Another useful estimate can be derived if there is at least one equilibrium on $W^c_{loc}(\mu)$. Let $E(\mu) = \sup\{y | g_0(y, 0, 0, \mu) = 0\}$. Set $y = E(\mu) + z$,

$$\dot{z}(t) = g_0(E(\mu) + z, 0, 0, \mu) - g_0(E(\mu), 0, 0, \mu) = 0(|z|).$$

Therefore, $\dot{z}(t) \le L \cdot |z| = L \cdot |y(t) - E(\mu)|$, where L is a small constant if μ and ρ are small, based on (6.1) and $a_1(0) = 0$. Thus, we have

$$\delta(|t|, \mu) \le C \cdot L|y(t) - E(\mu)|. \tag{6.2}$$

The hyperplane $\mathscr{C}(\mu)$ defined in section 2 is now only a point on $W^c_{loc}(\mu)$. Let its coordinate be $y = \mathscr{C}(\mu)$. The curve σ is also a point whose coordinate is $y = q_y(-\tau)$.

Theorem 6.1. *Assume that H_1)-H_5) are satisfied. Then there exist positive constants $\hat{\mu}$ and $\hat{\epsilon}$ such that for $|\mu| < \hat{\mu}$, all the solutions that are orbitally $\hat{\epsilon}$-near Γ_0 can be described as follows (cf. Figure 6.1):*

1) *If there is at least one equilibrium on $W^c_{loc}(\mu)$, $E(\mu) = sup\{y|g_o(y, 0, 0, \mu) = 0, |y| < \overline{\rho}\}$, if $\mathscr{C}(\mu) \le E(\mu)$ and if $\Phi(t, \mu, \mathscr{C}(\mu))$ approaches an equilibrium E_∞ on $W^c_{loc}(\mu)$, as $t \to +\infty$ then there is a unique heteroclinic solution $x(t)$ that stays in a $\hat{\epsilon}$-neighborhood of $q(t)$ and follows $q(t)$ only once. $x(t) \to (E(\mu), 0, 0)$ as $t \to -\infty$ and $x(t - \tau)$ approaches $x^+(t) = (\Phi(t, \mu, \mathscr{C}(\mu)), 0, 0) \in \mathbb{R}^{l+m+1}$ exponentially as $t \to +\infty$.*
2) *If there is at least one equilibrium on $W^c_{loc}(\mu)$ and if $\mathscr{C}(\mu) > E(\mu)$, then there is a unique simple periodic orbit $x(t)$ that is orbitally $\hat{\epsilon}$- near Γ_o. There also exists a heteroclinic solution $x^1(t)$, orbitally $\hat{\epsilon}$- near Γ_o. The solution $x^1(t)$ orbitally approaches $x(t)$ as $t \to +\infty$ and $x^1(t) \to (E(\mu), 0, 0)$ as $t \to -\infty$.*
3) *If there is no equilibrium on $W^c_{loc}(\mu)$, then there exists a unique simple periodic solution $x(t)$ that is orbitally $\hat{\epsilon}$-near Γ_o.*

The period of the periodic solution, denoted by ω, in cases 2) and 3), is close to t_o, determined by the equation

$$p(-t_o + 2\tau, \mu) = \mathscr{C}(\mu).$$

That is, $|\omega - t_o| \to 0$ as $\hat{\mu} \to 0$ and $\hat{\epsilon} \to 0$. The periodic orbits in both cases 2) and 3) are hyperbolic with ℓ (or m) unstable (or stable) characteristic values.

Proof. Let $\hat{\rho} = \sup\{p(-t_i + 2\tau, \mu)\}$ for any symbol $\{S_i\}^k_{h-1}$, $t_i = S_i$, $h \le i < k$. If $x(t)$ is orbitally $\hat{\epsilon}$-near Γ_0, $\mathscr{T}x = \{S_i\}^k_{h-1}$ and $|\mu| < \hat{\mu}$, then $\hat{\rho} \to 0$ if $\hat{\epsilon} \to 0$ and $\hat{\mu} \to 0$.

1) For the existence of a heteroclinic solution $x(t)$, let h=k=finite, $S_{h-1} = [0]$, $S_k = (\bar{y}, 0)$ with $\bar{y} = \mathscr{C}(\mu)$. There is only one bifurcation equation

$$G_k(\{S_i\}^k_{h-1}, \mu) = d(\bar{y}, \mu) = 0, \tag{6.3}$$

since there is only one outer solution and there is no inner solution in this case. We have also used the fact that the error terms in (2.6) are all zero, since by our convention made there, $\delta_i = 0$ for all i not satisfying $h \le i \le k - 1$. The existence of the simple heteroclinic (or homoclinic) solution then follows from Theorem 2.1 and the fact that $\Phi(t, \mu, \bar{y})$ approaches an equilibrium on $W^c_{loc}(\mu)$ as $t \to +\infty$.

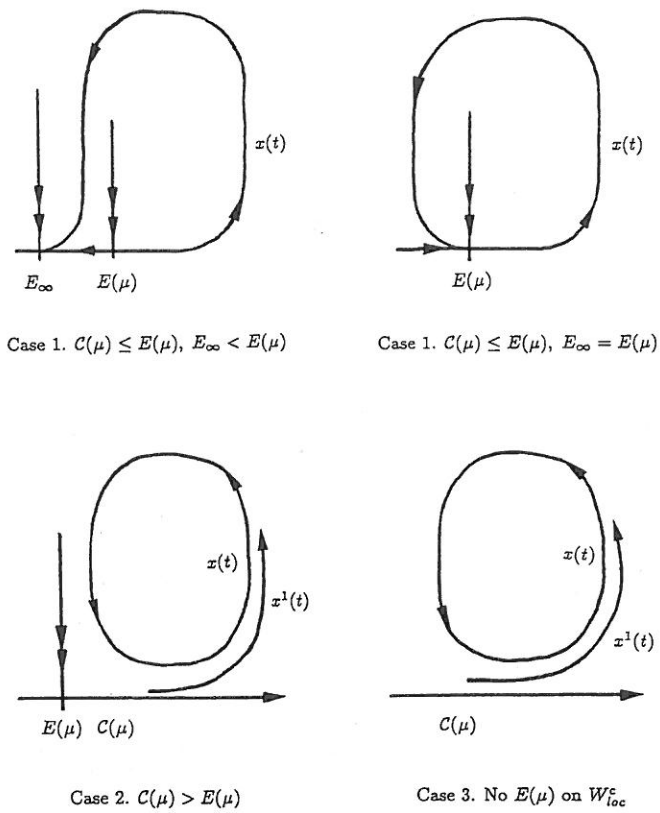

Figure 6.1.

We then prove that there is no periodic or other heteroclinic solution than the one corresponding to $S_{h-1} = 0$, $S_k = (\mathscr{C}(\mu), 0)$, $h = k$, except for a possible shift in time t. We will show that h=k=finite, i.e., there are no multiple heteroclinic solutions. Then (6.3) will imply that $\tilde{y} = \mathscr{C}(\mu)$. If not, i.e., $h < k$, then the set $h \leq i \leq k-1$ is nonempty. For those i, let $y_i = p(-t_i+2\tau, \mu)$. Recall that $p(-t_i+2\tau, \mu) = \Phi(-t_i+2\tau, \mu, q_y(-\tau))$. If $|\mu|$ is sufficiently small, then $q_y(-\tau) > E(\mu)$, and there is no equilibrium between them, thus $\mathscr{C}(\mu) \leq E(\mu) < y_j$, $h \leq j \leq k-1$. There exists y_i such that $2|y_i - E(\mu)| \geq \sup_j |y_j - E(\mu)|$. Observe that $d(y_i, \mu) = y_i - \mathscr{C}(\mu) \geq y_i - E(\mu)$. Also $\delta_i \leq CL|y_i - E(\mu)|$ and $\delta_{i-1} \leq CL|y_{i-1} - E(\mu)| \leq 2CL|y_i - E(\mu)|$. Observe that we must have $\bar{v} = 0$ if h is finite and $\bar{u} = 0$ if k is finite. The right hand side of (2.6) for such i is bounded by

$$C_1(\hat{\rho} + \hat{\mu})(CL|y_i - E(\mu)| + 2CL|y_i - E(\mu)|.$$

Therefore the right hand side of (2.6) cannot be greater than $d(y_i, \mu) = |y_i - \mathscr{C}(\mu)| \geq |y_i - E(\mu)|$ if $\hat{\epsilon}$ and $\hat{\mu}$ are sufficiently small, contradicting to (2.6). Therefore, h=k=finite. Case 1 has been proved.

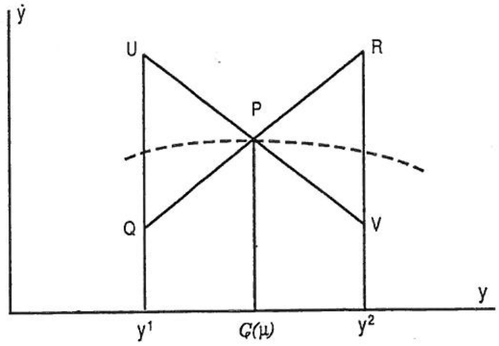

Figure 6.2.

2) Since $E(\mu) < \mathscr{C}(\mu) < q_y(-\tau)$, and since there is no equilibrium between $E(\mu)$ and $q_y(-\tau)$, there exists t_o such that $p(-t_o+2\tau, \mu) = \mathscr{C}(\mu)$. To show there is a periodic solution $x(t)$ with the period $\omega \simeq t_o$, we set $S_i = t_i = t_o$, for $i \in \mathbb{Z}$. We show the sequence $\{S_i\}_{-\infty}^{\infty}$ satisfies H_7). Let $g_o(\mathscr{C}(\mu), 0, 0, \mu) = Y > 0$. See Figure 6.2. In the $(y, \dot{y})$-coordinate plane, let the dotted line be the graph of $\dot{y} = g_o(y, 0, 0, \mu)$, which passes through $P = (\mathscr{C}(\mu), Y)$. Let $0 < \eta < 1$ and let $y^1 < \mathscr{C}(\mu) < y^2$ with $\mathscr{C}(\mu) - y^1 = y^2 - \mathscr{C}(\mu) = \eta Y$. Let Q and U be two points, vertically $(1-\eta)Y$ and $(1+\eta)Y$ above y^1. Let R and V be two points, vertically $(1+\eta)Y$ and $(1-\eta)Y$ above y^2. It is clear that QPR and UPV are two line segments with slopes 1 and -1 respectively. We may assume that the dotted line is between QPR and UPV if the Lipschitz number of g_o is bounded by 1. Let $-t^1 + 2\tau < -t_o + 2\tau < -t^2 + 2\tau$ be such that $p(-t^1 + 2\tau, \mu) = y^1$ and $p(-t^2 + 2\tau, \mu) = y^2$. We claim that $e_i = t^1 - t_o$ and $d_i = t_o - t^2$, $i \in \mathbb{Z}$ are the desired choice for H_7) to hold. First, recall that $y_i(\zeta) = p(-t_i - \zeta + 2\tau, \mu)$. Therefore, $d(p(-t_i - \zeta + 2\tau, \mu), \mu) = 0$ for $\zeta = 0$, $i \in \mathbb{Z}$. Next, since in Figure 6.2, the graph of g_0–the dotted line is above the line segments QPV, $\left|\frac{\partial}{\partial\zeta} d(y_i(\zeta), \mu)\right| = \left|\frac{\partial}{\partial\zeta} y_i(\zeta)\right| \geq (1-\eta)Y$. Also $\overline{\delta}_i = \sup\{|\dot{y}(-t_i - \zeta + 2\tau)/\dot{y}(0)|\} \leq \sup\{|g_0|\}/|\dot{y}(0)| \leq (1+\eta)Y/|\dot{y}(0)|$ for $-d_i < \zeta < e_i$, where $y(t) = \Phi(t, \mu, q_y(-\tau))$ and $|\dot{y}(0)|$ is bounded below uniformly if $|\mu|$ is small. It is clear that for any $0 < \eta < 1$, the right hand side of $(H_7; 1)$ is bounded by

$$
\begin{aligned}
&C(\hat{\rho} + \hat{\mu})\overline{\delta}_i \\
\leq &C(\hat{\rho} + \hat{\mu})(1 + \eta)Y/|\dot{y}(0)| \\
\leq &(1 - \eta)Y,
\end{aligned}
$$

provided that $\hat{\rho}$ and $\hat{\mu}$ are small. Thus, $(H_7; 1)$ is valid and the constant C_4 can be arbitrarily large if $\hat{\mu}$ and $\hat{\rho}$ are small. Third, $d(y_i(\zeta), \mu) = -\eta Y$ (or ηY) at $\zeta = -d_i$ (or $\zeta = e_i$), and $Y > 0$ for all sufficiently small $|\mu|$. Using the estimate for $\overline{\delta}_i$ obtained above, we find that $(H_7; 2)$ is also valid if $\hat{\rho}$ and $\hat{\mu}$ are sufficiently small. Finally, the smallness of $\hat{\mu}$ and $\hat{\rho}$ is guaranteed if $\hat{\mu}$ and $\hat{\epsilon}$ are sufficiently small.Therefore, both $(H_7; 1)$ and $(H_7; 2)$ are valid if $\hat{\epsilon}$ and $\hat{\mu}$ are small. (These conditions actually force t_0 to be large and $\hat{\rho}$ to be small, refer to the comments two paragraphs before Theorem 2.1.)

The existence of a periodic solution is obvious from Theorem 2.3 and the first part of Theorem 5.1. The hyperbolicity of the periodic orbit also comes from Theorem 5.1. Here again the largeness of C_4 is achieved by choosing small $\hat{\mu}$ and $\hat{\rho}$. Since $|\omega - t_o| \leq \max\{d_i, e_i\}$, $|y^1 - \mathcal{C}(\mu)| = |y^2 - \mathcal{C}(\mu)| = \eta Y$ and $g_o(y, 0, 0, \mu) > (1-\eta)Y$, we infer that $|\omega - t_o| \leq \eta Y/((1-\eta)Y) = \eta/(1-\eta)$. Since η can arbitrarily small provided that $\hat{\rho}$ and $\hat{\mu}$ are small, $|\omega - t_o| \to 0$ as $\hat{\mu} \to 0$ and $\hat{\varepsilon} \to 0$.

The existence of the heteroclinic solution $x^1(t)$, such that $x^1(t) \to x(t)$ as $t \to +\infty$, $x^1(t) \to (E(\mu), 0, 0)$ as $t \to -\infty$ is a consequence of Theorem 2.4, if we set $k^1 = k = +\infty$, $h^1 = 0$ and $S_{-1} = 0$.

We now show that in case 2, $x(t)$ and $x^1(t)$ obtained above are the only solutions in a neighborhood of $q(t)$. First, we want to rule out the possibility of having a solution $x^2(t)$ with $\mathcal{T}x^2(\cdot) = \{S_i\}_{h-1}^k$, with $k \neq +\infty$. If this were the case, the solution $x^2(\cdot) \in W_{loc}^{cs}$ for all sufficiently large t. We would have $S_k = (\bar{y}, 0)$. The solution is asymptotic to $\Phi(t, \mu, \bar{y})$, with a possible time shift. If $E(\mu) < \bar{y}$, the solution $\Phi(t, \mu, \bar{y})$ leaves W_{loc}^c in finite time. This is a contradiction. Also, we could not have $h = k$ or we would have $G_k(\{S_i\}, \mu) = d(\bar{y}, \mu)$, from the arguments in case 1. But $d(\bar{y}, \mu) \neq 0$ since $\bar{y} \leq E(\mu)$. Therefore $h \leq k-1$. We then infer that $\sup\{y_j | h \leq j \leq k-1\} \leq 2\mathcal{C}(\mu) - E(\mu)$ where $y_j = p(-t_j + 2\tau, \mu)$. If not, there exists $h \leq i \leq k-1$ such that $y_i = 2\mathcal{C}(\mu) - E(\mu) + \eta$ with some $\eta > 0$ and $y_i - E(\mu) \geq \frac{1}{2}\sup_j |y_j - E(\mu)|$. then $y_i - \mathcal{C}(\mu) = \mathcal{C}(\mu) - E(\mu) + \eta$, and $\frac{1}{2}(y_i - E(\mu)) = \mathcal{C}(\mu) - E(\mu) + \frac{1}{2}\eta$. Hence, $|d(y_i, \mu)| > \frac{1}{2}|y_i - E(\mu)|$. On the other hand, $\sup\{\delta_i | h \leq i \leq k-1\} \leq CL\sup\{|y_j - E(\mu)|\} \leq 2CL|y_i - E(\mu)|$ (cf. (6.2)). This contradicts (2.6) if $\hat{\mu}$ and $\hat{\rho}$ are small. Now that $\sup\{y_j | h \leq j \leq k-1\} \leq 2\mathcal{C}(\mu) - E(\mu)$, for $i = k$ the right hand side of (2.6) is bounded by $2CL(\mathcal{C}(\mu) - E(\mu))$. But $d(\bar{y}, \mu) \geq \mathcal{C}(\mu) - E(\mu)$. Since L is a small quantity, this contradicts (2.6). We have proved that $k = +\infty$.

We then have only two types of solutions i) $\mathcal{T}x(\cdot) = \{S_i\}_{-\infty}^{\infty}$ and ii) $\mathcal{T}x(\cdot) = \{S_i\}_h^{\infty}$. In the second type we will make a shift of indices so that $h = 0$. Let us prove that the heteroclinic solution $x^1(t)$ is the only solution with $h = 0$. The same argument will also show that the periodic solution $x(t)$ is the only one with $h = -\infty$.

For problems with one-dimensional center manifolds, system (4.1)-(4.4) can be simplified. For one reason $y_i^o = q_y(-\tau)$ is a given phase condition in (4.1) and (4.2) and can be dropped. For another reason, y_{i+1}^1 and y_i^1 in (4.3) and (4.4) will only affect the size of jump $\xi_*(w_{i+1}^1, v_{i+1}^1, \mu)$ etc. but not w_i^o and v_i^o. To show this, consider the generalized boundary value problem (3.4)-(3.6), with the solutions described in Lemma 3.3. We now have $w^1 = (y^1, u^1)$ with $y^1 \in \mathbb{R}$ and $\overrightarrow{\Delta} = (1, 0, 0)$. The solution for (3.4)-(3.6) actually satisfies the boundary conditions:

$$v(-\tau) = v^1, \quad u(\tau) = u^1, \quad y(\tau^-) = y^1 + \xi.$$

where ξ is the size of the jump in the y direction. Therefore, changing y^1 and ξ simultaneously but keeping $y^1 + \xi$ constant will not change $x(t)$ for $t \in [-\tau, \tau)$. We now have a reduced system (6.4) and (6.5) from (4.1)-(4.4). Here $0 \leq i < \infty$, and $v_o^1 = 0$ is given.

$$\begin{aligned} u_i^1 &= u^*(-t_i + 2\tau,\ t_i - 2\tau,\ u_i^o,\ v_i^o,\ \mu), \\ v_{i+1}^1 &= v^*(0,\ t_i - 2\tau,\ u_i^o,\ v_i^o,\ \mu), \end{aligned} \tag{6.4}$$

$$\begin{aligned} u_i^0 &= u_*(-\tau,\, u_{i+1}^1,\, v_{i+1}^1,\, \mu), \\ v_i^o &= v_*(\tau,\, u_i^1,\, v_i^1,\, \mu). \end{aligned} \tag{6.5}$$

More precisely, (6.5) comes from (4.3) and (4.4) by letting $y_i^1 \equiv 0$, $i \geq 0$. The jump functions are irrelevant for such y_i^1. After solving (6.4)-(6.5), we can compute the true y_i^1, $i \geq 0$ from the y–component of (4.1), and then the true jump size from the function $\xi_*(w_{i+1}^1, v_{i+1}, \mu)$, $i \geq 0$.

Observe now $y_i = p(-t_i + 2\tau, \mu)$ is a monotone function of t_i. Also, we can show that $\left|\frac{\partial y_i}{\partial t_i}\right| \geq Ce^{-\beta t_i}$, here β is a small constant related to $\mathrm{Lip}\{g_o(y, 0, 0, \mu)\}$. Thus, for $E(\mu) < y_i < q_y(-\tau)$, we can find the inverse function $t_i = t_i(y_i)$, with $\left|\frac{\partial t_i}{\partial y_i}\right| \leq Ce^{\beta t_i}$. We now substitute $t_i = t_i(y_i)$ into (6.4). System (6.4) and (6.5) is now parameterized by $\{y_i\}_0^\infty$.

Suppose we have two heteroclinic solutions $x^{(\nu)}(t)$, $\nu = 1, 2$, corresponding to $\{y_i^{(\nu)}\}$, $\nu = 1, 2$, $0 \leq i < \infty$. Let $y_i(\epsilon) = y_i^{(1)} + \epsilon\Delta y_i$ where $\Delta y_i = y_i^{(2)} - y_i^{(1)}$. Let $t_i^\epsilon = t_i(y_i(\epsilon))$. Since (6.4) and (6.5) are contractive, for each $0 < \epsilon < 1$, there exists a unique generalized solution, denoted by $\{u_i^1(\epsilon),\, v_{i+1}^1(\epsilon),\, u_i^o(\epsilon),\, v_i^o(\epsilon)\}_o^\infty$. We then have, from (6.4), and Lemma 3.4 which provides estimates on the derivatives of u^* and v^*,

$$\left|\frac{\partial u_i^1(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial v_{i+1}^1(\epsilon)}{\partial \epsilon}\right| \leq Ce^{-\alpha_1 t_i}\left|\frac{\partial t_i^\epsilon}{\partial \epsilon}\right| + Ce^{-\alpha_1 t_i}\left(\left|\frac{\partial u_i^o(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial v_i^0(\epsilon)}{\partial \epsilon}\right|\right).$$

From (6.5) and Lemma 3.3,

$$\left|\frac{\partial u_i^0(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial v_i^0(\epsilon)}{\partial \epsilon}\right| \leq C\left(\left|\frac{\partial u_i^1(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial v_i^1(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial u_{i+1}^1(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial v_{i+1}^1(\epsilon)}{\partial \epsilon}\right|\right).$$

Observing that $\left|\frac{\partial t_i^\epsilon}{\partial \epsilon}\right| \leq Ce^{\beta t_i}|\Delta y_i|$, we then have

$$\left\|\left\{\frac{\partial u_i^1(\epsilon)}{\partial \epsilon}\right\}\right\| + \left\|\left\{\frac{\partial v_{i+1}^1}{\partial \epsilon}\right\}\right\| \leq Ce^{(-\alpha_1+\beta)\hat{t}}\|\{\Delta y_i\}\|. \tag{6.6}$$

Here $\hat{t} = \inf\{t_i\}$, $\|\{\Delta y_i\}\| = \sup\{|\Delta y_i|,\ 0 \leq i < \infty\}$, etc. Since $\xi_*(w_i^1(\epsilon),\, v_i^1(\epsilon),\, \mu) = 0$ for $\epsilon = 0$ and $\epsilon = 1$, we have for each $0 \leq i < \infty$, there exists $0 < \epsilon = \epsilon_i < 1$ such that

$$\begin{aligned} 0 &= \frac{\partial \xi_*(y_i^1(\epsilon),\, u_i^1(\epsilon),\, v_i^1(\epsilon),\, \mu)}{\partial \epsilon} \\ &= \frac{\partial \xi_*}{\partial y_i^1}\frac{\partial y_i(\epsilon)}{\partial \epsilon} + \frac{\partial \xi_*}{\partial y_i^1}\frac{\partial y^\$(t_i^\epsilon)}{\partial \epsilon} + 0\left(\left|\frac{\partial u_i^1(\epsilon)}{\partial \epsilon}\right| + \left|\frac{\partial v_i^1(\epsilon)}{\partial \epsilon}\right|\right) \\ &= \frac{\partial \xi_*}{\partial y_i^1}\cdot \Delta y_i + 0(e^{-\alpha_1+2\beta)\hat{t}}\|\{\Delta y_j\}\|) \end{aligned}$$

Here we have used the fact $y_i^1(\epsilon) = y_i(\epsilon) + y^\$(-t_i^\epsilon + 2\tau)$ and $\frac{\partial y^\$}{\partial t_i} \leq Ce^{(-\alpha_1+\beta)t_i}$ (Lemma 3.4), and also (6.6). To remind the reader, $y_i(\epsilon) = p(-t_i^\epsilon + 2\tau, \mu)$, according to the definition of t_i^ϵ. Observe now

$$\frac{\partial \xi_*(y_i^1(\epsilon),\, u_i^1(\epsilon),\, v_i^1(\epsilon),\, \mu)}{\partial y} = \frac{\partial \xi_*(y_i(\epsilon), 0, 0, \mu)}{\partial y} + o(1),$$

since $|y^{\$}(-t_i^{\epsilon}+2\tau)|+|u_i^1(\epsilon)|+|v_i^1(\epsilon)| \to 0$ as $\hat{\mu}$ and $\hat{\rho} \to 0$. We then have

$$\left\{\left|\frac{\partial \xi_*(y_i(\epsilon),0,0,\mu)}{\partial y}\right| - o(1)\right\} |\Delta y_i| \le Ce^{(-\alpha_1+2\beta)\hat{t}} \|\{\Delta y_j\}\|$$

However, by Lemma 4.3, $\left|\frac{\partial \xi_*(y,0,0,\mu)}{\partial y}\right| = \left|\frac{\partial}{\partial y} d(y,\mu)\right| = 1$ and $\hat{t} \to \infty$ as $\hat{\mu}$ and $\hat{\rho} \to 0$. Therefore $|\Delta y_i| \le \frac{1}{2}\|\{\Delta y_j\}\|$ for small $\hat{\mu}$ and $\hat{\rho}$. This implies that $\Delta y_i = 0$, $0 \le i < \infty$. It also implies that $\mathcal{T}x^1(\cdot) = \mathcal{T}x^2(\cdot)$. From Theorem 2.1, $x^1 = x^2$. The uniqueness of solutions for $h = 0$, $k = +\infty$ has been proved. The case $h = -\infty$, $k = \infty$ can be considered similarly.

