9]; in a sense, this paper completes the study. In Figure 2.2, a heteroclinic solution connecting an equilibrium to a large periodic solution is depicted. Such solutions were overlooked in our previous work [4]. In this paper, we shall prove that heteroclinic solutions like that in Figure 2.2 always exist regardless of the dimension of the center manifold. (Similar solutions also exist in homoclinic bifurcation with a hyperbolic equilibrium). Moreover we shall show that when the center manifold is one dimensional, these heteroclinic solutions plus the solutions found in our early work are all the solutions that lie in a neighborhood of the homoclinic solution. The uniqueness of the solutions, which has not been discussed in previous works, is usually more difficult to prove than the existence of solutions.

The example with two-dimensional center manifold that we shall treat is that of Hopf bifurcation mentioned earlier. Deng and Sakamoto [10] studied this case by horseshoe maps. Our bifurcation equation approach allows us to treat some regions in parameter space that were not studied by them.

Our approach is related to a horseshoe map slightly different from that used by Deng and Sakamoto. In order to explain the difficulties in studying the Hopf/homoclinic bifurcation, we shall now construct this map. Suppose that in a neighborhood of the equilibrium, we have in cylindrical coordinates,

$$\begin{cases} \dot{z} = -z, \\ \dot{r} = \mu_1 r + a r^3, \ a > 0, \ \mu_1 \simeq 0, \\ \dot{\theta} = 1 + b r^2. \end{cases}$$

Notice that a Hopf bifurcation occurs as μ_1 passes 0. We assume that when $\mu = (\mu_1, \mu_2) = 0$, there is a heteroclinic orbit connecting the (r, θ)-plane to the positive z-axis.

Let $\Sigma_0 = \{z = 1\}$ and $\Sigma_1 = \{\theta = 0\}$, both transverse to the flow. Following the flow backwards in time a small strip σ_1 in Σ_1, which is narrow in z direction, is mapped to Σ_0 and becomes a small spiraling strip σ_0. This strip σ_0 is narrow in r direction and circles the z-axis once; see Figure 1.1. The forward flow takes σ_1 to meet Σ_0 again, with the image denoted by σ_1^1.

The Poincaré map: $\Sigma_0 \to \Sigma_0$ then maps σ_0 to σ_1^1, which is a typical horseshoe map. Fixed points in $\sigma_0 \cap \sigma_1^1$ correspond to simple periodic solutions of the original ODE, i.e, periodic solutions that follow the original homoclinic solution once. Suppose the image σ_1^1 depends on a second parameter μ_2, and is moving away from the z-axis as μ_2 increases. The horseshoe argument works well until σ_1^1 is nearly tangent to σ_0.

The reduced problem on the center manifold that we will derive is asymptotically a horseshoe map with σ_0 and σ_1^1 being curves of zero width. The investigation of the

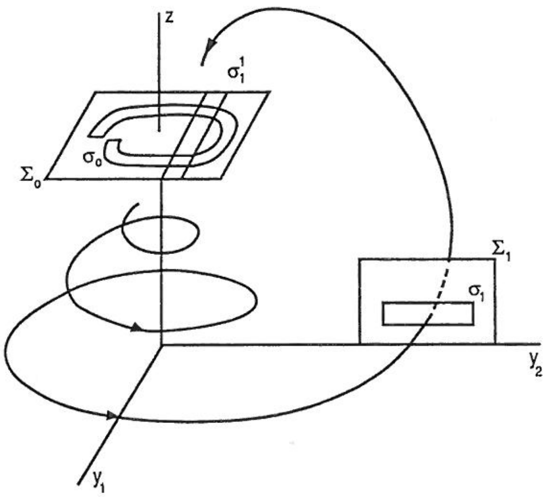

Figure 1.1.

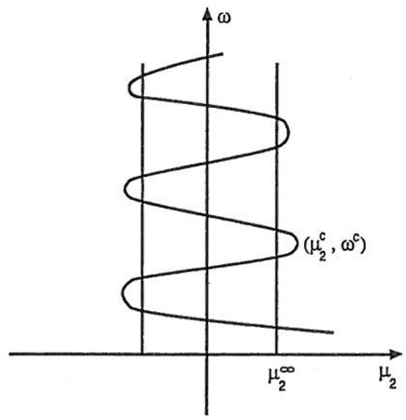

Figure 1.2.

difficult region relies on estimates on the error term (1.3) and its derivatives up to the third order.

Figure 1.2 is a bifurcation diagram for simple periodic solutions in the (ω, μ_2)-plane for $\mu_1 < 0$, where ω is the period of the solution. When $0 \leq \mu_2 \leq \mu_2^\infty$, there are infinitely many simple periodic solutions with the periods shown by the intersections of the sinusoidal curve with μ_2=constant. When $\mu_2 > \mu_2^\infty$, simple periodic solutions with large periods disappear and only finitely many of them remain. Specifically, when $\mu_2 \to \mu_2^c$ from the left, two simple periodic solutions with periods $\omega_1 > \omega^c$ and $\omega_2 < \omega^c$ coalesce and disappear at the quadratic turning point (μ_2^c, ω^c). We will show that (i) $\mu_2^c > \mu_2^\infty$; (ii) $\mu_2^c \to \mu_2^\infty$ monotonously as $\omega^c \to \infty$. The bifurcation of homoclinic solutions is not plotted in Figure 1.2, but is related to it. There are two simple homoclinic solutions connecting the limit cycle $r = \sqrt{\frac{a}{-\mu_1}}$ to itself when $0 \leq \mu_2 < \mu_2^\infty$. At $\mu_2 = \mu_2^\infty$, the two homoclinic solutions coalesce and disappear when $\mu_2 > \mu_2^\infty$.

When μ_2 passes through μ_2^c, much complicated bifurcation may happen. For example, some aperiodic solutions may coalesce pairwise and disappear, and a diagram like Figure 1.2 will be proved in this paper. We will also show that double periodic solutions disappear into simple periodic solutions by a reverse period doubling bifurcation. The method in [7] is employed for this purpose. However, the detailed bifurcation structure for other kind of solutions remains unknown. Related work can be found in [31], [22] and [23]. Recently, Hirschberg and Knobloch [18] have done detailed analytical and numerical study on a simplified Silnikov-Hopf system where all the error terms are being truncated. On the contrary, one of the major feature of this paper is to verify that error terms do not affect the bifurcation diagram.

This paper is organized as follows. In section 2 we will state our main hypotheses and general existence theorems for solutions that stay near the homoclinic orbit, with no restriction on the dimension of the center manifold. Theorem 2.1 describes how the bifurcation equations can be constructed. We then describe how the bifurcation equations can be asymptotically reduced to the local center manifold with very small errors. Theorem 2.2 gives estimates on the error terms of the reduction. Theorem 2.3 assures the existence of nondegenerate solutions, a notion that is related to the transversality condition required when proving chaos by the horseshoe method, but also applies when there is no horseshoe, e.g., when the local center manifold is one dimensional. Theorem 2.4 describes all the solutions that stay near a nondegenerate solution as $t \to +\infty$ or as $t \to -\infty$, or even for a finite time. The importance of Theorem 2.4 will be seen when we treat bifurcation with one-dimensional center manifold in section 6.

We present some basic definitions and technical lemmas in section 3. A technical problem when working near a nonhyperbolic equilibrium is that the variation of constant formula does not have a convergence factor $e^{-\lambda(t-s)}$. A useful method for dealing with this problem is to subtract a nearby solution on the center manifold from the solution under consideration and evaluate the difference. See [16], [4], and [9]. This also motivates the use of the asymptotic projection of local flows to the center manifold. The error of our projection approaches zero if the time a solution stays near the equilibrium approaches infinity. Such projection is achieved by an invariant foliation of the local center-stable manifold using stable fibers, and an invariant foliation of the local center-unstable manifold using unstable fibers [4]. Invariant foliation has also been used in [20]. If we could construct a smooth local invariant foliation of which the fibers were transverse to the local center manifold, we would have constructed the exact projection of local flows to the local center manifold. Unfortunately such a foliation may not be C^1 smooth. Lemma 3.6 is a useful tool for proving the hyperbolicity of periodic solutions constructed by the Shadowing Lemma [14]. It may be further generalized to prove hyperbolicity of aperiodic solutions near a homoclinic orbit.

The main results in section 2 are proved in section 4. For a given sequence $\{t_j\}_{-\infty}^{\infty}$, the solution is expressed as the union of sequences of inner and outer solutions. At the adjacent points, the outer and inner solutions have to match. The bifurcation equations are derived from that, following the idea of the Shadowing Lemma. The proof of the existence of a genuine solution, Theorem 2.3, uses degree theory on a truncated finite system, and also the continuity of bifurcation equations with respect to the sequence $\{t_j\}_{-\infty}^{\infty}$ in the product topology.

In section 5 we shall study simple periodic solutions. Theorem 5.1 shows the hyperbolicity of nondegenerate simple periodic solutions. The idea here is also borrowed from the Shadowing Lemma. Theorem 5.2 says that simple periodic solutions form an one parameter family. Similar results for hyperbolic equilibria are known, cf. [12] and [19].

In section 6 we discuss homoclinic bifurcation with a one-dimensional center manifold (Theorem 6.1). We will describe all the possible solutions near the homoclinic orbit for any type of bifurcation on the local center manifold. The novelty comparing to our early work is a proof that the solutions obtained here are complete.

In section 7, we study the local flows when the equilibrium undergoes a nondegenerate Hopf bifurcation. From the results in section 2 it is clear that sharp estimates on the contraction rate of $\Phi(t, \mu, y_0)$, $t < 0$ are crucial in our method. We show, in Theorem 7.5, 2), that the global rate of contraction is $C|\Phi(t, \mu, y_0)|$ for $t < 0$. However there is a foliation of C^3 curves outside (not including) the limit cycle if $\mu_1 < 0$ (or equilibrium if $\mu_1 \geq 0$) such that when y_0 moves along such a curve, the contraction rate is much sharper (Theorem 7.5, 3)). These curves are transverse to the flow which is roughly $\dot{\theta} = 1$ in the polar coordinates. The proof uses the method of averaging followed by a verification that the truncation of higher order terms does not affect the results. The averaging process in Lemma 7.2 is nonstandard since the leading coefficients are θ-dependent. Notice it is proved in Lemma 7.3 that the flow for the truncated system is C^3 conjugate to the original one.

In section 8 we study the existence of nondegenerate periodic and aperiodic solutions near Γ_0. Theorem 8.4 is a general existence theorem for such solutions. Due to the sharper estimates obtained in section 7 we are able to narrow down the degenerate region of μ_2 that contains μ_2^c to a small strip so that it is clear that simple periodic solutions with longer periods disappear earlier if μ_2 increases from $\mu_2 = 0$. (Theorem 8.3 and Figure 1.2). Notice that the degenerate strips are densely packed near μ_2^{∞}. This is why all the estimates in this paper have to be carefully rendered.

In section 9, Theorem 9.1 and 9.2, we study how simple or double periodic solutions as well as some aperiodic solutions disappear when μ_2 is near μ_2^c. As mentioned earlier in this introduction, these occur in the region of μ_2 where the horseshoe map is degenerate. Second order derivatives of the bifurcation functions are computed in order to extract information near that region. The region studied in section 9 overlaps that in section 8 leaving no gap between the two cases. Theorem 9.3 uses the idea of [7] to prove that double periodic solutions disappear to a simple periodic solution through inverse period doubling. Care has been taken to show the size of the bifurcation region so that the proof is some what different from that of [7].

To avoid tedious tracing of different constants, we use the symbol "term $1 \leq C$term 2" in the sense of "there exits a uniform constant C with term $1 \leq C$term 2", but not in the sense that C is always the same constant.

A solution $x_1(t)$ for system (1.1) is said to be orbitally ϵ-near another solution $x_2(t)$ if there exists an $\epsilon > 0$ such that for any t_1, we can find t_2, such that

$$|x_1(t_1) - x_2(t_2)| \leq \epsilon.$$

$x_1(t)$ is said to be orbitally approaching $x_2(t)$ as $t \to \infty$, if for any given $\epsilon > 0$, we can find T, such that for any $t_1 > T$, there exists t_2, such that

$$|x_1(t_1) - x_2(t_2)| \leq \epsilon.$$

2. Hypotheses, a Reduction Principle and Basic Existence Theorems

We study bifurcations near a homoclinic solution $x = q(t)$, $\mu = 0$, of the following system,

$$\dot{x}(t) = f(x(t), \mu) \tag{2.1}$$

where $f \in C^5$, $x \in \mathbb{R}^{l+m+n}$, $\mu = (\mu_1, \mu_2)$ with $\mu_1 \in \mathcal{M}$, $\mu_2 \in \mathbb{R}$, where $\mathcal{M}$ is an open set in a Banach space. To simplify the illustration, we assume the global existence of solutions of (2.1). Let $f(0,0) = 0$, and $D_x f(0,0)$ be nonhyperbolic, having l (m or n) eigenvalues with positive (negative or zero) real parts. Let $W^{cu}_{loc}(0,0)$, $W^{cs}_{loc}(0,0)$, $W^{c}_{loc}(0,0)$, $W^{u}_{loc}(0,0)$ and $W^{s}_{loc}(0,0)$ be the local center unstable, center stable, center, unstable and stable manifolds for the augmented system

$$\begin{aligned} \dot{x} &= f(x, \mu) \\ \dot{\mu} &= 0 \end{aligned} \tag{2.2}$$

near $(x, \mu) = (0, 0)$. Define a $\overline{\mu}$ section of these manifolds by $W^{cu}_{loc}(\overline{\mu}) = W^{cu}_{loc}(0,0) \cap \{\mu = \overline{\mu}\}$, etc.. Let $T(t, \mu, x)$ be the flow generated by (2.1), we define global invariant manifolds by

$$W^c(\mu) = \bigcup_{t \in \mathbb{R}} T(t, \mu, W^c_{loc}(\mu)),$$

etc. The local invariant manifolds mentioned above are all C^5, cf [1], [3], [17], [29] and [5]. Those locally invariant manifolds may not be unique, but this will not affect our analysis. Using the spectral projections, we have a coordinate system

$$x = \begin{pmatrix} y \\ u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$$

where the neutral, unstable and stable eigenspaces are identified with $\mathbb{R}^n$, $\mathbb{R}^l$ and $\mathbb{R}^m$. There exist C^5 functions $h_1 : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^l$, $h_2 : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, and a small constant $\rho > 0$ such that

$$\begin{aligned} W^{cs}_{loc}(\mu) &= \{x | u = h_1(v, y, \mu), \quad \max\{|y|, |v|, |\mu|\} < \rho\}, \\ W^{cu}_{loc}(\mu) &= \{x | v = h_2(u, y, \mu), \quad \max\{|y|, |u|, |\mu|\} < \rho\}. \end{aligned}$$

By a C^5 change of variable $(y, u, v) \to (y^1, u^1, v^1)$:

$$\begin{aligned} u^1 &= u - h_1(v, y, \mu), \\ v^1 &= v - h_2(u, y, \mu), \\ y^1 &= y, \end{aligned}$$

we may assume that

$$W^{cs}_{loc}(\mu) = \{x|u = 0\},$$
$$W^{cu}_{loc}(\mu) = \{x|v = 0\},$$
$$W^{c}_{loc}(\mu) = \{x|u = 0,\ v = 0\},$$

for all $|\mu| < \rho$. Here the super-scripts on the new variables are dropped. For the convenience of typing we shall write $x = (y, u, v)$ instead of $\begin{pmatrix} y \\ u \\ v \end{pmatrix}$. We say y, u and v are projections of x to its y, u and v components and shall be denoted by $(y, u, v) = (x_y, x_u, x_v)$. It is known that $W^{cs}_{loc}(\mu)$ is invariantly fibered by C^5 submanifolds $W^s(x, \mu)$, and $W^{cu}_{loc}(\mu)$ is invariantly fibered by C^5 submanifolds $W^u(x, \mu)$. These manifolds pass through x and depend C^4 on x, cf. [11] and [4]. Incidentally, $W^s(0, 0)$ and $W^u(0, 0)$ are the local strongly stable and unstable manifolds when $(x, \mu) = (0, 0)$. These fibers have the form:

$$W^s((y_0, 0, 0),\ \mu) = \{x|y = y_0 + h_3(v,\ y_0,\ \mu),\ u = 0, \quad \max\{|v| + |\mu|\} < \rho\},$$
$$W^u((y_0, 0, 0),\ \mu) = \{x|y = y_0 + h_4(u,\ y_0,\ \mu),\ v = 0, \quad \max\{|u| + |\mu|\} < \rho\}.$$

Here $h_3(v,\ y_0,\ \mu)$ is C^5 in v and jointly C^4 in all the variables while $h_4(u,\ y_0,\ \mu)$ is C^5 in u and jointly C^4 in all the variables. The function h_i, $i = 3, 4$ is globally defined, $h_3(0,\ y,\ \mu) = 0$, $h_4(0,\ y,\ \mu) = 0$ and $Lip(h_i)$ is $O(\rho)$. After a C^4 change of variable $(y,\ u,\ v) \to (y^1,\ u^1,\ v^1)$, which is implicitly defined by:

$$u = u^1,$$
$$v = v^1,$$
$$y = y^1 + h_3(v^1,\ y^1,\ \mu) + h_4(u^1,\ y^1,\ \mu),$$

we have that

$$W^s((y_0, 0, 0),\ \mu) = \{x|y = y_0,\ u = 0\},$$
$$W^u((y_0, 0, 0),\ \mu) = \{x|y = y_0,\ v = 0\}.$$

Again the super-scripts are dropped. We then assume that for $|\mu| < \mu_0$, a small positive constant, and $x \in \mathbb{O}$, a small neighborhood of $0 \in \mathbb{R}^{l+m+n}$, all the invariant manifolds and the foliations mentioned above exist, and the change of coordinates has been made. Equation (2.1) can be written in the new coordinates as

$$\begin{aligned} \dot{y} &= A_0 y + g_0(y, u, v, \mu), \\ \dot{u} &= A_1 u + g_1(y, u, v, \mu), \\ \dot{v} &= A_2 v + g_2(y, u, v, \mu). \end{aligned} \tag{1.1}$$

Here A_0 is an $n \times n$ matrix with $Re\sigma(A_0) = 0$, A_1 is an $l \times l$ matrix with $Re\sigma(A_1) \geq \alpha_0 > 0$ and A_2 is an $m \times m$ matrix with $Re\sigma(A_2) \leq -\alpha_0 < 0$, $D_x g_i(0, 0, 0, 0) = 0$, $i = 0, 1, 2$. Moreover, because $W^{cu}_{loc}(\mu)$, $W^{cs}_{loc}(\mu)$, $W^u((y, 0, 0),\ \mu)$ and $W^s((y, 0, 0),\ \mu)$ are locally invariant, we have $g_1(y, 0, v, \mu) = 0$, $g_2(y, u, 0, \mu) = 0$, and $g_0(y, 0, 0, \mu) =$

$g_0(y, u, 0, \mu) = g_0(y, 0, v, \mu)$ for $|y| + |u| + |v| + |\mu| < \rho$. Consequently, we have

$$\begin{aligned} g_1(y, u, v, \mu) &= O(|u|), \\ g_2(y, u, v, \mu) &= O(|v|), \\ g_0(y, u, v, \mu) - g_0(y, 0, 0, \mu) &= O(|u||v|). \end{aligned} \tag{2.3}$$

The validity of the above estimates are the main reason for the change of variables we have performed so far. In such coordinates, if (y, u, v) is a solution that stays near the origin for a long time, then $u(t)$ decays exponentially in backward time and $v(t)$ decays exponentially in forward time. Consequently, g_0 is almost independent of (u, v). We say a function x(t) decays (grows) exponentially as $t \to \infty$, if there exists $0 < \alpha < \alpha_0$ such that $|x(t)| \leq Ce^{-\alpha t}$ (or $|x(t)| \geq Ce^{\alpha t} > 0$) as $t \to +\infty$. Here the number α is close to α_0.

The following hypotheses will be used in this paper.

H1) For $\mu = 0$, (2.1) has a homoclinic orbit $\Gamma_0 := \cup\{q(t)|t \in \mathbb{R}\}$, such that Γ_0 is on $W^c_{loc}(0)$ as $t \to -\infty$ and is on $W^s_{loc}(0)$ as $t \to +\infty$.
Let $\tau > 0$ be a large constant with $q(-\tau) = (q_y(-\tau), q_u(-\tau), q_v(-\tau)) \in W^c_{loc}(0) \cap \mathbb{O}$ and $q(\tau) \in W^s_{loc}(0) \cap \mathbb{O}$. Let $\Phi(t, \mu, y)$ be the restrictions of $T(t, \mu, x)$ to $W^c_{loc}(\mu) \sim \mathbb{R}^n$.

H2) There is a small neighborhood $\mathcal{U}$ of $q_y(-\tau)$ on $W^c_{loc}(\mu)$ such that for $|\mu| < \mu_0$, $D_y\Phi(t, \mu, y)$ is a contraction for $-\bar{t} < t < -\bar{\bar{t}} < 0$ and $y \in \mathcal{U}$. Here $\mu_0 > 0$ is a constant. $\bar{t} = \bar{t}(\mu)$ is either a large constant or $+\infty$, $\bar{\bar{t}} > 0$. There is a function $0 < \delta(|t|, \mu) < 1$ such that $|D_y\Phi(t, \mu, y)| \leq \delta(|t|, \mu)$ for $-\bar{t} < t < -\bar{\bar{t}}$ and $y \in \mathcal{U}$. Assume that $\delta(|t|, \mu) = C(r(t) + |\mu|)$ where $r(t) = |\Phi(t, \mu, y)|$ and C does not depend on t, μ or y.

H3) $T_{q(t)}W^{cu}(0) \cap T_{q(t)}W^s(0) = \text{span}\{\dot{q}(t)\}$. Here T_xW denotes the tangent space of a manifold W at $x \in W$.

H4) $W^{cu}(0)$ and $W^{cs}(0)$ intersect transversely along Γ_0.

H5) The flow on $W^c_{loc}(\mu)$ depends only on μ_1.
The linearized equation of (2.1) at $\mu = 0$, around $q(t)$ is

$$\dot{x}(t) - D_x f(q(t), 0)x(t) = 0. \tag{2.4}$$

Let $\psi(t)$ be a nonzero solution of the adjoint equation

$$\dot{x}(t) + D_x f^*(q(t), 0)x(t) = 0 \tag{2.5}$$

of (2.4) with $\psi(0) \perp \{T_{q(0)}W^{cu}(0) + T_{q(0)}W^s(0)\}$.

H6) $\int_{-\infty}^{\infty} \psi(t) D_{\mu_2} f(q(t), 0)dt \neq 0$.

Remark. In H1), it is more natural to assume that the homoclinic orbit Γ_0 is tangent to $W^c_{loc}(0)$ as $t \to -\infty$. It can be proved that we can always choose the nonunique $W^c_{loc}(0)$ to contain Γ_0. Details will not be given here.

In H2), $-\bar{t}(\mu) < t < -\bar{\bar{t}}$ is needed to ensure that $T(t, \mu, x)$ stays in a small neighborhood of $x = 0$. One can see from the example $\dot{y} = \mu_1 + y^2$, $T(t, \mu, x)$ will leave any neighborhood of $x = 0$ if $t \to -\infty$ for any fixed $\mu_1 > 0$.

$\delta(|t|, \mu)$ represents the rate of contraction in backward time for initial point $y \in \mathcal{U}$. In fact for the theorems of this paper to hold, we do not need the entire local center manifold to be weakly expanding.

Hypothesis H4) does not depend on the choice of $W^{cs}(0)$ and $W^{cu}(0)$ though those manifolds are not unique. See [4], Appendix A for a discussion on this matter.