3) Exactly like in Case 2, we can show that there is no solution corresponding to a symbol $\{S_i\}_{h-1}^{k}$ with k being finite. There exist a periodic solution $x(t)$ and a heteroclinic solution $x^1(t)$ that connects W^c_{loc} to the periodic solution $x(t)$. Moreover $x(t)$ (or $x^1(t)$) is unique among all the solutions with symbols $\{S_i\}_{-\infty}^{\infty}$ (or$\{S_i\}_{-1}^{\infty}$, with $S_{-1} = 0$). The proofs of those are analogous to Case 2 and will not be repeated. The solution $x^1(t)$ is not mentioned in the theorem and has to be ruled out when proving Case 3 of the theorem. However, since there is no equilibrium on $W^c_{loc}(\mu)$, the solution $x^1(t)$ will leave $W^{cu}_{loc}(\mu)$ as $t \to -\infty$. Thus, the only solution that is orbitally $\hat{\epsilon}$-near Γ_o is the periodic solution $x(t)$.

The proof of Theorem 6.1 has been completed. □

7. Estimates Related to a Nondegenerate Hopf Bifurcation

This section does not depend on any of the previous sections. Partially for this reason we assume that all the variables defined in this section are local, independent of the other sections. For example, u and v are not variables on $W^u_{loc}(\mu)$ and $W^s_{loc}(\mu)$, l and m are not dimensions of $W^u_{loc}(\mu)$ and $W^s_{loc}(\mu)$, etc. By doing so we can ease the burden of choosing notations. We refer to [3] and [13] for basic technique used in this section.

Consider the following system in $\mathbb{R}^2$

$$\begin{aligned} \dot{z}_1 &= \mu z_1 - z_2 + h.o.t. \\ \dot{z}_2 &= z_1 + \mu z_2 + h.o.t. \end{aligned} \tag{7.1}$$

where $y = (z_1, z_2)$ and h.o.t.'s are C^∞ functions of y defined in a neighborhood of zero. After a standard averaging procedure, cf. [6], Theorem 2.2, we assume that (7.1) can be written in the polar coordinates as

$$\begin{aligned} \dot{R} &= \mu R + aR^3 + cR^5 + f_1(R,\theta,\mu) \\ \dot{\theta} &= 1 + bR^2 + dR^4 + f_2(R,\theta,\mu) \end{aligned} \tag{7.2}$$

where $(y_1, y_2) = (R\cos\theta, R\sin\theta)$, $D_r^i D_\theta^j f_1 = O(R^{7-i})$, $i \le 7$ and $D_r^i D_\theta^j f_2 = O(R^{6-i})$, $i \le 6$. The coefficients a, b, c, d are μ-dependent. Assume that $a > 0$ so that the system is weakly expansive when $\mu = 0$.

The equilibrium $R = 0$ is stable when $\mu < 0$ and unstable when $\mu \ge 0$. When $\mu < 0$, and unstable limit cycle $L(\mu)$ is cieated. Let $R = R^*(\theta,\mu)$ be the equation for $L(\mu)$, $\mu < 0$. For convenience let $L(\mu)$ be the equilibrium and $R^*(\theta,\mu) = 0$ when $\mu \ge 0$.

Lemma 7.1. *$R^*(\theta, \mu) = R_\mu + \tilde{R}(\theta, \mu)$ where $R_\mu = \sqrt{\frac{-\mu}{a}}$, $\mu < 0$ and $|D^i\tilde{R}(\theta, \mu)| \leq C|\mu|^{1.5}$, $i \geq 0$.*

Proof. Let $R = R_\mu + \rho(\theta)$ be the equation for $L(\mu)$. Then from (7.2)

$$\frac{d\rho}{d\theta} = -2\mu\rho + O(R_\mu\rho^2 + |\rho|^3 + R_\mu^4|\rho| + R_\mu^5) \tag{7.3}$$

To see this, observe $|\mu| = O(R_\mu^2)$ therefore $|\dot{R}| \leq C(R_\mu + |\rho|)^3$ from (7.2). We then have

$$\frac{d\rho}{d\theta} = \dot{R}/\dot{\theta} = \dot{R} + O((R_\mu + |\rho|)^5).$$

The error term above is bounded by the error term in (7.3). For the similar reason,

$$\begin{aligned}\dot{R} &= \mu(R_\mu + \rho) + a(R_\mu + \rho)^3 + O((R_\mu + |\rho|)^5)\\ &= (\mu\rho + 3aR_\mu^2\rho) + (\mu R_\mu + aR_\mu^3) + O(R_\mu\rho^2 + |\rho|^3 + R_\mu^4|\rho| + R_\mu^5)\\ &= -2\mu\rho + 0 + O(R_\mu\rho^2 + |\rho|^3 + R_\mu^4|\rho| + R_\mu^5).\end{aligned}$$

This leads to (7.3). Since the period of $\rho(\theta)$ is $T = 2\pi$, we have

$$(1 - e^{-2\mu T})\rho(\theta) = \int_\theta^{\theta+T} e^{-2\mu(\theta+T-\tau)} O(R_\mu\rho^2 + |\rho|^3 + R_\mu^4|\rho| + R_\mu^5)(\tau)d\tau. \tag{7.4}$$

$$|\mu|\rho(\theta) \leq C(|\mu|^{2.5} + |\mu|^2|\rho|_\infty + |\mu|^{0.5}|\rho|_\infty^2 + |\rho|_\infty^3).$$

$$\frac{\rho(\theta)}{|\mu|^{1.5}} \leq C(1 + |\mu|\frac{|\rho|_\infty}{|\mu|^{1.5}} + |\mu| \cdot \frac{|\rho|_\infty^2}{|\mu|^3} + |\mu|^2\frac{|\rho|_\infty^3}{|\mu|^{4.5}}).$$

Based on this, the integral equation (7.4) maps the family of 2π-periodic functions with $|\rho|_\infty/|\mu|^{1.5} \leq 2C$ into itself if $|\mu|$ is sufficiently small. By the contraction mapping principle, (7.4) admits a solution $|\rho(\theta)| \leq 2C|\mu|^{1.5}$. The estimate for $D\rho(\theta)$ follows from equation (7.3) and the estimate for $|\rho(\theta)|$. The estimates for higher order derivatives are more involved. First by differentiating (7.2) we can prove that $(\frac{d}{dt})^\nu R(t) = O(|\mu|^{1.5})$ inductively. Here we need to use $D_\theta^j f_1 = O(R^7)$. Next apply $(\frac{d}{d\theta})^\nu = (\frac{1}{1+bR^2+dR^4+f_2}\frac{d}{dt})^\nu$ to $\frac{dR}{d\theta} = \frac{\mu R+\cdots+f_1}{1+bR^2+dR^4+f_2}$, the numerator is a sum in which each term is either bounded by $O(R^i)$, $i \leq \nu$ or $D_\theta^j f_1$ and the denominator is bounded below if $|\mu|$ and R are small. Details will be left to the readers. □

When $\mu \geq 0$, it can be seen from (7.2) that $R(t) \to 0$ monotonously as $t \to -\infty$. When $\mu < 0$, picking an initial point outside the limit cycle $L(\mu)$, the orbit of $(R(t), \theta(t))$ approaches $L(\mu)$ as $t \to -\infty$, but $R(t)$ does not decay monotonously. For our convenience, we make a change of variable ($R = R^*(\theta, \mu) + r, \theta = \theta$). In the new coordinates,

$$\begin{aligned}\dot{r}(t) &= a_1(\theta)r + a_2(\theta)r^2 + a_3(\theta)r^3 + \ldots + g_1(r, \theta),\\ \dot{\theta}(t) &= b_0(\theta) + b_1(\theta)r + b_2(\theta)r^2 + \cdots + g_2(r, \theta).\end{aligned} \tag{7.5}$$

The coefficients and g_i, $i = 1, 2$ are μ-dependent. In the sequel, we often suppress the parameter μ for the typing convenience. Since

$$\begin{aligned}\dot r(t) &= \dot R(t)- \\ &\frac{\mu R^*(\theta,\mu) + aR^*(\theta,\mu)^3 + cR^*(\theta,\mu)^5 + f_1(R^*(\theta,\mu),\theta)}{1 + bR^*(\theta,\mu)^2 + dR^*(\theta,\mu)^4 + f_2(R^*(\theta,\mu),\theta)} \\ &\cdot \{1 + bR^2 + dR^4 + f_2(R,\theta,\mu)\},\end{aligned} \tag{7.6}$$

we can show that

$$\dot r(t) = [\mu(R^*(\theta,\mu)+r)+a(R^*(\theta,\mu)+r)^3]-[\mu R^*(\theta,\mu)+aR^*(\theta,\mu)^3]+O((|\mu|^{0.5}+r)^4 \cdot r).$$

To prove this, observe that

$$\begin{aligned}&\frac{1 + bR^2 + dR^4 + f_2(R,\theta,\mu)}{1 + bR^{*2} + dR^{*4} + f_2(R^*,\theta,\mu)} - 1 \\ =&\frac{b(R^2 - R^{*2}) + d(R^4 - R^{*4}) + \cdots}{1 + bR^{*2} + dR^{*4} + f_2(R^*,\theta,\mu)} \\ =&\frac{O(r(|\mu|^{0.5} + r))}{1 + O(|\mu|)}.\end{aligned}$$

Also $\mu R^* + aR^{*3} + cR^{*5} + f_1(R^*, \theta, \mu) = O(|\mu|^{1.5})$, from Lemma 7.1. Thus we can replace the second row of (7.6) by $\mu R^* + aR^{*3} + cR^{*5} + f_1(R^*, \theta, \mu)$ with an error bounded by $O(r(|\mu|^{0.5} + r)|\mu|^{1.5})$. The desired estimate then follows from

$$|R^5 - R^{*5}| + |f_1(R,\theta,\mu) - f_1(R^*,\theta,\mu)| = O((|\mu|^{0.5} + r)^4 \cdot r),$$

which can be verified easily.

Using Lemma 7.1, we now have the following estimates for the coefficients:

$$\begin{aligned}&a_1(\theta) = -2\mu + O(|\mu|^2),\quad a_2(\theta) = 3aR_\mu + O(|\mu|^{1.5}),\quad a_3(\theta) = a + 0(|\mu|),\\ &b_0(\theta) = 1 + 0(|\mu|),\quad \text{and}\quad |D^k a_j(\theta)| + |D^k b_j(\theta)| = 0(|\mu|),\quad 1 \le k.\end{aligned} \tag{7.7}$$

The last estimates is due to the fact that $R^*(\theta, \mu) = \text{constant} + O(|\mu|^{1.5})$.

Since the polar coordinates respect the symmetry group $(R, \theta) \to (-R, \theta + \pi)$, (7.2) is invariant under the same symmetry group. For the same reason, $R^*(\theta + \pi, \mu) = -R^*(\theta, \mu)$, and (7.5) is invariant when changing (r, θ) to $(-r, \theta + \pi)$. Thus, for $i \ge 0$,

$$\begin{aligned}&a_{2i}(\theta+\pi) = -a_{2i}(\theta),\quad a_{2i+1}(\theta+\pi) = a_{2i+1}(\theta),\\ &b_{2i}(\theta+\pi) = b_{2i}(\theta)\quad \text{and}\quad b_{2i+1}(\theta+\pi) = -b_{2i+1}(\theta),\\ &\overline{a_{2i}(\theta)} = \overline{b_{2i+1}(\theta)} = 0.\end{aligned} \tag{7.8}$$

Here and in the sequel, $\overline{f(\theta)} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$.

We now use the method of averaging to remove the θ-dependence of all the coefficients $a_i(\theta)$ up to $i = 2m$ and $b_i(\theta)$ up to $i = 2m - 1$.

Lemma 7.2. *There is a change of variable* $(r, \theta) \to (\rho, \varphi)$, *defined by*

$$\begin{aligned}\theta &= \varphi + u_0(\varphi) + u_1(\varphi)\rho + u_2(\varphi)\rho^2 + \cdots = \varphi + u^*(\rho,\varphi)\\ r &= \rho + v_1(\varphi)\rho + v_2(\varphi)\rho^2 + v_3(\varphi)\rho^3 + \cdots = \rho + \rho v^*(\rho,\varphi)\end{aligned}$$

So that system (7.5) with properties (7.7) and (7.8) becomes

$$\begin{aligned}\dot\rho &= \overline{a}_1\,\rho + \overline{a}_3\,\rho^3 + \cdots \overline{a}_{2m-1}\,\rho^{2m-1} + h_1(\rho, \varphi),\\ \dot\varphi &= \overline{b}_0 + \overline{b}_2\,\rho^2 + \cdots + \overline{b}_{2m-2}\,\rho^{2m-2} + h_2(\rho, \varphi).\end{aligned} \tag{7.9}$$

where $h_1 = 0(\rho^{2m+1})$, $h_2 = O(\rho^{2m})$. $\quad \overline{u_i(\varphi)} = 0,\ \overline{v_i(\varphi)} = 0.$

$$\begin{aligned}&u_{2j+1}(\varphi + \pi) = -u_{2j+1}(\varphi), \quad u_{2j}(\varphi + \pi) = u_{2j}(\varphi),\\ &v_{2j+1}(\varphi + \pi) = v_{2j+1}(\varphi), \quad v_{2j}(\varphi + \pi) = -v_{2j}(\varphi)\\ &\overline{a}_1 = -2\mu + 0(|\mu|^2),\ \overline{a}_3 = a + 0(|\mu|) \quad \textit{and} \quad \overline{b}_0 = 1 + 0(|\mu|).\end{aligned}$$

Here $\overline{a}_i$ *and* $\overline{b}_i$ *are constants which are related to but may not equal to* $\overline{a_i(\theta)}$ *and* $\overline{b_i(\theta)}$. *Also* $D^k u_i(\varphi) = 0(|\mu|)$, $D^k v_i(\varphi) = 0(|\mu|)$ *for* $0 \le k$.

There are numerous literatures concerning the method of averaging. [6, 3, 13]. However the case in Lemma 7.2 is nonstandard because $a_1(\theta)$ and $b_0(\theta)$ in (7.5) are θ-dependent. Besides, we need to obtain estimates on $u_j(\varphi)$, $v_j(\varphi)$, $\overline{a}_j$ and $\overline{b}_j$. Some details of the proof are necessary.

Proof. To eliminate the θ-dependence of $b_0(\theta)$, let

$$\begin{cases}\theta = \varphi + u(\varphi),\\ r = r.\end{cases}$$

We are looking for φ so that the following equation is valid,

$$\begin{aligned}\dot\varphi &= [1 + Du(\varphi)]^{-1}\cdot b_0(\varphi + u(\varphi)) + 0(r)\\ &= \overline{b}_0 + 0(r).\end{aligned}$$

$$Du(\varphi) = \frac{b_0(\varphi + u(\varphi)) - \overline{b}_0}{\overline{b}_0} = \frac{1 + \tilde b_0(\varphi + u(\varphi)) - \overline{b}_0}{\overline{b}_0}, \tag{7.10}$$

where $|D^k\tilde b_0|_\infty = 0(|\mu|)$ for $k \ge 0$. The function $u(\varphi)$ and the parameter $\overline{b}_0$ can be solved by the Lyapunov-Schmidt reduction and the Implicit Function Theorem, resulting a 2π-periodic solution with $\overline{u(\varphi)} = 0$. More precisely, let Π be the Banach space of continuous 2π-periodic functions with the supremum norm. Let Π^0 be the subspace of Π that consists of functions whose average is zero. For $f \in \Pi$, define $Pf = f - \overline{f}$. P is a projection from Π to Π^0. For each $f \in \Pi^0$, there is a unique solution $g \in \Pi^0$ to the equation $Dg = f$, denoted by $g = \mathcal{K}f$, with $|\mathcal{K}| \le C$. See [15], Chapt. 8 for the details. We now project the right hand side of (7.10) into Π^0 and its complementary. (7.10) is solvable only if

$$(I - P)\frac{1 + \tilde b_0(\varphi + u(\varphi)) - \overline{b}_0}{\overline{b}_0} = 0.$$

This leads to the second equation of the following system:

$$\begin{aligned}&u - \mathcal{K}P\left(\frac{1 + \tilde b_0(\varphi + u(\varphi)) - \overline{b}_0}{\overline{b}_0}\right) = 0,\\ &\overline{b}_0 - \overline{1 + \tilde b_0(\varphi + u(\varphi))} = 0\end{aligned}$$

We can now solve $(u, \overline{b}_0)$ by the Implicit Function Theorem in the product space $\Pi^0 \times \mathbb{R}$ when μ is small, since $(u, \overline{b}_0) = (0, 1)$ is a solution when $\mu = 0$ and the linearized system is clearly surjective in $\Pi^0 \times \mathbb{R}$. From the Implicit Function Theorem we also find that $D^k u(\varphi) = O(|\mu|)$ for $0 \geq k$ and $\overline{b}_0 - 1 = O(|\mu|)$. Since

$$\begin{aligned} Du(\varphi + \pi) &= [b_0(\varphi + \pi + u(\varphi + \pi)) - \overline{b}_0]/\overline{b}_0 \\ &= [b_0(\varphi + u(\varphi + \pi)) - \overline{b}_0]/\overline{b}_0, \end{aligned}$$

by the uniqueness of the solution $u(\varphi + \pi) = u(\varphi)$.

Observe that after substituting $\theta = \varphi + u(\varphi)$, properties (7.7) and (7.8) will not be disturbed. More precisely, Let $\hat{a}_i(\varphi) = a_i(\varphi + u(\varphi))$, then

$$\begin{aligned} \hat{a}_i(\varphi + \pi) &= a_i(\varphi + \pi + u(\varphi + \pi)) \\ &= a_i(\varphi + \pi + u(\varphi)) \\ &= \pm a_i(\varphi + u(\varphi)) \\ &= \pm \hat{a}_i(\varphi). \end{aligned}$$

The plus or minus sign depends on the evenness or oddness of i. Similarly we can define $\hat{b}_i(\varphi)$ and verify that (7.8) is satisfied if $b_i(\theta)$ is replaced by $\hat{b}_i(\varphi)$. The verification of (7.7) after replacing $a_i(\theta)$ and $b_i(\theta)$ by $\hat{a}_i(\varphi)$ and $\hat{b}_i(\varphi)$ uses the estimates on $D^k u(\varphi)$.

To eliminate the θ-dependence of $a_1(\theta)$, let

$$\begin{cases} r = \rho + \rho v(\theta), \\ \theta = \theta. \end{cases}$$

We have

$$\begin{aligned} \dot{r} &= a_1(\theta)[\rho + \rho v(\theta)] + O(\rho^2) \\ &= \dot{\rho}[1 + v(\theta)] + \rho D v(\theta)\dot{\theta}. \end{aligned}$$

We are looking for $v(\theta)$ so that the following is achieved:

$$\begin{aligned} \dot{\rho} &= \rho\{-\overline{b}_0 D v(\theta) + a_1(\theta)[1 + v(\theta)]\}(1 + v(\theta))^{-1} + 0(\rho^2) \\ &= \overline{a}_1 \rho + 0(\rho^2). \end{aligned}$$

Discarding the high order terms, we have

$$\overline{b}_0 D v(\theta) = (a_1(\theta) - \overline{a}_1)(1 + v(\theta)) = (-2\mu + \tilde{a}_1(\theta) - \overline{a}_1)(1 + v(\theta)), \tag{7.11}$$

where $|D^k \tilde{a}_1(\theta)| = 0(|\mu|)$ for $k \geq 1$ and $|\tilde{a}_1(\theta)| = 0(|\mu|^2)$ from (7.7). In order to have a solution for (7.11), the average of the right hand side of (7.11) must be zero. Therefore

$$(I - P)\{(-2\mu + \tilde{a}_1(\theta) - \overline{a}_1)(1 + v(\theta))\} = 0.$$

This leads to the second equation of the following system:

$$\begin{aligned} &v - \mathcal{K}P\{(-2\mu + \tilde{a}_1(\theta) - \overline{a}_1)(1 + v(\theta))/\overline{b}_0\} = 0, \\ &\overline{a}_1 + 2\mu - \overline{\tilde{a}_1(\theta)[1 + v(\theta)]} = 0. \end{aligned}$$

Observe that when $\mu = 0$, $(v, \overline{a}_1) = (0, 0)$ is a solution and also the linearized system is surjective in $\Pi^0 \times \mathbb{R}$. We can now use the Implicit Function Theorem to solve the above system for μ being small, resulting a 2π-periodic solution $v(\varphi)$ with $\overline{v(\varphi)} = 0$. The Implicit Function Theorem also implies that $|v| = O(|\mu|)$ and $\overline{a}_1 = O(|\mu|)$. Differentiating (7.11), we have,

$$|D^\alpha v(\theta)| = 0(|\mu|) \quad \text{for} \quad 0 \leq k.$$

The fact $v(\theta + \pi) = v(\theta)$ follows from the uniqueness of the solution.

After substituting $r = \rho + \rho v(\theta)$ into (7.5), the properties (7.7) and (7.8) are preserved. More precisely, the new coefficients are $\hat{a}_i(\theta) = a_i(\theta)(1+v(\theta))^i$, $\hat{b}_i(\theta) = b_i(\theta)(1+v(\theta))^i$. Since $|v| = O(|\mu|)$, $\hat{a}_i(\theta) = a_i(\theta)(1+O(|\mu|))$. From this, we verify that (7.7) is preserved for the new coefficients. Since $v(\theta + \pi) = v(\theta)$, it is easy to verify that (7.8) is also preserved.

We now use an induction argument. Assume that for $k \geq 1$,

$$\begin{aligned} \dot{r} &= \overline{a}_1 r + \overline{a}_2 r^2 + \cdots + \overline{a}_k\, r^k + a_{k+1}(\theta) r^{k+1} + 0(r^{k+2}), \\ \dot{\theta} &= \overline{b}_0 + \overline{b}_1 r + \cdots + \overline{b}_{k-1} r^{k-1} + b_k(\theta) r^k + 0(r^{k+1}), \end{aligned} \tag{7.12}$$

where h.o.t.'s are θ-dependent. Let

$$\begin{cases} \theta = \varphi + r^k u(\varphi), \\ r = r. \end{cases}$$

We are looking for $u(\varphi)$ so that the equation for $\dot{\varphi}$ is φ-independent up to order r^k.

$$\begin{aligned} \dot{\varphi} &= \overline{b}_0 + \overline{b}_1 r + \cdots + \overline{b}_{k-1} r^{k-1} + [b_k(\varphi) - \overline{b}_0 Du(\varphi) - k\overline{a}_1 u(\varphi)] r^k + 0(r^{k+1}) \\ &= \overline{b}_0 + \overline{b}_1 r + \cdots \overline{b}_{k-1} r^{k-1} + \overline{b}_k r^k + 0(r^{k+1}). \end{aligned}$$

$$\overline{b}_0 Du(\varphi) = b_k(\varphi) - k\overline{a}_1 u(\varphi) - \overline{b}_k, \tag{7.13}$$

where $\overline{b}_0 = 1 + 0(|\mu|)$.

In order to have a solution, the averaging of the right hand side of (7.13) must be zero. Assume that $\overline{u(\varphi)} = 0$, then

$$\overline{b}_k = \overline{b_k(\varphi)}.$$

It follows from (7.8) that $\overline{b}_k = 0$ if k is odd. Consider $D(b_k(\varphi) - \overline{b}_k) = O(|\mu|) \in \Pi^0$, cf. (7.7). We have $b_k(\varphi) - \overline{b}_k = \mathcal{K}D(b_k(\varphi) - \overline{b}_k) = O(|\mu|)$. Since $\overline{a}_1 = 0(|\mu|)$, knowing that when $\mu = 0$, $u = 0$ is a solution for the equation

$$u - \mathcal{K}[b_k(\varphi) - k\overline{a}_1 u(\varphi) - \overline{b}_k]/\overline{b}_0 = 0,$$

and the linearized equation is surjective in Π^0, we can solve $u \in \Pi^0$ by the Implicit Function Theorem. Since $b_k(\varphi) - \overline{b}_k = 0(|\mu|)$, from (7.13), it is easy to see that

$$|D^\alpha u(\varphi)|_\infty \leq c|\mu|, \quad 0 \leq \alpha.$$

By the uniqueness of the solution, $u(\varphi + \pi) = \pm u(\varphi)$, taking the – sign if k is odd.

To eliminate the θ-dependence of $a_{k+1}(\theta)$, set

$$\begin{cases} r = \rho + \rho^{k+1} v(\theta), \\ \theta = \theta. \end{cases}$$

$\dot{\theta} = \overline{b}_0 + \cdots + \overline{b}_k\rho^k + h.o.t.$ is obvious. We are looking for $v(\theta)$ so that the θ-dependence of the $(k+1)$-th term in $\dot{\rho}$ will be dropped. Therefore,

$$\begin{aligned}\dot{\rho} &= \overline{a}_1\rho + \overline{a}_2\rho^2 + \cdots + \overline{a}_k\rho^k + \rho^{k+1}[a_{k+1}(\theta) - \overline{b}_0 Dv(\theta) - k\overline{a}_1 v(\theta)] + 0(\rho^{k+2})\\ &= \overline{a}_1\rho + \cdots + \overline{a}_k\rho^k + \overline{a}_{k+1}\rho^{k+1} + 0(\rho^{k+2}).\end{aligned}$$

$$\overline{b}_0 Dv(\theta) = a_{k+1}(\theta) - k\overline{a}_1 v(\theta) - \overline{a}_{k+1}. \tag{7.14}$$

In order to solve (7.14), the average of the right hand side must be zero. Thus

$$\overline{a}_{k+1} = \overline{a_{k+1}(\theta)}.$$

We then have $D(a_{k+1} - \overline{a}_{k+1}) = O(|\mu|) \in \Pi^0$ and $a_{k+1} - \overline{a}_{k+1} \in \Pi^0$. Therefore

$$a_{k+1} - \overline{a}_{k+1} = \mathcal{K}O(|\mu|) = O(|\mu|).$$

The solution of (7.14) is obtained by solving

$$v - \mathcal{K}\{(a_{k+1}(\theta) - k\overline{a}_1 v(\theta) - \overline{a}_{k+1})/\overline{b}_0\} = 0.$$

Since when $\mu = 0$, $v = 0$ is a solution and the linearized equation is surjective in Π^0, we can solve $v \in \Pi^0$ by the Implicit Function Theorem when μ is small. The solution $v = O(|\mu|)$ by the Implicit Function Theorem again. Differentiating (7.14), we have

$$|D^\alpha v(\theta)|_\infty \le c|\mu|, \quad \text{for} \quad 0 \le \alpha.$$

It follows from (7.8) that $\overline{a}_{k+1} = 0$ if k is odd. By the uniqueness of the solution, $v(\varphi + \pi) = \pm v(\varphi)$, taking the – sign if k is odd.