The function $\dot{q}(t)$ is obviously a solution for the linear equation (2.4). Hypothesis H4) is equivalent to assuming that $\dot{q}(t)$ is the only solution of (2.4), up to a scalar factor, that does not grow exponentially as $t \to \pm\infty$.

We shall see in section 3 that up to a scalar factor $\psi(t)$ is the only solution of (2.5) that decays exponentially as $t \to -\infty$ and does not grow exponentially as $t \to +\infty$. Therefore the integral in H6) converges. To show this, observe that $x = 0$ is always an equilibrium of (2.1) for $\mu_1 = 0$, and $q(t) \to 0$ exponentially as $\to +\infty$,thus $D_{\mu 2} f(q(t), 0) \to D_{\mu 2} f(0, 0) = 0$ exponentially as $t \to +\infty$. Hypothesis H6) is equivalent to saying that the intersection of $W^{cu}(0)$ and $W^{s}(0)$ breaks transversely as μ_2 moves away from $\mu_2 = 0$. The latter is usually more difficult to verify than H6). See (5.6) for more details.

To justify H5), we may set $\mathcal{M}$ to be the set of all the C^5 vector fields which are defined in a neighborhood of $x = 0$ and are C^5 close to the constant vector $f(0, 0)$. The parameter μ_2 can be the distance between $W^{cu}(\mu)$ and $W^{s}(\mu)$. The basic setting in this paper is to assume that the flow on $W^{c}_{loc}(\mu)$ is fixed and the bifurcation diagram depends only on the change of μ_2.

We will be interested in a solution $x(t)$ that orbitally stays near Γ_0 and traverses around Γ_o at least once. We say such solution $x(t)$ is orbitally near Γ_o. If in addition $x(t)$ stays in a $\hat{\varepsilon}$-neighborhood of Γ_o, we say $x(t)$ is orbitally $\hat{\varepsilon}$-near Γ_o. A homoclinic (heteroclinic, periodic) solution that traverses around Γ_o only once will be called a simple homoclinic (heteroclinic, periodic) solution. Otherwise it is called a multiple homoclinic (heteroclinic, periodic) solution.

Let Σ be a codimension-one surface intersecting Γ_o transversely at $x = q(\tau)$. Let the orbit of $x(t)$ and Γ_o be close to each other. $x(t)$ may hit Σ infinitely or finitely many times. Let $\{t_i | h \le i < k\}$ be a sequence of times that $x(t)$ spent from Σ to Σ. If h (or k) is finite, then $x(t)$ stays in the neighborhood $\mathcal{O}$ of $x = 0$ as $t \to -\infty$ (or $+\infty$). We consider the case $h = -\infty$ and $k = \infty$ first. In such cases the time sequence will be denoted by $\{t_i\}_{-\infty}^{\infty} = \mathcal{T}(x(\cdot))$. We will also assume that $x(\tau) \in \Sigma$ and t_o is the first time $T(t, \mu, x(\tau))$ hit Σ to fix the phase of $x(t)$. $\mathcal{T}$ is well defined, and is continuous in the sense that $\mathcal{T}(\overline{x}(\cdot)) \to \mathcal{T}(\overline{\overline{x}}(\cdot))$ coordinatewise if $\overline{x}(t) \to \overline{\overline{x}}(t)$ uniformly in any compact subset of $\mathbb{R}$.

We then consider the case h and/or k are finite. Let $\zeta_1 = \inf\{t | x(t) \in \Sigma\}$ and/or $\zeta_2 = \sup\{t | x(t) \in \Sigma\}$. We have $x(\zeta_1 - 2\tau) \in W^{cu}_{loc}(\mu)$ and/or $x(\zeta_2) \in W^{cs}_{loc}(\mu)$. In other words, $x(t)$ satisfies the boundary conditions $x_v(\zeta_1 - 2\tau) = 0$ and/or $x_u(\zeta_2) = 0$. However, these boundary conditions do not imply that $x(t)$ will stay in $\mathcal{O}$ for $t \le \zeta_1 - 2\tau$ and/or $t \ge \zeta_2$. As we will see that given $x_v(\zeta_1 - 2\tau) = \tilde{v}$ and $x_w(\zeta_2) = (\tilde{y}, \tilde{u})$, and the time sequence $\{t_i\}_h^{k-1}$ that $x(t)$ spent from Σ to Σ for $\zeta_1 \le t \le \zeta_2$, there can be only one such $x(t)$ for $\zeta_1 - 2\tau \le t \le \zeta_2$. We therefore extend the domain of the mapping $\mathcal{T}$ to a solution $x(t)$, which is close to the orbit of $q(t)$ only for $\zeta_1 - 2\tau \le t \le \zeta_2$.

We define $\mathcal{T}(x(\cdot)) = \{S_i\}_{h-1}^{k}$ with the convention that $S_i = t_i \in \mathbb{R}^+$ if $h \le i \le k - 1$; $S_{h-1} = \tilde{v}$ if h is finite and $x_v(\zeta_1 - 2\tau) = \tilde{v}$; $S_k = (\tilde{y}, \tilde{u})$ if k=finite and $x_w(\zeta_2) = (\tilde{y}, \tilde{u})$. The definition will be fully justified after we show that $\mathcal{T}$ is in fact one-to-one. It is also clear that for each fixed i, S_i is an element of a finite dimensional linear space and the set of symbols $\{S_i\}_{h-1}^{k}$ is in a linear product space. Uniform and product topologies can be defined in the product space in the usual way.

However describing the range of $\mathcal{T}$ and constructing the inverse $\mathcal{T}^{-1}$ are not always easy. Silnikov studied these problems by introducing a sequence of equations satisfied by $\{t_i\}_{-\infty}^{\infty}$ and μ. We shall follow his idea. The bifurcation functions we obtain are some geometric quantities that can be asymptotically projected to $W^c_{loc}(\mu)$. Our method also bears much resemblance to the horseshoe method employed by [10].

Before constructing the bifurcation functions, we will present a heuristic argument as a motivation. Consider in $\mathbb{R}^2$ a homoclinic solution $q(t)$ asymptotic to a hyperbolic equilibrium. The homoclinic solution typically breaks when adding a perturbation $\mu \neq 0$ to the equation. However, let $\vec{\Delta}$ be a vector transverse to $TW^{cu} + TW^s$ at $q(\tau)$ and let us allow solutions to have a jump along $\vec{\Delta}$ direction after hitting Σ. Such a generalized homoclinic solution always uniquely exists and depends continuously on μ. This is merely the rephrasing of Melnikov's method, see [13]. However, considering a homoclinic solution as a periodic solution of infinite period, we infer that for each large period $\omega > 0$, and $|\mu| < \mu_o$, there must be a unique periodic solution $x(t)$ with period ω if we allow $x(t)$ to have a jump along $\vec{\Delta}$ after hitting Σ. We then further infer that if $\{S_i\}_{h-1}^{k}$ is a sequence of symbols and $|\mu| < \mu_o$, there must exist a unique $x(t)$ with $\mathcal{T}(x(\cdot)) = \{S_i\}_{h-1}^{k}$ provided that we allow $x(t)$ to have a jump $\xi_i\vec{\Delta}$ each time it hits Σ. Here ξ_i, $h \le i \le k$ is a sequence of real numbers. The generalized solution $x(t)$ as well as ξ_i, $h \le i \le k$ depend on $\{S_i\}_{h-1}^{k}$ and μ. $\xi_i = 0$, $h \le i \le k$ is the desired bifurcation equation for $x(t)$ to be a genuine solution. As a convention if $h = -\infty$ or $k = +\infty$, by $h \le i$ or $i \le k$, we really mean $-\infty < i$ or $i < +\infty$. All the theorems in the sequel are phrased for the case h and k are both finite. If h and/or k are $-\infty$ and/or $+\infty$, statements concerning indices $i \le h$ and/or $i \ge k$ should be neglected.

The following observation is useful throughout this paper. If $x(t)$ is orbitally $\hat{\epsilon}$-near Γ_o for all $t \in \mathbb{R}$ and $\mathcal{T}x(\cdot) = \{S_i\}_{h-1}^{k}$, then $S_{h-1} = \tilde{v} = 0$ and $S_k = (\tilde{y}, 0)$. Also if $|\mu| \to 0$ and $\hat{\epsilon} \to 0$, then $\tilde{y} \to 0$ and $\hat{t} = \inf\{t_i : h \le i \le k-1\} \to \infty$. On the other hand, if $|\mu| \to 0$, $\tilde{y} \to 0$ and $\hat{t} \to +\infty$, then $x(t)$, $\zeta_1 \le t \le \zeta_2$, is orbitally $\hat{\epsilon}$-near Γ_o, with $\hat{\epsilon} \to 0$. The proof of those facts uses Lemma 3.4 and will be left to the readers.

There are many ways to choose Σ and $\vec{\Delta}$. For our convenience we shall specify the one to be used in the sequel. Let $\tau > 0$ be a large constant so that $q(-\tau) \in \mathcal{O}$ and $q(\tau) \in \mathcal{O}$. Let σ be a $n-1$ dimensional surface on $W^c_{loc}(\mu)$ transverse to $\dot{q}_y(-\tau)$. See Figure 2.1. The hyperplane $\Sigma_o = \{x + q(-\tau) | x = (y, u, v),\ y \in \sigma,\ u \in \mathbb{R}^l,\ v \in \mathbb{R}^m\}$ intersects Γ_o transversely at $q(-\tau)$. Let $T(2\tau, \mu, \Sigma_o) = \Sigma$. Assume H4) holds. Then $W^{cu}(0) \cap W^{cs}(0)$ is a n-dimensional submanifold whose intersection with Σ is a (n-1)-dimensional submanifold, denoted by $\tilde{\mathcal{C}}(\mu)$. The tangent space of $\tilde{\mathcal{C}}(\mu)$ is linearly independent of $TW^s(\mu)$ at $q(\tau)$. We will show that assertion for $\mu = 0$, then it will be true for all $|\mu| < \mu_o$. Suppose the assertion were not true for $\mu = 0$, i.e. we could find a nonzero $\vec{\Delta}_o \in TW^s(0) \cap T\tilde{\mathcal{C}}(0) \subset TW^{cu}(0) \cap TW^s(0)$. From H3), $\vec{\Delta}_o = c\dot{q}(\tau)$. Therefore $\dot{q}(\tau) \in T\Sigma$. This is a contradiction to Σ being transverse to Γ_o. Let Π be a projection from $W^{cs}_{loc}(\mu)$ onto $W^c_{loc}(\mu)$ parallel to $W^s(x, \mu)$, $x \in W^{cu}_{loc}(\mu)$, i.e., $\Pi(y, 0, v) = y$ in the local coordinates. Let $\mathcal{C}(\mu) = \Pi\tilde{\mathcal{C}}(\mu)$. It can be shown that $\mathcal{C}(\mu)$ is (n-1)-dimensional and is diffeomorphic to $\tilde{\mathcal{C}}(\mu)$, based on the property $T\tilde{\mathcal{C}}(\mu) \cap TW^s(x, \mu) = \{0\}$. We then choose $\Delta_1 \perp T\mathcal{C}(0)$ at $y = 0$ on $W^c_{loc}(0)$. Let $\vec{\Delta} = (\Delta_1, 0, 0)$ be a vector in $\mathbb{R}^{l+m+n}$. We claim that $\vec{\Delta} \pitchfork (TW^{cu}(0) + TW^s(0))$ at $q(\tau)$. If not, we would have $\vec{\Delta} \in TW^{cu}(0) + TW^s(0)$. Then $TW^{cu}(0) \cap TW^{cs}(0)$ would be at least $(n+1)$-dimensional, since we already know

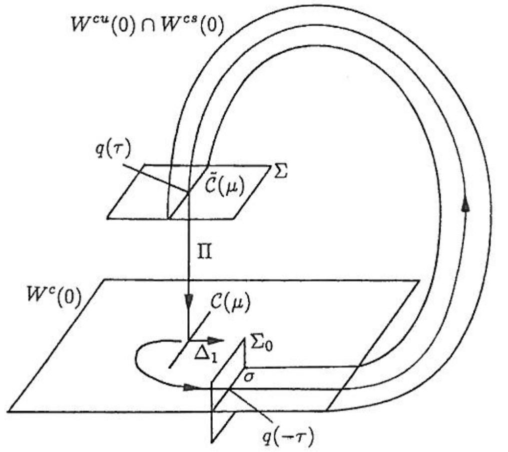

Figure 2.1.

that $T\tilde{\mathscr{C}}(\mu) \oplus \text{span}\{\dot{q}(\tau)\} \subset TW^{cu}(0) \cap TW^{cs}(0)$. This contradicts H4). Denote $p(t, \mu) = \Phi(t, \mu, q_y(-\tau))$, $t \le 0$.

Theorem 2.1. *Assume that H1)-H4) are satisfied. Then there are positive constants $\hat{\mu}$, $\hat{\delta}$, $\hat{\eta}_1$, $\hat{\rho}$ and $\hat{\epsilon}$ with the following property. Let $\{S_i\}_{h-1}^k$ be a sequence of symbols as described before, with $\bar{t}(\mu) \le t_i - 2\tau \le \bar{t}(\mu)$, $\delta(t_i - 2\tau, \mu) < \hat{\delta}$ and $|p(-t_i + 2\tau, \mu)| < \hat{\rho}$ if $h \le i < k$, $S_{h-1} = (\bar{v})$ with $|\bar{v}| < \hat{\eta}_1$ and/or $S_k = (\bar{y}, \bar{u})$ with $|\bar{y}| + |\bar{u}| < \hat{\eta}_1$ if h and/or k are finite. Also assume that $|\mu| < \hat{\mu}$. Then there exists a unique piecewise continuous solution $x(t)$ for (2.1), $\zeta_1 - 2\tau \le t \le \zeta_2$ that is orbitally $\hat{\epsilon}$-near Γ_o and satisfies $\mathcal{T}x(\cdot) = \{S_i\}_{h-1}^k$. $x(t)$ has jumps $\xi_i\vec{\Delta}$, $h \le i \le k$, each occurring when it hits Σ. That is, for each $h \le i \le k$ there exists $s_i \in \mathbb{R}$ such that $x(s_i^-) \in \Sigma$ and $x(s_i^-) - x(s_i) = \xi_i\vec{\Delta}$. Also, $x(s_i) = x(s_i^+)$. Denote*

$$\xi_i = G_i(\{S_j\}, \mu), \ h \le i \le k.$$

$\{G_i\}_h^k$ is C^3 in $\{S_j\}_{h-1}^k$ where both are equipped with the uniform topology. If H5) is satisfied and μ_1 is fixed, $\{G_i\}_h^k$ is C^3 in $\{S_j\}_{h-1}^k$ and μ_2 in the uniform topology. Moreover, if $\{S_i\}_{h-1}^k \to \{\overline{S_i}\}_{h-1}^k$ in the product topology and $\mu_2 \to \overline{\mu}_2$ with $\mu_1 = \overline{\mu}_1$ fixed, then $G_i(\{S_j\}, \mu) \to G_i(\{\overline{S_j}\}, \overline{\mu})$, $h \le i \le k$ and $x(t) \to \overline{x}(t)$ uniformly with respect to t in every compact subset of $[\zeta_1 - 2\tau, \zeta_2]$ that does contain jump points of $\overline{x}(t)$. In addition assume that $\bar{y}$ is such that $T(t, \mu, (\bar{y}, 0, 0)) \in W_{loc}^{cs}(\mu) \cap \mathbb{O}$ for all $t \ge 0$, and ν_2 is such that $T(t, \mu, (y, 0, 0)) \in W_{loc}^{c}(\mu) \cap \mathbb{O}$ for all $|y - q_y(-\tau)| \le \nu_2$ and $t \le 0$. Also assume that $|\bar{y}| < \tilde{\nu}_1$, $|\mu| < \tilde{\mu}$ and $\inf\{t_i\} > \tilde{t}$, with $\tilde{\nu}_1$, $\tilde{\mu}$ small and $\tilde{t}$ large. Then there exists a unique $x(t)$ with $\mathcal{T}x(\cdot) = \{S_i\}_{h-1}^k$, where $S_{h-1} = 0$ and $S_k = (\bar{y}, 0)$. The solution $x(t)$ can be defined for $t \in \mathbb{R}$ and is $\hat{\epsilon}$-near Γ_o for all $t \in \mathbb{R}$. Moreover, $x(t + \zeta_2) \to T(t, \mu, (\bar{y}, 0, 0))$ as $t \to +\infty$.

The bifurcation functions $G_i(\{S_j\}, \mu)$, $h \le i \le k$ are defined by a rather complicated procedure, therefore it is desirable to find approximations to such functions. To each small

$y \in \mathbb{R}^n$, there is a unique $y_1 \in \mathscr{C}(\mu)$ such that $y_1 - y = \xi\Delta_1$. We define $d(y, \mu) = \xi$. Let $\{S_i\}_{h-1}^k$ be a sequence of symbols. Define $y_i = p(-t_i + 2\tau, \mu)$, for $h \leq i \leq k - 1$ and $y_k = \tilde{y}$, where $S_k = (\tilde{y}, \tilde{u})$. In Theorem 2.2, (2.6) and (2.7) we show that $d(y_i, \mu)$ is a good approximation for $G_i(\{S_j\}, \mu)$, $h \leq i \leq k$. We give a short geometric explanation assuming that the y-equation of (1.1), in a neighborhood of $x = 0$, does not depend on (u, v). The solution x shall take time t_i from Σ to Σ and it will take time 2τ from Σ_0 to Σ. The rest of the time $t_i - 2\tau$ is spent close to the origin to move from Σ to Σ_0. Now q goes from $p(-t_i + 2\tau, \mu) = q(-\tau - (t_i - 2\tau))$ to $q(-\tau) \in \Sigma_0$, when $\mu = 0$. Let $\{s_i\}$ be the time sequence when x hits Σ. If $x(s_{i-1})$ is on Σ at the starting point s_{i-1} of the time interval of length t_{i-1}, it will be very close to σ when it hits Σ_0 due to the strong contraction in the v-variable. Therefor, $x(s_i)$ has to be very close to $\mathscr{C}(\mu)$. On the other hand, if $y(s_{i+1} - 2\tau)$ is near $q(-\tau)$, $y(s_{i+1} - t_{i+1})$ must be very close to $p(-t_i + 2\tau, \mu)$ due to the strong contraction on $W^c(0)$ in backwards time. Therefore, the jump between $x(s_i^-)$ to $x(s_i^+)$, along the $\vec{\Delta}$ direction, is almost equal to the jump between $\mathscr{C}(\mu)$ to $p(-t_i + 2\tau, \mu)$. In terms of the definitions made:

$$\xi_i\vec{\Delta} = y(s_i^-) - y(s_i^+) \approx d(p(-t_i + 2\tau), \mu)\Delta_1 = d(y_i, \mu)\Delta_1,$$

or $\xi_i \approx d(y_i, \mu)$.

We consider a one-parameter family of symbols indexed by ζ. Let $\{\Delta t_i\}_h^k$ be a uniformly bounded sequence of real numbers. We define $t_k = 0$ and $t_i(\zeta) = t_i + \zeta\Delta t_i$ $h \leq i \leq k$. Let $y_i(\zeta) = p(-t_i(\zeta) + 2\tau, \mu)$ for $h \leq i \leq k - 1$ and $y_k(\zeta) = g(t_k(\zeta))$ where $g \in C^1(\mathbb{R}, \mathbb{R}^n)$ with $g(0) = \tilde{y}$. Let $u_k(\zeta)$ and $v_h(\zeta)$ be C^1 functions of ζ with ranges in $\mathbb{R}^l$ and $\mathbb{R}^m$ respectively. Let $S_i(\zeta) = t_i(\zeta)$, $h \leq i \leq k - 1$, $S_{h-1}(\zeta) = v_h(\zeta)$ and $S_k(\zeta) = (y_k(\zeta), u_k(\zeta))$. Let $\dot{v}_h(\zeta)$, $\dot{u}_k(\zeta)$ and $\dot{y}_k(\zeta)$ denote derivatives of corresponding C^1 functions. Let $I_{i,j} = 1$ for $i = j$ and $I_{i,j} = 0$ for $i \neq j$. In Theorem 2.2, (2.8), we compare derivatives of $d(y_i(\zeta), \mu)$, ; $h \leq i \leq k$ and $G_i(\{S_j(\zeta)\}, \mu)$, $h \leq i \leq k$ with respect to ζ.

Theorem 2.2. *Assume that H1)-H4) are satisfied. Let $x(t)$ be the unique piecewise continuous solution corresponding to a sequence of symbols $\{S_i\}_{h-1}^k = \mathcal{T}x(\cdot)$ as in Theorem 2.1. Let $\delta_i = \delta(t_i - 2\tau, \mu)$ for $h \leq i \leq k - 1$ and $\delta_i = 0$ otherwise. Let $\rho_i = |y_i|$ for $h \leq i \leq k - 1$, $\rho_k = |\tilde{y}|$, $\rho_i = 0$ otherwise.*

Then if $\hat{\mu}$ and $\hat{\delta}$ in Theorem 2.1 are sufficiently small, we have

$$\begin{aligned}|G_i(\{S_j\}, \mu) - d(y_i, \mu)| \leq {} & C_1(\delta_i(\rho_{i-1} + \rho_i + \rho_{i+1} + |\mu|) \\ & + \delta_{i-1}(\rho_{i-2} + \rho_{i-1} + \rho_i + |\mu|)) \\ & + C_1(I_{i,h}|\tilde{v}| + I_{i,h+1}\delta_h|\tilde{v}| + I_{i,k}|\tilde{u}| + I_{i,k-1}\delta_{k-1}|\tilde{u}|), \ h \leq i \leq k,\end{aligned} \tag{2.6}$$

$$\left|\frac{\partial}{\partial\mu_2}G_i(\{S_j\}, \mu) - \frac{\partial}{\partial\mu_2}d(0, \mu)\right| \leq C_2(\rho_i + \rho_{i-1} + I_{i,h}|\tilde{v}| + I_{i,k}|\tilde{u}| + |\mu|), \quad h \leq i \leq k. \tag{2.7}$$

$$
\begin{aligned}
&\left|\frac{\partial}{\partial\zeta}G_i(\{S_j(\zeta)\},\mu)-\frac{\partial}{\partial\zeta}d(y_i(\zeta),\mu)\right| \\
&\quad\le C_3(\overline{\delta}_i(\overline{\rho}_{i-1}+\overline{\rho}_i+\overline{\rho}_{i+1}+|\mu|)+\overline{\delta}_{i-1}(\overline{\delta}_{i-2}+\overline{\delta}_{i-1}+\overline{\delta}_i))\|\{\Delta t_j\}\| \\
&\quad+C_3((\overline{c\delta})^{i-h}|\dot{v}_h(\zeta)|+(\overline{c\delta})^{k-i}|\dot{u}_k(\zeta)|+I_{i,h}\delta_h|v_h(\zeta)||\Delta t_h|+I_{i,k}\delta_k|u_k(\zeta)||\Delta t_k|)
\end{aligned}
\tag{2.8}
$$

where $\overline{\delta}_i=\delta(t_i(\zeta)-2\tau,\mu)$, $\overline{\rho}_i=|y_i(\zeta)|$ *for* $h\le i\le k-1$, $\overline{\rho}_k=|y_k(\zeta)|$, $\overline{\delta}_k=|\frac{d}{d\zeta}g(t_k(\zeta))|$, *and* $\overline{\delta}=\sup_i\{\overline{\delta}_i\}$.