We obtain $u^*(\rho, \varphi)$ and $v^*(\rho, \varphi)$ by constructing the composition of all the change of variables derived above. □

Consider the change of variable related to the averaging process in Lemma 7.2 where $\tilde{R}(\theta, \mu)$ is introduced in Lemma 7.1. For

$$\begin{aligned} z_1 &= (R_\mu + \tilde{R}(\theta, \mu) + \rho + \rho v^*(\rho, \varphi))\cos(\varphi + u^*(\rho, \varphi)),\\ z_2 &= (R_\mu + \tilde{R}(\theta, \mu) + \rho + \rho v^*(\rho, \varphi))\sin(\varphi + u^*(\rho, \varphi))\end{aligned}$$

where $\theta = \varphi + u^*(\rho, \varphi)$, define $(Z_1, Z_2) = ((R_\mu + \rho)\cos\varphi, (R_\mu + \rho)\sin\varphi)$. Let $\overrightarrow{Z} = (Z_1, Z_2)$ and $\overrightarrow{z} = (z_1, z_2)$.

Lemma 7.3. *The mapping $(z_1, z_2) \to (Z_1, Z_2)$ is a C^3 diffeomorphism if $|\mu| \le \mu_0$ and $0 \le \rho \le \rho_0$. $\overrightarrow{Z} = \overrightarrow{z} + 0((R_\mu + \rho)^3)$, $D_{\overrightarrow{z}}\overrightarrow{Z} = I + 0((R_\mu + \rho)^2)$, $|D^2_{\overrightarrow{z}}\overrightarrow{Z}| = 0(R_\mu + \rho)$, and $|D^3_{\overrightarrow{z}}\overrightarrow{Z}| = 0(1)$. These estimates are uniform with respect to $|\mu| \le \mu_0$ and $\rho \le \rho_0$. The periodic orbit $L(\mu)$ in the z-plane is mapped to $\rho = 0$ in the Z-plane.*

Proof. Let $R_\mu + \rho = r$. The Jacobian matrix $\frac{\partial(\rho,\varphi)}{\partial(Z_1,Z_2)} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r}\end{pmatrix}$. Let $\overline{z}_1 = z_1 - Z_1 = (R_\mu + \rho)[\cos(\varphi + u^*(\rho, \varphi)) - \cos\varphi] + (\tilde{R}(\theta, \mu) + \rho v^*(\rho, \varphi))\cos(\varphi + u^*(\rho, \varphi))$.

We can show $\overline{z}_1 = 0(r^3)$, since $\cos(\varphi + u^*) - \cos\varphi = 0(u^*) = 0(|\mu|) = 0(R_\mu^2) = 0(r^2)$, and $\tilde{R}(\theta, \mu) + \rho v^*(\rho, \varphi) = 0(|\mu|^{1.5} + |\mu|\rho) = 0((|\mu|^{0.5} + \rho)^3) = 0(r^3)$. Similarly,

$\overline{z}_2 = z_2 - Z_2 = 0(r^3)$ and

$$\left(\frac{\partial \overline{z}}{\partial \varphi}, \frac{\partial^2 \overline{z}}{\partial \varphi^2}, \frac{\partial^3 \overline{z}}{\partial \varphi^3}\right) = 0(r^3),$$
$$\left(\frac{\partial \overline{z}}{\partial \rho}, \frac{\partial^2 \overline{z}}{\partial \rho \partial \varphi}, \frac{\partial^3 \overline{z}}{\partial \rho \partial \varphi^2}\right) = 0(r^2),$$
$$\left(\frac{\partial^2 \overline{z}}{\partial \rho^2}, \frac{\partial^3 \overline{z}}{\partial \rho^2 \partial \varphi}\right) = 0(r),$$
$$\frac{\partial^3 \overline{z}}{\partial \rho^3} = 0(1).$$

Here $\overline{z} = \overline{z}_1$ or $\overline{z}_2$. The estimates on $u^*(\rho, \varphi)$ and $v^*(\rho, \varphi)$ from Lemma 7.2 have been employed. Observe that

$$\frac{\partial \overline{z}_1}{\partial Z_1} = \frac{\partial \overline{z}_1}{\partial \varphi}\frac{\partial \varphi}{\partial Z_1} + \frac{\partial \overline{z}_1}{\partial \rho}\frac{\partial \rho}{\partial Z_1}$$
$$\frac{\partial^2 \overline{z}_1}{\partial Z_1^2} = \frac{\partial^2 \overline{z}_1}{\partial \varphi^2}\left(\frac{\partial \varphi}{\partial Z_1}\right)^2 + \frac{\partial \overline{z}_1}{\partial \varphi}\frac{\partial^2 \varphi}{\partial Z_1^2} + 2\frac{\partial^2 \overline{z}_1}{\partial \rho \partial \varphi}\frac{\partial \rho}{\partial Z_1}\frac{\partial \varphi}{\partial Z_1} + \frac{\partial \overline{z}_1}{\partial \rho}\frac{\partial^2 \rho}{\partial Z_1^2} + \cdots$$
$$\frac{\partial^3 \overline{z}_1}{\partial Z_1^3} = \frac{\partial^3 \overline{z}_1}{\partial \varphi^3}\left(\frac{\partial \varphi}{\partial Z_1}\right)^3 + 3\frac{\partial^2 \overline{z}_1}{\partial \varphi^2}\frac{\partial^2 \varphi}{\partial Z_1^2}\frac{\partial \varphi}{\partial Z_1} + \frac{\partial \overline{z}_1}{\partial \varphi}\frac{\partial^3 \varphi}{\partial Z_1^3} + \cdots$$

Some factors have a singularity at $r = 0$, like

$$\frac{\partial \varphi}{\partial Z} = 0\left(\frac{1}{r}\right), \ \frac{\partial^2 \varphi}{\partial Z^2} = 0\left(\frac{1}{r^2}\right), \ \frac{\partial^3 \varphi}{\partial Z^3} = 0\left(\frac{1}{r^3}\right), \ \frac{\partial^2 \rho}{\partial Z^2} = 0\left(\frac{1}{r}\right)$$
$$\text{and} \quad \frac{\partial^3 \rho}{\partial Z^3} = 0\left(\frac{1}{r^2}\right).$$

But the singularity will be canceled. A similar situation occurs to $\overline{z}_2$, and the derivatives involving Z_2. □

Consider the case $\mu < 0$. Let $m = 3$ in system (7.9) as described in Lemma 7.2. Without loss of generality, let $\overline{b}_0 = 1$ after rescaling the time. We can rewrite (7.9) as

$$\begin{aligned} \dot{\rho} &= f(\rho) + h_1(\rho, \varphi), \\ \dot{\varphi} &= g(\rho) + h_2(\rho, \varphi), \end{aligned} \tag{7.15}$$

where $f(\rho) = a_1\rho + a_3\rho^3 + a_5\rho^5$, $a_1 = -2\mu + 0(\mu^2) \geq 0$ and $a_3 = a + 0(\mu) > 0$, $g(\rho) = 1 + b_2\rho^2 + b_4\rho^4$, $h_1 = 0(\rho^7)$ and $h_2 = 0(\rho^6)$. Here we have dropped the bars for convenience.

In order to treat $\mu \geq 0$ similarly, let $\rho = R$, $R^*(\theta, \mu) = R_\mu = 0$ and $\varphi = \theta$ when $\mu \geq 0$. We can rewrite (7.2) into (7.15) when $\mu \geq 0$ with $h_1 = f_1$, $h_2 = f_2$, $a_1 = \mu \geq 0$ and $a_3 = a > 0$.

Observe that we always have $a_1 \geq 0$ and $a_3 > 0$ for small μ. For such f and any positive C, if μ and $0 \leq \rho$ are small, we have

$$C_1 f(\rho) \leq f(C\rho) \leq C_2 f(\rho), \tag{7.16}$$

where the positive constants C_1 and C_2 depend on C. In fact, if $a_5 = 0$, then (7.16) is valid since

$$\min\{C, C^3\}(a_1\rho + a_3\rho^3) \leq f(C\rho) \leq \max\{C, C^3\}(a_1\rho + a_3\rho^3).$$

But $a_5\rho^5$ is a higher order term, thus adding it will not destroy (7.16) if ρ and μ are small. Consider (7.15) both for $\mu < 0$ and $\mu \geq 0$. Let $\rho(t) = \rho^*(t, \rho_0, \varphi_0)$ and $\varphi(t) = \varphi^*(t, \rho_0, \varphi_0)$ be the solution map for (7.15) with $(\rho(0), \varphi(0)) = (\rho_0, \varphi_0)$.

Lemma 7.4. *Let $\rho_0 > 0$ be sufficiently small. There exists a constant $C > 0$, depending on ρ_0 and $|\alpha|$ but not on $t \leq 0$, such that*

$$|D^\alpha \rho^*(t, \rho_0, \varphi_0)| \leq C\ f(\rho(t)),$$
$$|D^\alpha \varphi^*(t, \rho_0, \varphi_0)| \leq C.$$

Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index with $D^{\alpha_1}D^{\alpha_2} = \frac{\partial^{|\alpha|}}{\partial\rho_0^{\alpha_1}\partial\varphi_0^{\alpha_2}}$.

Proof. Let $\rho_1(t) = \rho_1^*(t, \rho_0)$ be the solution for equation $\dot{\rho}_1 = f(\rho_1(t))$ with $\rho_1(0) = \rho_0$. Our assumption on f implies that $f(\rho) > c\rho^3$ for some positive c if $\rho > 0$ and μ are both small. This fact is used throughout the proof. Since $f(\rho) > 0$ and $h_1(\rho, \varphi)$ is of higher order, both $\rho(t)$ and $\rho_1(t)$ approach zero monotonously as $t \to -\infty$. Since $\rho(t) = \rho^*(t, \rho_0, \varphi_0)$ is monotonous, for each $0 < \rho \leq \rho_0$ there is an inverse for $\rho = \rho(t)$, denoted by $t = t(\rho)$. Given $\overline{\rho} < \rho_0$, let $\rho_1(t_1) = \rho(t_2) = \overline{\rho}$ for $t_1 < 0$ and $t_2 < 0$. $t_1 = \int_{\rho_0}^{\overline{\rho}} \frac{d\rho}{f(\rho)}$ and $t_2 = \int_{\rho_0}^{\overline{\rho}} \frac{d\rho}{f(\rho)+h_1(\rho,\varphi(t(\rho)))}$.

$$|t_1 - t_2| = \int_{\overline{\rho}}^{\rho_0} \frac{|h_1(\rho, \varphi(t(\rho))|d\rho}{f(\rho)\cdot[f(\rho) + h_1(\rho, \varphi(t(\rho)))]}$$
$$\leq C\int_{\overline{\rho}}^{\rho_0} \frac{\rho^7 d\rho}{\rho^6} \leq C\rho_0^2,$$

where C does not depend on $\overline{\rho}$.

Let $t \leq 0$ and $\rho^*(t, \rho_0, \varphi_0) = \rho(t) \geq \rho_1^*(t, \rho_0) = \rho_1(t)$. Then

$$|\rho_1(t) - \rho(t)| \leq C\rho_0^2[f(\rho(t)) + h_1(\rho(t), \varphi(t))] \leq C\rho_0^2 \cdot f(\rho(t)) \tag{7.17}$$

Here $C\rho_0^2$ is the bound of the extra time for $\rho^*(t, \rho_0, \varphi_0)$ to reach $\rho_1(t)$ and the other factor is the bound for $\dot{\rho}(t)$. When $\rho(t) < \rho_1(t)$, we can prove, similar to (7.17),

$$|\rho_1(t) - \rho(t)| \leq C\rho_0^2 f(\rho_1(t)).$$

This implies that $\rho(t) \geq \rho_1(t) - C\rho_0^2 f(\rho_1(t))$, and $f(\rho(t)) \geq f(\rho_1(t) - C\rho_0^2 f(\rho_1(t)))$. On the other hand, if ρ_0 and ρ_1 are small, $\rho_1 - C\rho_0^2 f(\rho_1) > 0.5\rho_1$,

$$f(\rho_1(t)) < f(2(\rho_1(t) - C\rho_0^2 f(\rho_1(t)))) < cf(\rho_1(t) - C\rho_0^2 f(\rho_1(t))),$$

cf. (7.16). Therefore, (7.17) is still valid.

Let $A(t) = Df(\rho_1(t))$ and $B(t) = Df(\rho(t)) + \frac{\partial}{\partial\rho}h_1(\rho(t), \varphi(t))$. Then $|A(t) - B(t)| \leq C|\rho_1(t) - \rho(t)| + C\ \rho^6(t) \leq cf(\rho(t))$, by (7.17).

Let $T(t, s)$, $t \leq s \leq 0$ be the solution map for $\dot{\rho} = A(t)\rho$. It is easy to verify that $|T(t, s)| = f(\rho_1(t))/f(\rho_1(s)) \leq cf(\rho(t))/f(\rho(s))$. In fact, when ρ and μ are small, we have

$$\frac{1}{2}\rho \le \rho \pm C\rho_0^2 f(\rho) \le 2\rho.$$

$$\begin{aligned} f(\rho_1(t))/f(\rho_1(s)) &\le f(\rho(t) + C\rho_0^2 f(\rho(t)))/f(\rho(s) - C\rho_0^2 f(\rho(s))) \\ &\le f(2\rho(t))/f(\tfrac{1}{2}\rho(s)) \le Cf(\rho(t))/f(\rho(s)). \end{aligned}$$

Let $\dot{\xi}(t) = B(t)\xi(t)$, $t \le s \le 0$. Then

$$\xi(t) = T(t,s)\xi(s) + \int_s^t T(t,\tau)[B(\tau) - A(\tau)]\xi(\tau)d\tau.$$

$$|\xi(t)| \le c\frac{f(\rho(t))}{f(\rho(s))}|\xi(s)| + \int_t^s \frac{cf(\rho(t))}{f(\rho(\tau))} \cdot c \cdot f(\rho(\tau))|\xi(\tau)|d\tau.$$

$$\frac{|\xi(t)|}{f(\rho(t))} \le c\frac{|\xi(s)|}{f(\rho(s))} + \int_{\rho(t)}^{\rho(s)} c\frac{|\xi(\tau)|}{f(\rho(\tau))} \cdot d\rho.$$

Here we have changed the variable $|d\tau| = \frac{|d\rho|}{f(\rho)+h_1(\rho,\varphi(t(\rho))} \le c\,\frac{|d\rho|}{f(\rho)}$. By the Gronwall inequality, it is elementary to show if $\rho(s) < \rho_0$ is sufficiently small, then $\frac{|\xi(t)|}{|\xi(s)|} \le \frac{cf(\rho(t))}{f(\rho(s))}$. Let $T_{h_1}(t,s)$ be the solution map for $\dot{\xi}(t) = B(t)\xi(t)$, $t \le s \le 0$. We have shown that $|T_{h_1}(t,s)| \le cf(\rho(t))/f(\rho(s))$.

For a continuous function $\xi(t)$, $t \le 0$, define

$$\|\xi\|_f = \sup_{t \le 0} \|\xi(t)/f(\rho(t))\|,$$
$$\|\xi\|_\infty = \sup_{t \le 0} |\xi(t)|.$$

Since $\varphi(t) = \varphi(0) + \int_0^t \{g(\rho(s)) + h_2(\rho(s), \varphi(s))\}ds$,

$$\frac{\partial\varphi(t)}{\partial\rho_0} = \int_0^t \frac{\partial}{\partial\rho}\{g + h_2\}\frac{\partial\rho(s)}{\partial\rho_0}ds + \int_0^t \frac{\partial h_2}{\partial\varphi} \cdot \frac{\partial\varphi(s)}{\partial\rho_0}ds.$$

By the change of variable $|ds| \le c\frac{|d\rho|}{f(\rho)}$, we have

$$\left\|\frac{\partial\varphi}{\partial\rho_0}\right\|_\infty \le c\rho^2(0)\left\|\frac{\partial\rho}{\partial\rho_0}\right\|_f + c\rho^4(0)\left\|\frac{\partial\varphi}{\partial\rho_0}\right\|_\infty \le 2c\rho^2(0)\left\|\frac{\partial\rho}{\partial\rho_0}\right\|_f, \tag{7.18}$$

if $c\rho^4(0) < \frac{1}{2}$. To prove the first inequality in (7.18), we have used $|\frac{\partial}{\partial\rho}\{g + h_2\}| \le c\rho(s) \le c\rho(0)$ and $|\frac{\partial}{\partial\varphi}h_2|/f(\rho) \le c\rho^6(s)/\rho^3(s) \le \rho^3(0)$. The length of the integration is $|\rho(0) - \rho(t)|$ that provides another factor $\rho(0)$. On the other hand,

$$\frac{\partial}{\partial t}\frac{\partial\rho(t)}{\partial\rho_0} = \frac{\partial}{\partial\rho}\{f + h_1\}\frac{\partial\rho(t)}{\partial\rho_0} + \frac{\partial h_1}{\partial\varphi}\frac{\partial\varphi(t)}{\partial\rho_0}.$$

$$\frac{\partial\rho(t)}{\partial\rho_0} = T_{h_1}(t,0) + \int_0^t T_{h_1}(t,s) \cdot \frac{\partial h_1}{\partial\varphi} \cdot \frac{\partial\varphi(s)}{\partial\rho_0}ds.$$

Observe that

$$|T_{h_1}(t,0)| \le cf(\rho(t))/f(\rho(0)),$$

$$|T_{h_1}(t,s)||\frac{\partial h_1}{\partial \varphi}|/f(\rho(s)) \le cf(\rho(t))/f^2(\rho(s))\rho^7(s) \le cf(\rho(t))\rho(0).$$

After changing the variable $|ds| \le c\frac{|d\rho|}{f(\rho)}$, we have

$$\begin{aligned} \left|\frac{\partial \rho(t)}{\partial \rho_0}\right| &\le c\,\frac{f(\rho(t))}{f(\rho(0))} + c\,f(\rho(t))\cdot\rho^2(0)\cdot\left\|\frac{\partial \varphi}{\partial \rho_0}\right\|_\infty, \\ \left\|\frac{\partial \rho}{\partial \rho_0}\right\|_f &\le c + c\,\rho^2(0)\left\|\frac{\partial \varphi}{\partial \rho_0}\right\|_\infty. \end{aligned} \tag{7.19}$$

The additional power in $\rho^2(0)$ comes from the fact that the length of the integration is bounded by $\rho(0)$. Combining (7.18) and (7.19), we have

$$\left\|\frac{\partial \rho}{\partial \rho_0}\right\|_f + \left\|\frac{\partial \varphi}{\partial \rho_0}\right\|_\infty \le C.$$

Similarly, we can show that

$$\left\|\frac{\partial \rho}{\partial \varphi_0}\right\|_f + \left\|\frac{\partial \varphi}{\partial \varphi_0}\right\|_\infty \le C.$$

We can finish the rest of the proof by induction. Assume that the estimates for derivatives up to $|\alpha|$ have been proved. Let $l(t) = D^\alpha \rho(t)$ and $m(t) = D^\alpha \varphi(t)$, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial \rho_0^{\alpha_1}\partial \varphi_0^{\alpha_2}}$. Applying D^α to (7.15), we have

$$\begin{aligned} \frac{\partial}{\partial t} l(t) =& B(t)l(t) + \frac{\partial h_1}{\partial \varphi} m(t) \\ &+ \sum_{\substack{k\ge 2 \\ |\beta_1|+\cdots+|\beta_k|=|\alpha|}} \frac{d^k f(\rho(s))}{d\rho^k}\cdot C_{\beta_1\cdots\beta_k}\{D^{\beta_1}\rho(t)\cdot D^{\beta_2}\rho(t)\cdots D^{\beta_k}\rho(t)\} + \text{h.o.t.} \end{aligned}$$

Here h.o.t. contains derivatives of h_1 and products of $D^\beta\rho(t)$ and $D^\beta\varphi(t)$, $|\beta| \le |\alpha|$. To see this, consider

$$D_\xi[f(\rho) + h_1(\rho,\varphi)] = D_\rho[f(\rho) + h_1(\rho,\varphi)]D_\xi\rho + D_\varphi h_1(\rho,\varphi)D_\xi\varphi,$$

where $\xi = \rho_0$ or φ_0. Differentiate above $|\alpha| - 1$ times, and use the product rule. If all the differentiations are on ρ, we have the term $B(t)l(t)$. If all the differentiations are on φ, we have $\frac{\partial h_1}{\partial \varphi} m(t)$; The terms under the summation sign are obtained if we differentiate $f(\rho)$ more than once. They are all bounded by $c[f(\rho(t))]^2$ since $k \ge 2$. Some of the terms in *h.o.t.* contain a factor $D^\nu D^2_\rho h_1(\rho,\varphi)\{D^{\beta_1}\rho(t)\cdot D^{\beta_2}\rho(t)\cdots\}$ and are bounded by $c[f(\rho(t))]^2$ for the same reason. Others have ρ derivatives of h_1 at most once and are bounded by $O(\rho(t)^6) \le O[f(\rho(t))^2]$. Thus using the variation of constants formula we have

$$\begin{aligned} |l(t)| &\le c\frac{f(\rho(t))}{f(\rho(0))}\,|l(0)| + c\rho^2(0)f(\rho(t))\,\|m\|_\infty + \int_{\rho(t)}^{\rho(0)} c\frac{f(\rho(t))}{f(\rho(s))}\cdot f^2(\rho(s))\cdot\frac{d\rho}{f(\rho(s))} \\ &\le c\,f(\rho(t)) + c\,\rho^2(0)f(\rho(t))\cdot\|m\|_\infty. \end{aligned}$$

Here we have used the estimate

$$\begin{aligned}
&\int_{\rho(t)}^{\rho(0)} |T_{h_1}(t,s)| \left|\frac{\partial h_1}{\partial \varphi} m(s)\right| \frac{|d\rho|}{f(\rho(s))} \\
\le & C\int_{\rho(t)}^{\rho(0)} \frac{f(\rho(t))}{f(\rho(s))}\frac{\rho^7(s)}{f(\rho(s))}\|m\|_\infty d\rho \\
\le & c\rho^2(0) f(\rho(t))\|m\|_\infty .
\end{aligned}$$

Also $l(0) = 0$ from its definition.

$$\|l\|_f \le c + c\,\rho^2(0)\ \|m\|_\infty . \tag{7.20}$$

Similarly, applying D^α to (7.15) and integrating, we have

$$\begin{aligned}
m(t) =& m(0) + \int_0^t \frac{\partial}{\partial \rho}\{g+h_2\} l(s)ds + \int_0^t \frac{\partial h_2}{\partial \varphi}\cdot m(s)ds \\
&+ \int_0^t \{ \sum_{\substack{k\ge 2 \\ |\beta_1|+\cdots+|\beta_k|=|\alpha|}} \frac{d^k g(\rho(s))}{d\rho^k}\cdot C_{\beta_1\cdots\beta_k}\{D^{\beta_1}\rho(s) D^{\beta_k}\rho(s)\} + \text{h.o.t.}\} ds
\end{aligned}$$

The terms under the summation have at least two factors $D^{\beta_1}\rho(s)D^{\beta_2}\rho(s)$, each is bounded by $cf\rho(s))$, (induction assumption), and are bounded by $c\,f^2(\rho(s))$. The h.o.t. contains terms that either have a factor

$$|D^\nu D_\rho^2 h_2 \cdot D^{\beta_1}\rho(s)D^{\beta_2}\rho(s)| \le cf^2(\rho(s)) \le c\rho^5(s),$$

or a factor $|D_\varphi^\nu D_\rho^k h_2| \le c\rho^5(s)$ where $k \le 1$. The last integral is then bounded by $\int_{\rho(t)}^{\rho(0)} \frac{c\rho^5 d\rho}{f(\rho)} \le C$. By its definition, $m(0) = 0$. Also

$$\begin{aligned}
\left|\int_0^t \frac{\partial}{\partial\rho}(g+h_2) l(s)ds\right| \le & \int_{\rho(t)}^{\rho(0)} c\rho \frac{|l(s)|}{f(\rho(s))} d\rho \\
\le & c\rho^2(0)\|l\|_f . \\
\left|\int_0^t \frac{\partial h_2}{\partial\varphi} m(s)ds\right| \le & \|m\|_\infty \int_{\rho(t)}^{\rho(0)} c\rho^6 \frac{d\rho}{f(\rho)} \\
\le & c\rho^4(0)\|m\|_\infty .
\end{aligned}$$

We have derived

$$|m(t)| \le c + c\,\rho^2(0)\ \|l\|_f + c\,\rho^4(0)\ \|m\|_\infty .$$

If $c\,\rho^4(0) < \frac{1}{2}$, we have

$$|m(t)| \le c + c\,\rho^2(0)\ \|l\|_f , \tag{7.21}$$

with a larger c. Combining (7.20) and (7.21) we have $\|l\|_f + \|m\|_\infty \le C$ □

Theorem 7.5 is the main result of this section. It verifies the assumption H2) of section 2. Moreover, it provides sharper estimates of the rate of contraction in backward time when the initial point is on a curve σ. Let $y(t) = \Phi(t, \mu, y_0)$ be the solution for system (7.1), $y(0) = y_0$. Let (R, θ) be the polar coordinates for $y \in \mathbb{R}^2$,

$R = R^*(\theta, \mu)$ be the periodic solution if $\mu < 0$ and $R^*(\theta, \mu) = 0$ if $\mu \geq 0$, and let $r = R - R^*(\theta, \mu)$.

Theorem 7.5. *Assume that* $|\mu| < \mu_0$, $|y_0| < \eta_0$, y_0 *is outside the limit cycle if* $\mu < 0$ *and* $y_0 \neq 0$ *if* $\mu \geq 0$. *Let* $t \leq 0$. *Then,*
1) $r(t)$ *is monotonously decreasing and approaches zero at* $t \to -\infty$. *Moreover there exists* $c > 1$ *such that*

$$c^{-1} r(r^2 + |\mu|) < r'(t) < c r(r^2 + |\mu|). \tag{7.22}$$

2)
$$|D^{\alpha}_{y_0} \Phi(t, \mu, y_0)| \leq c|y(t)|, \quad for \quad 1 \leq |\alpha| \leq 3, \tag{7.23}$$

where $\alpha = (\alpha_1, \alpha_2)$ *and* $D^{\alpha}_{y_0} = \frac{\partial^{|\alpha|}}{(\partial y_{01})^{\alpha_1}(\partial y_{02})^{\alpha_2}}$,
3) *Passing through* y_0, *there is a* C^3 *curve* $\sigma = \{y(\zeta) : \zeta \in \mathbb{R}\}$, *equipped with the metric induced from* $\mathbb{R}^2$, *such that when* $y \in \sigma$, *we have*

$$|D^k_y \Phi(t, \mu, y)| \leq c\, r(r^2 + |\mu|) \leq c\, r'(t), \qquad 1 \leq k \leq 3.$$

The constants C in 2) and 3) depend on y_0, *and are uniformly bounded above if any* $\epsilon_0 > 0$ *is given so that* $\epsilon > |y_0| - R^*(\theta, \mu) \geq \epsilon_0$ *in 2) and* $\epsilon > |y(\zeta)| \geq \epsilon_0$ *in 3).*

Proof. 1) When $\mu \geq 0$, (7.22) can easily be proved from (7.2). When $\mu < 0$, from (7.5), since $a_1(\theta) = -2\mu + 0(|\mu|^2) > 0$, $a_3(\theta) = a + 0(|\mu|) > 0$ and $a_2(\theta) = 3a \cdot R_\mu + 0(|\mu|^{1.5}) > 0$, we have

$$a_1(\theta) r + a_2(\theta) r^2 + a_3(\theta) r^3 > c^{-1}(|\mu| + r^2) r$$

for some $c > 0$. The other inequality of (7.22) follows from $a_2(\theta) \leq c|\mu|^{0.5}$ and $2a_2(\theta) r^2 \leq r(a_2^2(\theta) + r^2)$.
2) Consider $\mu < 0$ first. To prove the theorem we only need to consider the flow on the (Z_1, Z_2) plane, due to Lemma 7.3. Let $t < 0$ be fixed and let the solution for (7.15) be $\rho(t) = P(\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\varphi)$, $\varphi(t) = \Phi(\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\varphi)$, with $(\rho(0), \varphi(0)) = (\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\varphi)$ being the initial condition. Using the Taylor expansion, we have

$$P(\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\varphi) = P_0 + \epsilon P_1 + \frac{1}{2}\epsilon^2 P_2 + \frac{1}{6}\epsilon^3 P_3 + \cdots,$$
$$\Phi(\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\rho) = \Phi_0 + \epsilon\Phi_1 + \frac{1}{2}\epsilon^2\Phi_2 + \frac{1}{6}\epsilon^3\phi_3 + \cdots,$$

where $F_0 = F(\rho, \varphi)$, $F_1 = F_\rho\Delta\rho + F_\varphi\Delta\varphi$, $F_2 = F_{\rho\rho}(\Delta\rho)^2 + 2F_{\rho\varphi}\Delta\rho \cdot \Delta\rho + F_{\varphi\varphi}(\Delta\varphi)^2$, $F_3 = F_{\rho\rho\rho}(\Delta\rho)^3 + 3F_{\rho\rho\varphi}(\Delta\rho)^2(\Delta\varphi) + 3F_{\rho\varphi\varphi}(\Delta\rho)(\Delta\varphi)^2 + F_{\varphi\varphi\varphi}(\Delta\varphi)^3$, where F can be P or Φ. Let the solution map in the Z-plane be $(Z_1(z_1 + \epsilon\Delta z_1,\ z_2 + \epsilon\Delta z_2),\ Z_2(z_1 + \epsilon\Delta z_1,\ z_2 + \epsilon\Delta z_2))$ for that fixed $t < 0$. It suffices to estimate derivatives of Z_1 since the ones regarding Z_2 can be obtained similarly. Consider $Z_1(z_1 + \epsilon\Delta z_1,\ z_2 + \epsilon\Delta z_2) = [R_\mu + P(\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\varphi)]\cos(\Phi(\rho + \epsilon\Delta\rho,\ \varphi + \epsilon\Delta\varphi))$. (Cf. Lemma 7.3.)

Comparing powers of ϵ^1 on both sides we have

$$\frac{\partial Z_1}{\partial z_1}\Delta z_1 + \frac{\partial Z_1}{\partial z_2}\Delta z_2 = (R_\mu + P_0)(-\sin\Phi_0 \cdot \Phi_1) + P_1 \cdot \cos\Phi_0 \leq C\,|Z|\,(|\Delta\varphi| + |\Delta\rho|).$$

Here we have used Lemma 7.4, $|P_\rho| + |P_\varphi| \leq C f(P_0) \leq C\,|Z|$ and $|R_\mu + P_0| = |Z|$. Also $|\Delta\rho| + |\Delta\varphi| \leq C(|\Delta z_1| + |\Delta z_2|)$. The derivatives of Z_2 can be estimated similarly. This proves (7.23) in the Z-plane when $|\alpha| = 1$.

Comparing powers of ϵ^2, we have

$$\frac{\partial^2 Z_1}{\partial z_1^2}(\Delta z_1)^2 + 2\frac{\partial z_1^2}{\partial z_1 \partial z_2}\Delta z_1 \Delta z_2 + \frac{\partial Z_1^2}{\partial z_2^2}(\Delta z_2)^2 = \frac{1}{2}P_2 \cdot \cos\Phi_0 - P_1\Phi_1 \sin\Phi_0$$
$$- \frac{1}{2}(R_\mu + P_0)[\Phi_2 \sin\Phi_0 + \Phi_1^2 \cos\Phi_0].$$

From Lemma 7.4 again, $|P_2| + |P_1\Phi_1| \le c\, f(P_0)(|\Delta\varphi| + |\Delta\rho|)^2 \le c(R_\mu + P_0)(|\Delta\varphi| + |\Delta\rho|)^2$. Derivatives of Z_2 can be handled similarly. This proves the case $|\alpha| = 2$.

Similarly, comparing powers of ϵ^3 and using Lemma 7.4 to show $|P_3| \le c\, f(P_0)(|\Delta\varphi| + |\Delta\rho|)^3$ and $|\Phi_3| \le c(|\Delta\varphi| + |\Delta\rho|)^3$, we have the desired estimates on $|D^\alpha Z_i|$, $i = 1, 2$, $|\alpha| = 3$.

When $\mu \ge 0$, there is no need to do the averaging and to change the z-plane to the Z-plane. Since $R_\mu = 0$, $R^*(\theta, \mu) = 0$ and $u^*(\rho, \varphi) = v^*(\rho, \varphi) = 0$, (7.2) can still be written as (7.15), from Lemma 7.4 we still have the same estimates on (P_i, Φ_i), $0 \le i \le 3$. Therefore (7.23) can be proved analogously.

3) Based on the remark above, we will prove property 3) on the Z-plane for both $\mu < 0$ and $\mu \ge 0$. Let the initial condition of system (7.15) be on a curve $(\rho_0(\zeta), \varphi_0(\zeta))$, $\zeta \in \mathbb{R}$, and the solution be $\rho(t) = \rho^*(t, \rho_0(\zeta), \varphi_0(\zeta))$, $\varphi(t) = \varphi^*(t, \rho_0(\zeta), \varphi_0(\zeta))$. Then,

$$\frac{\partial\varphi(t)}{\partial\zeta} = \left\{1 + \int_0^t \left[\frac{\partial}{\partial\rho}(g + h_2)\frac{\partial\rho(s)}{\partial\varphi_0} + \frac{\partial h_2}{\partial\varphi}\frac{\partial\varphi(s)}{\partial\varphi_0}\right] ds\right\} \frac{d\varphi_0(\zeta)}{d\zeta}$$
$$+ \int_0^t \left[\frac{\partial}{\partial\rho}(g + h_2)\frac{\partial\rho(s)}{\partial\rho_0} + \frac{\partial h_2}{\partial\varphi}\frac{\partial\varphi(s)}{\partial\rho_0}\right] ds\, \frac{d\rho_0(\zeta)}{d\zeta}.$$

We choose the initial condition so that $\frac{\partial\varphi(-\infty)}{\partial\zeta} = \lim_{t\to-\infty}\frac{\partial\varphi(t)}{\partial\zeta} = 0$ as $t \to -\infty$. Thus, let the initial curve be defined by an ordinary differential equation

$$\frac{d\varphi_0}{d\rho_0} = -\frac{\int_0^{-\infty}\left[\frac{\partial}{\partial\rho}(g + h_2)\frac{\partial\rho(s)}{\partial\rho_0} + \frac{\partial h_2}{\partial\varphi}\frac{\partial\varphi(s)}{\partial\rho_0}\right] ds}{1 + \int_0^{-\infty}\left[\frac{\partial}{\partial\rho}(g + h_2)\frac{\partial\rho(s)}{\partial\varphi_0} + \frac{\partial h_2}{\partial\varphi}\cdot\frac{\partial\varphi(s)}{\partial\varphi_0}\right] ds}.$$

Observing $|\frac{\partial\rho(s)}{\partial\rho_0}| \le cf(\rho(s))$, $|\frac{\partial h_2}{\partial\varphi}| \le c\rho^6(s)$, and using the change of variable $|ds| \le \frac{|d\rho|}{cf(\rho(s))} \le \frac{|d\rho|}{c\rho^3(s)}$, we can verify that the integrals converge uniformly. By the similar argument, the right hand side is a C^∞ function of (ρ_0, φ_0). Therefore, the local existence of a C^∞ solution is guaranteed. Since

$$\frac{\partial\varphi(t)}{\partial\zeta} = \frac{\partial\varphi(t)}{\partial\zeta} - \frac{\partial\varphi(-\infty)}{\partial\zeta},$$

$$\frac{\partial\varphi(t)}{\partial\zeta} = \int_{-\infty}^t \left[\frac{\partial}{\partial\rho}(g + h_2)\frac{\partial\rho(s)}{\partial\varphi_0} + \frac{\partial h_2}{\partial\varphi}\frac{\partial\varphi(s)}{\partial\varphi_0}\right] ds \cdot \frac{d\varphi_0(\zeta)}{d\zeta}$$
$$+ \int_{-\infty}^t \left[\frac{\partial}{\partial\rho}(g + h_2)\frac{\partial\rho(s)}{\partial\rho_0} + \frac{\partial h_2}{\partial\varphi}\frac{\partial\varphi(s)}{\partial\rho_0}\right] ds \cdot \frac{d\rho_0(\zeta)}{d\zeta}. \tag{7.24}$$

We have the following estimate,

$$\frac{\partial\varphi(t)}{\partial\zeta} \le c\rho^2(t)\left(|\frac{d\varphi_0}{d\zeta}| + |\frac{d\rho_0}{d\zeta}|\right). \tag{7.25}$$

In fact, let ψ_0 denote ρ_0 or φ_0, and let $[\ ]_1$ and $[\ ]_2$ denote the first and the second $[\ ]$ in (7.24). Change the variable $|ds| \le \frac{|d\rho|}{cf(\rho)} \le \frac{|d\rho|}{c\rho^3(s)}$. From Lemma 7.4, $|\frac{\partial\rho(s)}{\partial\psi_0}| \le c\, f(\rho(s))$, and $|\frac{\partial\varphi(s)}{\partial\psi_0}| \le c$. Also $\frac{\partial}{\partial\rho}(g+h_2) \le c\rho(s)$ and $\frac{\partial h_2}{\partial\varphi} \le c\rho^6(s) \le c\rho(s)f(\rho(s))$. Therefore

$$
\begin{aligned}
&|\int_{-\infty}^{t} [\ ]_1 ds + \int_{-\infty}^{t} [\ ]_2 ds| \\
\le& c\int_0^{\rho(t)} \rho f(\rho)\frac{d\rho}{f(\rho)} \\
\le& c\rho^2(t).
\end{aligned}
$$

This proves (7.25). It is easy to show,

$$
\frac{\partial\rho(t)}{\partial\zeta} \le \left|\frac{\partial\rho(t)}{\partial\rho_0}\cdot\frac{d\rho_0(\zeta)}{d\zeta} + \frac{\partial\rho(t)}{\partial\varphi_0}\cdot\frac{d\varphi_0}{d\zeta}\right| \le c\, f(\rho(t))\left(|\frac{d\varphi_0}{d\zeta}| + |\frac{d\rho_0}{d\zeta}|\right).
$$

Let the solution in the Z-plane be $(Z_1(z_1(\zeta), z_2(\zeta)), Z_2(z_1(\zeta), z_2(\zeta)))$. Using (7.25) and the fact that $\left(|\frac{d\varphi_0}{d\zeta}| + |\frac{d\rho_0}{d\zeta}|\right)$ is uniformly bounded if the initial points are chosen from a small compact segment of σ that contains y_0, we have

$$
\begin{aligned}
\frac{\partial Z_1}{\partial\zeta} &= \frac{\partial}{\partial\zeta}\,(R_\mu + \rho(t))\cos\varphi(t) \\
&= (R_\mu + \rho(t))(-\sin\varphi(t))\,\frac{\partial\varphi(t)}{\partial\zeta} + \frac{\partial\rho(t)}{\partial\zeta}\,\cos\varphi(t). \\
\left|\frac{\partial Z_1}{\partial\zeta}\right| &\le c|R_\mu + \rho(t)|\rho^2(t) + c\, f(\rho(t)) \\
&\le c(R_\mu^2 + \rho^2(t))\,\rho(t) + c\, f(\rho(t)) \\
&\le c(|\mu| + \rho^2(t))\,\rho(t) + c\, f(\rho(t)).
\end{aligned}
$$

In the above we have used $R_\mu^2 + \rho^2 > 2R_\mu\rho$. Recall that $r = \rho(1 + v^*(\rho, \varphi))$ when $\mu < 0$ and $r = \rho$ when $\mu \ge 0$, $\rho \le cr$. Since $f(\rho) \le c(|\mu| + \rho^2)\rho$, property 3), $k = 1$ then follows. It can be seen that the constants in the above proof depend on the initial curve σ but are uniform with respect to $t < 0$ and sufficiently small μ if initial points are chosen from a compact segment of σ.

Differentiating (7.24) and using the same method, we have

$$
\left|\frac{\partial^2\varphi(t)}{\partial\zeta^2}\right| \le c\rho^2(t)\left(|\frac{d\varphi_0}{d\zeta}|^2 + |\frac{d\rho_0}{d\zeta}|^2 + |\frac{d^2\varphi_0}{d\zeta^2}| + |\frac{d^2\rho_0}{d\zeta^2}|\right). \tag{7.26}
$$

In fact, from (7.24) we have

$$
\begin{aligned}
\frac{\partial^2\varphi(t)}{\partial\zeta^2} =& \int_{-\infty}^{t}\frac{\partial}{\partial\zeta}[\ ]_1 ds\cdot\frac{d\varphi_0(\zeta)}{d\zeta} + \int_{-\infty}^{t}\frac{\partial}{\partial\zeta}[\ ]_2 ds\cdot\frac{d\rho_0(\zeta)}{d\zeta} \\
&+ \int_{-\infty}^{t}[\ ]_1 ds\cdot\frac{d^2\varphi_0(\zeta)}{d\zeta^2} + \int_{-\infty}^{t}[\ ]_2 ds\cdot\frac{d^2\rho_0(\zeta)}{d\zeta^2}.
\end{aligned}
$$

By the same estimates that derive (7.25) from (7.24) we can show that the last two integrals are bounded by

$$
c\rho^2(t)\left(|\frac{d^2\varphi_0}{d\zeta^2}| + |\frac{d^2\rho_0}{d\zeta^2}|\right).
$$

To estimate the first two integrals, consider

$$\frac{\partial}{\partial\zeta}[\,] = \frac{\partial}{\partial\varphi_0}[\,]\frac{d\varphi_0(\zeta)}{d\zeta} + \frac{\partial}{\partial\rho_0}[\,]\frac{d\rho_0(\zeta)}{d\zeta},$$

where [ ] may be $[\,]_1$ or $[\,]_2$. Using $ab \leq \frac{1}{2}(a^2+b^2)$, to prove (7.26), we only need to show that

$$\sum_{j=1}^{2}\int_{-\infty}^{t}\left(\left|\frac{\partial}{\partial\varphi_0}[\,]_j\right| + \left|\frac{\partial}{\partial\rho_0}[\,]_j\right|\right) ds \leq c\rho^2(t). \tag{7.27}$$

Let ψ_0 denote φ_0 or ρ_0, $X = \frac{\partial(g+h_2)}{\partial\rho}$ or $\frac{\partial h_2}{\partial\varphi}$, $Y = \frac{\partial\rho}{\partial\psi_0}$ or $\frac{\partial\varphi}{\partial\psi_0}$. Then in the simplified notations, we have,

$$\begin{aligned}\frac{\partial}{\partial\psi_0}[\,] &= \frac{\partial}{\partial\psi_0}\sum XY\\ &= \sum\left(\frac{\partial}{\partial\psi_0}X\cdot Y + X\frac{\partial}{\partial\psi_0}Y\right).\\ \frac{\partial}{\partial\psi_0}X &= \frac{\partial X}{\partial\rho}\frac{\partial\rho}{\partial\psi_0} + \frac{\partial X}{\partial\varphi}\frac{\partial\varphi}{\partial\psi_0}\end{aligned}$$

Since $\frac{\partial\rho}{\partial\psi_0} = O(f(\rho)) = O(\rho)$, and $\frac{\partial\varphi}{\partial\psi_0} = O(1)$, if $X = \frac{\partial}{\partial\rho}(g+h_2) = O(\rho)$, then

$$\left|\frac{\partial}{\partial\psi_0}X\right| \leq c\cdot\rho.$$

Here $\frac{\partial X}{\partial\rho} = O(1)$ and $\frac{\partial X}{\partial\varphi} = O(\rho)$ are used. We have shown that $\left|\frac{\partial}{\partial\psi_0}X\right|$ has the same type of estimates as X. If $X = \frac{\partial}{\partial\varphi}h_2 = O(\rho^6)$, then

$$\left|\frac{\partial}{\partial\psi_0}X\right| \leq c\cdot\rho^6.$$

Here we have used $\frac{\partial X}{\partial\rho} = O(\rho^5)$ and $\frac{\partial X}{\partial\varphi} = O(\rho^6)$. We have shown in this case $\left|\frac{\partial}{\partial\psi_0}X\right|$ also has the same type of estimates as X. From Lemma 7.4, we see that $\frac{\partial Y}{\partial\psi_0}$ has the same type of estimates as Y. Thus $\frac{\partial}{\partial\psi_0}\sum XY$ has the same type of estimates as $\sum XY$. Therefore, the method that derives (7.25) from (7.24) also yields (7.27). (7.26) has been proved.

We now write $\frac{\partial^2}{\partial\zeta^2}Z_1$ as

$$\begin{aligned}\frac{\partial^2}{\partial\zeta^2}(R_\mu+\rho(t))\cos\varphi(t) &= \frac{\partial^2}{\partial\zeta^2}(R_\mu+\rho(t))\cos\varphi(t) + 2\frac{\partial}{\partial\zeta}(R_\mu+\rho(t))\frac{\partial}{\partial\zeta}\cos\varphi(t)\\ &\quad + (R_\mu+\rho(t))\frac{\partial^2}{\partial\zeta^2}\cdot\cos\varphi(t).\end{aligned}$$

The derivatives of $\rho(t)$ are dominated by $cf(\rho(t))$ by Lemma 7.4. If initial points are chosen from a compact small segment of σ containing y_0, then there is a uniform constant c such that $|\frac{\partial\varphi(t)}{\partial\zeta}| + |\frac{\partial^2\varphi(t)}{\partial\zeta^2}| \leq c\,\rho^2(t)$. We finally have

$$\left|(R_\mu+\rho(t))\frac{\partial^2}{\partial\zeta^2}\cos\varphi(t)\right| \leq c(R_\mu+\rho(t))\cdot\rho^2(t) \leq c\,\rho(|\mu|+\rho^2),$$

where the constants are uniform if the initial points are in a small segment of σ. This proves property 3) for $k = 2$.

The case $k = 3$ can be proved similarly and will not be given here. □

8. Interaction of Homoclinic Bifurcation and Hopf Bifurcation

We now study bifurcation of a homoclinic solution $q(t)$ that approaches a 2-dimensional center manifold as $t \to -\infty$. From now on $\mu = (\mu_1, \mu_2)$ is used to denote two parameters. We assume that all the Hypotheses in section 2 are satisfied and all the notations in section 2 will be followed. Suppose that the system is locally written as (1.1) with the flow on $W^c_{loc}(\mu)$ described by (7.1) and (7.2). Based on Theorem 7.5, Hypothesis H_2) is satisfied with $\delta(|t|, \mu) \le c|y(t)|$. The constant c is uniform for $y_0 \in \mathcal{U}$ which is a neighborhood of $q_y(-\tau)$ and is uniformly bounded away from $L(\mu)$. Also $-\bar{t} = -\infty$ because the solution $\Phi(t, \mu, y)$ will never leave $W^c_{loc}(\mu)$ as $t \to -\infty$. Therefore Theorems 2.1 - 2.4 are valid.

Because of Theorem 7.5, 3), the estimates in Corollary 3.5 can be improved if the condition $y(0) = q_y(-\tau) + y_0 \in \sigma$ is imposed, where σ is the curve passing through $q_y(-\tau)$ as described by Theorem 7.5, 3), and $(y(t), u(t), v(t))$ is the solution of the boundary value problem as in Corollary 3.5. In fact, when $y(0) \in \sigma$, we have

$$|D^i_{y(0)}(y(-t_0) - \Phi(-t_0, \mu, q_y(-\tau)))| \le C\delta^{\perp}(t_0, \mu)), \qquad 0 \le i \le 3$$

where $\delta^{\perp}(t_0, \mu) = cr(r^2 + |\mu|)$ and $r = |y| - R^*(\theta, \mu)$, $(|y|, \theta)$ are the polar coordinates for $y(-t_0)$ and $R^*(\theta, \mu)$ is the periodic solution when $\mu < 0$ and $R_*(\theta, \mu) = 0$ and when $\mu \ge 0$.

For this same reason, when constructing Σ_0 and Σ in Theorem 2.1, if we choose σ to be the curve as in Theorem 7.5, 3), we can improve the estimates in Theorem 2.2, namely, replacing $\delta(t, \mu)$ by $\delta^{\perp}(t, \mu)$. We will also need some new estimates on $D^\alpha G_i(\{S_j\}, \mu)$, $|\alpha| = 2, 3$.

Theorem 8.1. *Let $\{S_j\}^k_{h-1}$ be a symbol sequence as in Theorem 2.1. $S_j = t_j \in \mathbb{R}^+$ if $h \le j \le k-1$. $S_{h-1} = [\bar{v}]$ and $S_k = [\bar{y}, \bar{u}]$ with $\bar{v} = 0$ and $\bar{u} = 0$. Assume that the curve σ is chosen from that of Theorem 7.5, 3). Let $\delta^{\perp}_i = \delta^{\perp}(t_i - 2\tau, \mu)$ for $h \le i \le k-1$ and $\delta^{\perp}_i = 0$ otherwise; $y_i = p(-t_i + 2\tau, \mu)$ and $\rho_i = |y_i|$ for $h \le i \le k-1$, $\rho_k = |\bar{y}|$ and $\rho_i = 0$ otherwise. Then*

$$\begin{aligned}&|G_i(\{S_j\}, \mu) - d(y_i, \mu)|\\ &\quad \le C_1(\delta^{\perp}_i(\rho_{i-1} + \rho_i + \rho_{i+1} + |\mu|) + \delta^{\perp}_{i-1}(\rho_{i-2} + \rho_{i-1} + \rho_i + |\mu|)), \quad h \le i \le k,\end{aligned} \tag{8.1}$$

Let $t_i \in \mathbb{R}^+$ for $h \le i \le k-1$, $t_k = 0$, $t_i(\zeta) = t_i + \zeta\Delta t_i$ for $h \le i \le k$, $y_i(\zeta) = p(-t_i(\zeta) + 2\tau, \mu)$ for $h \le i \le k-1$, $y_k(\zeta) = g(t_k(\zeta))$ where $g(t)$ is a $\mathbb{R}^n$ valued C^3 function with $g(0) = \bar{y}$, $\bar{\rho}_i = |y_i(\zeta)|$ for $h \le i \le k-1$. Let $\{S_i(\zeta)\}^k_{h-1}$ be a symbol sequence with $S_i(\zeta) = t_i(\zeta)$ for $h \le i \le k-1$, $S_{h-1}(\zeta) = [0]$ and $S_k(\zeta) = [y_k(\zeta), 0]$.