We shall now consider a one-parameter family of symbols $\{S_i\}_{h-1}^k$ to be used in H7) and in Theorem 2.3 that is different from those used in (2.8). Let $\{S_i\}_{h-1}^k$ be a sequence of symbols with $S_i=t_i$, $0\le\overline{\overline{t}}<t_i<\overline{t}$, $h\le i\le k-1$, $S_{h-1}=\tilde{v}$ and $S_k=(\tilde{y},\tilde{u})$. Let $t_i(\zeta)=t_i+\zeta$ for $h\le i\le k$, where $t_k=0$. Let $y_i(\zeta)=p(-t_i(\zeta)+2\tau,\mu)$, $h\le i\le k-1$ and $y_k(\zeta)=g(t_k(\zeta))$. Let $S_i(\zeta)=t_i(\zeta)$, $h\le i\le k-1$, $S_{h-1}(\zeta)=\tilde{v}$ and $S_k(\zeta)=(y_k(\zeta),\tilde{u})$. The differences are: (i) we now assume $\Delta t_i=1$ for $h\le i\le k$, so that they are not used in the definition of symbols; (ii) $S_{h-1}(\zeta)$ does not depend on ζ now. We make the following hypothesis.

H7) There are positive constants d_i and e_i, $\sup_i\{d_i+e_i\}<\infty$, such that $d(y_i(\zeta_i),\mu)=0$ for some $-d_i<\zeta_i<e_i$, $h\le i\le k$. Also for $-d_i<\zeta<e_i$, $h\le i\le k$,

$$
\begin{aligned}
\left|\frac{\partial}{\partial\zeta}d(y_i(\zeta),\mu)\right| &> C_4[\overline{\delta}_i(\overline{\rho}_{i-1}+\overline{\rho}_i+\overline{\rho}_{i+1}+|\mu|)+\overline{\delta}_{i-1}(\overline{\delta}_{i-2}+\overline{\delta}_{i-1}+\overline{\delta}_i)] \\
&\quad+C_3(I_{i,h}\overline{\delta}_h|\tilde{v}|+I_{i,k}\overline{\delta}_k|\tilde{u}|)
\end{aligned}
\tag{H7;1}
$$

where C_3 is the constant in (2.8) and $C_4>C_3$, $\overline{\delta}_i=\sup\{\delta(t-2\tau,\mu)|t\in(t_i-d_i,t_i+e_i)\}$, $h\le i\le k-1$, $\overline{\delta}_k=\sup\{|\dot{y}_k(\zeta)|;-d_k<\zeta<c_k\}$, and $\overline{\rho}_i=\sup_\zeta\{|y_i(\zeta)|\}$. At the end points $\zeta=-d_i$ and $\zeta=e_i$, $h\le i\le k$, we have the inequality

$$
\begin{aligned}
|d(y_i(\zeta),\mu)| &> C_1\{\delta_i(\rho_{i-1}+\rho_i+\rho_{i+1}+|\mu|)+\delta_{i-1}(\rho_{i-2}+\rho_{i-1}+\rho_i+|\mu|) \\
&\quad+I_{i,h}|\tilde{v}|+I_{i,h+1}\delta_h|\tilde{v}|+I_{i,k}|\tilde{u}|+I_{i,k-1}\delta_{k-1}|\tilde{u}|)\},
\end{aligned}
\tag{H7;2}
$$

where $\delta_i=\delta(t_i(\zeta)-2\tau,\mu)$, $i\le h\le k-1$, $\delta_k=|\dot{y}_k(\zeta)|$ and $\rho_i=|y_i(\zeta)|$, $h\le i\le k$, all evaluated at $\zeta=-d_i$ and $\zeta=e_i$. Also C_1 is the constant in (2.6).

Theorem 2.3. *Let H1)-H4) and H7) be satisfied. Then if* $\hat{\mu}$, $\hat{\delta}$, $\hat{\eta}_1$ *and* $\hat{\rho}$, *as in Theorem 2.1, are small, then there exists a unique sequence* $\{\zeta_i^\infty\}_h^k$, $-d_i<\zeta_i^\infty<e_i$, *such that*

$$
G_i(\{S_j(\zeta_j^\infty)\},\mu)=0,\quad h\le i\le k.
$$

Hence, we have a unique C^1 *solution* $x(t)$ *of (2.1) in a neighborhood of the orbit of* $q(t)$, *with*

$$
\mathcal{T}x(\cdot)=\{S_i(\zeta_i^\infty)\}_{h-1}^k
$$

Solutions whose symbols $\{S_i\}_{h-1}^{k}$ satisfy H7) are called nondegenerate for the reason that $y_i(\zeta_i)$ lies on an arc $\mathfrak{a}_i = \{y_i(\zeta), \zeta \in (-d_i, e_i)\}$ that intersects $\mathscr{C}(\mu)$ transversely. $\{\mathfrak{a}_i\}_h^k$ is called a nondegenerate sequence of arcs.

Theorem 2.4. *If $\{S_i\}_{h-1}^{k}$ is a given symbol sequence that satisfies H7), and if $h \le h^1 \le k^1 \le k$, then $\{S_i\}_{h^1-1}^{k^1}$ is also a sequence of symbols that satisfies H7) provided that $S_{h^1-1} = \tilde{v}_1$, $S_{k^1} = (y_{k^1}, \tilde{u}_1)$, $\tilde{v}_1$ and $\tilde{u}_1$ are sufficiently small and $y_{k^1} = p(-t_{k^1} + 2\tau, \mu)$. Therefore from a nondegenerate solution $x(t)$, we can construct another nondegenerate solution $x^1(t)$, with $\mathcal{T}(x^1(\cdot)) = \{S_i\}_{h^1-1}^{k^1}$. Moreover, if $k^1 = +\infty$ (or $h^1 = -\infty$), then orbitally $x^1(t) \to x(t)$ as $t \to +\infty$ (or as $t \to -\infty$).*

Theorem 2.4, when $k^1 = +\infty$, describes a m-dimensional local stable manifold for the orbit of $x(t)$. If $\tilde{v}_1 = 0$, then $x^1(t)$ is a heteroclinic solution connecting $W^{cu}_{loc}(\mu)$ to the orbit of $x(t)$. The solution $x^1(t)$ may stay in a neighborhood of $\mathbb{O}$ as $t \to -\infty$ if its y projection does so. The existence of such a heteroclinic solution is clear in lower dimensional cases, see Figure 2.2. However such orbits have been overlooked in higher dimensional cases in our early paper [4]. Results like Theorem 2.4 are also true if the equilibrium is hyperbolic.

Theorem 2.4 can also be improved to show the existence of a $l + n - 1$ dimensional unstable manifold for $x(t)$. To do so, we need a local coordinate near $\bar{y} \in \mathscr{C}(\mu)$. See

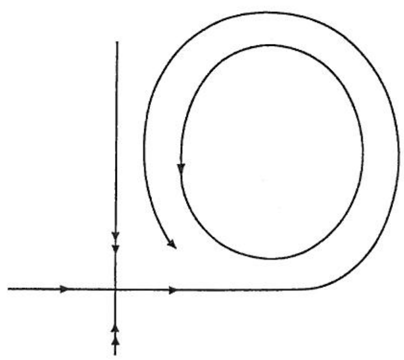

Figure 2.2.

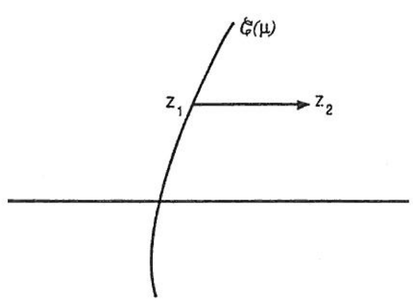

Figure 2.3.

Figure 2.3. Let $y \in \mathbb{R}^n$ be near $\bar{y}$. Define a local coordinate $y = (z_1, z_2)$, so that $z_2 = 0$ implies $y \in \mathcal{C}(\mu)$ and $z_2 \in \mathbb{R}$ measures the distance of y to $\mathcal{C}(\mu)$ along the flow on $W^c_{loc}(\mu)$. We then try to construct a solution $x^1(t)$ corresponding to a given small $(z_1, \tilde{u}_1)$, leaving z_2 to be determined by $t_k(\zeta)$ according to the transversality of $\mathfrak{a}_k$ and $\mathcal{C}(\mu)$. Details will not be presented in this paper.

Results in Theorem 2.4 should be compared with Theorem 5.1 where hyperbolicity of simple periodic solutions will be discussed. Because of the hyperbolicity of the periodic solution $x^2(t)$, we actually have $x^1(t) \to x^2(t)$ in the usual sense, not only orbitally.

3. Preliminaries

Let X(t,s) be the principal matrix solution for the linear equation

$$\dot{x}(t) - D_x f(q(t), 0)x(t) = h(t) \tag{3.1}$$

X(t,s) (or equation (3.1)) is said to have an exponential trichotomy on an interval J if there exist constants $-\alpha < \nu - \epsilon < \nu + \epsilon < \beta$ and $K > 0$ and continuous projections $I = P_c(t) + P_u(t) + P_s(t)$, for all $t \in J$. Furthermore

$$X(t, s)P_i(s) = P_i(t)X(t, s), \text{ for } i = c, u, s \text{ and } t, s, inJ. \tag{3.2}$$

And for all $t, s \in J$, and $t \geq s$ we have

$$\begin{aligned} |X(t, s)P_s(s)| &\leq Ke^{-\alpha(t-s)}, \\ |X(s, t)P_u(t)| &\leq Ke^{-\beta(t-s)}, \\ |X(t, s)P_c(s)| &\leq Ke^{(\nu+\epsilon)(t-s)}, \\ |X(s, t)P_c(t)| &\leq Ke^{(-\nu+\epsilon)(t-s)}. \end{aligned} \tag{3.3}$$

The adjoint equation of (3.1) is (2.5) which has a principal matrix solution $Y(t, s) = [X(t, s)^*]^{-1} = X(s, t)^*$. If $X(t, s)$ has an exponential trichotomy on J, then $Y(t, s)$ also has an exponential trichotomy on J with projections $P^*_u(t)$, $P^*_s(t)$ and $P^*_c(t)$, $t \in J$. Properties similar to (3.2) and (3.3) also hold for the adjoint equation, i.e.,

$$Y(t, s)P^*_i(s) = P^*_i(t)Y(t, s), \ i = c, u, s. \tag{3.2*}$$

And for $t \geq s$,

$$\begin{aligned} |Y(s, t)P^*_s(t)| &\leq Ke^{-\alpha(t-s)}, \\ |Y(t, s)P^*_u(s)| &\leq Ke^{-\beta(t-s)}, \\ |Y(s, t)P^*_c(t)| &\leq Ke^{(\nu+\epsilon)(t-s)}, \\ |Y(t, s)P^*_c(s)| &\leq Ke^{(-\nu+\epsilon)(t-s)}. \end{aligned} \tag{3.3*}$$

Lemma 3.1. *(3.1) has an exponential trichotomy on* $[a, +\infty)$ *(or* $(-\infty, a]$*) where* $a \in \mathbb{R}$ *is any constant. Moreover, let* $\eta > 0$ *be so small that if* $\lambda \in \sigma\{D_x f(0, 0)\}$ *and* $-\eta \leq Re\lambda \leq \eta$*, then* $Re\lambda = 0$*. Then we can choose* $\nu = 0$, $-\alpha < -\eta < -\epsilon < \epsilon < \eta < \beta$ *in (3.3), provided that the constant* $K > 0$ *is sufficiently large. Furthermore* dim $P_i(t) =$ dim $W^i(0))$ *where* $i = u, c, s$.

Proof. See [16] (Lemma 4.3).

Now that (3.1) has exponential trichotomies on $(-\infty, t]$ and $[t, +\infty)$ for any $t \in \mathbb{R}$, we use $P_i^{\pm}(t)$, $i = c, u, s$ to indicate projections related to the right or left intervals. Clearly $\mathcal{R}(P_c^-(t) + P_u^-(t)) = T_{q(t)}W^{cu}(0)$ or $\mathcal{R}P_s^+(t) = T_{q(t)}W^s(0)$ if $|t|$ is large so that $q(t) \in W_{loc}^{cu}(0)$ or $W_{loc}^s(0)$. From a construction of [16], the above are valid for all $t \in \mathbb{R}$. From H_3), $\mathcal{R}(P_c^-(t) + P_u^-(t)) + \mathcal{R}P_s^+(t)$ is of codimension one, and $\psi(0) \perp \{\mathcal{R}(P_c^-(0) + P_u^-(0)) + \mathcal{R}P_s^+(0)\}$. It follows that $\psi(0) \in \mathcal{R}P_s^{-*}(0) \cap \mathcal{R}(P_c^{+*}(0) + P_u^{+*}(0))$. In fact, for each $\phi \in \mathbb{R}^{l+m+n}$, $\langle \psi(0), P_s^+(0)\phi \rangle = 0$. Thus, $\langle P_s^{+*}(0)\psi(0), \phi \rangle = 0$. This implies that $\psi(0) \in \mathcal{R}((P_c^{+*}(0) + P_u^{+*}(0))$. Similarly, one can show $\psi(0) \in \mathcal{R}P_s^{-*}(0)$. Thus, $\psi(t) = Y(t, 0)\psi(0) \in \{\mathcal{R}P_s^{-*}(t), t \leq 0\} \cap \{\mathcal{R}(P_c^{+*}(t) + P_u^{+*}(t)), t \geq 0\}$. Moreover $|\psi(t)| \leq Ce^{\gamma t}$ for $t \leq 0$ and $|\psi(t)| < Ce^{\gamma t}$ for $t > 0$. Here γ is any constant with $0 < \gamma < \alpha_o$, α_o is known from (1.1) and C depends on γ. We have just proved

Lemma 3.2. *There exists a unique (up to scalar multiples) solution $\psi(t)$ of the adjoint equation (2.5) such that $|\psi(t)| \leq C_1 e^{\gamma t}$ for $t \leq 0$ and $|\psi(t)| \leq C_2 e^{\gamma t}$ for $t \geq 0$. Here $0 < \gamma < \alpha_o$ and C_i depends on γ. Moreover*

$$\psi(t) \perp (T_{q(t)}W^{cu}(0) + T_{q(t)}W^s(0)), \ t \in \mathbb{R}.$$

Let $\tau > 0$ be a large fixed constant so that $q(\tau) \in W_{loc}^s(0) \cap \mathbb{O}$ and $q(-\tau) \in W_{loc}^{cu}(0) \cap \mathbb{O}$. Solutions of (2.1) that are near the orbit of $q(t)$, $-\tau \leq t \leq \tau$ shall be considered as solutions of the boundary value problem

$$\begin{cases} \dot{x}(t) = f(x, \mu), & -\tau \leq t \leq \tau \\ v(-\tau) = v^1 \\ w(\tau) = w^1 \end{cases} \tag{3.4}$$

where $x = (y, u, v) = (w, v)$ and $w = (y, u)$. $v^1 \in \mathbb{R}^m$, $w^1 \in \mathbb{R}^{l+n}$ are given small vectors. Obviously $q(t)$, $-\tau \leq t \leq \tau$ is a solution for (3.4) when $v^1 = 0$, $w^1 = 0$ and $\mu = 0$. Let $\mathcal{S}_1 = \{x | v = v^1\}$ be the initial manifold and $\mathcal{S}_2 = \{x | w = w^1\}$ be the terminal manifold. Observe that $T(2\tau, 0, \mathcal{S}_1)$ does not intersect $\mathcal{S}_2$ transversely when $v^1 = 0$, $w^1 = 0$ and $\mu = 0$. We impose a phase condition at $t = -\tau$,

$$x(-\tau) \in \Sigma_o. \tag{3.5}$$

We consider generalized solutions by allowing the solution to have a jump at $t = \tau$,

$$x(\tau^-) - x(\tau) = \xi \overrightarrow{\Delta}, \ \xi \in \mathbb{R}. \tag{3.6}$$

Here Σ_o is a codimension-one hyperplane transverse to $\dot{q}(-\tau)$, $q(-\tau) \in \Sigma_o$, and $\overrightarrow{\Delta}$ is transverse to $TW^{cu}(0) + TW^s(0)$ at $q(\tau)$ and $x(\tau^-) = \lim_{t \uparrow \tau} x(t)$.

Lemma 3.3. *There exist $\mu_o > 0$ and $\epsilon > 0$ such that for $|\mu| < \mu_o$ and $|w^1| + |v^1| < \epsilon$, the generalized boundary value problem (3.4), (3.5) and (3.6) admits a unique solution $x(t)$ and ξ. Moreover $x(t)$ and ξ depend C^r on v^1, w^1 and μ if $f \in C^r$. Let $x = (y, u, v) = (w, v)$. Denote the solution by $x(t) = (w_*(t, w^1, v^1, \mu) + q_w(t), v_*(t, w^1, v^1, \mu) + q_v(t))$ and $\xi = \xi_*(w^1, v^1, \mu)$. Then*

$$\max_{-\tau \leq t \leq \tau} \{|w_*(t, w^1, v^1, \mu)| + |v_*(t, w^1, v^1, \mu)|\} + |\xi_*(w^1, v^1, \mu)| \leq C(|w^1| + |v^1| + |\mu|).$$

Proof. Recall that $q_v(-\tau) = 0$ and $q_w(\tau) = 0$. We need to find $w_o \in \mathbb{R}^{l+n}$ and $v_o \in \mathbb{R}^m$ so that

$$T(2\tau, \mu, (w_o + q_w(-\tau), v^1)) - (w^1, v_o + q_v(\tau)) = \xi\vec{\Delta}.$$

Rewrite it as

$$\begin{aligned} &X(\tau, -\tau)(w_o, 0) - (0, v_o) - \xi\vec{\Delta} \\ &= (w^1, 0) - X(\tau, -\tau)(0, v^1) - H(w_o, v^1, \mu). \end{aligned} \tag{3.7}$$

where $H(w_o, v^1, \mu) = T(2\tau, \mu, (w_o+q_w(-\tau), v^1)) - q(\tau) - X(\tau, -\tau)(w_o, v^1) = 0(|\mu| + |w_o|^2 + |v^1|^2)$. Observe that $\mathcal{S}_1 = W^{cu}_{loc}(0)$ and $\mathcal{S}_2 = W^s_{loc}(0)$ when $\mu = 0$, $v^1 = 0$ and $w^1 = 0$. Therefore $\vec{\Delta} \pitchfork \{T_{q(\tau)}\mathcal{S}_2 + X(\tau, -\tau)T_{q(-\tau)}\mathcal{S}_1\}$ by our construction of $\vec{\Delta}$. The left hand side of (3.7) is surjective from $\mathbb{R}^{n+l} \times \mathbb{R}^m \times \mathbb{R}^1$ onto $\mathbb{R}^{l+m+n}$ with the kernel spanned by $(w_o, v_o, \xi) = (\dot{q}_w(-\tau), \dot{q}_v(\tau), 0)$, since $X(\tau, -\tau)\dot{q}(-\tau) = \dot{q}(\tau)$, $\dot{q}_v(-\tau) = 0$ and $\dot{q}_w(\tau) = 0$. Since Σ_o is transverse to $\dot{q}(-\tau) = (\dot{q}_w(-\tau), 0)$, clearly the linearized equation of (3.7) has a unique solution if we require that $(w_o, v^1) \in \Sigma_o$. The rest of the proof follows from the implicit function theorem. □

As pointed out in the introduction, we normally cannot fully project local flows to the local center manifold, i.e., g_0 in (1.1) will depend on (u, v). However, we can choose a coordinate system so that the first equation in (1.1) depends very little on (u, v). See (2.3). In such coordinates, we expect that each solution that stays in a neighborhood of the equilibrium for a long time is very close to a solution on the local center manifold in their y-coordinates. This will be proved in Lemma 3.4. Let $\mathcal{O}$ be a small neighborhood of $x = 0$ where all the local invariant manifolds and invariant foliations as described in section 2 exist. Assume that the desired change of coordinates as in section 2 has been made so that (2.3) is valid. Let α_0 be the constant introduced after (1.1).

Lemma 3.4.

i) *Let β and α_1 be constants with $0 < \beta < \alpha_1 < \alpha_0$. Then we can always choose $\mathcal{O}$ to be sufficiently small so that the following property is valid. Let $-T$ be the first time $(p(t, \mu), 0, 0)$ hits the boundary of $\mathcal{O}$. There is a small neighborhood $\mathcal{O}'$ of the orbit of $(p(t, \mu), 0, 0)$, $-T \leq t \leq 0$ such that each solution $x(t)$ of (1.1), $-t_o \leq t \leq 0$, that stays in $\mathcal{O}'$ can be written in the form*

$$x(t) = (y_c(t) + y^\$(t), u^\$(t), v^\$(t)), \quad -t_o \leq t \leq 0.$$

Here $y_c(t) = \Phi(t, \mu, x_y(0))$; $y^\$(0) = 0$; and $t_o \leq T$. Moreover

$$\begin{aligned} |y^\$(t)| &\leq Ce^{-\alpha_1 t_o - \beta t}, \\ |u^\$(t)| &\leq Ce^{\alpha_1 t}, \\ |v^\$(t)| &\leq Ce^{-\alpha_1(t+t_o)}, \end{aligned}$$

where $C = C(\beta, \mathcal{O}')$ is independent of μ and t_o if $|\mu| < \mu_o$ is satisfied.

ii) *There exist positive μ_o, ϵ_o and $\bar{t}$ such that for $|\mu| < \mu_o$ and $|y_o| + |u_o| + |v_o| < \epsilon_o$, the solution $x(t)$, $-t_o \leq t \leq 0$, as described in i) exists, with $y_c(0) = q_y(-\tau) + y_o$, $u(0) = u_o$ and $v(-t_o) = v_o$, provided that $\bar{t} < t_o < T$ (possibly $T = +\infty$). Furthermore, $y^\$(t)$, $u^\$(t)$ and $v^\$(t)$ are C^{r-2} functions of (t_o, t, y_o, u_o, v_o) if $f \in C^r$. Moreover,* *let ν be a multi-index with $0 \le |\nu| \le r-2$, D^ν be the differentiation with respect to (t_0, t, y_0, u_0, v_0). Suppose $0 < \beta < \alpha_1 - |\nu|\beta - \beta$. Then*

$$|D^\nu y^\$(t)| \le Ce^{-\alpha_1 t_o - (|\nu|+1)\beta t},$$
$$|D^\nu u^\$(t)| \le Ce^{\alpha_1 t},$$
$$|D^\nu v^\$(t)| \le Ce^{-\alpha_1 (t+t_o)},$$

Proof. i) The proof of the estimates on $u^\$(t)$ and $v^\$(t)$ uses integral equations, Gronwall's inequality and (2.3). Details will not be given here. The exponential estimate on the center component $y^\$(t)$ looks surprising at first sight, but it relies on the choice $y^\$(0) = 0$. Assume that the estimates on $|u^\$(t)|$ and $|v^\$(t)|$ are valid, then from (1.1) and (2.3), we have

$$|y^\$(t)| = |\int_0^t e^{A_o(t-s)}\{g_0(y_c(s) + y^\$(s),\ u^\$(s),\ v^\$(s),\ \mu) - g_0(y_c(s),\ 0,\ 0,\ \mu)\}\, ds|$$
$$\le C_1 |\int_0^t e^{\beta(s-t)/2}\{L|y^\$(s)| + C|u^\$(s)||v^\$(s)|\}\, ds|.$$

Here C_1 depends on $\beta > 0$, L is the Lipschitz number of g_0. Using Gronwall's inequality on

$$|e^{\beta t/2} y^\$(t)| \le C_1 |\int_0^t \{Le^{\beta s/2}|y^\$(s)| + Ce^{\beta s/2 - \alpha_1 t_0}\}\, ds|,$$

we obtain the estimate on $|y^\$(t)|$ provided that $C_1 L < \beta/2$. The latter can be achieved if the neighborhood $\mathcal{O}$ is sufficiently small. The proof of (ii) uses the contraction the mapping principle in weighted Banach spaces and can be found in [4].