Let $\overline{\delta}_i^{\perp} = \delta^{\perp}(t_i(\zeta) - 2\tau, \mu)$ *for* $h \le i \le k-1$, $\overline{\delta}_i^{\perp} = 0$ *otherwise. Then*

$$\begin{aligned}
&\left| \frac{\partial^{\nu}}{\partial \zeta^{\nu}} G_i(\{S_j(\zeta)\}, \mu) - \frac{\partial^{\nu}}{\partial \zeta^{\nu}} d(y_i(\zeta), \mu) \right| \\
&\qquad \le C_3 (\overline{\delta}_i^{\perp} (\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) \\
&\qquad + \overline{\delta}_{i-1}^{\perp} (\overline{\rho}_{i-2} + \overline{\rho}_{i-1} + \overline{\rho}_i)) \; \|\{\Delta t_j\}\|^{\nu}, \quad 1 \le \nu \le 3, \; h \le i \le k
\end{aligned} \tag{8.2}$$

$$\left| \frac{\partial^2}{\partial \zeta \partial \mu_2} G_i(\{S_j(\epsilon)\}, \mu) \right| \le C_5 (\overline{\rho}_i + \overline{\rho}_{i-1}) \; \|\{\Delta t_j\}\|, \qquad h \le i \le k \tag{8.3}$$

Proof. In the proof of (2.6) and (2.8), we find that the δ_i term in (4.5), (4.7), (4.8) and (4.10) can be replaced by $\delta_i^{\perp}$. Also, all the $\delta_i' s$ in Lemma 4.2 can be replaced by $\delta_i^{\perp}$. Estimate (8.1) then follows from (2.6). Similarly, estimate (8.2), for $\nu = 1$, follows from (2.8).

Let $\{\mathscr{X}_i\}_h^{k-1}$ be the fixed point of (4.1)-(4.4) corresponding to the symbol sequence $\{S_i(\zeta)\}_{h-1}^{k}$. Differentiating (4.1)-(4.4) with respect to ζ, we have

$$\sup_i |D_{\zeta}^{\nu} \mathscr{X}_i| \le C \; \|\{\Delta t_i\}\|^{\nu}, \qquad \nu = 1, 2. \tag{8.4}$$

From (4.1) and (4.2),

$$\begin{aligned}
&\frac{\partial^2}{\partial \zeta^2} w_i^1 = \left(\frac{\partial^2 w^*}{\partial t^2} + 2 \frac{\partial^2 w^*}{\partial t \partial T} + \frac{\partial^2 w^*}{\partial T^2} \right) \Bigg| \cdot (\Delta t_i)^2 + \cdots, \\
&\qquad\qquad\qquad\qquad\qquad\qquad T = t_i(\zeta) - 2\tau, \; t = -T, \\
&\frac{\partial^2}{\partial \zeta^2} v_{i+1}^1 = \frac{\partial^2 v^*}{\partial T^2} \Bigg|_{T = t_i(\zeta) - 2\tau} \cdot (\Delta t_i)^2 + \cdots,
\end{aligned} \tag{8.5}$$

where the dotted terms involve products of $D^{\nu} w^*$, or $D^{\nu} v^*$ and $D_{\zeta}^{\nu}\{\mathscr{X}_i\}$, $\nu = 1, 2$. From Lemma 3.4, $D^{\nu} w^*$ and $D^{\nu} v^*$ are bounded by $\overline{\delta}_i$. Using (8.5) we have

$$\left| \frac{\partial^2 w_i^1}{\partial \zeta^2} \right| + \left| \frac{\partial^2 v_{i+1}^1}{\partial \zeta^2} \right| \le C \overline{\rho}_i \|\{\Delta t_j\}\|^2, \qquad h \le i \le k-1.$$

Here we have used Theorem 7.5, 2). Substituting the above into (4.3)-(4.4), and using the C^2 boundedness of w_* and v_*, we have

$$\left| \frac{\partial^2 w_i^0}{\partial \zeta^2} \right| + \left| \frac{\partial^2 v_i^0}{\partial \zeta^2} \right| \le C (\overline{\rho}_{i+1} + \overline{\rho}_i + \overline{\rho}_{i-1}) \|\{\Delta t_j\}\|^2, \qquad h \le i \le k-1. \tag{8.6}$$

Substituting into (8.5) again, we have

$$\begin{aligned}
&\left| \frac{\partial^2 w_i^1}{\partial \zeta^2} - \left(\frac{\partial^2 w^*}{\partial t^2} + 2 \frac{\partial^2 w^*}{\partial t \partial T} + \frac{\partial^2 w^*}{\partial T^2} \right) \Bigg|_{T = t_i(\zeta) - 2\tau, \, t = -T} (\Delta t_i)^2 \right. \\
&+ \left| \frac{\partial^2 v^*}{\partial \zeta^2} - \frac{\partial^2 v^*}{\partial T^2} \right|_{T = t_i(\zeta) - 2\tau} (\Delta t_i)^2 \\
&\le C \overline{\delta}_i^{\perp} (\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1}) \|\{\Delta t_j\}\|^2, \qquad h \le i \le k-1.
\end{aligned} \tag{8.7}$$

In fact, the dotted terms in (8.5) are bounded by terms like $|D^\nu w^*||D^\eta_\zeta w_i^0|$, $|D^\nu v^*||D^\eta_\zeta w_i^0|$, $|D^\nu w^*||D^\eta_\zeta v_i^0|$ and $|D^\nu v^*||D^\eta_\zeta v_i^0|$. Since $y_i^0 \in \sigma$, (cf. (3.5) and the definition of Σ_0 before Theorem 2.1), using Theorem 7.5, 3), and the improved version of Corollary 3.5, the first factors are always bounded by $c\overline{\delta}_i^\perp$ and (8.6) can be used to estimate the second factors.

Differentiating $\xi_*(w_i^1, v_i^1, \mu)$ with respect to ζ, we have

$$\frac{\partial^2 \xi_*}{\partial y^2}\left(\frac{\partial y_i^1}{\partial \zeta}\right)^2 + \frac{\partial \xi_*}{\partial y}\frac{\partial^2 y_i^1}{\partial \zeta^2} + \cdots .$$

The $\cdots$ represents terms that contain derivatives of u_i^1 and v_i^1. By Corollary 3.5, in (8.7), derivatives of u^* and v^* with respect to t and T are bounded by $ce^{-\alpha_1 t_i} \le c\overline{\delta}_i^\perp \overline{\rho}_i$. The last estimate can be proved by the method used in the Remark following Corollary 3.5. Therefore

$$\left|\frac{\partial^2 u_i^1}{\partial \zeta^2}\right| + \left|\frac{\partial^2 v_{i+1}^1}{\partial \zeta^2}\right| \le c\overline{\delta}_i^\perp(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1})\|\{\Delta t_j\}\|^2.$$

From (4.17) we can derive

$$\left|\frac{\partial u_i^1}{\partial \zeta}\right| + \left|\frac{\partial v_{i+1}^1}{\partial \zeta}\right| \le c\overline{\delta}_i^\perp(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1})\|\{\Delta t_j\}\|.$$

Using the above to estimate the dotted terms, we have

$$\begin{aligned}\frac{\partial^2 \xi_*(w_i^1, v_i^1, \mu)}{\partial \zeta^2} &= \frac{\partial \xi_*}{\partial y}\cdot\frac{\partial^2 y_i^1}{\partial \zeta^2} + \frac{\partial^2 \xi_*}{\partial y^2}\left(\frac{\partial y_i^1}{\partial \zeta}\right)^2 \\ &+ 0(\overline{\delta}_i^\perp(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) + \overline{\delta}_{i-1}^\perp(\overline{\rho}_{i-2} + \overline{\rho}_{i-1} + \overline{\rho}_i))\|\{\Delta t_j\}\|^2, \qquad h \le i \le k.\end{aligned} \tag{8.8}$$

We now derive error estimates when replacing y_i^1 by $y_i(\zeta)$ in the above. The error estimate of replacing $\frac{\partial y_i^1}{\partial \zeta}$ by $\frac{\partial y_i(\zeta)}{\partial \zeta}$ has been obtained in section 4 – (in the proof of (4.20)). The error estimate of replacing $\frac{\partial^2 y_i^1}{\partial \zeta^2}$ by $\frac{\partial^2 y_i(\zeta)}{\partial \zeta^2}$ can be derived by using (8.7). First replace $\frac{\partial^2 y_i^1}{\partial \zeta^2}$ in (8.8) by

$$\frac{\partial^2 y^*}{\partial t^2} + 2\frac{\partial^2 y^*}{\partial t \partial T} + \frac{\partial^2 y^*}{\partial T^2}.$$

The error is negligible according to (8.7). We then replace y^* by $y_i(\zeta) = p(-t_i(\zeta)+2\tau, \mu)$ in the above. The error is bounded by $O(\sum_\nu(|D^\nu(y^* - \Phi)| + |D^\nu(\Phi - p)|)$, where D^ν is the differentiation with respect to t and T, $\Phi = \Phi(-t_i(\zeta) + 2\tau, \mu, y_i^0 + q_y(-\tau))$. By Lemma 3.4, $|D^\nu(y^* - \Phi)| = |D^\nu y^\$| \le ce^{-\alpha_1 t_i} \le c\overline{\delta}_i^\perp \overline{\rho}_i$. We can show $|D^\nu(\Phi - p)| \le c\overline{\delta}_i^\perp |y_i^0|$. In fact,

$$|D_t(\Phi(-t_i(\zeta) + 2\tau, \mu, y_i^0 + q_y(-\tau)) - p(-t_i(\zeta) + 2\tau, \mu)|) \le C|\Phi - p| \le \overline{\delta}_i^\perp |y_i^0|.$$

Here the differential equations for $\Phi(t)$ and $p(t)$ are used and the initial conditions are on σ with a distance equals $|y_i^0|$, cf. Theorem 7.5, 3). Similarly we can handle $|\nu| > 1$. From the estimate between (4.19) and (4.20), we have $|y_i^0| \le c(\overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|)$. Therefore replacing y_i^1 by $y_i(\zeta)$ in (8.8) will not worsen the estimate. Finally, we show that replacing the arguments (y_i^1, u_i^1, v_i^1) in $\frac{\partial \xi_*}{\partial y}$, $\frac{\partial^2 \xi_*}{\partial y^2}$ by $(y_i(\zeta), 0, 0, \mu)$ will not worsen the error estimate in (8.8). In fact,

$$|\frac{\partial^2 y_i^1}{\partial \zeta^2}| + |\frac{\partial y_i^1}{\partial \zeta}| \leq c\|\{\Delta t_j\}\|^2.$$

Also

$$\begin{aligned}|y_i^1 - y_i(\zeta)| &\leq |y^\$| + |\Phi - p| \leq c\overline{\delta}_i^\perp(\overline{\rho}_{i-1} + \overline{\rho}_i + |\mu|),\\ |u_i^1| + |v_i^1| &= |u^*| + |v^*| \leq ce^{-\alpha_1 t_i} \leq c\overline{\delta}_i^\perp \overline{\rho}_i.\end{aligned}$$

After replacing the arguments in (8.8) by $((y_i(\zeta), 0, 0, \mu)$, we have

$$\begin{aligned}\frac{\partial^2 \xi_*(w_i^1, v_i^1, \mu)}{\partial \zeta^2} &= \frac{\partial \xi_*}{\partial y} \cdot \frac{\partial^2 y_i(\zeta)}{\partial \zeta^2} + \frac{\partial^2 \xi_*}{\partial y^2}\left(\frac{\partial y_i(\zeta)}{\partial \zeta}\right)^2\\ &+ 0(\overline{\delta}_i^\perp(\overline{\rho}_{i-1} + \overline{\rho}_i + \overline{\rho}_{i+1} + |\mu|) + \overline{\delta}_{i-1}^\perp(\overline{\rho}_{i-2} + \overline{\rho}_{i-1} + \overline{\rho}_i))\|\{\Delta t_j\}\|^2, \qquad h \leq i \leq k.\end{aligned}$$

Observe that the first two terms in the right hand side are equal to

$$\frac{\partial^2 \xi_*(y_i(\zeta), 0, 0, \mu)}{\partial \zeta^2} = \frac{\partial^2 d(y_i(\zeta), \mu)}{\partial \zeta^2}.$$

See Lemma 4.3 for the relation of ξ_* and d. The proof of (8.2) for $\nu = 2$ has been completed. The case $\nu = 3$ can be proved similarly.

The proof of (8.3) is similar and will be omitted. Notice that $|\mu|$ does not enter the r.h.s. of (8.3). This is because (8.3) is not as sharp as (8.2), since $\overline{\delta}_i^\perp < \overline{\rho}_i$.

We now study the intersection of $\mathscr{C}(\mu)$ and the orbit of $p(-t + 2\tau, \mu)$, $t \leq 0$ in the region $|\mu| < \hat{\mu}$ and $|y| < \hat{\rho}$. See Figure 8.1. Assume that $y = (z_1, z_2)$ with the z_1-axis being orthogonal to $\mathscr{C}(0)$. The graph of $\mathscr{C}(\mu)$ can be written as $z_1 = \overline{z}_1 + \varphi(\overline{z}_1, z_2, \mu)$ where $\overline{z}_1$ is the coordinate of the intersection of $\mathscr{C}(\mu)$ with the z_1-axis. Thus, $\varphi(\overline{z}_1, 0, \mu) = 0$, $D_{z_2}\varphi(0, 0, 0) = 0$. When $(\mu_1, \mu_2) = (0, 0)$, we have $\overline{z}_1(\mu) = 0$, (the existence of the homoclinic orbit Γ_0), and $\frac{\partial \overline{z}_1(0)}{\partial \mu_2} \neq 0$, (cf. H6) and (5.6), since $\overline{z}_1 = d(0, \mu))$. Let us make a change of parameters $(\mu_1, \tilde{\mu}_2) = (\mu_1, \overline{z}_1(\mu))$. In the new parameters, $\overline{z}_1(\mu) = \mu_2$, where the $\tilde{}$ is dropped for simplicity. Let us make the near identity change of variables $\Psi(\overline{y}) = y$:

$$\begin{aligned}z_1 &= \overline{z}_1 + \varphi(\overline{z}_1, \overline{z}_2, \mu),\\ z_2 &= \overline{z}_2.\end{aligned}$$

We then have $y = \overline{y} + 0(|\overline{y}|^2)$ with the inverse $\overline{y} = y + 0(|y|^2)$. Moreover $D_{\overline{y}}y = I + 0(|\overline{y}|)$ and $D_y\overline{y} = I + 0(|y|)$. In the new coordinates, $C(\mu) = \{(\overline{z}_1, \overline{z}_2)|\overline{z}_1 = \mu_2\}$. It can be verified that system (7.1) still has the form

$$\begin{aligned}\overline{z}_1' &= \mu_1\overline{z}_1 - \overline{z}_2 + hot.,\\ \overline{z}_2' &= \overline{z}_1 + \mu_1\overline{z}_2 + hot.\end{aligned}$$

Let the flow of the above equation be $\overline{\Phi}(t, \mu, \overline{y})$. In the polar coordinates, let $(\overline{z}_1, \overline{z}_2) = (\overline{R}\cos\overline{\theta}, \overline{R}\sin\overline{\theta})$. Let $\overline{R} = \overline{R}^*(\overline{\theta}, \mu)$ be the periodic orbit if $\mu < 0$ and zero when $\mu \geq 0$. Let $\overline{r} = \overline{R} - \overline{R}^*(\overline{\theta}, \mu)$. Let $\overline{\sigma} = \Psi^{-1}(\sigma)$.

Lemma 8.2. *i) There is a unique smooth curve* $\mathcal{L} = \{(\overline{z}_1, \overline{z}_2) | \overline{z}_2 = L^*(\overline{z}_1, \mu)\}$ *such that if* $(\overline{z}_1, \overline{z}_2) \in \mathcal{L}$, *then* $\dot{\overline{z}}_1(t) = 0$. *Moreover,* $L^*(\overline{z}_1, \mu) = \mu_1 \overline{z}_1 + O(\overline{z}_1^2)$, $\partial_1 L^*(\overline{z}_1, \mu) = \mu_1 + 0(\overline{z}_1)$, *and* $|\partial_1^2 L^*(\overline{z}_1, \mu)| \leq C$.
ii) The two open segments of the curve $\mathcal{L}$, *corresponding to* $\overline{z}_1 > 0$ *and* < 0, *are also smoothly parameterized by* $\overline{R}$ *with*

$$c^{-1} \leq \pm \frac{d\overline{R}}{d\overline{z}_1} \leq c.$$

Here the "+" (or "−") sign is taken if $\overline{z}_1 > 0$ *(or* < 0*). If* $\mu_1 < 0$, *then outside the loop of the periodic solution* $(z_1^p(t), z_2^p(t))$, *the two segments of* $\mathcal{L}$ *are smoothly parameterized by* $\overline{r}$, *with*

$$c^{-1} \leq \pm \frac{d\overline{r}}{d\overline{z}_1} \leq c.$$

Here the "+" (or "−") sign is taken if $\overline{z}_1 > \max_t(z_1^p(t))$ *(or* $\overline{z}_1 < \min_t(z_1^p(t))$*).*
iii) $|D_{\overline{y}}^{\alpha} \overline{\Phi}(t, \mu, \overline{y})| \leq C |\overline{y}(t)|$ *for* $1 \leq |\alpha| \leq 3$. *If* $\overline{y} \in \overline{\sigma}$ *and* $1 \leq k \leq 3$, *then* $|D_{\overline{y}}^k \overline{\Phi}(t, \mu, \overline{y})| \leq C\overline{r}(\overline{r}^2 + |\mu|) \leq c\overline{r}'(t)$.

Proof. i) By the implicit function theorem, we can solve $\overline{z}_2$ as a function of $\overline{z}_1$ from $\dot{\overline{z}}_1 = \mu_1 \overline{z}_1 - \overline{z}_2 + hot. = 0$.
ii) Consider $\overline{z}_1 > 0$ only. Since $\overline{R} = (\overline{z}_1^2 + \overline{z}_2^2)^{0.5}$, we have

$$\begin{aligned} \frac{\partial \overline{R}}{\partial \overline{z}_1} &= (\overline{z}_1^2 + \overline{z}_2^2)^{-0.5} (\overline{z}_1 + \overline{z}_2 \frac{d\overline{z}_2}{d\overline{z}_1}) \\ &= (1 + O(\mu_1^2 + \overline{z}_1^2)), \end{aligned}$$

based on the estimates in i) of this lemma. Suppose now $\mu_1 < 0$ and $\overline{z}_1 > 0$. From $\overline{\theta} = \arctan \frac{\overline{z}_2}{\overline{z}_1}$, on the two segments, we have

$$\frac{\partial \overline{\theta}}{\partial \overline{z}_1} = \frac{1}{1 + (\overline{z}_2 / \overline{z}_1)^2} \frac{[(\partial \overline{z}_2 / \partial \overline{z}_1) \overline{z}_1 - \overline{z}_2]}{\overline{z}_1^2}.$$

The terms in the [ ] are bounded by $c\overline{z}_1^2$. Thus, $\frac{\partial \overline{\theta}}{\partial \overline{z}_1} = O(1)$. From Lemma 7.1, we have $\frac{\partial \overline{R}^*}{\partial \overline{\theta}} = O(|\mu|)$. Therefore,

$$\begin{aligned} \frac{\partial \overline{r}}{\partial \overline{z}_1} &= \frac{\partial (\overline{R} - \overline{R}^*(\overline{\theta}, \mu))}{\partial \overline{z}_1} \\ &= [1 + O(\mu_1^2 + \overline{z}_1^2)] - \frac{\partial \overline{R}^*}{\partial \overline{\theta}} \frac{\partial \overline{\theta}}{\partial \overline{z}_1} \\ &= 1 + O(|\mu_1| + \overline{z}_1^2). \end{aligned}$$

iii) The proof follows from the fact $\overline{\Phi}(t, \mu, \overline{y}) = \Psi^{-1} \Phi(t, \mu, \Psi(\overline{y}))$ and Theorem 7.5. Each straight line $\overline{\theta}$=constant is mapped into a curve by the near identity mapping $y = \Psi(\overline{y})$. It is tedious but elementary to show that

$$C^{-1} r(r^2 + |\mu|) \leq \overline{r}(\overline{r}^2 + |\mu|) \leq Cr(r^2 + |\mu|).$$ □

Due to Lemma 8.2 the estimates in Theorem 8.1 are still valid after the change of variables. From now on we will drop the bars and assume that in the original coordinate

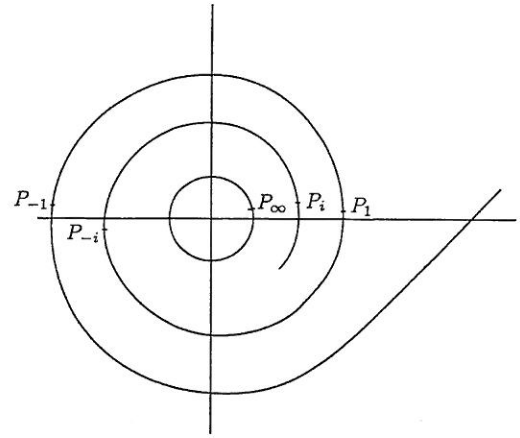

Figure 8.1.

system $\mathscr{C}(\mu) = \{(z_1, z_2)|z_1 = \mu_2\}$ to simplify the illustration. In a small neighborhood of $y = 0$, on the curve $p(t, \mu) = \Phi(t, \mu, q_y(-\tau))$ we label the points which have vertical tangent line, (cf. Lemma 8.2, i)), in the order of decreasing t by

$$P_{-1}, P_1, P_{-2}, P_2, \dots P_{-i}, P_i, P_{-i-1}, \dots .$$

Let $P_i = (z_1(i), z_2(i))$ and $P_{-i} = (z_1(-i), z_2(-i))$ in the rectangular coordinates.

$$z_1(-1) < z_1(-2) < \cdots < z_1(-\infty) \le z_1(+\infty) < \cdots < z_1(2) < z_1(1).$$

We have written indices in parentheses for the convenience of typing. $z_1(-\infty) = z_1(+\infty) = 0$ when $\mu_1 \ge 0$; $(z_1(\pm\infty), z_2(\pm\infty))$ are points on the periodic orbit, with $z_2(\pm\infty) = \mathscr{L}(z_1(\pm\infty))$, when $\mu_1 < 0$. See Figure 8.1. Let $|P_i| = (z_1^2(i) + z_2^2(i))^{\frac{1}{2}}$.

The notion of nondegenerate arcs has been defined in Theorem 2.3. See section 2, hypothesis H_7). Because Theorem 8.1 is sharper than Theorem 2.2, we will modify Hypothesis H_7). Observe here $h = -\infty$ and $k = +\infty$. Define $t_i(\zeta)$, $y_i(\zeta)$ and $S_i(\zeta)$ as in section 2, H_7).

$H_7)'$ There are positive constants d_i and e_i, $\sup_i\{d_i + e_i\} < \infty$ such that for $-d_i < \zeta < e_i$, $-\infty < i < \infty$,

$$\left|\frac{\partial}{\partial\zeta} d(y_i(\zeta), \mu)\right| > C_4[\overline{\delta}_i^{\perp}(\overline{\rho}_{i-1}+\overline{\rho}_i+\overline{\rho}_{i+1}+|\mu|)+\overline{\delta}_{i-1}^{\perp}(\overline{\rho}_{i-2}+\overline{\rho}_{i-1}+\overline{\rho}_i)]. \qquad (\mathrm{H}_7; 1)'$$

Here $C_4 > C_3$ where C_3 is the constant in (8.2). $\overline{\delta}_i^{\perp} = \sup\{\overline{\delta}^{\perp}(t - 2\tau, \mu)|t \in (t_i - d_i, t_i + e_i)\}$ and $\overline{\rho}_i = \sup_\zeta\{|y_i(\zeta)|\}$.

At the end points $\zeta = -d_i$ and $\zeta = e_i$, assume

$|d(y_i(\zeta), \mu)| > C_1\{\delta_i^{\perp}(\rho_{i-1} + \rho_i + \rho_{i+1} + |\mu|) + \delta_{i-1}^{\perp}(\rho_{i-2} + \rho_{i-1} + \rho_i + |\mu|)$. $(\mathrm{H}_7; 2)'$

Here $\delta_i^{\perp} = \delta^{\perp}(t_i(\zeta) - 2\tau, \mu)$ and $\rho_i = |y_i(\zeta)|$, evaluated at $\zeta = -d_i$ and e_i.

Condition $\mathrm{H}_7)'$ has a simple geometric meaning when considering simple periodic solutions, that is, the point $p(-t_i + 2\tau, \mu)$ must not be near the vertical turning point of the spiraling orbit $p(t, \mu)$, $t \le 0$. It follows that certain restrictions on μ_2 must be

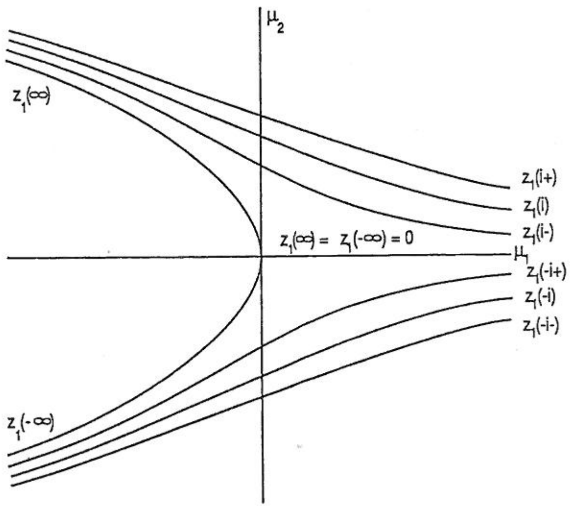

Figure 8.2.

imposed since $\mu_2 = z_1(i)$ where $P_i = (z_1(i), z_2(i))$ is the turning point. Theorem 8.3 gives some estimates on the distance of μ_2 and $z_1(i)$ in order to have a nondegenerate periodic solution. The results are illustrated in Figure 8.2, where i, $i+$, $i-$ are indices. Recall we use $\{\omega\}_p$ to denote a periodic symbol $\{t_i\}_{-\infty}^{\infty}$ with $t_i = \omega$ for $-\infty < i < \infty$.