Corollary 3.5. *For each $|\mu| < \mu_o$, $|y_o| + |u_o| + |v_o| < \epsilon_o$, let $x(t) = (y(t), u(t), v(t))$, $-t_o \le t \le 0$, be the unique solution of (1.1) with the boundary conditions $y(0) = q_y(-\tau) + y_o$, $u(0) = u_o$ and $v(-t_o) = v_o$. Then $u(-t_o)$, $y(-t_o)$ and $v(0)$ are C^3 functions of $(t_o, \mu_2, y_o, u_o, v_o)$ if $f \in C^5$. Moreover, for $0 \le |\nu| \le 3$,*

$$|D^\nu(y(-t_o) - q_y(-\tau - t_0))| \le C\delta(t_o, \mu)$$
$$|D^\nu u(-t_o)| \le Ce^{-\alpha_1 t_o}$$
$$|D^\nu v(0)| \le Ce^{-\alpha_1 t_o}$$

provided that $\delta(t_o, \mu) > Ce^{(-\alpha_1 + 4\beta)t_o}$.

Remark. We need to consider up to the third order derivatives of $(u(t), v(t), y(t))$. Denote $\alpha = \alpha_1 - 4\beta$. Then, for example, $|y^\$(-t_0)| \le Ce^{-\alpha t_o}$. For each given constant C, we can choose t_o large enough such that $\delta^\nu(t_o, \mu) > Ce^{-\alpha t_o}$, $\nu = 1, 2$. This will be useful in proving Theorem 2.2, (2.8), in section 4. A brief proof goes like this. Let $y(t) = \Phi(t, \mu, y(0))$, $-t_0 \le t \le 0$. Let $\eta(t) = y(-t_0 + t)$, $0 \le t \le t_0$. Then

$$\eta' = A_0\eta + g_0(\eta, 0, 0, \mu)$$

with $|g_0| \leq c|\mu| + \epsilon|\eta|$. Here ϵ is small if $|\mu|$ and $|\eta|$ are small. Since $Re\sigma(A_0) = 0$, for any $\beta > 0$, using the variation of constant formula,

$$\begin{aligned}|\eta(t)| \leq & Ce^{\beta t}|\eta(0)| + C\int_0^t e^{\beta(t-s)}|g_0(\eta(s), 0, 0, \mu)|ds \\ \leq & Ce^{\beta t}|\eta(0)| + C\int_0^t e^{\beta(t-s)}(|\mu| + \epsilon|\eta(s)|)ds.\end{aligned}$$

Using Gronwall's inequality on

$$e^{-\beta t}|\eta(t)| \leq C|\eta(0)| + \frac{C}{\beta}|\mu| + C\epsilon\int_0^t e^{-\beta s}|\eta(s)|ds.$$

we find that

$$e^{-\beta t}|\eta(t)| \leq (C|\eta(0)| + \frac{C}{\beta}|\mu|)e^{c\epsilon t}.$$

It $|\eta(t_0)|$ is uniformly bounded away from zero, then

$$\begin{aligned}e^{-(c\epsilon+\beta)t_0} \leq & C_1(|\eta(0)| + |\mu|) \\ \leq & C_2\delta(t_0, \mu).\end{aligned}$$

The desired result follows if $2(c\epsilon + \beta) < \alpha$.

The solution of the boundary value problem in Corollary 3.5 shall be denoted by $x^*(t, t_o, w_o, v_o, \mu) = (w^*(t, t_o, w_o, v_o, \mu), v^*(t, t_o, w_o, v_o, \mu))$ where $w = (y, u)$.

Consider a Banach space Z which splits into 2 linear subspaces

$$Z = U \oplus V.$$

Let $A : Z \to Z$ be a linear bounded operator. Let A^{-1} be uniquely defined at least on U (i.e. for any $u \in U$, there exits a unique $u_1 \in Z$ such that $Au_1 = u$), and $A^{-1} : U \to Z$ be bounded. Assume that $A : V \to Z$ and $A^{-1} : U \to Z$ are contractions with

$$\begin{aligned}|Av| \leq \lambda|v|, \quad & \text{for } v \in V, \\ |A^{-1}u| \leq \lambda|u|, \quad & \text{for } u \in U.\end{aligned}$$

where the constant $0 < \lambda < 1$. Let P_u be the projection with the range being U and the kernel being V. Let $P_v = I - P_u$. Assume that $\max\{|P_u|, |P_v|\} \leq M$.

Lemma 3.6. *If $4M^2\lambda^2 < 1$, then there exist invariant subspaces $\overline{U}$ and $\overline{V}$ under A^{-1} and A respectively, i.e.*

i) *$A^{-1}\overline{U} \subset \overline{U}$ and $A\overline{V} \subset \overline{V}$.*
Moreover,

ii) *$|A\overline{v}| \leq 8M\lambda|\overline{v}|$ for $\overline{v} \in \overline{V}$ and $|A^{-1}\overline{u}| \leq 8M\lambda|\overline{u}|$ for $\overline{u} \in \overline{U}$;*

iii) *$\overline{U} = (I + S_u)U$ and $\overline{V} = (I + S_v)V$, where $S_u : U \to Z$ and $S_v : V \to Z$ with $|S_u| \leq 2M\lambda^2$ and $|S_v| \leq 2M\lambda^2$.*

iv) *Let $\overline{P}_u$ be the projection with the range $\overline{U}$ and the kernel $\overline{V}$. Let $\overline{P}_v = I - \overline{P}_u$. Then*

$$\begin{aligned}\overline{P}_u &= (I + S_u)P_u(I + S_uP_u + S_vP_v)^{-1}, \\ \overline{P}_v &= (I + S_v)P_v(I + S_uP_u + S_vP_v)^{-1}\end{aligned}$$

and

$$\max\{|\overline{P}_u|, |\overline{P}_v|\} \le M(1+2M\lambda^2)(1-4M^2\lambda^2)^{-1}.$$

Proof. For each linear operator $S : V \to Z$, $|S| \le \frac{1}{2M}$, let $S_1 : V \to Z$ be defined as

$$S_1 = -A^{-1}P_u(I+SP_v)^{-1}A. \tag{3.8}$$

Obviously $|S_1| \le 2M\lambda^2 < \frac{1}{2M}$. Also, the map $S \to S_1$ is an contraction with the rate $4M^2\lambda^2 < 1$. Therefore, there exists a unique fixed point, denoted by S_v, with $|S_v| \le 2M\lambda^2$.

Let $(I+S_v)V = \overline{V}$. For each $v \in V$, let

$$v_1 = (I_v + P_vS_v)^{-1}P_vAv.$$

where I_v is the identity: $V \to V$ and $(I_v + P_vS_v)^{-1} : V \to V$.

Obviously $v_1 \in V$. We shall show that

$$A(I+S_v)v = (I+S_v)v_1. \tag{3.9}$$

To this end, first observe that

$$P_v(I+S_vP_v)^{-I} = (I_v+P_vS_v)^{-1}P_v.$$

Applying $(I+S_v)$ to both sides, we have

$$I - P_u(I+S_vP_v)^{-1} = (I+S_v)(I_v+P_vS_v)^{-1}P_v.$$

Applying both sides to Av, we obtain

$$Av - P_u(I+S_vP_v)^{-1}Av = (I+S_v)v_1.$$

From (3.8), the left hand side is precisely $A(I+S_v)v$. (3.9) has been proved. Based on (3.9) we have $A\overline{V} \subset \overline{V}$. This proves half of i).

$$\begin{aligned}|A(I+S_v)v| &\le |I+S_v||v_1| \\ &\le 2|v_1| \le 2(2M\lambda)|v| \\ &\le 4M\lambda \cdot 2|(I+S_v)v| \\ &\le 8M\lambda|(I+S_v)v|\end{aligned}$$

This proves half of ii).

Similarly, we can define S_u as the fixed point for the equation

$$S = -AP_v(I+SP_u)^{-1}A^{-1},$$

where $S : U \to Z$ and $|S| \le \frac{1}{2M}$. Let $\overline{U} = (I+S_u)U$. A^{-1} on $\overline{U}$ is well defined. It is understood that for $u \in U$, $A^{-1}(I+S_u)u = A^{-1}u + A^{-1}S_uu$, while $A^{-1}S_u = -P_v(I+S_uP_u)^{-1}A^{-1}$. We can show

$$A^{-1}(I+S_u)u = (I+S_u)u_1,$$

where $u_1 = (I_u + P_uS_u)^{-1}P_uA^{-1}u$ and I_u is the identity: $U \to U$. The rest of the proof follows from those for $\overline{V}$ and S_v. This proves i)-iii). iv) can be verified directly. It is easy to see $\mathcal{R}\overline{P}_u = (I + S_u)U$. Let $v \in V$. We can show $\overline{P}_u(I + S_v)v = 0$. Details are left as an exercise. Thus $\overline{V} \subset \text{Kernel}\,\overline{P}_u$. Similarly $\overline{U} \subset \ker \overline{P}_v$. Therefore $\overline{P}_u^2 = \overline{P}_u(I - \overline{P}_v) = \overline{P}_u$. □

Definition 3.7. The angle Θ, $0 \le \Theta \le \frac{\pi}{2}$ between two linear subspaces U and V is defined as follows

$$\sin \Theta = \inf\{\text{dist}(u, V),\ \text{dist}(v, U) | u \in U,\ v \in V,\ |u| = |v| = 1\}.$$

It can be shown that $M \le C\Theta^{-1}$. This is very useful in proving Lemma 5.4.

4. Proof of the Main Results in 2.

Proof of Theorem 2.1. The proof is given only for the case where both h and k are finite. Obvious changes can be made for the cases where h and/or k are $-\infty$ and/or $+\infty$. For example, if $h = -\infty$ and $k = +\infty$, $h \le i \le k-1$ should be changed to $-\infty < i < +\infty$, and any statement concerning $i = -\infty$ and $i = +\infty$ should be ignored.

We are given a sequence of symbols $\{S_i\}_{h-1}^{k}$ and are seeking for a solution whose orbit is the union of those of

$$x_h(t),\ x^h(t), \cdots,\ x_i(t),\ x^i(t), \cdots,\ x^{k-1}(t),\ x_k(t).$$

where $x^i(t)$, $-t_i + 2\tau \le t \le 0$, $h \le i \le k-1$ is in $\mathcal{O}$ and shall be called the inner solution and $x_i(t)$, $-\tau \le t \le \tau$, $h \le i \le k$ is orbitally near $q(t)$, and shall be called the outer solution. These solutions are described by the boundary value problems as in Lemma 3.3 and Corollary 3.5. In particular, the boundary value problem that describes inner solutions is sometimes called Silnikov's problem [8] and the boundary value problem that describes outer solutions admits a jump $x_i(\tau^-) - x_i(\tau) = \xi_i \overrightarrow{\Delta}$ and is related to Melnikov's function which measures the gap between $W^{cu}(\mu)$ and $W^s(\mu)$. Also a phase condition $x_i(-\tau) \in \Sigma_o$ applies. Let w_i^o and v_i^o be the w and v components of $x^i(0) - q(-\tau)$ and $x^i(-t_i + 2\tau) - q(\tau)$ respectively. Let w_i^1 and v_i^1 be the w and v components of $x_i(\tau) - q(\tau)$ and $x_i(-\tau) - q(-\tau)$ respectively. We then have

$$x^i(t) = x^*(t,\ t_i - 2\tau,\ w_i^o,\ v_i^o,\ \mu)$$
$$x_i(t) = x_*(t,\ w_i^1,\ v_i^1,\ \mu) + q(t).$$

The outer and inner solutions have to match at common points. This leads to the following system

$$w_i^1 = w^*(-t_i + 2\tau,\ t_i - 2\tau,\ w_i^o,\ v_i^o,\ \mu), \tag{4.1}$$
$$v_{i+1}^1 = v^*(0,\ t_i - 2\tau,\ w_i^o,\ v_i^o,\ \mu), \tag{4.2}$$
$$w_i^o = w_*(-\tau,\ w_{i+1}^1,\ v_{i+1}^1,\ \mu), \tag{4.3}$$
$$v_i^o = v_*(\tau,\ w_i^1,\ v_i^1,\ \mu), \tag{4.4}$$

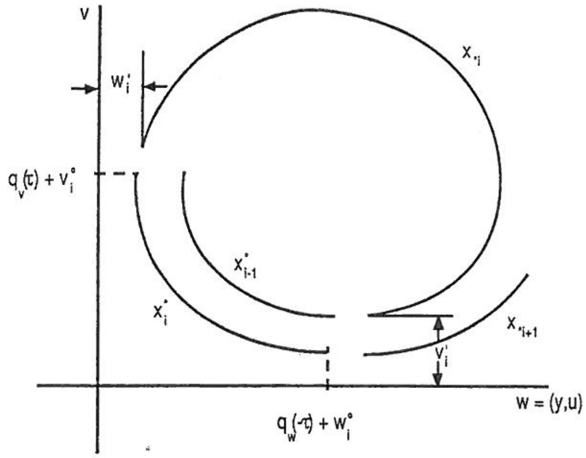

Figure 4.1.

where $h \leq i \leq k-1$, $x^* = (w^*, v^*)$ and $x_* = (w_*, v_*)$. $w_k^1 = (\bar{y}, \bar{u})$ and $v_h^1 = \bar{v}$ are given boundary conditions. Also a phase condition

$$y_*(-\tau, w_i^1, v_i^1, \mu) \in \sigma, \qquad h \leq i \leq k,$$

must be satisfied. However such a constraint is implied in the construction of the functions w_* and v_*, see Lemma 3.3.

To continue the proof of Theorem 2.1, We need the following lemma.

Lemma 4.1. *System (4.1)-(4.4) with $w_k^1 = (\bar{y}, \bar{u})$ and $v_h^1 = \bar{v}$ admits a unique solution $\{w_i^1, v_{i+1}^1, w_i^o, v_i^o\}_h^{k-1}$ if $|\bar{y}| < \hat{\eta}_1$, $|\bar{u}| + |\bar{v}| < \hat{\eta}_2$ and $\inf\{t_i\} > \hat{t}$, $\hat{\mu}$, $\hat{\delta}$, and $\hat{\rho}$ are small and $\hat{t}$ is large. The solution depends C^3 on $\{S_i\}_{h-1}^k$ and μ_2 in the uniform topology. It also depends continuously on $\{S_i\}_{h-1}^k$ and μ_2 in the product topology if $h = -\infty$ and/or $k = +\infty$.*

Proof. We are looking for $\mathscr{X}_i = \{w_i^1, v_{i+1}^1, w_i^o, v_i^o\}$, $h \leq i \leq k-1$. Let $\mathbb{O}(\epsilon_1, \epsilon_2) = \{\mathscr{X}_i | \, |w_i^1| + |v_{i+1}^1| < \epsilon_1, \, |w_i^o| + |v_i^o| < \epsilon_2, \, h \leq i \leq k-1\}$. The right hand side of (4.1)-(4.4) maps $\{\mathscr{X}_i\}_h^{k-1}$ to the left hand side, denoted by $\mathscr{F}\{\mathscr{X}_i\} = \{\overline{\mathscr{X}}_i\} = \{\overline{w}_i^1, \overline{v}_{i+1}^1, \overline{w}_i^o, \overline{v}_i^o\}$. Let $\overline{v}_h^1 = \bar{v}$ and $\overline{w}_k^1 = (\bar{y}, \bar{u})$ so that $\mathscr{F}$ preserves the boundary conditions. Since $w_i^1 = 0$, $v_i^1 = 0$ and $\mu = 0$ imply that $w_* = 0$ and $v_* = 0$, see Lemma 3.3, we have $|\overline{w}_i^o| + |\overline{v}_i^o| < C_1(\hat{\mu} + \epsilon_1)$ if $|\bar{y}| + |\bar{u}| + |\bar{v}| < \epsilon_1$, $|\mu_1| < \hat{\mu}$ and $\{\mathscr{X}_i\}_h^{k-1} \in \mathbb{O}(\epsilon_1, \epsilon_2)$. From Corollary 3.5, where the time t_0 is large because $\hat{t}$ is large, also from (4.1) and (4.2), if $\delta(t_i - 2\tau, \mu) < \hat{\delta}$, then $|\overline{w}_i^1| + |\overline{v}_{i+1}^1| < C_2(\hat{\rho} + \hat{\delta})$, $h \leq i \leq k-1$. The system

$$C_1(\hat{\mu} + \epsilon_1) < \epsilon_2,$$
$$C_2(\hat{\rho} + \hat{\delta}) < \epsilon_1,$$

has many solutions if $\hat{\mu}$, $\hat{\rho}$ and $\hat{\delta}$ are small. Also ϵ_1 and ϵ_2 can be arbitrarily small. For such ϵ_1 and ϵ_2, $\mathscr{F}$ maps $\mathbb{O}(\epsilon_1, \epsilon_2)$ into itself. We show that $\mathscr{F}^2$ is contractive uniformly with respect to $\{S_i\}_{h-1}^k$ and μ. To this end, let $\{\overline{\mathscr{X}}_i\}_h^{k-1}$ and $\{\overline{\overline{\mathscr{X}}}_i\}_h^{k-1}$ be in $\mathbb{O}(\epsilon_1, \epsilon_2)$.

Let $\{\Delta\mathscr{X}_i\}_h^{k-1} = \{\overline{\mathscr{X}}_i - \overline{\overline{\mathscr{X}}}_i\}_h^{k-1}$ and $\{\Delta\mathscr{X}_i(j)\}_h^{k-1} = \mathscr{F}^j\{\overline{X}_i\}_h^{k-1} - \mathscr{F}^j\{\overline{\overline{\mathscr{X}}}_i\}_h^{k-1}$, $j \geq 0$. For brevity, let $\delta_i = \delta(t_i - 2\tau, \mu)$. We can choose $\hat{t}$ sufficiently large so that $\delta_i > e^{-\alpha t_i}$, as in Corollary 3.5 holds. From (4.1), (4.2) and Corollary 3.5, we have estimate (4.5) for all $h \leq i \leq k-1$.

$$|\Delta w_i^1(1)| + |\Delta v_{i+1}^1(1)| \leq C\delta_i(|\Delta w_i^o| + |\Delta v_i^o|). \tag{4.5}$$

From Lemma 3.3, the derivatives of w_* and v_* are uniformly bounded. We can then derive (4.6) from (4.3) and (4.4).

$$|\Delta w_i^o(1)| + |\Delta v_i^o(1)| \leq C(|\Delta w_{i+1}^1| + |\Delta v_{i+1}^1| + |\Delta w_i^1| + |\Delta v_i^1|). \tag{4.6}$$

Here $\Delta w_k^1 = \overline{w}_k^1 - \overline{\overline{w}}_k^1$ and $\Delta v_h^1 = \overline{v}_h^1 - \overline{\overline{v}}_h^1$ are given from the boundary conditions. Repeat the same argument once more, for $h \leq i \leq k-1$,

$$|\Delta w_i^1(2)| + |\Delta v_{i+1}^1(2)| \leq C\delta_i(|\Delta w_i^1| + |\Delta v_i^1| + |\Delta w_{i+1}^1| + |\Delta v_{i+1}^1|), \tag{4.7}$$

where again Δw_k^1 and Δv_h^1 are given. The estimate for $|\Delta w_i^o(2)| + |\Delta v_i^o(2)|$ is not so easy to obtain due to the complexity of the right hand side of (4.6). From (4.5), if $i \neq h$ or $k-1$, we have

$$|\Delta w_{i+1}^1(1)| \leq C\delta_{i+1}(|\Delta w_{i+1}^o| + |\Delta v_{i+1}^o|),$$
$$|\Delta v_i^1(1)| \leq C\delta_{i-1}(|\Delta w_{i-1}^o| + |\Delta v_{i-1}^o|).$$

Those estimates, together with (4.5), allow us to derive from (4.6),

$$\begin{aligned}|\Delta w_i^o(2)| + |\Delta v_i^o(2)| \leq C\{&\delta_i(|\Delta w_i^o| + |\Delta v_i^o|) + \delta_{i+1}(|\Delta w_{i+1}^o| + |\Delta v_{i+1}^o|)\\ &+ \delta_{i-1}(|\Delta w_{i-1}^o| + |\Delta v_{i-1}^o|) + I_{i,h}|\Delta v_h^1| + I_{i,k-1}|\Delta w_k^1|\},\end{aligned} \tag{4.8}$$

if $i \neq h$ or $k-1$. If $i = h$, then $|\Delta v_i^1(1)| = \Delta v_h^1$. If $i = k-1$, then $|\Delta w_{i+1}^1(1)| = \Delta w_k^1$. All these cases can be unified into (4.8) by letting $\delta_k = \delta_{h-1} = 0$, $I_{i,j} = 1$ if $i = j$ and $I_{i,j} = 0$ if $i \neq j$.

Let $\hat{\delta}$ be small, so that $C\delta_i < \frac{1}{8}$ for all $h \leq i \leq k-1$. Then $\mathscr{F}^2$ is a uniform contraction in $\mathbb{O}(\epsilon_1, \epsilon_2)$ and admits a unique fixed point $\{\mathscr{X}_i^\infty\}_h^{k-1}$, denoted by $\{\mathscr{X}_i(\{S_i\}, \mu)\}_h^{k-1}$, which is also the unique fixed point for $\mathscr{F}$, and is C^3 in $\{S_i\}_{h-1}^k$ and μ_2 in the supremum norm if the vector field is C^5.

We now derive Lemma 4.2 which will be used to show that $\{\mathscr{X}_i\}_h^{k-1}$ depends on $\{S_i\}_{h-1}^k$ continuously in the product topology.

Let $\mathscr{Y}_i = (w_i^1, v_{i+1}^1)$, $h \leq i \leq k-1$. Define $\{\overline{\mathscr{Y}}_i\}_h^{k-1} = \mathscr{A}\{\mathscr{Y}_i\}_h^{k-1}$ by $\overline{\mathscr{Y}}_i = (\overline{w}_i^1, \overline{v}_{i+1}^1)$, where $\{\overline{w}_i^1, \overline{v}_{i+1}^1\}_h^{k-1}$ is the first 2 components of $\{\overline{w}_i^1, \overline{v}_{i+1}^1, \overline{w}_i^o, \overline{v}_i^o\}_h^{k-1} = \mathscr{F}^2\{w_i^1, v_{i+1}^1, w_i^o, v_i^o\}_h^{k-1}$, here w_i^o, v_i^o can be any vectors so that $\{w_i^1, v_{i+1}^1, w_i^o, v_i^o\}_h^{k-1} \in \mathbb{O}(\epsilon_1, \epsilon_2)$. From (4.1) and (4.2), $\overline{\mathscr{Y}}_i$ depends on $(w_i^0(1), v_i^0(1))$. The latter, from (4.3) and (4.4), depends only on $(w_i^1, v_i^1, w_{i+1}^1, v_{i+1}^1)$. Thus $\overline{\mathscr{Y}}_i$ does not depend on $\{w_i^o, v_i^o\}_h^{k-1}$ and $\mathscr{A}$ is well defined. From (4.7), $\mathscr{A}$ is a contraction with the fixed point $\{\mathscr{Y}_i^\infty\}_h^{k-1} = \{w_i^{1\infty}, v_{i+1}^{1\infty}\}_h^{k-1}$. The fixed point can be obtained by an iteration scheme starting from any $\{\mathscr{Y}_i(0)\}_h^{k-1} = \{w_i^1(0), v_{i+1}^1(0)\}_h^{k-1}$ with $|w_i^1(0)| + |v_{i+1}^1(0)| \leq \epsilon_1$. Let $\{\mathscr{Y}_i(j)\}_h^{k-1} = \mathscr{A}^j\{\mathscr{Y}_i(0)\}_h^{k-1}$. We now use (4.7) to derive an estimate on $|\mathscr{Y}_i(j) - \mathscr{Y}_i(j+1)|$. Let $\overline{\mathscr{X}}_i = \mathscr{Y}_i(0)$, $\overline{\overline{\mathscr{X}}}_i = \mathscr{Y}_i(1)$. Then from (4.7), if $i \neq h$ or $k-1$, then

$$\begin{aligned}|\mathcal{Y}_i(j) - \mathcal{Y}_i(j+1)| &\le |w_i^1(2j) - w_i^1(2j+2)| + |v_{i+1}^1(2j) - v_{i+1}^1(2j+2)| \\ &\le C\delta_i\{|w_i^1(2j) - w_i^1(2j-2)| + |v_{i+1}^1(2j) - v_{i+1}^1(2j-2)| \\ &\quad + |w_{i+1}^1(2j) - w_{i+1}^1(2j-2)| + |v_i^1(2j) - v_i^1(2j-2)|\}.\end{aligned}$$

Therefore, for $h \le i \le k-1$,

$$|\mathcal{Y}_i(j) - \mathcal{Y}_i(j+1)| \le C\delta_i \sum_{\nu=i-1}^{i+1} |\mathcal{Y}_\nu(j) - \mathcal{Y}_\nu(j-1)|.$$

The above is valid for $i = h$ and $i = k-1$ if the conventions $\delta_i = 0$, and $\mathcal{Y}_i(j) - \mathcal{Y}_i(j-1) = 0$ for $i \notin [h, k-1]$ are made.

It is an unfortunate fact that the right hand side of the above contains different indices ν. Further iteration of the above will increase the indices set if j increases. We can prove, by induction that

$$|\mathcal{Y}_i(j) - \mathcal{Y}_i(j+1)| \le \sum_{\nu=i-j}^{i+j} \overline{C}^j a(i,j,\nu)|\mathcal{Y}_\nu(1) - \mathcal{Y}_\nu(0)|,$$

where $\overline{C} = 3C$, $\delta = \sup \delta_i$ and

$$\begin{aligned}&a(i,j,i) = \delta^j, \\ &a(i,j,\nu) = (\delta_i \cdots \delta_{\nu-1})\delta^{j+i-\nu}, \quad \text{if } \nu > i, \\ &a(i,j,\nu) = (\delta_i \cdots \delta_{\nu+1})\delta^{j+\nu-i}, \quad \text{if } \nu < i.\end{aligned}$$

Here the conventions $\delta_i = 0$ and $\mathcal{Y}_i(j) - \mathcal{Y}_i(j-1) = 0$ for $i \notin [h, k-1]$ are assumed. The assertion is certainly true for $j = 1$. If it is true for the index $j \ge 1$, then for the index $j+1$, we have

$$\begin{aligned}|\mathcal{Y}_i(j+1) - \mathcal{Y}_i(j+2)| &\le C\delta_i \sum_{\xi=i-1}^{i+1} |\mathcal{Y}_\xi(j) - \mathcal{Y}_\xi(j+1)| \\ &\le C\delta_i \sum_{\xi=i-1}^{i+1}\{\sum_{\nu=\xi-j}^{\xi+j} \overline{C}^j a(\xi,j,\nu)|\mathcal{Y}_\nu(1) - \mathcal{Y}_\nu(0)|\} \\ &\le \sum_{\nu=i-j-1}^{i+j+1} C\overline{C}^j\{\sum_{i-1\le\xi\le i+1 \text{ and } \nu-j\le\xi\le\nu+j} \delta_i a(\xi,j,\nu)|\mathcal{Y}_\nu(1) - \mathcal{Y}_\nu(0)|\}.\end{aligned}$$

For a fixed ν, the summation inside the braces contains at most three terms corresponding to $\xi = i-1$, i and $i+1$. In all the three cases, one can verify directly that

$$\delta_i a(\xi,j,\nu) \le a(i,j+1,\nu).$$

Replacing $\delta_i a(\xi,j,\nu)$ in the summation by $a(i,j+1,\nu)$ at most three times and let $3C = \overline{C}$, the assertion for the index $j+1$ then follows easily. This completes the induction argument.

Adding from $j = 0$ to ∞, we have proved

Lemma 4.2. *If* $\overline{C}\delta < 1$ *then*

$$|\mathcal{Y}_i(0) - \mathcal{Y}_i^{\infty}| \leq \frac{1}{1-\overline{C}\delta}\Big\{\sum_{\nu=1}^{\infty} \overline{C}^{\nu}(\delta_i \cdots \delta_{i+\nu-1})|\mathcal{Y}_{i+\nu}(1) - \mathcal{Y}_{i+\nu}(0)|$$
$$+ \sum_{\nu=-1}^{-\infty} \overline{C}^{|\nu|}(\delta_i \cdots \delta_{i+\nu+1})|\mathcal{Y}_{i+\nu}(1) - \mathcal{Y}_{i+\nu}(0)|$$
$$+ |\mathcal{Y}_i(1) - \mathcal{Y}_i(0)|\Big\}.$$

for $h \leq i \leq k-1$. *Here* $\delta_i = 0$ *and* $\mathcal{Y}_i(1) - \mathcal{Y}_i(0) = 0$ *for* $i \notin [h, k-1]$.

It follows immediately from Lemma 4.2 that if $|\mathcal{Y}_i(0) - \mathcal{Y}_i(1)| \to 0$ for each $h \leq i \leq k-1$, then $|\mathcal{Y}_i(0) - \mathcal{Y}_i^{\infty}| \to 0$ for each $h \leq i \leq k-1$.

We now prove the last part of Lemma 4.1. First, let $\overline{\mu}$ be fixed and let $\{S_i\}_{h-1}^k \to \{\overline{S}_i\}_{h-1}^k$ in the product topology, we show

$$\mathcal{X}_i(\{S_j\}, \overline{\mu}) \to \mathcal{X}_i(\{\overline{S}_j\}, \overline{\mu}), \quad \text{for all} \quad h \leq i \leq k-1 \tag{4.9}$$

Let $\{\mathcal{X}_i(\{S_j\}, \overline{\mu})\}_h^{k-1}$ be the fixed point of $\mathcal{F}\{\mathcal{X}_i\} = \{\mathcal{X}_i\}$ corresponding to $\{S_j\}_{h-1}^k$ and $\overline{\mu}$.

Let us consider $\mathcal{X}_i(0) = \mathcal{X}_i(\{S_j\}, \overline{\mu})$ as a zero-th approximation for $\mathcal{X}_i(\{\overline{S}_j\}, \overline{\mu})$. Using (4.1)–(4.4), it is not hard to show $\mathcal{X}_i(2) - \mathcal{X}_i(0) \to 0$ as $\{S_j\}_{h-1}^k \to \{\overline{S}_j\}_{h-1}^k$ for $h \leq i \leq k-1$, where $\{\mathcal{X}_i(2)\}_h^{k-1} = \mathcal{F}^2\{\mathcal{X}_i(0)\}_h^{k-1}$. We like to remind the readers that here the mapping $\mathcal{F}$ is defined using the symbols $\{\overline{S}_i\}_{h-1}^k$ but not $\{S_i\}_{h-1}^k$. Let the first two components of $\mathcal{X}_i(0)$ be $\mathcal{Y}_i(0)$. We then have $\mathcal{Y}_i(1) - \mathcal{Y}_i(0) \to 0$, $h \leq i \leq k-1$. From Lemma 4.2, we have $(w_i^1(0), v_{i+1}^1(0)) \to (w_i^{1\infty}, v_i^{1\infty})$ $h \leq i \leq k-1$. This proves (4.9). Observe that if $\mu_1 = \overline{\mu}_1$ is fixed

$$\sup_i |\mathcal{X}_i(\{S_j\}, \mu) - \mathcal{X}_i(\{S_j\}, \overline{\mu})| \leq C|\mu_2 - \overline{\mu}_2|,$$

where C does not depend on $\{S_i\}_{h-1}^k$, since $\sup_i \left|\frac{\partial \mathcal{X}_i}{\partial \mu_2}\right|$ is bounded uniformly with respect to $\{S_i\}_{h-1}^k$ and μ. The proof of Lemma 4.1 has been completed.

To prove Theorem 2.1, observe that $G_i(\{S_j\}, \mu) = \xi_*(w_i^1, v_i^1, \mu)$, cf. Lemma 3.3. Therefore $\{G_i\}_h^k$ is C^3 in $\{S_j\}_{h-1}^k$ and μ_2 in the uniform topology. Also $\{G_i\}_h^k$ is continuous in $\{S_j\}_{h-1}^k$ and μ_2 in the product topology since $\{w_i^1, v_i^1\}$ is.

We now prove the last part of Theorem 2.1. If $h \neq -\infty$ and/or $k \neq +\infty$, solve (4.1)–(4.4) with $v_h^1 = 0$ and/or $w_k^1 = (\bar{y}, 0)$ where $\bar{y}$ satisfies the condition in Theorem 2.1. Our solution depends continuously on μ, $\bar{y}$ and $\{t_i\}_h^{k-1}$. If $\bar{y} = 0$ and $t_i = +\infty$ for $h \leq i \leq k-1$, one can verify that $\mathcal{X}_i = 0$, $h \leq i \leq k-1$ is the solution to (4.1)-(4.4). Thus if $|\bar{y}| < \tilde{\nu}_1$, $|\mu| < \tilde{\mu}$ and $\inf\{t_i\} > \tilde{t}$, with $\tilde{\nu}_1$, $\tilde{\mu}$ small and $\tilde{t}$ large, we have that $|\mathcal{X}_i|$, $h \leq i \leq k-1$ is small. Therefore $|y_h^1| < \nu_2$ where ν_2 is the constant in Theorem 2.1. We then define $x^-(t) = T(t, \mu, (w_h^1, 0) + q(-\tau))$ for $t \leq 0$ and/or $x^+(t) = T(t, \mu, (\bar{y}, 0, v_k^1) + q(\tau))$, $t \geq 0$. The solution $x^+(t)$ is on $W_{loc}^{cs}(\mu)$, and $x^+(t) \to T(t, \mu, (\bar{y}, 0, 0))$ as $t \to +\infty$ exponentially since $x^+(0) \in W^s((\bar{y}, 0, 0), \mu)$. Since $|y_h^1| < \nu_2$, $\tilde{x}(t) = T(t, \mu, (y_h^1, 0, 0) + q(-\tau)) \in W_{loc}^c(\mu) \cap \mathcal{O}$ for $t \leq 0$. Now the choice of our coordinate system guarantees that $\tilde{x}(0)$ and $x^-(0)$ belong to the same unstable fiber, therefore, $x^-(t)$ approaches $\tilde{x}(t)$ exponentially as $\to -\infty$. We then define $x(t)$, $t \in \mathbb{R}$, orbitally as the union of $x^-(t)$, $x_h(t)$, $x^h(t)$, $\cdots$, $x_k(t)$, $x^+(t)$. The proof of Theorem 2.1 has been completed.

Proof of Theorem 2.2. Like in the proof of Theorem 2.1, let $\mathscr{X}_i = \{w_i^1, v_{i+1}^1, w_i^o, v_i^o\}$. Let $\{\mathscr{X}_i(0)\}_h^{k-1} = \{0\}_h^{k-1}$, $\{\mathscr{X}_i(j)\}_h^{k-1} = \mathscr{F}\{\mathscr{X}_i(j-1)\}_h^{k-1}$, $j \geq 1$, where $\mathscr{F}$ is defined by the right hand side of (4.1)-(4.4). It is also understood that $v_h^1(j) = \bar{v}$ and $w_k^1(j) = (\bar{y}, \bar{u})$ for all $j \geq 0$. The proof of Theorem 2.2 is based on a very simple idea. Since μ, δ_i and ρ_i are small, the first iterate $\{\mathscr{X}_i(1)\}_h^{k-1}$ must be an excellent approximation for the fixed point $\{\mathscr{X}_i^\infty\}_h^{k-1}$ of $\mathscr{F}$.

Let $\eta_o(i) = |w_i^o(0) - w_i^o(1)| + |v_i^o(0) - v_i^o(1)|$ and $\eta_1(i) = |w_i^1(0) - w_i^1(1)| + |v_{i+1}^1(0) - v_{i+1}^1(1)|$, $h \leq i \leq k-1$, and $\eta_1(i) = 0$ for $i < h$ and $i \geq k$. Let $I_{i,j} = 1$ for $i = j$ and $=0$ for $i \neq j$. Substitute $\mathscr{X}_i(0) = 0$ into the right hand side of (4.1) to (4.4). Since w_* and v_* are C^1 bounded, from (4.3) and (4.4), $\eta_o(i) = |w_i^o| + |v_i^o| = O(\mu)$ if $i \neq h$ or $k-1$. If $i = h$ or $k-1$, then $v_h^1 = \bar{v}$ or $w_k^1 = \bar{w}$. Thus

$$\eta_o(i) \leq C(|\mu| + I_{i,h}|\bar{v}| + I_{i,k-1}|\bar{w}|).$$

Using Lemma 3.4, from (4.1) and (4.2),

$$\begin{aligned}\eta_1(i) &= |w_i^1| + |v_{i+1}^1| = |w^*| + |v^*| \\ &\leq |y_c(-t_i + 2\tau) + y^\$(-t_i + 2\tau)| + |u^\$(-t_i + 2\tau)| + |v^\$(0)|.\end{aligned}$$

Since $|y_c(-t_i + 2\tau)| = |p(-t_i + 2\tau, \mu)| = |y_i| = \rho_i$, and $|u^\$| + |v^\$| + |y^\$| \leq Ce^{-\alpha t_o}$ (Lemma 3.4), and since $e^{-\alpha t_o} < \delta_i = C(\rho_i + |\mu|)$, cf. H2), we have

$$\eta_1(i) \leq C(\rho_i + \delta_i) \leq C(\rho_i + |\mu|), \; h \leq i \leq k-1.$$

From (4.5), we have

$$|w_i^1(1) - w_i^1(2)| + |v_{i+1}^1(1) - v_{i+1}^1(2)| \leq C\delta_i\eta_o(i), \;\; h \leq i \leq k-1.$$

From (4.7)

$$|w_i^1(2) - w_i^1(3)| + |v_{i+1}^1(2) - v_{i+1}^1(3)| \leq C\delta_i[\eta_1(i-1) + \eta_1(i) + \eta_1(i+1)],$$

for $h \leq i \leq k-1$. Thus

$$|w_i^1(1) - w_i^1(3)| + |v_{i+1}^1(1) - v_{i+1}^1(3)| \leq C\delta_i(\rho_{i-1} + \rho_i + \rho_{i+1} + I_{i,h}|\bar{v}| + I_{i,k-1}|\bar{u}| + |\mu|). \tag{4.10}$$

Here, when $i = k-1$, the term $|\bar{w}| = |\bar{y}| + |\bar{u}| = \rho_k + |\bar{u}|$ and can be replaced by $|\bar{u}|$. Let $\mathscr{Y}_i(0) = \{w_i^1(1), v_{i+1}^1(1)\}$, $h \leq i \leq k-1$. Then $\mathscr{Y}_i(1) = \{w_i^1(3), v_{i+1}^1(3)\}$. From the proof of Lemma 4.1, $\sup_i |\mathscr{Y}_i(0) - \mathscr{Y}_i(1)| \leq 2\epsilon_1$. We infer, from Lemma 4.2 that

$$\begin{aligned}|\mathscr{Y}_i(0) - \mathscr{Y}_i^\infty| \leq \frac{1}{1 - \overline{C}\delta}&(|\mathscr{Y}_i(0) - \mathscr{Y}_i(1)| \\ &+ \overline{C}\delta_i \sum_{|\nu - i| = 1} |\mathscr{Y}_\nu(0) - \mathscr{Y}_\nu(1)| + \overline{C}^2\delta_i(\delta_{i-1} + \delta_{i+1})\frac{4\epsilon_1}{1 - \overline{C}\delta})\end{aligned} \tag{4.11}$$

The last term of (4.11) comes from adding up terms in Lemma 4.2 with indices $|\nu - i| \geq 2$. Combining this with (4.10), we find that

$$|w_i^1(1) - w_i^{1\infty}| + |v_{i+1}^1(1) - v_{i+1}^{1\infty}| \leq C\delta_i(\rho_{i-1} + \rho_i + \rho_{i+1} + I_{i,h}|\bar{v}| + I_{i,k-1}|\bar{u}| + |\mu|). \tag{4.12}$$

We explain how the right hand side of (4.12) dominates all the terms in (4.11). Due to the fact that $\delta_\nu < C(\rho_\nu + |\mu|)$, $\nu = i-1,\ i+1$, the last term in (4.11) is bounded by $C\delta_i(\rho_{i-1} + \rho_{i+1} + |\mu|)$. From (4.10), $|\mathcal{Y}_\nu(0) - \mathcal{Y}_\nu(1)| \leq C\delta_\nu$, $\nu = i-1,\ i+1$. Thus, $\delta_i \sum_{|\nu - i| = 1} |\mathcal{Y}_\nu(0) - \mathcal{Y}_\nu(1)| \leq C\delta_i(\delta_{i-1} + \delta_{i+1})$. Those terms can be added to the right hand side of (4.10) that dominates $|\mathcal{Y}_i(0) - \mathcal{Y}_i(1)|$. We now have

$$\begin{aligned} G_i(\{S_j\}, \mu) &= \xi_*(w_i^{1\infty}, v_i^{1\infty}, \mu) \\ &= \xi_*(w_i^1(1), v_i^1(1), \mu) + \sum_{\nu = i, i-1} 0(\delta_\nu(\rho_{\nu-1} + \rho_\nu + \rho_{\nu+1} + |\mu|)) \\ &\quad + 0(I_{i,h}|\bar{v}| + I_{i,h+1}\delta_h|\bar{v}| + I_{i,k}|\bar{u}| + I_{i,k-1}\delta_{k-1}|\bar{u}|). \end{aligned}$$

Recall the constant $\hat{t}$ introduced in Lemma 4.1. Choosing $\hat{t}$ so large that $e^{-\alpha t_i} \leq C\delta_i(\rho_i + |\mu|)$, we have, for $h+1 \leq i \leq k$, $|v_i^1(1)| = |v^*(0,\ t_{i-1} - 2\tau,\ 0,\ 0,\ \mu)| \leq Ce^{-\alpha_1 t_{i-1}} \leq C\delta_{i-1}(\rho_{i-1} + |\mu|)$. Similarly, $|u_i^1(1)| \leq C\delta_i(\rho_i + |\mu|)$ and $y_i^1(1) = p(-t_i + 2\tau,\ \mu) + 0(\delta_i(\rho_i + |\mu|))$, $h \leq i \leq k-1$. Therefore,

$$\begin{aligned} \xi_*(w_i^1(1), v_i^1(1), \mu) &= \xi_*(y_i, 0, 0, \mu) + 0(\delta_i(\rho_i + |\mu|) + \delta_{i-1}(\rho_{i-1} + |\mu|)) \\ &\quad + 0(I_{i,h}|\bar{v}| + I_{i,k}|\bar{u}|), \quad h \leq i \leq k. \end{aligned}$$

Hence

$$\begin{aligned} G_i(\{S_j\}, \mu) &= \xi_*(y_i, 0, 0, \mu) \\ &\quad + 0(\delta_i(\rho_{i-1} + \rho_i + \rho_{i+1} + |\mu|) + \delta_{i-1}(\rho_{i-2} + \rho_{i-1} + \rho_i + |\mu|)) \\ &\quad + 0(I_{i,h}|\bar{v}| + I_{i,h+1}\delta_h|\bar{v}| + I_{i,k}|\bar{u}| + I_{i,k-1}\delta_{k-1}|\bar{u}|) \end{aligned}$$

Lemma 4.3. $\xi_*(y, 0, 0, \mu) = d(y, \mu)$.

Proof. Consider the boundary value problem (3.4) with $v_1 = 0$, $w_1 = (y, 0)$. Since $v_1 = 0$, $x(-\tau) \in W_{loc}^{cu}(\mu)$. It is also required that $x(-\tau) \in \Sigma_o$. Thus $x(-\tau) \in W_{loc}^{cu}(\mu) \cap \Sigma_o$ and $x(\tau^-) \in W^{cu}(\mu) \cap \Sigma$. Now $x(\tau) = (y, 0, \bar{v})$ where $\bar{v}$ is a vector in $\mathbb{R}^m$, and $x(\tau^-) - x(\tau) = (\xi_* \cdot \Delta_1, 0, 0)$ where $\Delta_1 \in \mathbb{R}^n$ is given in section 2 right before Theorem 2.1. It is now clear that $x(\tau^-)$ is in $W^{cu}(\mu) \cap W^{cs}(\mu)$ and $x(\tau^-) \in W^{cu}(\mu) \cap W^{cs}(\mu) \cap \Sigma = \mathcal{C}(\mu)$. Finally the projection Π kills the vector $\bar{v}$, thus $\xi_*\Delta_1$ is the vector between y and $\mathcal{C}(\mu)$ on $W_{loc}^c(\mu)$. In fact,

$$\xi_*\Delta_1 = \Pi(x(\tau^-) - x(\tau)) = \Pi(x(\tau^-)) - y = d(y, \mu),$$

since $\Pi(x(\tau^-)) \in \mathcal{C}(\mu)$. □

Estimate (2.6) follows from Lemma 4.3.

As in Lemma 4.2, let $\mathcal{Y}_i = (w_i^1, v_{i+1}^1)$. Since

$$\left|\frac{\partial w^*}{\partial w_i^o}\right| + \left|\frac{\partial w^*}{\partial v_i^o}\right| + \left|\frac{\partial v^*}{\partial w_i^o}\right| + \left|\frac{\partial v^*}{\partial v_i^o}\right| \leq C\delta_i, \quad h \leq i \leq k-1,$$

and since $\frac{\partial w^*}{\partial \mu_2} = 0$, $\frac{\partial v^*}{\partial \mu_2} = 0$, cf. H2), from (4.1) and (4.2) we have

$$|D_{\mu_2}\mathcal{Y}_i| \le C\delta_i(|\frac{\partial w_i^o}{\partial \mu_2}| + |\frac{\partial v_i^o}{\partial \mu_2}|).$$

But from Lemma 4.1, $|D_{\mu_2}\mathcal{X}_i^\infty| \le C$. Therefore $|D_{\mu_2}\mathcal{Y}_i^\infty| \le C\delta_i$, for $h \le i \le k-1$. We can now write

$$\frac{d\xi_*(w_i^1, v_i^1, \mu)}{d\mu_2} = \frac{\partial \xi_*(w_i^1, v_i^1, \mu)}{\partial \mu_2} + 0(\delta_i + \delta_{i-1})$$

for $h \le i \le k$. Here (w_i^1, v_i^1) is used for $(w_i^{1,\infty}, v_i^{1,\infty})$ in Lemma 4.1. Since $|w_i^1|+|v_{i+1}^1| \le |w_i^1 - w_i^1(1)|+|v_{i+1}^1 - v_{i+1}^1(1)|+|w_i^1(1)|+|v_{i+1}^1(1)|$, using (4.12) and $\eta_1(i) \le C(\rho_i+|\mu|)$, also using $\delta_i \le C(\rho_i + |\mu|)$, we have

$$|w_i^1| + |v_{i+1}^1| \le C(\rho_i + |\mu|), \quad h \le i \le k-1. \tag{4.12a}$$

We then have

$$\frac{\partial \xi_*(w_i^1, v_i^1, \mu)}{\partial \mu_2} = \frac{\partial \xi_*(0, 0, \mu)}{\partial \mu_2} + 0(\rho_i + \rho_{i-1} + I_{i,h}|\bar{v}| + I_{i,k}|\bar{u}| + |\mu|).$$

Here $|v_h^1| = |\bar{v}|$ or $|w_k^1| = |\bar{u}| + \rho_k$ is included in the right hand side if $i = h$ or $i = k$. The proof of (2.7) has been completed.