If P and Q are two points on the orbit of $p(t, \mu)$, then segment of the orbit of $p(t, \mu)$ that connects P and Q is denoted by $\widetilde{PQ}$.

Theorem 8.3. *In Figure 8.2, there are three continuous curves of μ_1 that lie above $z_1(+\infty)$,*

$$0 \le z_1(+\infty) < z_1(i-) < z_1(i) < z_1(i+)$$

with the following properties: For $0 \le \mu_2 \le z_1(i-)$, there are two nondegenerate simple periodic solutions with symbols $\{\omega_1(i)\}_p$ and $\{\omega_2(i)\}_p$ respectively. Also $p(-\omega_1(i) + 2\tau, \mu) \in \widetilde{P_iP_{-i}}$ and $p(-\omega_2(i) + 2\tau, \mu) \in \widetilde{P_iP_{-i-1}}$. Here $\omega_1(i)$ and $\omega_2(i)$ are smooth functions of μ_2 with $\omega_1(i) < \omega_2(i)$ and

$$\frac{\partial \omega_1(i)}{\partial \mu_2} > 0 \text{ and } \frac{\partial \omega_2(i)}{\partial \mu_2} < 0.$$

For $\mu_2 \ge z_1(i+)$, there is no simple ω-periodic solution with $p(-\omega + 2\tau, \mu) \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}$. Moreover

$$z_1(i-) > z_1((i+1)+).$$

Similar results also hold for the curves

$$z_1(-i-) < z_1(-i) < z_1(-i+) < z_1(-\infty) \le 0.$$

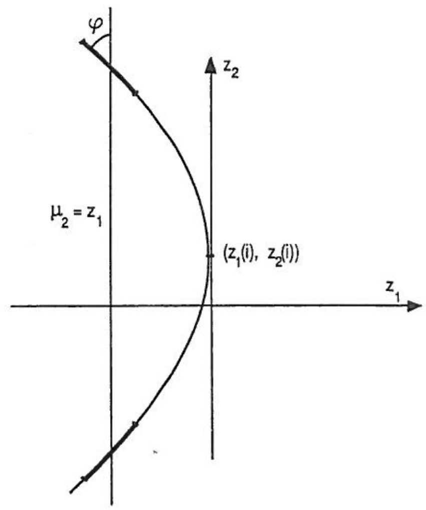

Figure 8.3.

The nondegenerate periodic solutions corresponding to $0 \leq \mu_2 \leq z_1(i-)$ *are all hyperbolic with* $l+1$ *(or* m*) characteristic values outside (or inside) the unit circle in the complex plane.*

Proof. For $y \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}$, if $R = |P_i|$, $r = R - R^*(\theta_0, \mu)$ and μ are small enough, then

$$\begin{aligned} 0.5R &\leq |y| \leq 1.5R, \\ 0.5r &\leq |y| - R^*(\theta, \mu) \leq 1.5r \\ \delta^{\perp}(|t|, \mu) &< c_1 \delta^{\perp}, \end{aligned} \tag{8.9}$$

where (R, θ_0) is the polar coordinates for P_i, t is such that $p(-t + 2\tau, \mu) \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}$, and $\delta^{\perp} = cr(|\mu| + r^2)$. In fact, let $\rho(t) = |y|$, $\rho(0) = R = |P_i|$, and $\rho(t)$ satisfy (7.2). If $|t| \leq 2\pi$ and $\rho(t) \leq 1.5R$, then

$$\begin{aligned} |\rho(t) - \rho(0)| &= |\int_0^t (\mu\rho + a\rho^3 + \cdots)dt| \\ &\leq 2\pi c(|\mu| + R^2)R \\ &\leq 0.5R. \end{aligned}$$

The last inequality is valid if $|\mu|$ and R are small. We then find that $\rho(t) \leq 1.5R$ is always valid for $|t| \leq 2\pi$. The second estimate concerning $|y| - R^*(\theta, \mu)$ can be proved similarly, using the fact that $|y| - R^*(\theta, \mu)$ satisfies (7.5). The third estimate follows from the second easily.

Let s be the arc length from P_i. Let φ be the angle between $\dot{y}(t)$ and $\mathscr{C}(\mu)$ using polar coordinates, we can show that for $y \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}$, $\frac{d\varphi}{dt} = 1 + 0(|R|^2)$ and

$|\frac{ds}{dt}| = |\dot{y}(t)| = R + O((|\mu_1| + R^2)R)$. In fact, let $\rho(t) = |y|$, then

$$\begin{aligned}|\dot{y}(t)| &= [(\frac{d\rho}{dt})^2 + (\rho\frac{d\theta}{dt})^2]^{0.5} \\ &= [(\mu_1\rho + a\rho^3 + \cdots)^2 + \rho^2(1 + b\rho^2 + \cdots)^2]^{0.5} \\ &= \rho + O(\rho(\mu_1^2 + \rho^2)).\end{aligned}$$

The estimate for $|\frac{ds}{dt}|$ then follows from $|\rho(t) - R| \leq 2\pi c(|\mu| + R^2)R$. On the other hand, using the polar coordinates (ρ, θ), we can write $\varphi = \theta - \arctan(\frac{d\rho}{\rho d\theta}) = \theta - \arctan(\frac{\mu_1 + a\rho^2 + \cdots}{1 + b\rho^2 + \cdots})$. Thus

$$\begin{aligned}\frac{d\varphi}{dt} &= (1 + b\rho^2 + \cdots) - O(\frac{d}{dt}(\frac{\mu_1 + a\rho^2 + \cdots}{1 + b\rho^2 + \cdots})) \\ &= 1 + O(\rho^2).\end{aligned}$$

From $\varphi(s) = \int_0^s \frac{d\varphi/dt}{ds/dt} ds = \int_0^s \frac{1+0(R^2)}{(1+O(|\mu_1|+R^2)R} ds$, we have

$$\frac{c^{-1}|s|}{R} \leq |\varphi(s)| \leq \frac{c|s|}{R}. \tag{8.10}$$

If $|\varphi(s)| \leq \frac{3\pi}{4}$, then there exists $c > 0$ such that

$$c^{-1}|\varphi(s)| \leq |\sin\varphi(s)| \leq c|\varphi(s)|.$$

The change in the z_1-coordinate can now be estimated in terms of s, by using (8.10),

$$\frac{c^{-1}s^2}{R} \leq |z_1(i) - z_1| = |\int_0^s \sin\varphi(s)ds| \leq \frac{cs^2}{R}, \tag{8.11}$$

provided that $|\varphi(s)| \leq \frac{3\pi}{4}$. Thus, from (8.10) and (8.11),

$$C^{-1}\sqrt{\frac{z_1(i) - z_1}{R}} \leq |\varphi(s)| \leq C\sqrt{\frac{z_1(i) - z_1}{R}} \tag{8.12}$$

We then have an estimate for $\frac{\partial}{\partial t}d(y(t), \mu)$ for points not too far away from P_i (i.e., $|\varphi(s)| \leq \frac{3\pi}{4}$),

$$\begin{aligned}c^{-1}\sqrt{(z_1(i) - z_1)R} &\leq \left|\frac{\partial}{\partial t}d(y(t), \mu)\right| = |\sin\varphi(s)| \cdot \left|\frac{dy}{dt}\right| \\ &\leq c|\varphi(s)| \cdot R \leq c\sqrt{(z_1(i) - z_1)R}.\end{aligned} \tag{8.13}$$

We now choose nondegenerate arcs of intervals: $y_i(\theta)$, $-d_i \leq \theta \leq e_i$, on $\widetilde{P_iP_{-i}}$ or $\widetilde{P_iP_{-i-1}}$. With $y_i(\theta)$ on such curves, we have $\overline{\rho}_i < 1.5R$ and $\overline{\delta}_i^{\perp} < c\delta^{\perp}$. Therefore, we can find a constant $\overline{c}$, such that the terms on the right hand side of $(H_{7;1})'$ and $(H_{7;2})'$ are all strictly bounded by $\overline{c}(|\mu| + R)\delta^{\perp}$. We claim for $0 \leq \mu_2 \leq z_1(i) - 2\overline{c}(|\mu| + R)\delta^{\perp}$, we can form two nondegenerate arcs of intervals consisting of intersections of $\widetilde{P_iP_{-i}}$ (or $\widetilde{P_iP_{-i-1}}$) with $|z_1 - \mu_2| \leq \overline{c}(|\mu| + R)\delta^{\perp}$. The two arcs are depicted by bold curves in Figure 8.3. At the two end points of each arc, $d(y(t), \mu) = \overline{c}(|\mu| + R)\delta^{\perp} >$ r.h.s of $(H_{7;2})'$. Also, on those arcs, $|z_1(i) - z_1| \geq \overline{c}(|\mu| + R)\delta^{\perp}$. If $|\mu|$ and R are sufficiently small, from (8.13),

$$\begin{aligned}\left|\frac{\partial}{\partial t}d(y(t),\mu)\right| &\geq c^{-1}\sqrt{(z_1(i)-z_1)R}\\ &\geq c^{-1}(\bar{c}(|\mu|+R)\delta^{\perp}\cdot R)^{\frac{1}{2}}\\ &> \bar{c}(|\mu|+R)\delta^{\perp}\\ &\geq \text{r.h.s. } of (H_{7;1})'.\end{aligned}$$

From Theorem 2.3, with H_7) replaced by H_7)′; there will be two periodic solutions with periods ω_1 and ω_2 as asserted by the theorem. Assume that $p(-\omega_1+2\tau,\mu)\in\widetilde{P_iP_{-i}}$ and $p(-\omega_2+2\tau,\mu)\in\widetilde{P_iP_{-i-1}}$. From Figure 8.1, it is easy to see that $\frac{d}{dt}d(p(-\omega_1+2\tau,\mu),\mu)<0$. Let $G(\omega,\mu)$ be a brief notation for the bifurcation function $G(\{\omega\}_p,\mu)$. Since $p(-\omega_1+2\tau,\mu)$ is on a nondegenerate arc, condition $(H_{7;1})'$ applies. Also by (8.2),

$$\begin{aligned}\left|\frac{\partial}{\partial\omega}[G(\omega_1,\mu)-d(p(-\omega_1+2\tau,\mu))]\right| &\leq \bar{c}\delta^{\perp}(|\mu|+R)\\ &< -\frac{d}{dt}d(p(-\omega_1+2\tau,\mu),\mu).\end{aligned}$$

Similar estimates also hold for the bifurcation function $G(\omega_2,\mu)$. Therefore,

$$\frac{\partial G(\omega_1,\mu)}{\partial\omega}<0 \quad\text{and}\quad \frac{\partial G(\omega_2,\mu)}{\partial\omega}>0.$$

Also hypothesis H_6) implies $\frac{\partial G}{\partial\mu_2}>0$. See Theorem 2.2, (2.7) and the proof of Theorem 5.2. Therefore $\frac{\partial\omega_1}{\partial\mu_2}>0$ and $\frac{\partial\omega_2}{\partial\mu_2}<0$.

The proof of the hyperbolicity of $x(t)$ is similar to the proof of Theorem 5.1, which depends on Lemma 3.6 and Lemma 5.4. However, some refinement of Lemma 5.4 is needed. The angles between π_τ, $\pi_{\omega+\tau}$ and $\text{span}\{\dot{x}(\tau)\}$ there are essentially controlled by the angle between $\dot{p}(-t+2\tau,\mu)$ and $T\mathscr{C}(\mu)$. That angle is now bounded below by $|\varphi(s)|\geq c\sqrt{(z_1(i)-z_1)/R}$, (see (8.12), which is not bounded away from zero when R and $|\mu|$ approach zero. Therefore, the projections defined by the splitting

$$\pi_\tau\oplus\pi_{\omega+\tau}\oplus\text{span}\{\dot{x}(\tau)\}$$

are bounded by $M\to\infty$ as $R\to 0$ and $|\mu|\to 0$. In fact, since $|\sqrt{(z_1(i)-z_1)/R}\geq c\sqrt{\delta^{\perp}\cdot(R+|\mu|)/R}$, we have $M\leq c\sqrt{R/(\delta^{\perp}(R+|\mu|))}$. However, the rate of contraction can be shown to be $\lambda\leq c\delta^{\frac{1}{\perp}}$ due to our choice of the curve σ in $W^c(0)$. Thus, $4M^2\lambda^2\leq cR\delta^{\perp}/(R+|\mu|)\to 0$ as R and $\mu\to 0$. The condition $4M^2\lambda^2<1$ of Lemma 3.6 is satisfied, and the result of Theorem 5.1 still holds. (Other arguments, including various other subspaces and angles, in the proof of Theorem 5.1 go through analogously.)

Set $z_1(i-)=z_1(i)-2\bar{c}(|\mu|+R)\delta^{\perp}$. We have proved that for $0\leq\mu_2\leq z_1(i-)$, there exist two hyperbolic simple periodic solutions. We now set $z_1(i+)=z_1(i)+\bar{c}(|\mu|+R)\delta^{\perp}$. If $\mu_2>z_1(i+)$, then for any ω with $p(-\omega+2\tau,\mu)\in\widetilde{P_iP_{-i}}\cup\widetilde{P_iP_{-i-1}}$,

$$d(p(-\omega+2\tau,\mu),\mu)>\bar{c}(|\mu|+R)\delta^{\perp}.$$

From Theorem 8.1, (8.1), $G(\omega,\mu)\neq 0$.

We then show $z_1(i-)>z_1((i+1)+)$. We first show that there is a constant $c_o>0$ such that $|z_1(i+1)-z_1(i)|>c_o r(|\mu|+r^2)$. Due to (7.22) and the fact that the time spent from P_i to P_{i+1} is nearly 2π, we have $r_i-r_{i+1}>cr(|\mu|+r^2)$. The desired estimate follows from Lemma 8.2, ii). Since $z_1(i-) = z_1(i) - 2\bar{c}(|\mu| + R)\delta^{\perp}$ and $z_1((i+1)+) = z_1(i+1) + \bar{c}(|\mu| + R_1)(|\mu| + r_1^2)r_1 \le z_1(i+1) + \bar{c}(|\mu| + R)\delta^{\perp}$, where $P_{i+1} = (R_1, \theta_1)$ in polar coordinates and $r_1 = R_1 - R^*(\theta_1, \mu)$, it is clear that if $3\bar{c}(|\mu| + R) < c_o$ then $z_1(i-) > z_1((i+1)+)$. Finally, since $z_1(i-) > z_1((i+1)+)$, we have $z_1(i-) > z_1(+\infty)$. □

One of the consequences of Theorem 8.3 is that for each fixed μ_1, if $z_1(-\infty) \le \mu_2 \le z_1(\infty)$, then there are infinitely many simple or periodic solutions which are orbitally near Γ_o, and if $\mu_2 > z_1(\infty)$ or $\mu_2 < z_1(-\infty)$, there are only finitely many simple periodic solutions. When μ_2 passes the interval $(z_1(i-), z_1(i+))$, a pair of simple periodic solutions with almost equal periods disappear. Also solutions with longer periods disappear first since $z_1((i+1)+) < z_1(i-)$.

The following theorem describes various chaotic and nonchaotic solutions that are orbitally $\bar{\epsilon}$-near the homoclinic orbit Γ_0 and traverse a tubular neighborhood of Γ at least once.

Theorem 8.4. *Assume that the restriction of system (1.1) to a local 2-dimensional center manifold satisfies (7.1) and (7.2). Then there exits positive constant $\hat{\mu}$ such that if $|\mu| \le \hat{\mu}$ then the following properties hold.*
(a) If $\mu_1 \ge 0$, then system (1.1) has a homoclinic solution connecting the equilibrium 0 to itself if and only if $\mu_2 = 0$. Also if $\mu_1 < 0$ then system (1.1) has a homoclinic solution connecting $L(\mu)$ to $L(\mu)$ if and only if $z_1(-\infty) \le \mu_2 \le z_1(+\infty)$. If one of the two equalities is valid, then there is only one such homoclinic solution, otherwise there are exactly two such solutions. There are (infinitely many) heteroclinic solutions connecting $L(\mu)$ to 0 if and only if $\mu_1 < 0$ and $z_1(-\infty) < \mu_2 < z_1(+\infty)$.
(b) System (1.1) has infinitely many periodic and aperiodic solutions.

Proof. To study a homoclinic (heteroclinic) solution from $L(\mu)$ to $L(\mu)$ or 0, we set $h = k$, $\bar{v} = 0$ and $\bar{u} = 0$. The symbols are $S_{h-1} = [0]$ and $S_k = S_h = [\bar{y}, 0]$. From Theorem 8.1, we have

$$G_h(\{S_j\}, \mu) = d(\bar{y}, \mu),$$

since the error term in (8.1) is zero here. If $z_1(-\infty) \le \mu_2 \le z_1(+\infty)$, then there exists at least one $\bar{y} = (\bar{z}_1, \bar{z}_2) \in L(\mu)$ such that $\bar{z}_1 = \mu_2$. The latter implies that $d(\bar{y}, \mu) = 0$. The genuine solution obtained from such $\bar{y}$ asymptotically approaches $x(t) = T(t, \mu, (\bar{y}, 0, 0))$ as $t \to \infty$. See the last part of Theorem 2.1 and its proof. The orbit of $x(t)$ is $L(\mu)$. If now $\mu_1 < 0$ and $z_1(-\infty) < \mu_2 < z_1(+\infty)$, then there exists infinitely many $\bar{y} = (\bar{z}_1, \bar{z}_2)$ inside the loop of $L(\mu)$, with $\bar{z}_1 = \mu_2$. The solution obtained from such $\bar{y}$ asymptotically approaches $x(t) = T(t, \mu, (\bar{y}, 0, 0))$ as $t \to \infty$. However $x(t)$ now approaches 0 as $t \to \infty$.

To construct periodic or aperiodic solutions, we show that for $0 \le \mu_2 \le \hat{\mu}$, we can choose sequences of real constants $\{t_i\}_{-\infty}^{\infty}$ such that $\inf\{t_i\} = \hat{t}$ is sufficiently large and satisfies the condition $z_1^i = \mu_2$, where $y_i = (z_1^i, z_2^i) = p(-t_i + 2\tau, \mu)$. And we can construct nondegenerate arcs so that condition $(H_7; 1)'$ and $(H_7; 2)'$ are satisfied.

An index sequence $\{j(i)\}_{i=-\infty}^{\infty}$ is said to be slowly varying associated to a constant $K > 1$ if for each $i \in \mathbb{N}$,

$$
\begin{aligned}
|P_{j(i\pm 1)}| &\le K|P_{j(i)}|,\\
|P_{j(i\pm 1)}| - R^*(\theta_{j(i\pm 1)}, \mu) &\le K(|P_{j(i)}| - R^*(\theta_{j(i)}, \mu)),
\end{aligned}
\tag{8.14}
$$

where $(|P_j|, \theta_j)$ is the polar coordinates for P_j. Slowly varying index sequences exist, for example, if $|j(i+1) - j(i)| \le 1$, then (8.9) shows that $\{j(i)\}_{i=-\infty}^{\infty}$ is slowly varying. In fact, based on (8.9), for any $y \in \widetilde{P_jP_{-j}} \cup \widetilde{P_jP_{-j-1}}$ where $j = j(i \pm \nu)$, $\nu = 0, 1, 2$, we have, with possibly a larger K,

$$
\begin{aligned}
|y| &\le K|P_{j(i)}|,\\
|y| - R^*(\theta, \mu) &\le K(|P_{j(i)}| - R^*(\theta_{j(i)}, \mu)),
\end{aligned}
\tag{8.15}
$$

where $(|y|, \theta)$ is the polar coordinates for y. Let $\{\mathfrak{a}_i\}_{-\infty}^{\infty}$ be a sequence of arcs with $\mathfrak{a}_i \subset \widetilde{P_{j(i)}P_{-j(i)}}$ or $\widetilde{P_{j(i)}P_{-j(i)-1}}$. each $\mathfrak{a}_i$ is defined by $p(-t_i(\zeta) + 2\tau, \mu)$, $\zeta \in (-d_i, e_i)$, where $t_i(\zeta) = t_i + \zeta$. Due to (8.15), the right hand side of $(\mathrm{H}_7; 1)'$ and $(\mathrm{H}_7; 2)'$ are bounded by $\hat{C}\delta^{\perp}_{j(i)}(\rho_{j(i)} + |\mu|)$. Here $\rho_{j(i)} = |P_{j(i)}|$; $\delta^{\perp}_{j(i)} = cr(r^2 + |\mu|)$ is the rate of contraction evaluated at $P_{j(i)}$; $\hat{C}$ is related to C_1, C_4 and K, and is uniform if μ and $\max\{|P_{j(i)}|\}$ are small.

The rest of the proof is arranged in the reversed order so that the introducing of Δz will not be a surprise. Consider now $0 \le \mu_2 \le \hat{\mu}$, where $\hat{\mu} > 0$ is a small constant which may be further adjusted later. (The case $-\hat{\mu} \le \mu_2 \le 0$ can be treated similarly.) Let $z_1(i)$ be the z_1 coordinate for P_i. We can find i_1 so that for $i \le i_1$, we have

$$0 \le \hat{\mu} < z_1(i) - 2\hat{C}\delta^{\perp}_i(\rho + \hat{\mu}).$$

Such i_1 exists because the right hand side of the above is greater than $z_1(i+1)$, (see the proof of Theorem 8.3). For each $\mu_2 \le \hat{\mu}$, construct a slowly varying index sequence satisfying the condition $j(i) \le i_1$. We can easily verify that there are infinitely many ways to construct such sequences, periodic or aperiodic. Consider now the intersection of $z_1 = \mu_2$ with $\widetilde{P_{j(i)}P_{-j(i)}}$ or $\widetilde{P_{j(i)}P_{-j(i)-1}}$. We now define intervals $(-d_i, e_i)$ and t_i exactly like in the proof of Theorem 8.3, but replacing $\bar{c}\delta^{\perp}$ by $\hat{C}\delta^{\perp}_{j(i)}$. From the proof of Theorem 8.3, we see that if $\hat{\mu}$ and $\sup\{r\} \le \hat{r}$ are small, where $r = |y| - R^*(\theta, \mu)$ is evaluated at each $P_{j(i)}$, then

$$
\begin{aligned}
\Big|\frac{\partial}{\partial\zeta} d(y_i(\zeta), \mu)\Big| &> \hat{C}\delta^{\perp}_{j(i)}(\rho_{j(i)} + \hat{\mu}),\\
\big|d(y_i(\zeta), \mu)\big| &> \hat{C}\delta^{\perp}_{j(i)}(\rho_{j(i)} + \hat{\mu}).
\end{aligned}
\tag{8.16}
$$

Here in the first estimate $-d_i \le \zeta \le e_i$ and in the second estimate $\zeta = -d_i$ or e_i. The smallness of $\hat{r}$ imposes a lower bound on the sequence $j(i)$, say $j(i) \ge i_2$. The question is whether $i_2 < i_1$, that is, whether there is enough room to construct $j(i)$. To this end, consider a constant $\Delta z > 0$ such that $z_1 - z(\infty) \le \Delta z$ implies that $r \le \hat{r}$. Such Δz exists due to Lemma 8.2, ii). Due to (7.22) and the fact that the time spent from P_i to P_{i+1} is nearly 2π, $r_i - r_{i+1} \le cr(r^2 + \hat{\mu})$. Using Lemma 8.2, ii), we have $|z_1(i+1) - z_1(i)| < cr(r^2 + \hat{\mu})$. Therefore if Δz is small, by some elementary argument which will not be rendered here, there are more than two indices i with

$$\frac{\Delta z}{2} < z_1(i+1) < z_1(i) < \Delta z.$$

Slowly varying sequences, periodic or aperiodic, can be constructed from those indices. Let $\hat{\mu} = \frac{\Delta z}{2}$. Then since $z_1(i+1) < z_1(i) - 2\hat{C}\delta^{\perp}_{j(i)}(\rho_{j(i)} + \hat{\mu})$, we have

$$\mu_2 < z_1(i) - 2\hat{C}\delta^{\perp}_{j(i)}(\rho_{j(i)} + \hat{\mu}),$$
$$z_1(i) < z_1(\infty) + \Delta z.$$

The proof of part (b) of this theorem then follows from (8.16) and Theorem 2.3.

9. The Disappearance of Periodic and Aperiodic Solutions when μ_2 Passes Through Turning Points

In section 8 we discussed nondegenerate periodic and aperiodic solutions. but nothing has been said about how solutions disappear when μ_2 passes through each vertical turning points of the orbit $p(-t, \mu)$, $t \leq 0$. From the work of [22], [23] and [31], stable sinks appear (in terms of return map) before the disappearance of the two simple periodic solutions. But no result shows that the two solutions will disappear at the same time, and it is not obvious that no more than two periodic solutions can exist for each $\mu_2 \in (z_1(i-), z_1(i+))$. These problems will be discussed in Theorem 9.1.