Before proving (2.8), we will derive estimate (4.16) for future use. Let $\{\Delta t_i\}$ be defined as before Theorem 2.2. Let $\{\mathcal{X}_i\}_h^{k-1}$ be the fixed point of (4.1)–(4.4) corresponding to a symbol sequence $\{S_i(\zeta)\}_{h-1}^k$. Differentiating (4.1)–(4.4) with respect to ζ, we have for $h \le i \le k-1$.

$$\begin{aligned} \frac{\partial w_i^1}{\partial \zeta} &= -\frac{\partial w^*}{\partial t}\Delta t_i + \frac{\partial w^*}{\partial T}\Delta t_i + \frac{\partial w^*}{\partial w_i^o}\frac{\partial w_i^o}{\partial \zeta} + \frac{\partial w^*}{\partial v_i^o}\frac{\partial v_i^o}{\partial \zeta}, \\ \frac{\partial v_{i+1}^1}{\partial \zeta} &= \frac{\partial v^*}{\partial T}\Delta t_i + \frac{\partial v^*}{\partial w_i^o}\frac{\partial w_i^o}{\partial \zeta} + \frac{\partial v^*}{\partial v_i^o}\frac{\partial v_i^o}{\partial \zeta}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \frac{\partial w_i^o}{\partial \zeta} &= \frac{\partial w_*}{\partial w_{i+1}^1}\frac{\partial w_{i+1}^1}{\partial \zeta} + \frac{\partial w_*}{\partial v_{i+1}^1}\frac{\partial v_{i+1}^1}{\partial \zeta}, \\ \frac{\partial v_i^o}{\partial \zeta} &= \frac{\partial v_*}{\partial w_i^1}\frac{\partial w_i^1}{\partial \zeta} + \frac{\partial v_*}{\partial v_i^1}\frac{\partial v_i^1}{\partial \zeta}. \end{aligned} \tag{4.14}$$

Here $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial T}$ are used as ∂_1 and ∂_2 in the functions w^* and v^*.

After substituting (4.14) into (4.13), the right hand side of (4.13) defines a mapping $\mathcal{B} : \{\mathcal{Z}_i\}_h^{k-1}$ into the left hand side, where $\mathcal{Z}_i = \{\frac{\partial w_i^1}{\partial \zeta}, \frac{\partial v_{i+1}^1}{\partial \zeta}\}$. Fixed point of $\mathcal{B}$, denoted by $\{\mathcal{Z}_i^\infty\}_h^{k-1}$, can be sought by an iteration scheme, starting with $\{\mathcal{Z}_i(0)\}_h^{k-1} = \{0\}$. Let $\{\mathcal{Z}_i(j)\}_h^{k-1} = \mathcal{B}^j\{\mathcal{Z}_i(0)\}_h^{k-1}$. Then

$$|\mathcal{Z}_i(j) - \mathcal{Z}_i(j+1)| \le C\bar{\delta}_i \sum_{\nu=i-1}^{i+1} |\mathcal{Z}_\nu(j) - \mathcal{Z}_\nu(j-1)|,$$

where $\bar{\delta}_i = \delta(t_i(\zeta) - 2\tau, \mu)$ is evaluated at ζ, and $\mathcal{Z}_i(j) - \mathcal{Z}_i(j-1) = 0$ for $i \notin [h, k-1]$. We now have a system similar to the one in Lemma 4.2. We can derive, analogously, the following estimate

$$|\mathfrak{X}_i^\infty| = |\mathfrak{X}_i^\infty - \mathfrak{X}_i(0)| \le \frac{1}{1-\overline{C}\overline{\delta}}\Big\{\sum_{\nu=1}^{\infty}\overline{C}^{\nu}(\overline{\delta}_i\cdots\overline{\delta}_{i+\nu-1})|\mathfrak{X}_{i+\nu}(1)-\mathfrak{X}_{i+\nu}(0)| + \sum_{\nu=-1}^{-\infty}\overline{C}^{|\nu|}(\overline{\delta}_i\cdots\overline{\delta}_{i+\nu+1})|\mathfrak{X}_{i+\nu}(1)-\mathfrak{X}_{i+\nu}(0)| + |\mathfrak{X}_i(1)-\mathfrak{X}_i(0)|\Big\}, \tag{4.15}$$

for $h \le i \le k-1$, provided that $\overline{C}\overline{\delta} < 1$. Here $\overline{C} = 3C$, $\overline{\delta} = \sup \overline{\delta}_i$.

Recall that $\mathfrak{X}_i(0) = 0$, $h \le i \le k-1$, therefore, from (4.13) and (4.14), for $h \le i \le k-1$,

$$\mathfrak{X}_i(1) = \left\{-\frac{\partial w^*}{\partial t}\Delta t_i + \frac{\partial w^*}{\partial T}\Delta t_i,\ \frac{\partial v^*}{\partial T}\Delta t_i\right\} + 0(I_{i,h}\overline{\delta}_h|\dot{v}_h^1(\zeta)| + I_{i,k-1}\overline{\delta}_{k-1}|\dot{w}_k^1(\zeta)|)$$

The boundary terms are included since when $i = h$ or $k-1$, $\frac{\partial v_i^1}{\partial\zeta} = \dot{v}_h^1(\zeta)$ or $\frac{\partial w_{i+1}^1}{\partial\zeta} = \dot{w}_k^1(\zeta)$ is given by the definition of the symbols $\{S_i(\zeta)\}_{h-1}^{k}$. Substituting the above into (4.15), we have the estimate (4.16) showing how the effect of $\Delta t_{i+\nu}$ decays as $|\nu| \to \infty$.

$$\begin{aligned}|\frac{\partial w_i^1}{\partial\zeta}| + |\frac{\partial v_{i+1}^1}{\partial\zeta}| &= |\mathfrak{X}_i^\infty| \\ &\le C\Bigg\{\sum_{\nu=1}^{j}\overline{C}^{\nu}(\overline{\delta}_i\cdots\overline{\delta}_{i+\nu-1})\overline{\delta}_{i+\nu}\Delta t_{i+\nu} + \sum_{\nu=-1}^{-j}\overline{C}^{|\nu|}(\overline{\delta}_i\cdots\overline{\delta}_{i+\nu+1})\overline{\delta}_{i+\nu}\Delta t_{i+\nu} \\ &\quad + \overline{\delta}_i\Delta t_i \\ &\quad + \overline{C}^{i-h}(\overline{\delta}_i\cdots\overline{\delta}_{h+1})\overline{\delta}_h|\dot{v}_h^1(\zeta)| + \overline{C}^{k-i-1}(\overline{\delta}_i\cdots\overline{\delta}_{k-2})\overline{\delta}_{k-1}|\dot{u}_k^1(\zeta)| \\ &\quad + [(\overline{\delta}_i\cdots\overline{\delta}_{i+j}) + (\overline{\delta}_i\cdots\overline{\delta}_{i-j})]\frac{\overline{C}^{j+1}\overline{\delta}\|\{\Delta t_j\}\|}{1-\overline{C}\overline{\delta}}\Bigg\}, \qquad h \le i \le k-1.\end{aligned} \tag{4.16}$$

Recall in Theorem 2.2, we define $\overline{\delta}_k = |\frac{d}{dt}g(t)|$ and $y_k(\zeta) = g(\zeta\Delta t_k)$. Therefore, $|\dot{y}_k(\zeta)| \le \overline{\delta}_k|\Delta t_k|$. This allows us to drop the term involving $|\dot{y}_k(\zeta)|$ in $|\dot{w}_k^1(\zeta)|$ when deriving (4.16). By adjusting the constant C outside the $\{\cdots\}$–bracket, such term is either absorbed by the first term in the $\{\cdots\}$ if $i + j \ge k$, or by the last term in the $\{\cdots\}$ if $i + j < k$. From (4.16), by letting $j \to \infty$, we can easily derive that for $h \le i \le k-1$,

$$\left|\frac{\partial w_i^1}{\partial\zeta}\right| + \left|\frac{\partial v_{i+1}^1}{\partial\zeta}\right| \le C(\overline{\delta}_i\|\{\Delta t_j\}\| + (\overline{C}\overline{\delta})^{i-h+1}|\dot{v}_h(\zeta)| + (\overline{C}\overline{\delta})^{k-i}\quad |\dot{u}_k(\zeta)|).$$

Substituting into (4.14), we have for $h \le i \le k-1$

$$|\frac{\partial w_i^o}{\partial\zeta}| + |\frac{\partial v_i^o}{\partial\zeta}| \le C((\overline{\delta}_{i-1} + \overline{\delta}_i + \overline{\delta}_{i+1})\|\{\Delta t_j\}\| + (\overline{C}\overline{\delta})^{i-h}|\dot{v}_h(\zeta)| + (\overline{C}\overline{\delta})^{k-i-1}|\dot{u}_k(\zeta)|).$$

Substituting into (4.13), we have for $h \le i \le k-1$,

$$\begin{aligned}&|\frac{\partial w_i^1}{\partial\zeta} - (\frac{\partial w^*}{\partial T} - \frac{\partial w^*}{\partial t})\Delta t_i|_{T=t_i(\zeta)-2\tau,\,t=-T} + |\frac{\partial v_{i+1}^1}{\partial\zeta} - \frac{\partial v^*}{\partial T}\Delta t_i|_{T=t_i(\zeta)-2\tau,t=0} \\ &\le C\overline{\delta}_i\{(\overline{\delta}_{i-1} + \overline{\delta}_i + \overline{\delta}_{i+1})\|\{\Delta t_j\}\| + (\overline{C}\overline{\delta})^{i-h}|\dot{v}_h(\zeta)| + (\overline{C}\overline{\delta})^{k-i-1}|\dot{u}_k(\zeta)|\}\end{aligned} \tag{4.17}$$

If $\inf\{t_i(\zeta)\}$ is large, we have $Ce^{-\alpha t_i} \leq \overline{\delta}_i^2$. See the Remark following Corollary 3.5. Now that $w_i^1 = (y_c(-t_i(\zeta) + 2\tau, \mu) + y^\$(-t_i(\zeta) + 2\tau), u^*(-t_i(\zeta) + 2\tau))$, from Lemma 3.4,

$$\left|\frac{\partial u^*}{\partial t}\right| + \left|\frac{\partial u^*}{\partial T}\right| + \left|\frac{\partial y^\$}{\partial t}\right| + \left|\frac{\partial y^\$}{\partial T}\right| + \left|\frac{\partial v^*}{\partial T}\right| \leq Ce^{-\alpha t_i} \leq \overline{\delta}_i^2, \ h \leq i \leq k-1. \tag{4.18}$$

Observe that also,

$$\begin{aligned} &\left|\frac{\partial}{\partial t}[\Phi(-t_i(\zeta) + 2\tau, \ \mu, \ y_i^o(\zeta) + q_y(-\tau)) - p(-t_i(\zeta) + 2\tau, \ \mu)]\right| \\ &\leq C\overline{\delta}_i(\overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|). \end{aligned} \tag{4.19}$$

This is because the left hand side is bounded by

$$g_0(\Phi(-t_i(\zeta) + 2\tau, \ \mu, \ y_i^o(\zeta) + q_y(-\tau)), 0, 0, \mu) - g_0(p(-t_i(\zeta) + 2\tau, \ \mu), 0, 0, \mu).$$

See (1.1). Also g_0 is C^1 and $\Phi - p$ is bounded by $C\delta_i|y_i^o(\zeta)|$, (condition H2). From (4.3) and (4.12a), we also have

$$|y_i^o(\zeta)| \leq C(|w_{i+1}^1| + |v_{i+1}|^1 + |\mu|) \leq C(\overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|).$$

The desired estimate follows easily. We now derive, for $h \leq i \leq k$,

$$\begin{aligned} \frac{\partial \xi_*(w_i^1, \ v_i^1, \ \mu)}{\partial \zeta} &= \frac{\partial \xi_*(w_i^1, \ v_i^1, \ \mu)}{\partial y} \cdot \frac{\partial y_i(\zeta)}{\partial \zeta} \\ &\quad + 0([\overline{\delta}_i(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) + \overline{\delta}_{i-1}(\overline{\delta}_{i-2} + \overline{\delta}_{i-1} + \overline{\delta}_i)]\|\{\Delta t_j\}\|) \\ &\quad + 0((\overline{C\delta})^{i-h}|\dot{v}_h^1(\zeta)| + (\overline{C\delta})^{k-i}|\dot{u}_k^1(\zeta)|) \end{aligned} \tag{4.20}$$

Consider

$$\frac{\partial \xi_*(w_i^1, \ v_i^1, \ \mu)}{\partial \zeta} = \frac{\partial \xi_*}{\partial w_i^1}\frac{\partial w_i^1}{\partial \zeta} + \frac{\partial \xi_*}{\partial v_i^1}\frac{\partial v_i^1}{\partial \zeta}. \tag{4.21}$$

Observe that ξ_* is C^1 bounded. We only need to simplify $\frac{\partial w_i^1}{\partial \zeta}$ and $\frac{\partial v_i^1}{\partial \zeta}$. We obtain estimates for $|\frac{\partial v_i^1}{\partial \zeta} - \frac{\partial v^*}{\partial T}\Delta t_{i-1}|$ and $|\frac{\partial v^*}{\partial T}\Delta t_i|$ from (4.17) and (4.18). They are all bounded by the error terms in (4.20). Thus we can drop $\frac{\partial v_i^1}{\partial \zeta}$ in (4.21). For the similar reason, we can also drop $\frac{\partial u_i^1}{\partial \zeta}$ from (4.21). From (4.17) again, also using $\overline{\delta}_i \leq C(\overline{\rho}_i + |\mu|)$, we have $\frac{\partial y_i^1}{\partial \zeta} = (\frac{\partial y^*}{\partial T} - \frac{\partial y^*}{\partial t})\Delta t_i + \cdots$, with an error term bounded by the error terms in (4.20). From Lemma 3.4, $y^* = y^\$ + \Phi(-t_i(\zeta) + 2\tau, \ \mu, \ y_i^o(\zeta) + q_y(-\tau))$. Now that the derivatives of $y^\$$ are small, and Φ is independent of T, we can replace (4.21) by $\frac{\partial \xi_*}{\partial y}(\frac{\partial \Phi}{\partial t}\Delta t_i)$ with an error bounded by the error terms in (4.20). Using (4.19), we further replace $\frac{\partial \Phi}{\partial t}\Delta t_i$ by $\frac{\partial p(-t_i(\zeta)+2\tau, \ \mu)}{\partial t}\Delta t_i + O(\overline{\delta}_i(\overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|)|\Delta t_i|)$. However, $\frac{\partial p(-t_i(\zeta)+2\tau, \ \mu)}{\partial t}\Delta t_i = \frac{\partial y_i(\zeta)}{\partial \zeta}$. From here (4.20) follows easily. We now replace the argument in $\frac{\partial \xi_*(w_i^1, v_i^1, \mu)}{\partial y}$ to obtain $\frac{\partial \xi_*(y_i(\zeta), 0, 0, \mu)}{\partial y}$ that will introduce an error term $C(|u_i^1| + |v_i^1| + |y_i^1 - y_i(\zeta)|)|\frac{\partial y_i(\zeta)}{\partial \zeta}|$. Using (4.12a), $|y_i(\zeta)| \leq \overline{\rho}_i$ for $h < i < k$, and $|\frac{\partial y_i(\zeta)}{\partial \zeta}| \leq C\overline{\delta}_i|\Delta t_i|$, we find that the error term is negligible for $h < i < k$. Extra terms have to be introduced in the estimate when $i = h$ and $i = k$, since $u_k(\zeta)$ and $v_h(\zeta)$ are given boundary conditions. But $|y_k(\zeta)|$ is bounded by $\overline{\rho}_k$ and does not need to be listed separately.

$$\frac{\partial \xi_*(w_i^1, v_i^1, \mu)}{\partial \zeta} = \frac{\partial \xi_*(y_i(\zeta), 0, 0, \mu)}{\partial y}\frac{\partial y_i(\zeta)}{\partial \zeta}$$
$$+ 0([\overline{\delta}_i(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) + \overline{\delta}_{i-1}(\overline{\delta}_{i-2} + \overline{\delta}_{i-1} + \overline{\delta}_i)]\|\{\Delta t_j\}\|)$$
$$+ 0((\overline{C\delta})^{i-h}|\dot{v}_h(\zeta)| + (\overline{C\delta})^{k-i}|\dot{u}_k(\zeta)| + I_{i,h}\delta_h|v_h(\zeta)|\,|\Delta t_h|$$
$$+ I_{i,k}\delta_k|u_k(\zeta)|\,|\Delta t_k|)$$

Estimate (2.8) then follows from Lemma 4.3. □

Proof of Theorem 2.3. For any integer $N > 0$, let $\zeta_i = 0$ for $|i| > N$ and solve the finite system

$$G_i(\{S_j(\zeta_j)\}, \mu) = 0, \quad -N \le i \le N \tag{4.22}$$

We show that (4.22) has at least one solution $\{\zeta_i\}_{-N}^N$ with $-d_i < \zeta_i < e_i$, $-N \le i \le N$. To this end, we infer that from $(H_7; 2)$ and (2.6),

$$G_i(\{S_j(\zeta_j)\}, \mu) \neq 0,$$

for $\zeta_i = -d_i$ or e_i, $-N \le i \le N$. Thus, G_i changes sign when ζ_i moves from $-d_i$ to e_i. Assume that G_i moves from negative to positive, otherwise consider $-G_i = 0$. The mapping defined by the left hand side of (4.22)

$$G : \Pi_{-N}^N(-d_i, e_i) \to \mathbb{R}^{2N+1}$$

is homotopic to the identity map and the image of the boundary of $\Pi_{-N}^N(-d_i, e_i)$ does not intersect $0 \in \mathbb{R}^{2N+1}$ in the homotopy process. From the standard theory of degree, see [26], there exists at least one solution $\{\zeta_i^N\}_{-N}^N$ to (4.22).

Since $\sup_i\{d_i + e_i\} < \infty$, a subsequence of $\{\zeta_i^N\}_h^k$ can be found which approaches $\{\zeta_i^\infty\}_h^k$ as $N \to \infty$ in the product topology. Since $\{G_i\}_h^k$ is continuous in $\{S_i(\zeta_i)\}_{h-1}^k$ in the product topology, we have $G_i(\{S_j(\zeta_j^\infty)\}, \mu) = 0$ for $h \le i \le k$.

We now show that the solution is unique. If not, assume that there are two solutions, $\{\zeta_i^\nu\}_h^k$, $\nu = 1, 2$, with $G_i(\{S_j(\zeta_j^\nu)\}, \mu) = 0$. Let $\Delta t_i = \zeta_i^2 - \zeta_i^1$. Then for each $h \le i \le k$, there exists $0 < \zeta = \zeta_i < 1$ such that

$$\frac{\partial}{\partial \zeta} G_i(\{S_j(\zeta_j^1 + \zeta \Delta t_j)\}, \mu) = 0.$$

From (2.8), we would have

$$|\frac{\delta}{\delta \zeta} d(y_i(\zeta_i), \mu)\Delta t_i| \le \text{the r.h.s. of (2.8)}, \quad h \le i \le k,$$

where $\zeta_i = \zeta_i^1 + \zeta \Delta t_i$. Observe that $\dot{v}_h(\zeta) = 0$ and $\dot{u}_k(\zeta) = 0$ in the r.h.s. of (2.8), since $S_{h-1}(\zeta) = v_h(\zeta) = \tilde{v}$ and $S_k(\zeta) = (y_k(\zeta), \tilde{u})$. But from $(H_7; 1)$ we have

$$|\Delta t_i| \le \frac{C_3}{C_4}\,\|\{\Delta t_j\}\|,$$

This is a contradiction unless $\|\{\Delta t_j\}\| = 0$. □

Proof of Theorem 2.4. The existence of such a solution $x^1(t)$ is obvious by virtue of Theorem 2.3. Assuming that $k = k^1 = +\infty$, we show that orbitally $x^1(t) \to x(t)$ as

$t \to +\infty$. To this end, suppose that $x^\nu(t)$, $\nu = 1, 2$ are two solutions corresponding to $\{S_i(\zeta_i^\nu)\}_{h-1}^\infty$ where $-d_i < \zeta_i^\nu < e_i$, $\nu = 1, 2$. $S_i(\zeta_i^\nu) = t_i(\zeta_i^\nu)$ for $i \geq h$. $S_{h-1}(\zeta) = v_h(\zeta)$ with $v_h(0) = \bar{v}^1$, $v_h(1) = \bar{v}^2$, $\zeta_{h-1}^1 = 0$ and $\zeta_{h-1}^2 = 1$. Let $\Delta t_i = \zeta_i^2 - \zeta_i^1$ and $\zeta_i(\zeta) = \zeta_i^1 + \zeta \Delta t_i$. For each $h \leq i \leq k$ there exists $0 < \zeta = \zeta_i < 1$ such that

$$\frac{\partial}{\partial \zeta} G_i(\{S_j(\zeta_i(\zeta))\}, \mu) = 0.$$

From (2.8) and H_7) we have

$$\begin{aligned}
C_4(\overline{\delta}_i(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) + \overline{\delta}_{i-1}(\overline{\delta}_{i-2} + \overline{\delta}_{i-1} + \overline{\delta}_i))|\Delta t_i| \\
\leq C_3(\overline{\delta}_i(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) + \overline{\delta}_{i-1}(\overline{\delta}_{i-2} + \overline{\delta}_{i-1} + \overline{\delta}_i)]\|\{\Delta t_j\}\|) \\
+ C_3((\overline{C\delta})^{i-h}|\dot{v}_h(\zeta)| + I_{i,h}\delta_h|v_h(\zeta)\|\Delta t_h|)
\end{aligned}$$

For any $\eta > 0$, with $(C_3 + \eta)/C_4 < 1$ there is a large integer $N > 0$ such that for $i \geq h + N$,

$$C_4|\Delta t_i| \leq (C_3 + \eta)\ \|\{\Delta t_j\}\|$$

Thus, $|\Delta t_i| \leq \frac{C_3+\eta}{C_4}\ \|\{\Delta t_j\}\|$ if $i \geq h + N$. The process can be repeated infinitely often, therefore $|\Delta t_i| \to 0$ as $i \to \infty$.

We now use (4.16) to conclude that $\left|\frac{\partial w_i^1(\zeta)}{\partial \zeta}\right| + \left|\frac{\partial v_{i+1}^1(\zeta)}{\partial \zeta}\right| \to 0$ for all $0 < \zeta < 1$ as $i \to \infty$. In fact, for any given $\epsilon > 0$, we can choose j so large that the last term in (4.16), $[(\overline{\delta}_i \cdots \overline{\delta}_{i+j}) + (\overline{\delta}_i \cdots \overline{\delta}_{i-j})]\frac{\overline{C}^{j+1}\overline{\delta}\|\{\Delta t_j\}\|}{1-\overline{C}\overline{\delta}} < \frac{\epsilon}{4}$. Let j be fixed. Let $\epsilon_i = \sup\{|\Delta t_{i+\nu}| : |\nu| \leq j\}$. Then $\epsilon_i \to 0$ as $i \to \infty$. Thus if i is sufficiently large,

$$\sum_{\nu=1}^{j} + \sum_{\nu=-1}^{-j} \overline{C}^{|\nu|}(\overline{\delta}_i \cdots \overline{\delta}_{i+\nu})\Delta t_{i+\nu} < \frac{\epsilon}{4}.$$

Also $\overline{\delta}_i \Delta t_i < \frac{\epsilon}{4}$. The term involving the index k does not appear ($k = \infty$). Thus the right hand side of (4.16) approaches zero as $i \to \infty$. Therefore $|w_i^1(0) - w_i^1(1)| + |v_{i+1}^1(0) - v_{i+1}^1(1)| \to 0$ as $i \to \infty$. The assertion $x^1(t) \to x^2(t)$ as $t \to +\infty$ orbitally then follows from Lemmas 3.3 and 3.4. In fact, let the orbit of $x^j(t)$, $j = 1, 2$ be the union of those of $x_i^{j*}(t)$, $x_{i*}^j(t)$, $i = h, h+1, \cdots$, where $x_i^{j*}(t)$, $-t_i^j + 2\tau \leq t \leq 0$ is the inner solution and $x_{i*}^j(t)$, $-\tau \leq t \leq \tau$ is the outer solution as in the proof of Theorem 2.1. For the outer solutions, the boundary conditions are $v_i^1(j)$ at $t = -\tau$ and $w_i^1(j)$ at $t = \tau$. Thus, by Lemma 3.3, $|x_{i*}^1(t) - x_{i*}^2(t)| \to 0$ uniformly as $i \to \infty$. The inner solutions also depend continuously on the boundary conditions and on the length of the domains, by Lemma 3.4, therefore, $|x_i^{1*}(t) - x_i^{2*}(t)| \to 0$ uniformly in their common domain of t as $i \to \infty$. For t not in the common domain, the variation of $x_i^{1*}(t)$ or $x_i^{2*}(t)$ is small if $|t_i^1 - t_i^2|$ is small. Finally the case $h = -\infty$ can be proved similarly.

5. Simple Periodic Solutions

The notation $\{S_1, \cdots, S_\nu\}_P$ is used to denote a periodic symbol $\{S_i\}_{-\infty}^\infty$ with $S_{i+\nu} = S_i$, $i \in \mathbb{Z}$. Theorems 5.1 and 5.2 concern some general properties of simple periodic solutions.

Theorem 5.1. *(Hyperbolicity of nondegenerate simple periodic solutions). Assume that* $\{S_i\}_{-\infty}^{\infty}$ *and* μ *satisfy the hypotheses of Theorem 2.3 and* $S_i = t_i \equiv t_o$, $d_i \equiv d_o$ *and* $e_i \equiv e_o$ *for all* $i \in \mathbb{Z}$. *Then* $S_i(\zeta_i) = t_i(\zeta_i) = \omega$ *for all* $i \in \mathbb{Z}$, *and* $x(t)$, *with* $\mathcal{T}x = \{\omega\}_p$, *is a simple periodic solution with period* ω. *Moreover, if the constant* C_4 *as in* $(H_{7;1})$ *is sufficiently large, then the periodic solution* $x(t)$ *is hyperbolic with* $l + n - 1$ *unstable characteristic values and* m *stable characteristic values.*

Simple periodic solutions degenerate or not, form a one parameter connected family of periodic solutions. This was first observed by Glendining and Sparrow [12] in a simplified model. A proof for the case that the equilibrium is hyperbolic is given in [19]. We will show the same result for nonhyperbolic equilibria.

Theorem 5.2. *Assume that H1)–H6), are satisfied. Then for each* $\hat{\epsilon} > 0$, *there exist positive constants* $\hat{\mu}_1, \hat{\mu}_2, \hat{\delta}, \hat{t}$ *and* ρ *such that if* $|\mu_1| < \hat{\mu}_1$, $\hat{t} < \omega < \bar{t}$, $|p(-\omega+2\tau, \mu)| < \rho$, $\delta(\omega - 2\tau, \mu) < \hat{\delta}$, *then there exists a unique* $\mu_2 = \mu_2(\omega, \mu_1)$, $|\mu_2| < \hat{\mu}_2$, *such that (2.1) has a unique simple periodic solution* $x(t)$ *of period* ω, *which is orbitally* $\hat{\epsilon}$*-near* Γ. *Moreover* μ_2 *is a* C^3 *function of* ω.

Theorems 5.1 and 5.2 really do not rely on $H_4)$ since $H_4)$ is only used to construct the special Σ and $\overrightarrow{\Delta}$ which are used in Theorem 2.2.

Definition 5.3. We say a linear operator $A : X_1 \to X$ is a contraction modulo a vector $\mathbf{a}$ if we can find $\zeta \in \mathbb{R}$ such that

$$|Ax + \zeta \mathbf{a}| \le \lambda |x|, \; 0 \le \lambda < 1.$$

Here $X_1 \subset X$ are Banach spaces and λ is the rate of contraction (mod $\mathbf{a}$).

Proof of Theorem 5.1. Let $\mathcal{T}(x(\cdot)) = \{\omega\}_{-\infty}^{\infty}$ for all $i \in \mathbb{Z}$. Then $\mathcal{T}(x(\cdot + \omega)) = \{\omega\}_{-\infty}^{\infty}$. It follows from the uniqueness part of Theorem 2.1 that $x(\cdot + \omega) = x(\cdot)$. Therefore $x(t)$ is of period ω.

Let τ be the time when $x(\tau) \in \Sigma$. We shall linearize the equation around $x(t)$, $\tau \le t \le \tau + \omega$. As in section 4, we assume that the orbit of $x(t)$ is the union of those of $x^*(t)$, $-\omega + 2\tau \le t \le 0$ and $x_*(t)$, $-\tau \le t \le \tau$, i.e., $x(t) = x^*(t - \omega + \tau)$ for $\tau \le t \le \omega - \tau$ and $x(t) = x_*(t - \omega)$ for $\omega - \tau \le t \le \omega + \tau$. Let $T(t, s)$ with $\tau \le s \le t \le \omega + \tau$, $T^*(t, s)$ with $-\omega + 2\tau \le s \le t \le 0$ and $T_*(t, s)$ with $-\tau \le s \le t \le \tau$ be the principal matrix solutions for $\dot{z}(t) = D_x f(x(t), \mu) z(t)$, $\dot{z}(t) = D_x f(x^*(t), \mu) z(t)$ and $\dot{z}(t) = D_x f(x_*(t), \mu) z(t)$ respectively. Obviously we have

$$T(\tau + \omega, \tau) = T_*(\tau, -\tau) \cdot T^*(0, -\omega + 2\tau)$$

The proof is then based on Lemma 3.6. Roughly speaking, the unstable space U and the stable space V are almost $TW^{cu}(\mu)$ and $TW^s(\mu)$ except for vectors that are too close to span $\{\dot{x}(t)\}$. More precisely, we have the following result.

Lemma 5.4. *Let* $x(t)$ *be a nondegenerate simple periodic solution with period* ω. *Then there exist subspaces* π_τ *and* $\pi_{\omega+\tau}$ *such that*

$$\pi_\tau \oplus \pi_{\omega+\tau} \oplus span\{\dot{x}(\tau)\} = \mathbb{R}^{l+m+n}.$$

The projections defined by the above splitting are bounded by $C^{-1}(|p(-\omega+2\tau,\mu)|+|\mu|)^{-1}$ *where* C *is proportional to the constant* C_4 *in* $(H_{7;1})$. *Moreover,* $T(\omega+\tau,\tau)|_{\pi_\tau}$ *is a contraction modulo* $\dot{x}(\tau)$ *with the rate denoted by* λ_τ. $T(\tau,\omega+\tau)|_{\pi_{\omega+\tau}}$ *is a contraction modulo* $\dot{x}(\tau)$ *with the rate denoted by* $\lambda_{\omega+\tau}$. $\lambda_\tau+\lambda_{\omega+\tau}\le C_1\delta(\omega-2\tau,\mu)$.

Remark. The norms of the projections above are very large if $\delta(\omega-2\tau,\mu)$ is small. This is the major reason why the constants in Lemma 3.6 have to be evaluated carefully.

Proof of Lemma 5.4. Let the solution map of (2.1) be $T(t,\mu,x)$. Let $y_o=x_y^*(0)$, $u_o=x_u^*(0)$ and $v_o=x_v^*(-\omega+2\tau)$. For any given vectors Δy_o, Δu_o and Δv_o, consider a one-parameter boundary value problem as in Corollary 3.5 with the boundary data being $y_o+\zeta\Delta y_o$, $u_o+\zeta\Delta u_o$ and $v_o+\zeta\Delta v_o$, $\zeta\in\mathbb{R}$. Using the solution map, we have

$$\begin{aligned}&T(\omega-2\tau,\mu,\{w^*(-\omega+2\tau;\ \omega-2\tau,\ w_o+\zeta\Delta w_o,\ v_o+\zeta\Delta v_o,\ \mu),\ v_o+\zeta\Delta v_o\})\\&\quad=(w_o+\zeta\Delta w_o,\ v^*(0;\ \omega-2\tau,\ w_o+\zeta\Delta w_o,\ v_o+\zeta\Delta v_o,\ \mu)).\end{aligned}$$

Here $w^*=(y^*,u^*)$, $w_o+\zeta\Delta w_o=(y_o+\zeta\Delta y_o,u_o+\zeta\Delta u_o)$. Differentiating with respect to ζ at $\zeta=0$ and observing that $D_zT(t,\mu,(x(s)+z))|_{z=0}=T(t+s,s)$ and $T^*(0,-\omega+2\tau)=T(\omega-\tau,\tau)$, we have

$$\begin{aligned}T^*(0,-\omega+2\tau)\Big(&\frac{\partial y^*(-\omega+2\tau)}{\partial y_o}\Delta y_o+\frac{\partial y^*(-\omega+2\tau)}{\partial u_o}\Delta u_o+\frac{\partial y^*(-\omega+2\tau)}{\partial v_o}\Delta v_o,\\&\frac{\partial u^*(-\omega+2\tau)}{\partial y_o}\Delta y_o+\frac{\partial u^*(-\omega+2\tau)}{\partial u_o}\Delta u_o+\frac{\partial u^*(-\omega+2\tau)}{\partial v_o}\Delta v_o,\ \Delta v_o\Big)\\&=\Big(\Delta y_o,\ \Delta u_o,\ \frac{\partial v^*(0)}{\partial y_o}\Delta y_o+\frac{\partial v^*(0)}{\partial u_o}\Delta u_o+\frac{\partial v^*(0)}{\partial v_o}\Delta v_o\Big)\end{aligned}\tag{5.1}$$

I. Let $\Delta y_o=0$, $\Delta v_o=0$. Recall that $T^*(-\omega+2\tau,0)=[T^*(0,-\omega+2\tau)]^{-1}$.

$$T^*(-\omega+2\tau,0)(0,\ \Delta u_o,\ \frac{\partial v^*(0)}{\partial u_o}\Delta u_o)=\left(\frac{\partial y^*(-\omega+2\tau)}{\partial u_o}\Delta u_o,\ \frac{\partial u^*(-\omega+2\tau)}{\partial u_o}\Delta u_o,\ 0\right).$$

Define $\pi^u(0)=\left\{(0,\ \Delta u_o,\ \frac{\partial v^*(0)}{\partial u_o}\Delta u_o)|\Delta u_o\in\mathbb{R}^l\right\}$. The subspace $\pi^u(0)$ is close to $TW^u_{loc}(0,\mu)=\{(0,\ \Delta u_o,\ 0)|\Delta u_o\in\mathbb{R}^l\}$, since $\left|\frac{\partial v^*(0)}{\partial u_o}\right|\le Ce^{-\alpha\omega}$. Moreover, for $\Delta x\in\pi^u(0)$,

$$|T^*(-\omega+2\tau,0)\Delta x|\le Ce^{-\alpha\omega}|\Delta u_o|\le Ce^{-\alpha\omega}|\Delta x|,$$

Here estimates for $\left|\frac{\partial v^*(0)}{\partial u_o}\right|$, $\left|\frac{\partial y^*(-\omega+2\tau)}{\partial u_o}\right|$ and $\left|\frac{\partial u^*(-\omega+2\tau)}{\partial u_o}\right|$ are based on Lemma 3.4.

II. Let $\Delta u_o=0$, $\Delta v_o=0$. We then have

$$T^*(-\omega+2\tau,0)(\Delta y_o,\ 0,\ \frac{\partial v^*(0)}{\partial y_o}\Delta y_o)=\left(\frac{\partial y^*(-\omega+2\tau)}{\partial y_o}\Delta y_o,\ \frac{\partial u^*(-\omega+2\tau)}{\partial y_o}\Delta y_o,\ 0\right)$$

Define $\pi^y(0)=\{(\Delta y_o,\ 0,\ \frac{\partial v^*(0)}{\partial y_o}\Delta y_o)|\Delta y_o\perp\dot{q}_y(-\tau)\}$. Since $\left|\frac{\partial v^*(0)}{\partial y_o}\right|\le Ce^{-\alpha\omega}$, the space $\pi^y(0)$ is close to $\{(\Delta y_o,0,0)|\Delta y_o\perp\dot{q}_y(-\tau)\}$, a codimension-one subspace of $TW^c_{y_{loc}}(0,\mu)$. Moreover, for any $\Delta x\in\pi^y(0)$,

$$|T^*(-\omega+2\tau,0)\Delta x|\le C\delta(\omega,\mu)|\Delta y_o|\le C\delta(\omega,\mu)|\Delta x|.$$

The estimates for derivatives come from Corollary 3.5. Here H2) is involved and $-\omega + 2\tau \in (-\bar{t}, -\bar{\bar{t}})$ is required.

III. Let $\Delta y_o = 0$, $\Delta u_o = 0$. We have

$$\begin{aligned} T^*(0, -\omega + 2\tau)(&\frac{\partial y^*(-\omega + 2\tau)}{\partial v_o}\Delta v_o, \frac{\partial u^*(-\omega + 2\tau)}{\partial v_o}\Delta v_o, \Delta v_o) \\ &= (0, 0, \frac{\partial v^*(0)}{\partial v_o}\Delta v_o). \end{aligned} \tag{5.2}$$

We shall show in Proposition 5.5 that $|\dot{x}_v^*(-\omega + 2\tau)|$ is uniformly bounded below with respect to ω and μ.

Define $\pi^v(-\omega + 2\tau) = \{(\frac{\partial y^*(-\omega+2\tau)}{\Delta v_o}\Delta v_o, \frac{\partial u^*(-\omega+2\tau)}{\partial v_o}\Delta v_o, \Delta v_o) | \Delta v_o \perp \dot{x}_v^*(-\omega + 2\tau)\}$. $\pi^v(-\omega + 2\tau)$ is close to a subspace of $TW^s_{loc}(0, \mu)$, $\{(0, 0, \Delta v_o) | \Delta v_o \perp \dot{x}_v^*(-\omega + 2\tau)\}$. If $\Delta x \in \pi^v(-\omega + 2\tau)$, then

$$|T^*(0, -\omega + 2\tau)\Delta x| \leq Ce^{-\alpha\omega}|\Delta x|.$$

IV. Denote $x^*(t)$ by $(y(t), u(t), v(t))$ and $\dot{x}^*(t)$ by $(\Delta y(t), \Delta u(t), \Delta v(t))$. We now need the following proposition.

Proposition 5.5. *There is a constant $C_1 > 0$, independent of ω and μ, such that*

$$\begin{aligned} &|\Delta v(-\omega + 2\tau)| \geq C_1, \\ &|\Delta y(0)| \geq C_1, \\ &|u(0)| + |\Delta u(0)| + |v(-\omega + 2\tau)| + |\Delta v(-\omega + 2\tau)| \leq C_1^{-1}. \end{aligned}$$

Moreover, if $\delta(\omega - 2\tau, \mu)$ is small and ω is large, then there exists $C > 0$ such that

$$\delta^2(\omega - 2\tau, \mu)|\Delta y(-\omega + 2\tau)| \geq Ce^{-\alpha\omega}|\Delta y(0)|.$$

Proof. From (4.12), $\delta_i < C(\rho_i + |\mu|)$ and $\eta_1(i) \leq C(\rho_i + |\mu|)$, we have $|w_i^1| + |v_i^1| \leq C(\rho_i + \rho_{i-1} + |\mu|)$. Here of course $\rho_i = \rho$ for all $i \in \mathbb{Z}$. Therefore $|x(-\tau) - q(-\tau)| + |x(\tau) - q(\tau)| \leq C(\rho + |\mu|)$. Since $x^*(-\omega + 2\tau) = x(\tau)$ and $x^*(0) = x(-\tau)$, we have

$$\begin{aligned} &|x^*(-\omega + 2\tau) - q(\tau)| + |x^*(0) - q(-\tau)| \leq C(\rho + |\mu|), \\ &|\dot{x}^*(-\omega + 2\tau) - \dot{q}(\tau)| + |\dot{x}^*(0) - \dot{q}(-\tau)| \leq C(\rho + |\mu|). \end{aligned}$$

The second estimate above is obtained by equation (2.1) and the first estimate. Now that $q(-\tau) = (q_y(-\tau), q_u(-\tau), 0)$ and $q(\tau) = (0, 0, q_v(\tau))$, we obtain the upper bound

$$\begin{aligned} |u(0)| + |v(-\omega + 2\tau)| &\leq C(\rho + |\mu|) + |q_u(-\tau)| + |q_v(\tau)|, \\ |\Delta u(0)| + |\Delta v(-\omega + 2\tau)| &\leq C(\rho + |\mu|) + |\dot{q}_u(-\tau)| + |\dot{q}_v(\tau)|. \end{aligned}$$

Suppose now $C(\rho + |\mu|) < \frac{1}{2}|\dot{q}_y(-\tau)| + |\dot{q}_v(\tau)|$, then we have the lower bound

$$\begin{aligned} |\Delta y(0)| &\geq |\dot{q}_y(-\tau)| - C(\rho + |\mu|) \geq \frac{1}{2}|\dot{q}_y(-\tau)|, \\ |\Delta v(-\omega + 2\tau)| &\geq |\dot{q}_v(\tau)| - C(\rho + |\mu|) \geq \frac{1}{2}|\dot{q}_v(\tau)|. \end{aligned}$$

Let $t_0 = \omega - 2\tau$. Consider the linear variational equation for $\Delta y(t)$,

$$\frac{d}{dt}\Delta y(t) = A_0 \Delta y(t) + \frac{\partial g_0}{\partial y}\Delta y(t) + \frac{\partial g_0}{\partial u}\Delta u(t) + \frac{\partial g_0}{\partial v}\Delta v(t).$$

Here $g_0 = g_0(y(t), u(t), v(t), \mu)$. Therefore for $-t_0 \leq t \leq 0$ we have an integral equation

$$\Delta y(t) = e^{A_0(t+t_0)}\Delta y(-t_0) + \int_{-t_0}^{t} e^{A_0(t-s)}(\frac{\partial g_0}{\partial y}\Delta y(s) + \cdots)ds.$$

Using the estimates of $|u(t)|$ and $|v(t)|$ from Lemma 3.4, we obtain $|\Delta u(t)| \leq C|u(0)|e^{\alpha t}$ and $|\Delta v(t)| \leq C|v(-t_0)|e^{-\alpha(t+t_o)}$, since $g_1 = O(|u|)$ and $g_2 = O(|v|)$, cf. (2.3). Since $g_0(y, 0, 0, \mu) = g_0(y, u, 0, \mu) = g_0(y, 0, v, \mu)$, we have $\frac{\partial g_0}{\partial u} = O(|v|)$ and $\frac{\partial g_0}{\partial v} = O(|u|)$. We then have the integral inequality

$$|\Delta y(t)| \leq Ce^{\beta(t+t_0)}|\Delta y(-t_0)| + \int_{-t_0}^{t} Ce^{\beta(t-s)}L|\Delta y(s)|ds + \int_{-t_o}^{t} C|u(0)||v(-t_0)|e^{-\alpha t_o}ds.$$

Here $\beta > 0$ is a small constant, L is the Lipschitz number for g_0 which can be arbitrarily small if the neighborhood $\mathcal{O}$ and $|\mu|$ are small. Using Gronwall's inequality, we have

$$|\Delta y(0)| \leq Ce^{(\beta+CL)t_0}|\Delta y(-t_0)| + C|u(0)||v(-t_0)|e^{(\beta-\alpha)t_o}.$$

If t_0 is sufficiently large, the last term is bounded by $\frac{1}{2}|\Delta y(0)|$.(Recall $|\Delta y(0)| \geq C_1$). If also $CL \leq \beta$, then

$$|\Delta y(0)| \leq Ce^{2\beta t_o}|\Delta y(-t_0)|.$$

Choose now t_0 large so that $e^{(2\beta-\alpha)t_o} \leq C\delta^2(t_0, \mu)$. The last inequality of the proposition follows from this.

We then use the following obvious property,

$$T^*(0, -\omega + 2\tau)(\Delta y(-\omega + 2\tau), \Delta u(-\omega + 2\tau), \Delta v(-\omega + 2\tau)) = (\Delta y(0), \Delta u(0), \Delta v(0)). \tag{5.3}$$

Let $\Delta x = (\Delta y(-\omega+2\tau) - \frac{\partial y^*(-\omega+2\tau)}{\partial v_o}\Delta v(-\omega+2\tau), \Delta u(-\omega+2\tau) - \frac{\partial u^*(-\omega+2\tau)}{\partial v_o}\Delta v(-\omega+2\tau), 0)$. We then have

$$\begin{aligned} &T^*(0, -\omega + 2\tau)\Delta x \\ &\quad = (\Delta y(0), \Delta u(0), \Delta v(0) - \frac{\partial v^*(0)}{\partial v_o}\Delta v(-\omega + 2\tau)). \end{aligned} \tag{5.4}$$

Here we have employed (5.2). Define an one-dimensional space

$$\pi^y(-\omega + 2\tau) = \text{span}\{\Delta x\}.$$

Notice that $\left|\frac{\partial y^*(-\omega+2\tau)}{\partial v_o}\right| + \left|\frac{\partial u^*(-\omega+2\tau)}{\partial v_o}\right| \leq Ce^{-\alpha\omega}$. Also from the proposition, $|\Delta u(-\omega + 2\tau)| \leq Ce^{-\alpha\omega}|\Delta u(0)| \leq Ce^{-\alpha\omega}|\Delta y(0)|$, and $|\Delta v(-\omega + 2\tau)| \leq C|\Delta y(0)|$.
$Ce^{-\alpha\omega}|\Delta y(0)|/|\Delta y(-\omega + 2\tau)| \leq C(\delta(\omega, \mu))^2$, therefore $\pi^y(-\omega + 2\tau)$ is close to $\text{span}\{(\Delta y(-\omega + 2\tau), 0, 0)\}$, if ω is large and $|\mu|$ is small. The error terms are bounded by $C\delta^2(\omega, \mu)$.

We claim that $T^*(0, -\omega + 2\tau)|_{\pi^y(-\omega+2\tau)}$ is contractive modulo $\dot{x}^*(0)$. To this end, subtracting (5.3) from (5.4), we have

$$\begin{aligned}|T^*(0, -\omega + 2\tau)\Delta x - \dot{x}^*(0)| &\le \left|\left(0, 0, \frac{\partial v^*(0)}{\partial v_o}\Delta v(-\omega + 2\tau)\right)\right| \\ &\le Ce^{-\alpha\omega}|\Delta v(-\omega + 2\tau)| \\ &\le Ce^{-\alpha\omega}|\Delta y(0)| \\ &\le C\delta^2(w, \mu)|\Delta y(-\omega + 2\tau)| \\ &\le C\delta(\omega, \mu)|\Delta x|\end{aligned}$$

Therefore, the rate of contraction modulo $\dot{x}^*(0)$ is bounded by $C\delta(\omega, \mu)$.

We have constructed (cf. the lower bounds for $\dot{x}_y^*(-\omega + 2\tau)$ and $\dot{x}_v^*(-\omega + 2\tau)$ in Proposition 5.5),

$$\begin{aligned}\pi^v(-\omega + 2\tau) &\simeq \{(0, 0, \Delta v_o)|\Delta v_o \perp \dot{x}_v^*(-\omega + 2\tau)\} \stackrel{\text{def}}{=} \overline{\pi}^v(-\omega + 2\tau), \\ \pi^y(-\omega + 2\tau) &\simeq \text{span}\{(\Delta y(-\omega + 2\tau), 0, 0)|\Delta y(-\omega + 2\tau) \\ &= \dot{x}_y^*(-\omega + 2\tau)\} \stackrel{\text{def}}{=} \overline{\pi}^y(-\omega + 2\tau), \\ \pi^u(0) &\simeq \{(0, \Delta u_o, 0)|\Delta u_o \in \mathbb{R}^l\} \stackrel{\text{def}}{=} \overline{\pi}^u(0), \\ \pi^y(0) &\simeq \{(\Delta y_o, 0, 0)|\Delta y_o \perp \dot{q}_y(-\tau)\} \stackrel{\text{def}}{=} \overline{\pi}^y(0).\end{aligned}$$

Here "$\simeq$" means that the two spaces are arbitrarily close if $\omega > \omega_o$ is large and $|\mu| < \mu_o$ is small, i.e., any unit vector in one space corresponds to another one in another space with an error of $O(\delta^2(\omega, \mu))$.