A very general result concerning simple periodic solutions is given in Theorem 5.2, i.e., for each fixed μ_1, there is a curve $\mu_2 = \mu_2^*(\omega)$ in the (ω, μ_2)-plane such that $G(\omega, \mu_2) = 0$. See Figure 1.2. Theorem 8.3 shows that away from the turning points, $\mu_2^*(\omega)$ is monotone. Below, we will show that in the (ω, μ_2)-plane there exist isolated turning points $(\omega^c(i), \mu_2^c(i))$ with $z_1(i-) < \mu_2^c(i) < z_1(i+)$ such that if $\mu_2 \to \mu_2^c(i)$ from the left, periods $\omega_1(i, \mu_2)$ and $\omega_2(i, \mu_2)$ of two solutions with $\omega_1(i, \mu_2) < \omega_2(i, \mu_2)$, approach each other along the smooth curve $\mu_2 = \mu_2^*(\omega)$ and disappear at the nondegenerate quadratic turning point. According to Theorem 8.3, $z_1((i+1)+) < z_1(i-)$, therefore

$$z_1(\infty) < \mu_2^c(i+1) < \mu_2^c(i), \qquad i \in \mathbb{Z}.$$

Also $\mu_2^c(i) \to \mu_2^c(\infty) = z_1(\infty)$ as $i \to \infty$ since $z_1(i) \to z_1(\infty)$.

Let $\overline{\omega}_1$ and $\overline{\omega}_2$ be the periods of the two nondegenerate simple periodic solutions corresponding to $\mu_2 = z_1(i-)$. That is, $\overline{\omega}_1 = \omega_1(i, z_1(i-))$ and $\overline{\omega}_2 = \omega_2(i, z_1(i-))$, $p(-\overline{\omega}_1 + 2\tau, \mu) \in \widehat{P_i P_{-i}}$ and $p(-\overline{\omega}_2 + 2\tau, \mu) \in \widehat{P_i P_{-i-1}}$. Let $\varphi(s)$ be the angle between $\dot{p}(-\omega + 2\tau, \mu)$ and $\mathscr{C}(\mu)$. See Figure 8.3. Here $\varphi(0) = 0$ and s is the are length from P_i.

Theorem 9.1. *For $\overline{\omega}_1 < \omega < \overline{\omega}_2$ we have $z_1(i-) < \mu_2^*(\omega) < z_1(i+)$, and*

$$\frac{\partial^2 \mu_2^*(\omega)}{\partial \omega^2} \leq -CR < 0,$$

where $R = |P_i|$. The last estimate is valid if $|\varphi(s)| \leq \frac{\pi}{6}$.

Proof. Recall the definition of $z_1(i-)$ in the proof of Theorem 8.3. For $\mu_2 = z_1(i-) = z_1(i) - 2\overline{c}(|\mu| + R)\delta^{\perp}$, we have $z_1(i) - 3\overline{c}(|\mu| + R)\delta^{\perp} \leq z_1 \leq z_1(i) - \overline{c}(|\mu| + R)\delta^{\perp}$, where $(z_1, z_2) = p(-\overline{\omega}_2 + 2\tau, \mu)$ or $p(-\overline{\omega}_1 + 2\tau, \mu)$. Therefore, for $\overline{\omega}_1 < \omega < \overline{\omega}_2$, we have $z_1(i) - 3\overline{c}(|\mu| + R)\delta^{\perp} \leq z_1 \leq z_1(i)$ where $(z_1, z_2) = p(-\omega + 2\tau, \mu)$. We now use

(8.1) to estimate $|z_1 - \mu_2|$. When $G(\omega, \mu) = 0$, (8.1) implies that $|z_1 - \mu_2| = d(y_i, \mu) \leq \overline{c}(|\mu| + R)\delta^{\perp}$. Thus $z_1(i) - 4\overline{c}(|\mu| + R)\delta^{\perp} \leq \mu_2(\omega) \leq z_1(i) + \overline{c}(|\mu| + R)\delta^{\perp} = z_1(i+)$.

Let $\varphi(s)$ be the angle between $\dot{p}(-\omega + 2\tau, \mu)$ and $\mathscr{C}(\mu)$. From (8.12) we have

$$|\varphi(s)| \leq c\sqrt{\frac{z_1(i) - z_1}{R}}, \qquad \text{if } |\varphi(s)| \leq \frac{3\pi}{4}.$$

Now that $\sqrt{\frac{z_1(i)-z_1}{R}} \leq \sqrt{\frac{3\overline{c}(|\mu|+R)\delta^{\perp}}{r}} \to 0$ if $|\mu|$ and $R \to 0$, we have

$$|\varphi(s)| \leq \frac{\pi}{6}, \quad \text{for} \quad \overline{\omega}_1 < \omega < \overline{\omega}_2, \tag{9.1}$$

if $|\mu|$ and R are small.

Differentiating $G(\omega, \mu_2^*(\omega)) = 0$, we have

$$D_\omega G(\omega, \mu_2^*(\omega)) + D_{\mu_2} G(\omega, \mu_2^*(\omega)) \frac{\partial \mu_2}{\partial \omega} = 0.$$

Based on Theorem 8.1, (8.2), we have $|\widetilde{D_\omega} G - \widetilde{D_\omega} d| \leq \overline{\delta}^{\perp}(|\mu| + \overline{\rho})$. Also $|D_\omega d| \leq c\overline{\rho}$, from (8.13). Since $p(-\omega + 2\tau, \mu) \in \widetilde{P_i P}_{-i} \cup \widetilde{P_i P}_{-i-1}$, from (8.9), we see that $\overline{\overline{\delta}}^{\perp} < c\delta^{\perp} \leq cR$ and $\overline{\rho} < cR$. Therefore $|D_\omega G| \leq CR$. Also from (2.7) and (5.6) we have $|D_{\mu_2} G| \geq c > 0$. Thus

$$\left|\frac{\partial \mu_2}{\partial \omega}\right| \leq CR \tag{9.2}$$

Observe that

$$\frac{\partial^2}{\partial \omega^2} G(\omega, \mu_2^*(\omega)) + \frac{2\partial^2}{\partial \omega \partial \mu_2} G(\omega, \mu_2^*(\omega)) \frac{\partial \mu_2}{\partial \omega} + \frac{\partial^2}{\partial \mu_2^2} G(\omega, \mu_2^*(\omega)) \left(\frac{\partial \mu_2}{\partial \omega}\right)^2$$
$$+ \frac{\partial}{\partial \mu_2} G(\omega, \mu_2^*(\omega)) \frac{\partial^2 \mu_2}{\partial \omega^2} = 0.$$

From (8.3), since $\overline{\rho} < cR$, $\left|\frac{\partial^2 G}{\partial \omega \partial \mu_2}\right| \leq c(|\mu| + R)$. Since G is a C^2 function of μ_2, we have $\left|\frac{\partial^2}{\partial \mu_2^2} G\right| \leq c$. This together with (9.2) implies that

$$\left|\frac{\partial^2 \mu_2}{\partial \omega^2}\right| \geq c \left|\frac{\partial^2}{\partial \omega^2} G(\omega, \mu_2^*(\omega))\right| - c_1 R(|\mu| + R)$$

where $c > 0$. From (8.2)

$$\left|\frac{\partial^2}{\partial \omega^2} G(\omega, \mu_2^*(\omega))\right| \geq \left|\frac{\partial^2}{\partial \omega^2} d(p(-\omega + 2\tau, \mu), \mu)\right| - cR(|\mu| + R).$$

However

$$\left|\frac{\partial^2}{\partial \omega^2} d(p(-\omega + 2\tau, \mu), \mu)\right| = \left|\frac{\partial d}{\partial y} \cdot \frac{\partial^2 p(-\omega + 2\tau, \mu)}{\partial \omega^2}\right|$$
$$= |\ddot{z}_1| \geq |\dot{z}_2| - c(|\mu| R + R^2).$$

Since $|\dot{z}_2| = \left|\frac{dy}{dt}\right| \cdot |\cos(\varphi(s))| \geq cR$, if $|\varphi| < \frac{\pi}{6}$, we have $\left|\frac{\partial^2 G}{\partial \omega^2}\right| \geq cR$ if $|\varphi| \leq \frac{\pi}{6}$. This proves $\left|\frac{\partial^2 \mu_2^*(\omega)}{\partial \omega^2}\right| \geq cR > 0$ if $|\varphi| \leq \frac{\pi}{6}$. From (9.1), this includes the region $\overline{\omega}_1 < \omega < \overline{\omega}_2$.

By Theorem 8.3, $\frac{\partial \mu_2^*(\omega)}{\partial \omega} > 0$ (or < 0) if $\omega = \overline{\omega}_1$ (or $\overline{\omega}_2$). Therefore, there exists $\omega^c(i) \in (\overline{\omega}_1, \overline{\omega}_2)$ such that $\frac{\partial \mu_2^*(\omega)}{\partial \omega} = 0$ when $\omega = \omega^c(i)$. Thus, μ_2 is monotonously increasing for $\omega \in (\overline{\omega}_1, \omega^c(i))$ and decreasing for $\omega \in (\omega^c(i), \overline{\omega}_2)$. This proves $\mu_2 \geq z_1(i-)$ for $\overline{\omega}_1 \leq \overline{\omega} \leq \overline{\omega}_2$. □.

We will present an analogy of Theorem 9.1 on aperiodic or multiple periodic solutions.

Let μ_1 be fixed. Choose a sequence of arcs $\{\mathfrak{a}_i\}_{-\infty}^{\infty}$, with $\mathfrak{a}_i = \widetilde{P_{j(i)}P}_{-j(i)}$ or $\widetilde{P_{j(i)}P}_{-j(i)-1}$, where $j(i) \in \mathbb{Z}^+$ for $i \in \mathbb{Z}$. Assume that $j(0) = \max\{j(i) : i \in \mathbb{Z}\} < \infty$ and $j(i) < j(0)$ for $i \neq 0$. That means that $|P_{j(0)}|$ is minimal among the $|P_{j(i)}|$'s. Also assume that the index sequence $\{j(i)\}_{-\infty}^{\infty}$ is slowly varying, i.e., there exists a $K > 1$ such that (8.14) and (8.15) are satisfied for all $i \in \mathbb{Z}$. Based on Lemma 8.2, (8.14) and (8.15) imply that

$$\begin{aligned} |z_1(i \pm 1)| &\leq K|z_1(i)|, \\ |z_1(i \pm 1) - z_1(\infty)| &\leq K|z_1(i) - z_1(\infty)|, \quad i \in \mathbb{Z}, \end{aligned} \tag{9.3}$$

where $(z_1(i), z_2(i)) = P_{j(i)}$. For the convenience of typing, the notation here is different from that in section 8, where we set $(z_1(i), z_2(i)) = P_i$. Let $r_i = R_i - R^*(\theta_i, \mu)$, where (R_i, θ_i) are the polar coordinates for $P_{j(i)}$; $\delta_i^{\perp} = cr_i(|\mu| + r_i^2)$. If $\{t_i\}_{-\infty}^{\infty}$ is a sequence with $y_i = p(-t_i + 2\tau, \mu) \in \mathfrak{a}_i$, then from (8.1), (8.2), (8.14) and (8.15), there exists a $\overline{c} > 0$ such that

$$|G_i(\{t_j\}, \mu) - d(y_i(\epsilon), \mu)| < \overline{c}\delta_i^{\perp}(|\mu| + R_i), \tag{9.4}$$

$$\left| \frac{\partial^\nu}{\partial \epsilon^\nu} G_i(\{t_j(\epsilon)\}, \mu) - \frac{\partial^\nu}{\partial \epsilon^\nu} d(y_i(\epsilon), \mu) \right| < \overline{c}\delta_i^{\perp}(|\mu| + R_i)\|\{\Delta t_j\}\|^\nu, \tag{9.5}$$

where $1 \leq \nu \leq 3$, $\{\Delta t_i\}$ is a bounded sequence with $\|\{\Delta t_j\}\| \leq 1$. $t_i(\epsilon) = t_i + \epsilon \Delta t_i$ and $y_i(\epsilon) = p(-t_i(\epsilon) + 2\tau, \mu)$.

Let $z_1(0-) = z_1(0) - 2\overline{c}\delta_0^{\perp}(|\mu| + R_0)$ and $z_1(0+) = z_1(0) + \overline{c}\delta_0^{\perp}(|\mu| + R_0)$, where the constant $\overline{c}$ is taken from the right hand side of (9.4) and (9.5). We are able to show that for $0 \leq \mu_2 \leq z_1(0-)$, there exists a unique sequence $\{t_i\}_{-\infty}^{\infty}$ with $p(-t_i + 2\tau, \mu) \in \mathfrak{a}_i$ and $G_i(\{t_j\}, \mu) = 0$, $i \in \mathbb{Z}$. For $\mu_2 \geq z_1(0+)$ there does not exist such sequence. The proof is completely similar to that of Theorem 8.3 and will not be repeated. For the same reason, if $\{\overline{\mathfrak{a}}_i\}_{-\infty}^{\infty}$ is another sequence of arcs with $\overline{\mathfrak{a}}_i = \mathfrak{a}_i$ if $i \neq 0$ and $\overline{\mathfrak{a}}_0 = \widetilde{P_{j(0)}P}_{-j(0)}$ (or $\widetilde{P_{j(0)}P}_{-j(0)-1}$) if $\mathfrak{a}_0 = \widetilde{P_{j(0)}P}_{-j(0)-1}$ (or $\widetilde{P_{j(0)}P}_{-j(0)}$), then there exists a unique sequence $\{\overline{t}_i\}_{-\infty}^{\infty}$ with $p(-\overline{t}_i + 2\tau, \mu) \in \overline{\mathfrak{a}}_i$ and $G_i(\{\overline{t}_j\}, \mu) = 0$, $i \in \mathbb{Z}$.

The sequence $\{t_i(\mu_2)\}_{-\infty}^{\infty}$ and $\{\overline{t}_i(\mu_2)\}_{-\infty}^{\infty}$ obtained above are smooth functions of μ_2. Also $t_o(\mu_2)$ and $\overline{t}_o(\mu_2)$ are monotone functions of μ_2 and approach each other as $\mu_2 \to z_1(0-)$. The two solutions disappear in the interval $\mu_2 \in (z_1(0-), z_1(0+))$. Below we will show that they disappear by coalescing into one solution at a quadratic turning point similar to the situation described by Theorem 9.1.

First we will define a sequence of intervals $(-d_i, e_i)$, $i \in \mathbb{Z}$. Let $(z_1^i(\zeta), z_2^i(\zeta)) = y_i(\zeta) = \Phi(\zeta, \mu, P_{j(i)})$, $i \in \mathbb{Z}$. $-d_o$ and e_o are two values of ζ at which $y_0(\zeta)$, $|\zeta| < \frac{\pi}{2}$ is on the line $z_1 = z_1(0) - 3\overline{c}\delta_0^{\perp}(|\mu| + R_0)$. For $i \neq 0$, $-d_i$ and e_i are two values of ζ at which $y_i(\zeta)$ is on $\mathfrak{a}_i \cap \{z_1 = z_1(0) - 5\overline{c}\delta_i^{\perp}(|\mu| + R_i)\}$ or $\mathfrak{a}_i \cap \{z_1 = z_1(0) + 2\overline{c}\delta_i^{\perp}(|\mu| + R_i)\}$. We assume $-d_i < e_i$, though d_i and e_i, $i \neq 0$ are not always positive. Here we need to justify the existence of the last intersection. Since the distance of $z_1(i)$ to the z_1

coordinates of two adjacent turning points are bounded below by $c_1\delta_i^{\perp}$, see the end of the proof of Theorem 8.3,

$$z_1(i) - z_1(0) > c_1\delta_i^{\perp}, \quad i \neq 0. \tag{9.6a}$$

Thus, $z_1(i) > z_1(0) + 2\bar{c}\delta_i^{\perp}(|\mu| + R_i)$ if $i \neq 0$ and the intersection does exist. In fact, the main idea of Theorem 9.2 is that the arcs $\mathfrak{a}_i$, $i \neq 0$ are well transversal to the line $\{z_1 = \mu_2 : \mu_2 \in [\mu_m, \mu_M]\}$. This can be seen from the following. Based on (9.6a), $z_1(i) - z_1^i(\zeta) \geq z_1(i) - z_1(0) - 2\bar{c}\delta_i^{\perp}(|\mu| + R_i) \geq c_1\delta_i^{\perp} - 2\bar{c}\delta_i^{\perp}(|\mu| + R_i) \geq c\delta_i^{\perp}$ for $i \neq 0$ and $\zeta \in (-d_i, e_i)$. Replacing $z_1(i) - z_1$ in (8.13) by $c\delta_i^{\perp}$, we have

$$\left|\frac{\partial}{\partial\zeta} d(y_i(\zeta), \mu)\right| \geq c\sqrt{\delta_i^{\perp} R_i}, \text{ for } \zeta \in (-d_i, e_i),\ i \neq 0. \tag{9.6}$$

Theorem 9.2. *There exists $\hat{\mu} > 0$ such that the following holds. If $|\mu| < \hat{\mu}$ and $|z_1(0)| < \frac{\hat{\mu}}{2}$, then there exist C^2 functions $\zeta_i = \zeta_i(\zeta_o)$, $i \neq 0$ and $\mu_2 = \mu_2(\zeta_o)$, $\zeta_o \in (-d_o, e_o)$, such that*

$$G_i(\{t_j + \zeta_j\}, \mu_2) = 0$$

for all $i \in \mathbb{Z}$. Here t_i is the time that $p(-t_i + 2\tau, \mu) = P_{j(i)}$ for all $i \in \mathbb{Z}$. The solution we have found is unique if $\zeta_i \in (-d_i, e_i)$, $i \in \mathbb{Z}$ is satisfied. Moreover, $0 \leq \mu_2(-d_o) < z_1(0-)$, $0 \leq \mu_2(e_o) < z_1(0-)$ and

$$\frac{\partial^2 \mu_2(\zeta_o)}{\partial \zeta_o^2} \leq -cR < 0, \quad \textit{for all} \quad \zeta_o \in (-d_o, e_o). \tag{9.7}$$

Proof. Consider

$$G_i(\{t_j + \zeta_j\}, \mu) = 0,\ i \in \mathbb{Z},$$

where $\zeta_j \in (-d_j, e_j)$, $j \in \mathbb{Z}$. Define $\mu_m = z_1(0) - 4\bar{c}\delta_0^{\perp}(\hat{\mu} + R_0)$ and define $\mu_M = z_1(0) + \bar{c}\delta_0^{\perp}(\hat{\mu} + R_0)$ where $\hat{\mu} > 0$ and $\mu_M < \hat{\mu}$. The latter is valid since $z_1(0) < \hat{\mu}/2$ and $\delta_0^{\perp}$ are small. We now construct a mapping

$$\mathscr{C} : (\{\zeta_i\}, \mu_2) \to (\{\overline{\zeta}_i\}, \overline{\mu}_2).$$

For $\zeta_i \in [-d_i, e_i]$, $i \in \mathbb{Z}$ and $\mu_2 \in [\mu_m, \mu_M]$, we first solve $\overline{\mu}_2$ from

$$G_o(\{t_j + \zeta_j\}, \overline{\mu}_2) = 0. \tag{9.8}$$

We can show that when $\overline{\mu}_2$ moves from μ_m to μ_M, G_o changes sign. In fact, for $\overline{\mu}_2 = \mu_m$,

$$\begin{aligned} z_1^0(\zeta_0) &\geq z_1(0) - 3\bar{c}\delta_0^{\perp}(\hat{\mu} + R_0), \\ \mu_m &= z_1(0) - 4\bar{c}\delta_0^{\perp}(\hat{\mu} + R_0). \end{aligned}$$

Thus,

$$\begin{aligned} d(y_0(\zeta_0), \mu_m) &= \mu_m - z_1^0(\zeta) \\ &\leq -\bar{c}\delta_0^{\perp}(\hat{\mu} + R_0). \end{aligned}$$

However, form (9.4),

$$|G_0(\{t_j + \zeta_j\}, \mu_m) - d(y_0(\zeta_0), \mu_m)| < \bar{c}\delta_0^{\perp}(\hat{\mu} + R_0).$$

Therefore, $G_0(\{t_j + \zeta_j\}, \mu_m) < 0$. Similarly, from

$$\begin{aligned} z_1^0(\zeta_0) &\le z_1(0), \\ \mu_M &= z_1(0) + \bar{c}\delta_0^{\perp}(\hat{\mu} + R_0), \end{aligned}$$

we can show $G_0(\{t_j + \zeta_j\}, \mu_M) > 0$. Thus, there exists at least one $\overline{\mu}_2 \in (\mu_m, \mu_M)$ that solves (9.8). To prove the uniqueness of such $\overline{\mu}_2$, we show that $\frac{\partial G_o}{\partial \mu_2} > 0$. In fact, from (2.7),

$$\begin{aligned} \left|\frac{\partial}{\partial \mu_2}(G_0 - d)\right| &\le C(\hat{\mu} + R_0). \\ \frac{\partial}{\partial \mu_2} d(y_0(\zeta_0), \mu) &= 1. \end{aligned}$$

Thus,

$$\frac{\partial G_0}{\partial \mu_2} = 1 + O(\hat{\mu} + R_0) > 0, \tag{9.9}$$

if $\hat{\mu} + R_0$ is small. The unique solution $\overline{\mu}_2$ is a C^2 function of $\{\zeta_i\}_{-\infty}^{\infty}$.

We then solve $\overline{\zeta}_i$, $i \ne 0$ from

$$G_i(\{t_j + \tilde{\zeta}_j\}, \overline{\mu}_2) = 0, \ i \ne 0. \tag{9.10}$$

Here $\tilde{\zeta}_j = \zeta_j$ for $j \ne i$ and $\tilde{\zeta}_j = \overline{\zeta}_j$ for $j = i$. The solution of (9.10) exists since when moving $\overline{\zeta}_i$ from $-d_i$ to e_i, G_i changes sign. In fact, when $\overline{\zeta}_i = -d_i$,

$$\begin{aligned} z_1^i(-d_i) &= z_1(0) - 5\bar{c}\delta_i^{\perp}(\hat{\mu} + R_i), \\ \overline{\mu}_2 &\ge \mu_m = z_1(0) - 4\bar{c}\delta_i^{\perp}(\hat{\mu} + R_i), \\ d(y_i(-d_i), \mu) &= \mu_2 - z_1^i(-d_i) \ge \bar{c}\delta_i^{\perp}(\hat{\mu} + R_i), \\ G_i &> d(y_i(-d_i), \mu) - |G_i - d| > 0. \end{aligned}$$

The last estimate uses (9.4). Similarly, when $\overline{\zeta}_i = e_i$, using

$$\begin{aligned} z_1^i(\zeta_i) &= z_1(0) + 2\bar{c}\delta_i^{\perp}(\hat{\mu} + R_i), \\ \overline{\mu}_2 \le \mu_M &= z_1(0) + \bar{c}\delta_i^{\perp}(\hat{\mu} + R_i), \end{aligned}$$

we can show that $G_i < 0$. Therefore the solution for (9.10) exists. We now show $\frac{\partial G_i}{\partial \zeta_i} \ne 0$ so that the solution is unique. From (9.6) and (9.5), for $\overline{\zeta}_i \in (-d_i, e_i)$,

$$\begin{aligned} \left|\frac{\partial}{\partial \zeta} d(y_i(\overline{\zeta}_i), \mu)\right| &\ge c\sqrt{\delta_i^{\perp} R_i}, \\ \left|\frac{\partial}{\partial \zeta}(G_i - d)\right| &\le \bar{c}\delta_i^{\perp}(\hat{\mu} + R_i). \end{aligned}$$

Therefore, if $c\sqrt{R_i} > \bar{c}\sqrt{\delta_i^{\perp}}(\hat{\mu} + R_i)$, then $c\sqrt{\delta_i^{\perp} R_i} > \bar{c}\delta_i^{\perp}(\hat{\mu} + R_i)$. This implies that $\frac{\partial G_i}{\partial \zeta_i} \ne 0$. Let $\overline{\zeta}_o = \zeta_o$. The solution $\{\overline{\zeta}_i\}_{-\infty}^{\infty}$ is a C^2 function of $\{\zeta_i\}_{-\infty}^{\infty}$. It is also clear that $\mathscr{C} : \prod_i[-d_i, e_i] \times [\mu_m, \mu_M] \to \prod_i[-d_i, e_i] \times [\mu_m, \mu_M]$ is a C^2 mapping.