We have shown that $T^*(0, -\omega + 2\tau) : \pi^v(-\omega + 2\tau) \oplus \pi^y(-\omega + 2\tau) \to \mathbb{R}^{l+m+n}$ is a contraction modulo $\dot{x}^*(-\omega + 2\tau)$ and $T^*(-\omega + 2\tau, 0) : \pi^u(0) \oplus \pi^y(0)$ is a contraction. The rate of contractions is bounded by $C\delta(\omega, \mu)$.

Define $\pi_\tau = \pi^v(-\omega + 2\tau) \oplus \pi^y(-\omega + 2\tau)$ and $\pi_{w+\tau} = T_*(\tau, -\tau)\{\pi^u(0) \oplus \pi^y(0)\}$. It is easy to prove that they have the desired contraction properties under $T(\omega + \tau, \tau)$ and $T(\tau, \omega + \tau)$ respectively. In fact, $T(\omega + \tau, \tau) = T_*(\tau, -\tau)T^*(0, -\omega + 2\tau)$. Since $T^*(0, -\omega + 2\tau)$ is a contraction on $\pi^v(-\omega + 2\tau) \oplus \pi^y(-\omega + 2\tau)$, and $|T_*(\tau, -\tau)| \le C$, thus $T(\omega + \tau, \tau)$ is a contraction on π_τ. Also since $T_*(\tau, -\tau)$ is a homeomorphism $\pi^u(0) \oplus \pi^y(0) \to \pi_{w+\tau}$, let $x \in \pi_{w+\tau}$. then there exists $x^1 \in \pi^u(0) \oplus \pi^y(0)$, $C|x| \le |x^1| \le C^{-1}|x|$ with $x = T_*(\tau, -\tau)x^1$.

$$\begin{aligned}|T(\omega + \tau, \tau)x| &= |T^*(-\omega + 2\tau, 0)x^1| \\ &\le C\delta(-\omega + 2\tau, \mu)|x^1| \\ &\le C\delta(-\omega + 2\tau, \mu)|x|.\end{aligned}$$

Consider the solution of the boundary value problem from Corollary 3.5, with $w_0 = 0$, $v_0 = 0$. Let

$$\begin{aligned}w_1 &= w^*(-\omega + 2\tau, \omega - 2\tau, 0, 0, \mu), \\ v_1 &= v^*(0, \omega - 2\tau, 0, 0, \mu).\end{aligned}$$

It can be verified that $w_1 = (p(-\omega + 2\tau, \mu), 0)$ and $v_1 = 0$. Using the notation in the proof of Lemma 2.2, $w_1 = w_i^1(1)$ and $v_1 = v_i^1(1)$. Let $\overline{x}(t)$ be the solution of the generalized boundary value problem (3.4)-(3.6) with $v_l = 0, w_l = (p(-\omega + 2\tau, \mu), 0)$. Let $x(t)$ be the solution of (3.4)–(3.6) with the boundary values $(w_i^{l\infty}, v_i^{l\infty})$. From (4.12), we have

$$|\overline{x}(t) - x(t)| \leq C\delta(\omega - 2\tau, \mu)(\rho + |\mu|) \leq C(\rho + |\mu|)^2,$$

Due to the continuous dependence of solutions of the boundary value problems, Lemma 3.3. Let $\overline{T}(\tau, -\tau)$ be the solution map, from $-\tau$ to τ, of the linearized equation around $\overline{x}(t)$,

$$\dot{z}(t) = D_x f(\overline{x}(t), \mu) z(t), \quad -\tau \leq t \leq \tau.$$

Since solution maps depend C^1 on the vector fields, we have

$$|\overline{T}(\tau, -\tau) - T_*(\tau, -\tau)| \leq C(\rho + |\mu|)^2.$$

We shall give an estimate of the projections corresponding to the splitting

$$\pi_\tau \oplus \pi_{\omega+\tau} \oplus \mathrm{span}\{\dot{x}(\tau)\} = \mathbb{R}^{l+m+n}.$$

Observe that the estimates on the projections are equivalent to the estimates on the lower bound of angles between any subspace in the above to the sum of the other two subspaces. See the remark after Definition 3.7. We will show that those projections are bounded by $O(\rho+|\mu|)$. Since we have proved that the angles between $\pi^v(-\omega+2\tau)$ and $\overline{\pi}^v(-\omega+2\tau)$, $T(\tau, -\tau)\pi^u(0)$ and $\overline{T}(\tau, -\tau)\overline{\pi}^u(0)$, etc. are bounded below by $O((\rho + |\mu|)^2)$, it suffices to prove that

$$\{\overline{\pi}^v(-\omega + 2\tau) \oplus \overline{\pi}^y(-\omega + 2\tau)\} \oplus \{\overline{T}(\tau, -\tau)(\overline{\pi}^u(0) \oplus \overline{\pi}^y(0))\}$$
$$\oplus \ \mathrm{span}\{\dot{x}^*(-\omega + 2\tau)\} = \mathbb{R}^{l+m+n},$$

and the projections associated to the splitting are bounded by $C(\rho + |\mu|)^{-1}$. Also, $\mathrm{span}\{\dot{x}^*(-\omega + 2\tau)\}$ can be replaced by $\mathrm{span}\{(0, 0, \Delta v)\}$, since $|\Delta u(-\omega + 2\tau)|$ is small (see Proposition 5.5 for an estimate on $|\Delta u(0)|$, then use Lemma 3.4), and $(\Delta y(-\omega + 2\tau), 0, 0) \in \overline{\pi}^y(-\omega + 2\tau)$.

Let Δu_o^j, $1 \leq j \leq l$ be an orthonormal basis for $\mathbb{R}^l$ and Δv_o^j, $1 \leq j \leq m-1$ be an orthonormal basis for a codimension one subspace of $\mathbb{R}^m$ with $\Delta v_o^j \perp \dot{x}_v(\tau)$, $1 \leq j \leq m-1$. Here we point out that $x(\tau) = x^*(-\omega+2\tau)$, as seen in the beginning of the proof of Theorem 5.1. Therefore, the nonzeroness of $\dot{x}_v^*(-\omega + 2\tau)$ (Proposition 5.5) implies that $\dot{x}_v(\tau) \neq 0$. Let Δy_o^j, $1 \leq j \leq n-1$ be an orthonormal basis for a codimension one subspace of $\mathbb{R}^n$ with $\Delta y_o^j \perp \dot{q}_y(-\tau)$. Consider a matrix

$$\mathfrak{D} = (\mathbf{UYyVz}).$$

The column vectors of each block are as follows,

$$\mathbf{U} = \{\overline{T}(\tau, -\tau)(0, \Delta u_o^j, 0) \,|\, 1 \leq j \leq l\};$$
$$\mathbf{V} = \{(0, 0, \Delta v_o^j) |\ 1 \leq j \leq m-1,\ \Delta v_o^j \perp \Delta v(-\omega + 2\tau)\};$$
$$\mathbf{Y} = \{\overline{T}(\tau, -\tau)(\Delta y_o^j, 0, 0) |\ 1 \leq j \leq n-1,\ \Delta y_o^j \perp \dot{q}_y(-\tau)\};$$
$$\mathbf{y} = \{(\Delta y(-\omega + 2\tau)/\|\Delta y(-\omega + 2\tau)\|, 0, 0)\};$$
$$\mathbf{z} = (0, 0, \Delta v(-\omega + 2\tau))/\|\Delta v(-\omega + 2\tau)\|.$$

In the above, $\Delta y(-\omega+2\tau)\neq 0$ and $\Delta v(-\omega+2\tau)\neq 0$ (Proposition 5.5).

To prove Lemma 5.4, one needs to show $|\det\mathcal{D}|\geq C(\rho+|\mu|)$.

We claim that there is a unique $(l+n-1)\times(l+n-1)$ matrix $\mathcal{B}$ such that

$$(\mathbf{UY})\mathcal{B}=(\mathbf{U}_1\mathbf{Y}_1).$$

Here $\mathbf{Y}_1$ is an $(l+m+n)\times(n-1)$ matrix whose columns form an orthonormal basis of $T(W^{cu}(\mu)\cap\Sigma)\cap TW^{cs}_{loc}(\mu)=T\tilde{\mathcal{C}}(\mu)$, where $T\mathcal{C}(\mu)$ is defined in section 2. $\mathbf{U}_1$ is an $(l+m+n)\times l$ matrix whose column vectors form an orthonormal basis of $[TW^{cs}_{loc}(\mu)]^{\perp}=TW^{u}_{loc}(\mu)$.

In fact, the column vectors of $(\mathbf{UY})$ span the tangent space of $(W^{cu}(\mu)\cap\Sigma)$. We also claim that $|\det\mathcal{B}^{-1}|\geq C>0$ uniformly with respect to μ. First recall that $\overline{\pi}^{u}(0)\perp\overline{\pi}^{y}(0)$ and $\overline{T}(\tau,-\tau)$ is a homeomorphism when $\mu=0$. Thus, $|\det\mathcal{B}^{-1}|\geq C_0>0$. When μ is near zero, $\overline{T}(\tau,-\tau)$ and hence $(\mathbf{UY})$ depend continuously on μ. Also the solution $\mathcal{B}$ of the matrix equation depends continuously on μ. Finally, the inverse $\mathcal{B}^{-1}$ depends continuously on μ. Therefore $|\det\mathcal{B}^{-1}|\geq\frac{1}{2}C_0$ if μ is small. We then only need to show that

$$|\det(\mathbf{U}_1\mathbf{Y}_1\mathbf{y}\mathbf{V}\mathbf{z})|\geq C(\rho+|\mu|).$$

Observe that the vectors in $(\mathbf{Vz})$ form an orthonormal basis for $TW^{s}_{loc}(0,\mu)$. Therefore to compute the determinant, we can project $span\{\mathbf{Y}_1\}$ to the subspace $\{(y,u,v)|u=0,\ v=0\}$ along $TW^{s}_{loc}(0,\mu)$. However the projection of $T\tilde{\mathcal{C}}(\mu)$ yields $T\mathcal{C}(\mu)$. Both of these spaces are defined in section 2. We need an estimate on the angle of the subspaces span$\{\mathbf{y}\}$ and $T\mathcal{C}(\mu)$. Replacing $\frac{\Delta y(-\omega+2\tau)}{\|\Delta y(-\omega+2\tau)\|}$ by

$$\frac{\dot{p}(-\omega+2\tau,\mu)}{\|\dot{p}(-\omega+2\tau,\mu)\|},$$

we find the error is $0(e^{-\alpha\omega}/|\dot{p}(-\omega+2\tau,\mu)|)$. Using Proposition 5.5, we find that the error is negligible in proving the lemma.

The distance of the vector $\dot{p}(-\omega+2\tau,\mu)/|\dot{p}(-\omega+2\tau,\mu)|$ to the linear space $T\mathcal{C}(\mu)$ is $|\frac{\partial}{\partial\omega}d(p(-\omega+2\tau,\mu),\mu)|/|\dot{p}(-\omega+2\tau,\mu)|\geq C_4[\delta(\omega-2\tau,\mu)(|p(-\omega+2\tau,\mu)|+|\mu|)]/|\dot{p}(-\omega+2\tau,\mu)|$, from $(H_{7;1})$. Here $\overline{\delta}_i$ in H7) can be replaced by $\delta(\omega-2\tau,\mu)$. Since $|\dot{p}(-\omega+2\tau,\mu)|\leq C\delta(\omega-2\tau,\mu)|\dot{p}(0,\mu)|$, cf. H2), and $|\dot{p}(0,\mu)|$ is bounded above uniformly with respect to small μ, the angle between $\dot{p}/|\dot{p}|$ and $T\mathcal{C}(\mu)$ is bounded below by $C(|p|+|\mu|)$. This proves Lemma 5.4.

Let $X=\pi_\tau\oplus\pi_{\omega+\tau}$. For each $z\in X$, there are unique $\zeta_1(z)$ and $\zeta_2(z)\in\mathbb{R}$ such that

$$T(\tau+\omega,\tau)z+\zeta_1(z)\dot{x}(\tau)\in X,$$
$$T(\tau,\tau+\omega)z+\zeta_2(z)\dot{x}(\tau)\in X.$$

Define $Az=T(\tau+\omega,\tau)z+\zeta_1(z)\dot{x}(\tau)$ and $A^{-1}z=T(\tau,\tau+\omega)z+\zeta_2(z)\dot{x}(\tau)$. Since the angle between X and span$\{\dot{x}(\tau)\}$ is bounded below from zero, it follows that $A:\pi_\tau\to X$ and $A^{-1}:\pi_{\omega+\tau}\to X$ are also contractions with the rates bounded by $C_1\delta(\omega-2\tau,\mu)$. Details shall be left to the readers. We can now use Lemma 3.6 with $U=\pi_{\omega+\tau}$, $V=\pi_\tau$, $M=C^{-1}[|p(-\omega+2\tau,\mu)|+|\mu|]^{-1}$ and $\lambda=C_1\delta(\omega-2\tau,\mu)$. Recall that C is proportional to C_4, thus $4M^2\lambda^2<1$ if C_4 is sufficiently large. Observe that $\dim\pi_\tau=m$ and $\dim\pi_{\omega+\tau}=l+n-1$. The proof of Theorem 5.1 has been completed.

Proof of Theorem 5.2. From Theorem 2.1, for each $|\mu| < \hat{\mu}$, there is a unique piecewise continuous solution to (2.1), denoted by $x(t, \mu)$, such that $x_v(-\tau, \mu) = 0$ $x_w(\tau, \mu) = 0$ and $x(\tau^-, \mu) - x(\tau, \mu) = \xi\vec{\Delta}$. Here $\xi = \xi_*(w, v, \mu)$ with $w = 0$, $v = 0$ and $\xi_*(0, 0, \mu) = d(0, \mu)$. Let $\psi(\tau)$ be defined in section 2, cf. (2.5) and H6). It is known if $\varphi(t)$ is a solution for (2.4), then $\frac{d}{dt} < \psi(t), \varphi(t) >= 0$. Thus from $\psi(0) \perp \{T_{q(0)}W^{cu}(0) + T_{q(0)}W^s(0)\}$, we have $\psi(\tau) \perp \{T_{q(\tau)}W^{cu}(0) + T_{q(\tau)}W^s(0)\}$. Since $\vec{\Delta} \notin \{T_{q(\tau)}W^{cu}(0) + T_{q(\tau)}W^s(0)\}$, $< \psi(\tau), \vec{\Delta} > \neq 0$ and without loss of generality, $< \psi(\tau), \vec{\Delta} >= 1$. Thus,

$$\begin{aligned}\xi &= < \psi(\tau),\ x(\tau^-, \mu) - x(\tau, \mu) > \\ &= \int_{-\infty}^{\tau} \frac{\partial}{\partial t}(\psi(t)x(t, \mu))dt + \int_{\tau}^{\infty} \frac{\partial}{\partial t}(\psi(t)x(t, \mu))dt,\end{aligned}$$

where $\mu_1 = 0$ is fixed. Here we have used the fact that $\psi(t)x(t, \mu) \to 0$ as $t \to \pm\infty$. Therefore, if $\mu_1 = 0$,

$$\xi = \left(\int_{-\infty}^{\tau} + \int_{\tau}^{\infty}\right)\{-\psi(t)D_x f(q(t), 0)x(t, \mu) + \psi(t)f(x(t, \mu), \mu)\}dt.$$

$\frac{\partial}{\partial \mu}x(t, \mu)$ satisfies the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \mu}x(t, \mu)\right) = D_x f(x(t, \mu), \mu)\frac{\partial}{\partial \mu}x(t, \mu) + D_\mu f(x(t, \mu), \mu) \tag{5.5}$$

Since $x(\tau, \mu) \in W^s_{loc}(0)$, we have $x_w(t, \mu) = 0$, and $\frac{\partial}{\partial \mu}x_w(t, \mu) = 0$, for all $t \geq \tau$. Moreover, from (2.3), $D_\mu g_i(0, 0, v, \mu) = 0$ for $i = 0, 1$. Thus, for $t \geq \tau$, we only need to solve $\frac{\partial}{\partial \mu}x_v(t, \mu)$, denote it by $\overline{v}(t)$, in (5.5). The function $\overline{v}(t)$, $t \geq \tau$ satisfies the equation

$$\begin{aligned}\overline{v}(t) &= e^{A_2(t-\tau)}\overline{v}(\tau) + \int_{\tau}^{t} e^{A_2(t-s)}D_\mu g_2(0, 0, x_v(s), \mu)ds \\ &\quad + \int_{\tau}^{t} e^{A_2(t-s)}D_v g_2(0, 0, x_v(s), \mu)\overline{v}(s)ds.\end{aligned}$$

Since $D_\mu f(x(t, \mu), \mu) \to 0$ exponentially fast as $t \to +\infty$, therefore, we can use the decaying property of $e^{A_2(t-s)}$ and Gronwall's inequality to show that $\frac{\partial}{\partial \mu}x(t, \mu) \to 0$ exponentially fast as $t \to +\infty$. Details will be left to the readers. Similarly we can show that $\left|\frac{\partial}{\partial \mu}x(t, \mu)\right| \leq Ce^{-\gamma t}$ for $t < -\tau$, where $0 < \gamma < \alpha_o$. We can change the order of $\frac{\partial}{\partial \mu_2}$ and the integrations to obtain

$$\begin{aligned}\frac{\partial}{\partial \mu_2}\xi\Big|_{\mu=0} &= \left(\int_{-\infty}^{\tau} + \int_{\tau}^{\infty}\right)\{-\psi(t)D_x f(q(t), 0)\frac{\partial}{\partial \mu_2}x(t, \mu) \\ &\quad + \psi(t)D_x f(q(t), 0)\frac{\partial}{\partial \mu_2}x(t, \mu) + \psi(t)D_{\mu 2}f(q(t), 0)\}dt \\ &= \int_{-\infty}^{\infty} \psi(t)D_{\mu 2}f(q(t), 0)dt.\end{aligned}$$

By H6), we have

$$\frac{\partial}{\partial \mu_2}d(0, \mu)\Big|_{\mu=0} = \frac{\partial}{\partial \mu_2}\xi_*(0, 0, \mu)\Big|_{\mu=0} \neq 0. \tag{5.6}$$

For each ω and μ with $\sup(|\mu_1|, |\mu_2|) < \hat{\mu}$, $p(-\omega+2\tau, \mu) < \hat{\rho}$ and $\delta(\omega-2\tau, \mu) < \hat{\delta}$, there exists a unique piecewise continuous ω-periodic solution $x(t, \mu, \omega)$ of (2.1) with a jump $\xi\overrightarrow{\Delta} = x(\tau^-) - x(\tau)$. See Theorem 2.1 again. The size of jump is $G(\omega, \mu) = \xi_*(\omega_i^1, v_i^1, \mu)$ with $w_i^1 = w_o^1$ and $v_i^1 = v_o^1$ for all $i \in \mathbb{Z}$. Since

$$|w_i^1| + |v_i^1| \leq C\hat{\rho},$$
$$|\xi_*(w_i^1, v_i^1, v_i^1, \mu)|_{\mu_2=0} \leq C(\hat{\rho} + |\mu_1|).$$

Now (5.6) implies that

$$\left|\frac{\partial}{\partial \mu_2}\xi_*(w_i^1, v_i^1, \mu)\right| > \beta_o > 0$$

for some fixed $\beta_o > 0$ if $\hat{\rho}$ and $\hat{\mu}$ are small. Let $\mu = (\mu_1, \mu_2)$.

$$|\xi_*(w_i^1, v_i^1, (\mu_1, \pm\hat{\mu})) - \xi_*(w_i^1, v_i^1, (\mu_1, 0))| \geq \beta_o\hat{\mu}.$$

Thus, if $C(\hat{\rho} + |\mu_1|) < \beta_o\hat{\mu}$, $\xi_*(w_i^1, v_i^1, (\mu_1, \pm\hat{\mu}))$ changes sign. Thus, there exists a unique $|\mu_2| < \hat{\mu}$ such that $\xi_*(w_i^1, v_i^1, (\mu_1, \mu_2)) = 0$. We can show that μ_2 is a C^k function of ω by using the implicit function theorem locally. The proof of Theorem 5.2 has been completed.

6. Bifurcations of Homoclinic Solutions with One-Dimensional Local Center Manifolds

When the local center manifold is one-dimensional, we are able to give a complete description of all the solutions of system (1.1) that are orbitally near Γ_0. Here $\mu = (\mu_1, \mu_2)$ is a parameter. μ_1 is from a Banach space that determines the vector field near $x = 0$. Typical examples are $\mu_1 \in \mathbb{R}^1$ and $g_0(y, 0, 0, \mu) = \mu_1 + y^2$ (saddle-node), $g_0(y, 0, 0, \mu) = \mu_1 y + y^3$ (pitch-fork), and $g_0(y, 0, 0, \mu) = \mu_1 y + y^2$ (transcritical) on $W^c_{loc}(\mu)$. We assume that μ_2 does not affect the flow on $W^c_{loc}(\mu)$ if μ_1 is fixed. In fact, all we need to know about the flow on $W^c_{loc}(\mu)$ is whether there is no equilibrium or there is at least one equilibrium on $W^c_{loc}(\mu)$.

When $u = 0$ and $v = 0$, we write the first equation of (1.1) as

$$\dot{y} = g_0(y, 0, 0, \mu) = a_0(\mu) + a_1(\mu)y + a_2(\mu)y^2 + \dots \tag{6.1}$$

with $a_0(0) = a_1(0) = 0$. Assume $g_0(y, 0, 0, 0) > 0$ for $0 < y < \overline{\rho}$. Assume that $\tau > 0$ is sufficiently large so that $0 < q_y(-\tau) < \overline{\rho}$. Here $W^c_{loc}(\mu) = \{|y| < \overline{\rho}\}$. Then it is clear that for $|\mu| \leq \mu_0$, we can find a neighborhood $\mathcal{U}$ of $q_y(-\tau)$ on $W^c_{loc}(\mu)$ and $\bar{t}(\mu) > \bar{\bar{t}}(\mu)$ so that $\Phi(t, \mu, y)$, $y \in \mathcal{U}$, is contractive for $-\bar{t} < t < -\bar{\bar{t}}$, see H_2). If there is at least one equilibrium on $W^c_{loc}(\mu)$, let the one that is closest to $q_y(-\tau)$ be $y = E(\mu)$. Then $y(t) = \Phi(t, \mu, y) \to E(\mu)$ as $t \to -\infty$ and we set $\bar{t}(\mu) = +\infty$. If there is no equilibrium on $W^c_{loc}(\mu)$, then $\Phi(t, \mu, y)$ will leave $W^c_{loc}(\mu)$ as $t \to -\infty$, and the restriction $-\bar{t}(\mu) < t < \bar{t}(\mu)$ is used to ensure that $\Phi(t, \mu, y)$ stays in $W^c_{loc}(\mu)$.

We derive estimates on $\delta(|t|, \mu)$. Since $\dot{y}(0) \neq 0$ for $\mu = 0$, $|\dot{y}(0)|$ is bounded away from zero. From the definition in section 2, H2), $\delta(|t|, \mu) = |^{\dot{y}(t)}/\dot{y}(0)| \leq C|\dot{y}(t)|$. From (6.1) $|\dot{y}(t)| \leq c(\rho + |\mu|)$, where $\rho = |y(t)|$, therefore $\delta(|t|, \mu) < c(\rho + |\mu|)$ and $H_2)$ is valid.