We now show $\mathscr{E}$ is a uniform contraction for each fixed $\zeta_o \in (-d_o, e_o)$. Let $\mathscr{E} : \{\zeta_i + \Delta\zeta_i\} \times (\mu_2 + \Delta\mu_2) \to \{\overline{\zeta}_i + \Delta\overline{\zeta}_i\} \times (\overline{\mu}_2 + \Delta\overline{\mu}_2)$, where $\overline{\zeta}_0 = \zeta_0$, $\Delta\zeta_0 = \Delta\overline{\zeta}_0 = 0$. That is,

$$G_o(\{t_j + \zeta_j + \Delta\zeta_j\},\ \overline{\mu}_2 + \Delta\overline{\mu}_2) = 0, \tag{9.11}$$

$$G_i(\{t_j + \tilde{\zeta}_j + \Delta\tilde{\zeta}_j\},\ \overline{\mu}_2 + \Delta\overline{\mu}_2) = 0,\ i \neq 0. \tag{9.12}$$

See the definition for $\tilde{\zeta}_j$ after (9.10). For $0 \leq \epsilon \leq 1$, define $t_j(\epsilon) = t_j + \zeta_j + \epsilon\Delta\zeta_j$ and $\mu_2(\epsilon) = \overline{\mu}_2 + \epsilon\Delta\overline{\mu}_2$. Subtracting (9.8) from (9.11), we find ϵ_0 such that

$$\begin{aligned} \frac{d}{d\epsilon} G_0(\{t_j(\epsilon_0)\}, \mu_2(\epsilon_0)) &= 0. \\ \frac{\partial G_0}{\partial \mu_2}\Delta\overline{\mu}_2 + \sum_{j\neq 0} \frac{\partial G_0}{\partial \zeta_j}\Delta\zeta_j &= 0. \end{aligned}$$

The first term of the above is bounded below by

$$(1 + O(\hat{\mu} + R_0))|\Delta\overline{\mu}_2|.$$

See (9.9). For some fixed μ_2, the second term can be written as

$$\begin{aligned} &\frac{\partial}{\partial\epsilon} G_0(\{t_j(\epsilon_0)\}, \mu_2) \\ =&\frac{\partial}{\partial\epsilon}(G_0(\{t_j(\epsilon_0)\}, \mu_2) - d(y_0(\zeta_0), \mu)) \\ \leq& c\delta_0^{\perp}(\hat{\mu} + R_0)\|\{\Delta\zeta_j\}\|. \end{aligned}$$

Here we have used (9.5) and the fact that $y_0(\zeta_0)$ is independent of ϵ. Therefore,

$$|\Delta\overline{\mu}_2| \leq C\delta_0^{\perp}(\hat{\mu} + R_0)\|\{\Delta\ \zeta_j\}\|.$$

Now define $t_j(\epsilon) = t_j + \tilde{\zeta}_j + \epsilon\Delta\tilde{\zeta}_j$ and $\mu_2(\epsilon) = \overline{\mu}_2 + \epsilon\Delta\overline{\mu}_2$. Subtracting (9.10) from (9.12), we find ϵ_0 such that for $i \neq 0$,

$$\begin{aligned} &\frac{d}{d\epsilon} G_i(\{t_j(\epsilon_0)\}, \mu_2(\epsilon_0)) = 0. \\ &\frac{\partial G_i}{\partial \mu_2}\Delta\overline{\mu}_2 + \sum_{j\neq 0,\, j\neq i} \frac{\partial G_i}{\partial \zeta_j}\Delta\zeta_j + \frac{\partial G_i}{\partial \zeta_i}\Delta\overline{\zeta}_i = 0. \end{aligned}$$

Since $\frac{\partial G_i}{\partial \mu_2}$ is uniformly bounded, the first term is bounded by

$$C|\Delta\overline{\mu}_2| \leq C\delta_0^{\perp}(\hat{\mu} + R_0)\|\{\Delta\zeta_j\}\|.$$

For some fixed μ_2 and for $\Delta\zeta_i = 0$, the second term can be written as

$$\begin{aligned} &\frac{\partial}{\partial\epsilon} G_i(\{t_j(\epsilon_0)\}, \mu_2) \\ =&\frac{\partial}{\partial\epsilon}(G_i(\{t_j(\epsilon_0)\}, \mu_2) - d(y_i(\zeta_i), \mu)) \\ \leq& c\delta_i^{\perp}(\hat{\mu} + R_i)\|\{\Delta\zeta_j\}\|. \end{aligned}$$

Here we have used (9.5) and the fact that $y_i(\zeta_i)$ is independent of ϵ. Setting now $\Delta\zeta_j = 0,\ j \neq i$, the third term is bounded below by

$$
\begin{aligned}
&|\frac{\partial}{\partial\epsilon} d(y_i(\overline{\zeta}_i + \epsilon\Delta\overline{\zeta}_i), \mu)| - |\frac{\partial}{\partial\epsilon}(G_i - d)| \\
\geq & C\sqrt{\delta_i^{\perp} R_i}|\Delta\overline{\zeta}_i| - c\delta_i^{\perp}(\hat{\mu} + R_i)|\Delta\overline{\zeta}_i|.
\end{aligned}
$$

Since $\sqrt{\delta_i^{\perp} R_i} \gg \delta_i^{\perp}(\hat{\mu} + R_i)$, and $\delta_0^{\perp} < \delta_i^{\perp}$, $R_0 < R_i$, we have

$$
\begin{aligned}
|\Delta\overline{\zeta}_i| \leq & C\delta_i^{\perp}(\hat{\mu} + R_i)\|\{\Delta\zeta_j\}\| / \sqrt{\delta_i^{\perp} R_i} \\
= & C(\hat{\mu} + R_i)\sqrt{\frac{\delta_i^{\perp}}{R_i}}\, \|\{\Delta\zeta_j\}\|, \quad i \neq 0.
\end{aligned}
$$

Therefore $\mathscr{C}$ is a uniform contraction. Let the unique fixed point of $\mathscr{C}$ be denoted by $(\{\zeta_i(\zeta_o)\}_{-\infty}^{\infty}, \mu_2(\zeta_o))$, where all the functions are C^2 in ζ_o.

It is not difficult to verify that if $\zeta_o = -d_o$ and e_o, we have $\mu_2 \leq z_1(0-)$, and the solutions there are nondegenerate. Hence we have $\frac{\partial\mu_2}{\partial\zeta_o} > 0$(or < 0) if $\zeta_o = -d_o$(or e_o).

We now show that $\frac{\partial^2\mu_2}{\partial\zeta_o^2} \leq -cr < 0$ for $\zeta_o \in (-d_o, e_o)$. Differentiating $G_i(\{t_j + \zeta_j(s)\}, \mu_2(s)) = 0$ where $s = \zeta_o \in (-d_o, e_o)$, we have

$$
\sum_j \frac{\partial G_i}{\partial t_j}\frac{\partial\zeta_j(s)}{\partial s} + \frac{\partial G_i}{\partial\mu_2}\frac{\partial\mu_2}{\partial s} = 0. \tag{9.13}
$$

Let $i = 0$ first. Since $\frac{\partial G_0}{\partial\mu_2} = 1 + O(\hat{\mu} + R_0)$, from (9.13), we have

$$
\begin{aligned}
\left|\frac{\partial\mu_2}{\partial s}\right| & \leq C\left|\frac{\partial G_0}{\partial\mu_2}\cdot\frac{\partial\mu_2}{\partial s}\right| \leq C\left|\sum_j \frac{\partial G_i}{\partial t_j}\frac{\partial\zeta_j(s)}{\partial s}\right| \\
& \leq C\left|\frac{\partial}{\partial s} d(y_o(s), \mu)\right| + C\delta_0^{\perp}(\hat{\mu} + R_0)\|\left\{\frac{\partial\zeta_j}{\partial s}\right\}\| \\
& \leq C\left(\sqrt{\delta_0^{\perp}(\hat{\mu} + R_0)R_0} + \delta_0^{\perp}(\hat{\mu} + R_0)\|\left\{\frac{\partial\zeta_j}{\partial s}\right\}\|\right).
\end{aligned}
\tag{9.14}
$$

Here (9.5), with $\Delta t_j = \frac{\partial\zeta_j}{\partial s}$, is used to obtain the second to the last estimate and (8.13) is used to get the last estimate. Let $i \neq 0$ next.

$$
\begin{aligned}
\left|\frac{\partial}{\partial\zeta_i} d(y^i(\zeta_i), \mu)\right|\left|\frac{\partial\zeta_i}{\partial s}\right| & = \left|\frac{\partial}{\partial\zeta_i} d(y^i(\zeta_i), \mu)\cdot\frac{\partial\zeta_i}{\partial s}\right| \\
& \leq |\Sigma_j \frac{\partial G_i}{\partial t_j}\frac{\partial\zeta_j(s)}{\partial s}| + |error| \\
& \leq |\frac{\partial G_i}{\partial\mu_2}\frac{\partial\mu_2}{\partial s}| + |error| \\
& \leq C|\frac{\partial\mu_2}{\partial s}| + |error|,
\end{aligned}
\tag{9.15}
$$

where $|error| = C\delta_i^{\perp}(\hat{\mu} + R_i)\|\left\{\frac{\partial\zeta_j}{\partial s}\right\}\|$. Here (9.5) is used to obtain the third to the last estimate, with $\Delta t_j = \frac{\partial\zeta_j}{\partial s}$, (9.13) is used to obtain the second to the last estimate, and $\frac{\partial G_i}{\partial \mu_2} = 1 + O(\hat{\mu} + R_i)$ is used to get the last estimate. Substituting (9.14) in (9.15), observing that $\delta_0^{\perp} < \delta_i^{\perp}$, $R_0 < R_i$ and $\sqrt{\delta_i^{\perp} R_i} \gg \delta_i^{\perp}(\hat{\mu} + R_i)$, and using (9.6), we have

$$\| \left\{ \frac{\partial \zeta_i}{\partial s} \right\} \| \leq C, \quad i \in \mathbb{Z}.$$

Substituting into (9.14) and (9.15) again, we have

$$| \frac{\partial \zeta_i}{\partial s} | \leq C\sqrt{\hat{\mu} + R_i}, \quad i \neq 0. \tag{9.16}$$

$$\left| \frac{\partial \mu_2}{\partial s} \right| \leq C\sqrt{\delta_0^{\perp}(\hat{\mu} + R_0)R_0} \tag{9.17}$$

When deriving (9.17), we have used $\sqrt{\delta_0^{\perp}(\hat{\mu} + R_0)R_0} \gg \delta_0^{\perp}(\hat{\mu} + R_0)$. When deriving (9.16), we have used (9.6).

Differentiating (9.13), we have

$$\begin{aligned} &\sum_{i,j} \frac{\partial^2 G_\nu}{\partial t_i \partial t_j} \cdot \frac{\partial \zeta_i(s)}{\partial s} \cdot \frac{\partial \zeta_j(s)}{\partial s} + \sum_i \frac{\partial G_\nu}{\partial t_i} \cdot \frac{\partial^2 \zeta_i}{\partial s^2} \\ &+2 \sum_i \frac{\partial^2 G_\nu}{\partial \mu_2 \partial t_i} \cdot \frac{\partial \mu_2}{\partial s} \cdot \frac{\partial \zeta_i}{\partial s} + \frac{\partial^2 G_\nu}{\partial \mu_2^2} \left(\frac{\partial \mu_2}{\partial s} \right)^2 + \frac{\partial G_\nu}{\partial \mu_2} \cdot \frac{\partial^2 \mu_2}{\partial s^2} = 0, \quad \nu \in \mathbb{Z}. \end{aligned} \tag{9.18}$$

We first obtain an estimate on $\frac{\partial^2 \mu_2}{\partial s^2}$. Let $\nu = 0$ first. Since $\frac{\partial G_0}{\partial \mu_2} = 1 + O(\hat{\mu} + R_0)$,

$$\begin{aligned} \left| \frac{\partial^2 \mu_2}{\partial s^2} \right| &\leq C \left| \frac{\partial G_0}{\partial \mu_2} \frac{\partial^2 \mu_2}{\partial s^2} \right| \\ &\leq C |\text{The sum of the first 4 terms in (9.18)}|. \end{aligned}$$

Since G_0 is a C^2 function of μ_2, and $\frac{\partial \mu_2}{\partial s}$ has an estimate in (9.17),

$$\frac{\partial^2 G_0}{\partial \mu_2^2} \left(\frac{\partial \mu_2}{\partial s} \right)^2 \leq C\delta_0^{\perp} R_0(\hat{\mu} + R_0). \tag{9.19}$$

Let $t_i(\epsilon) = t_i + \zeta_i(s) + \epsilon \Delta_i$ where $\Delta_i = \frac{\partial \zeta_i(s)}{\partial s}$.

$$\begin{aligned} | \sum_i \frac{\partial^2 G_0}{\partial t_i \partial \mu_2} \Delta_i | &= | \frac{\partial^2 G_0(\{t_j(\epsilon)\}, \mu)}{\partial \mu_2 \partial \epsilon} | \\ &\leq C R_0 \| \{\Delta_j\} \|, \end{aligned}$$

due to (8.3). From (9.16), $\|\{\Delta_j\}\| \leq C$. Combining these with (9.17), we have

$$\begin{aligned} | \sum_i \frac{\partial^2 G_0}{\partial \mu_2 \partial t_i} \cdot \frac{\partial \mu_2}{\partial s} \cdot \frac{\partial \zeta_i}{\partial s} | &\leq C R_0 | \frac{\partial \mu_2}{\partial s} | \\ &\leq C R_0. \end{aligned} \tag{9.20}$$

Also

$$\sum_{i,j}\frac{\partial^2 G_0}{\partial t_i \partial t_j}\cdot\Delta_i\Delta_j = \frac{\partial^2 G_0(\{t_j(\epsilon)\},\mu)}{\partial\epsilon^2}$$
$$= \frac{\partial^2 d(y_0(\epsilon),\mu)}{\partial\epsilon^2} + O(\delta_0^{\perp}(\hat\mu + R_0)\|\{\Delta_j\}\|^2),$$

by (9.5). However,

$$\frac{\partial^2 d(y_0(t_0(\epsilon)),\mu)}{\partial\epsilon^2} = \frac{\partial^2}{\partial\epsilon^2}[\mu_2 - z_1^0(t_0(\epsilon))]$$
$$= \frac{\partial^2}{\partial t^2} z_1^0(t_0(\epsilon))|\Delta_0|^2.$$

Recall that $\Delta_0 = 1$. From the differential equation for $z_1(t)$, see the proof of Theorem 9.1, we have $|\frac{\partial^2}{\partial t^2} z_1^0(t_0(\epsilon))| \le CR_0$. Thus,

$$|\sum_{i,j}\frac{\partial^2 G_0}{\partial t_i \partial t_j}\cdot\frac{\partial\zeta_i(s)}{\partial s}\cdot\frac{\partial\zeta_j(s)}{\partial s}| \le CR_0 + C\delta_0^{\perp}(\hat\mu + R_0)\|\{\frac{\partial\zeta_j}{\partial s}\}\|^2 \tag{9.21}$$

Now let $t_i(\epsilon) = t_i + \zeta_i(s) + \epsilon\Delta_i$ where $\Delta_i = \frac{\partial^2\zeta_i}{\partial s^2}$. (Note $\Delta_0 = 0$.)

$$\sum_i \frac{\partial G_0}{\partial t_i}\cdot\frac{\partial^2\zeta_i}{\partial s^2} = \frac{\partial G_0(\{t_j(\epsilon)\},\mu)}{\partial\epsilon}$$
$$= \frac{\partial}{\partial\epsilon} d(y_0(t_0(\epsilon)),\mu) + O(\delta_0^{\perp}(\hat\mu + R_0)\|\{\Delta_j\}\|) \tag{9.22}$$
$$= O(\delta_0^{\perp}(\hat\mu + R_0)\|\{\frac{\partial^2\zeta_j}{\partial s^2}\}\|)$$

Here we use (9.5) to get the second to the last estimate, and the fact $\Delta_0 = 0$ to get the last estimate. Combining (9.19)–(9.22) we have proved

$$|\frac{\partial^2\mu_2}{\partial s^2}| \le C(R_0 + \delta_0^{\perp}(\hat\mu + R_0)\left\|\left\{\frac{\partial^2\zeta_j}{\partial s^2}\right\}\right\|). \tag{9.23}$$

We now set $\nu \ne 0$ in (9.18). From (9.23),

$$|\frac{\partial G_\nu}{\partial\mu_2}\cdot\frac{\partial^2\mu_2}{\partial s^2}| \le C(R_0 + \delta_0^{\perp}(\hat\mu + R_0)\left\|\left\{\frac{\partial^2\zeta_j}{\partial s^2}\right\}\right\|). \tag{9.24}$$

Similar to (9.19), (9.20) and (9.22), recalling $\delta_\nu^{\perp} > \delta_0^{\perp}$ and $R_\nu > R_0$, we can prove

$$|\frac{\partial^2 G_\nu}{\partial\mu_2^2}\left(\frac{\partial\mu_2}{\partial s}\right)^2| \le C\delta_\nu^{\perp}(\hat\mu + R_\nu)R_\nu. \tag{9.25}$$

$$|\sum_i \frac{\partial^2 G_\nu}{\partial\mu_2\partial t_i}\cdot\frac{\partial\mu_2}{\partial s}\cdot\frac{\partial\zeta_i}{\partial s}| \le CR_\nu. \tag{9.26}$$

$$|\sum_i \frac{\partial G_\nu}{\partial t_i} \cdot \frac{\partial^2 \zeta_i}{\partial s^2}| \geq |\frac{\partial}{\partial \epsilon} d(y_\nu(t_\nu(\epsilon))| - c\delta_\nu^\perp(\hat{\mu} + R_\nu)\|\{\frac{\partial^2 \zeta_j}{\partial s^2}\}\|. \tag{9.27}$$

Similar to (9.21), we can prove

$$\sum_{i,j} \frac{\partial^2 G_\nu}{\partial t_i \partial t_j} \cdot \frac{\partial \zeta_i(s)}{\partial s} \cdot \frac{\partial \zeta_j(s)}{\partial s} = \frac{\partial^2}{\partial \epsilon^2} d(y_\nu(t_\nu(\epsilon)) + O(\delta_\nu^\perp(\hat{\mu} + R_\nu)\|\{\frac{\partial \zeta_j}{\partial s}\}\|^2).$$

Similar to the case $\nu = 0$, we can prove that

$$|\frac{\partial^2}{\partial \epsilon^2} d(y_\nu(t_\nu(\epsilon))| \leq CR_\nu |\frac{\partial \zeta_\nu}{\partial s}|^2.$$

Using $\|\{\frac{\partial \zeta_j}{\partial s}\}\| \leq C$, we have

$$|\sum_{i,j} \frac{\partial^2 G_\nu}{\partial t_i \partial t_j} \cdot \frac{\partial \zeta_i(s)}{\partial s} \cdot \frac{\partial \zeta_j(s)}{\partial s}| \leq CR_\nu. \tag{9.28}$$

Combining (9.24)–(9.28), we have

$$\left|\frac{\partial}{\partial \zeta} d(y_\nu(\zeta), \mu)\right| \cdot \left|\frac{\partial^2 \zeta_\nu}{\partial s^2}\right| \leq C\left(R_\nu + \delta_\nu^\perp(\hat{\mu} + R_\nu)\left\|\left\{\frac{\partial^2 \zeta_j}{\partial s^2}\right\}\right\|\right)$$

Since $\sqrt{\delta_\nu^\perp R_\nu} \gg \delta_\nu^\perp(\hat{\nu} + R_\nu)$, using (9.6), we have

$$\left\|\left\{\frac{\partial^2 \zeta_i}{\partial s^2}\right\}_{i \neq 0}\right\| \leq c \sup_\nu \{\sqrt{\frac{R_\nu}{\delta_\nu^\perp}}\}.$$

We now prove a useful proposition.

Proposition. *There exists a uniform constant $C > 0$ such that*

$$\frac{R_i}{\delta_i^\perp} \leq C\frac{R_0}{\delta_0^\perp}.$$

Proof.. Let (R_i, θ_i) be the polar coordinates for $P_{j(i)}$. Let $R = R^*(\theta, \mu_1)$ be the equation for the limit cycle when $\mu_1 < 0$, and $R^* = 0$ when $\mu_1 \geq 0$. Recall that $\delta_i^\perp = cr_i(r_i^2 + |\mu|)$. It is easy to verify that

$$\frac{R^*(\theta_i, \mu_1) + r_0}{R^*(\theta_0, \mu_1) + r_0} = 1 + O(|\mu_1|),$$

based on the fact that when $\mu_1 < 0$, $R^*(\theta, \mu_1) = \sqrt{\frac{-\mu_1}{a}} + O(|\mu_1|^{1.5})$. See Lemma 7.1. Therefore,

$$\begin{aligned}\frac{R_i}{\delta_i^\perp} &= \frac{R^*(\theta_i, \mu_1) + r_i}{cr_i(r_i^2 + |\mu|)} \\ &\le \frac{R^*(\theta_i, \mu_1) + r_0}{cr_0(r_0^2 + |\mu|)} \\ &\le C\frac{R^*(\theta_0, \mu_1) + r_0}{cr_0(r_0^2 + |\mu|)} \\ &= C\frac{R_0}{\delta_0^\perp}.\end{aligned}$$

This proves the proposition.

Proof of Theorem 9.2 continued. We have

$$\left\| \left\{ \frac{\partial^2 \zeta_i}{\partial s^2} \right\}_{i \ne 0} \right\| \le C\sqrt{\frac{R_0}{\delta_0^\perp}}. \tag{9.29}$$

Substituting into (9.18) with $\nu = 0$ again, we have

$$\frac{\partial G_0}{\partial \mu_2} \cdot \frac{\partial^2 \mu_2}{\partial s^2} = -(\text{The sum of the other 4 terms in (9.18)}).$$

From the proof of (9.21),

$$\sum_{i,j} \frac{\partial^2 G_0}{\partial t_i \partial t_j} \cdot \Delta_i \Delta_j = \frac{\partial^2 d(y_0(\epsilon), \mu)}{\partial \epsilon^2} + O(\delta_0^\perp(\hat{\mu} + R_0)).$$

However, from the proof of Theorem 9.1, $\frac{\partial^2 d(y_0(\epsilon),\mu)}{\partial \epsilon^2} \ge CR_0 > 0$. Therefore,

$$\sum_{i,j} \frac{\partial^2 G_0}{\partial t_i \partial t_j} \cdot \Delta_i \Delta_j \ge CR_0 + O(\delta_0^\perp(\hat{\mu} + R_0)). \tag{9.30}$$

From (9.22) and (9.29),

$$\begin{aligned}\sum_i \frac{\partial G_0}{\partial t_i} \cdot \frac{\partial^2 \zeta_i}{\partial s^2} &= O(\delta_0^\perp(\hat{\mu} + R_0)\sqrt{\frac{R_0}{\delta_0^\perp}}) \\ &= O(\sqrt{\delta_0^\perp R_0}(\hat{\mu} + R_0))\end{aligned} \tag{9.31}$$

From (9.20) and (9.17),

$$\left| \sum_i \frac{\partial^2 G_0}{\partial \mu_2 \partial t_i} \cdot \frac{\partial \mu_2}{\partial s} \cdot \frac{\partial \zeta_i}{\partial s} \right| \le CR_0\sqrt{\delta_0^\perp R_0(\hat{\mu} + R_0)}. \tag{9.32}$$

Combining (9.30)–(9.32), and (9.19), we have

$$\begin{aligned}\frac{\partial G_0}{\partial \mu_2} \cdot \frac{\partial^2 \mu_2}{\partial s^2} &\le -CR_0 + O(\sqrt{\delta_0^\perp R_0}(\hat{\mu} + R_0)) \\ &\le -cR_0,\end{aligned}$$

with a smaller $c > 0$. The desired result (9.7) follows from $\frac{\partial G_o}{\partial \mu_2} \simeq 1$. □

Similar results also hold for multiple periodic solutions $x(t)$, $\mathcal{T}x(\cdot) = \{t_1, \cdots, t_\nu\}_p$, with $t_1 < \min\{t_2, \cdots, t_\nu\}$ and $p(-t_1 + 2\tau, \mu)$ is near some $P_{i(1)}$. The proof follows from that of Theorem 9.2 and will not be given here.

Let us summarize what has been done so far. For a nondegenerate solution, we have $\frac{\partial G_i(\{t_i\}, \mu_2)}{\partial t_i} \neq 0$ for all $i \in \mathbb{Z}$ and we can solve $t_i = t_i(\mu_2)$ from $G_i(\{t_i\}, \mu_2) = 0$, $i \in \mathbb{Z}$. However, if the system $G_i = 0$, $i \in \mathbb{Z}$ is degenerate and $i = i_0$ is the only index such that $\frac{\partial G_i(\{t_j\}, \mu_2)}{\partial t_i} \approx 0$, we can not solve $G_{i0} = 0$ to obtain $t_{i0} = t_{i0}(\mu_2)$, but by using $\frac{\partial G_{i0}}{\partial \mu_2} \neq 0$ we can solve $\mu_2 = \mu_2(t_{i0})$ from $G_{i0}(\{t_i\}, \mu_2) = 0$. This is the basic strategy used in Theorems 9.1 and 9.2. However, if more than one $t_i's$ are such that $\frac{\partial}{\partial t_i} d(p(-t_i + 2\tau, \mu), \mu) \simeq 0$ and thus $\frac{\partial G_i}{\partial t_i} \simeq 0$, we do not have enough extra parameters to play with. If the number of such degenerate $t_i's$ is ν, the problem will be reduced to solving ν equations with $\nu + 1$ variables, including μ_2, i.e., a bifurcation problem. In general, those bifurcation problems are difficult to solve. However, when $\nu = 2$, we can use Crandall-Rabinowitz's Theorem [7] to show how a double periodic solution bifurcates from a simple periodic solution when μ_2 is near a turning point of the curve $y = p(-t + 2\tau, \mu)$.

We return to the case described by Theorems 8.3-9.1. There P_i is a point on the curve $y = p(-t + 2\tau, \mu)$ such that $\frac{dy}{dt}$ is tangent to $\mathcal{C}(\mu) = \{\mu_2 = z_1(i)\}$. We have shown that there are $z_1(i-) < z_1(i) < z_1(i+)$ such that for $0 \leq \mu_2 \leq z_1(i-)$, we have two simple nondegenerate periodic solutions $x_1(t)$ and $x_2(t)$ with $\mathcal{T}x_1(\cdot) = \{\omega_1\}_p$ and $\mathcal{T}x_2(\cdot) = \{\omega_2\}_p$. Also, $p(-\omega_1 + 2\tau, \mu) \in \widetilde{P_iP_{-i}}$ and $p(-\omega_2 + 2\tau, \mu) \in \widetilde{P_iP_{-i-1}}$. In fact there is a nondegenerate double periodic solution $x_3(t)$ at the same μ_2 with $\mathcal{T}x_3(\cdot) = \{\omega_3, \omega_4\}_p$ such that $p(-\omega_3 + 2\tau, \mu) \in \widetilde{P_iP_{-i}}$ and $p(-\omega_4 + 2\tau, \mu) \in \widetilde{P_iP_{-i-1}}$. When μ_2 moves from $z_1(i-)$ to $z_1(i+)$, all three solutions $x_\nu(t)$, $\nu = 1, 2, 3$ disappear. We have shown that $x_1(t)$ and $x_2(t)$ approach each other and coalesce into one solution and disappear. In the (ω, μ_2)-plane $x_1(t)$ and $x_2(t)$ are related to a one parameter family of curves $\mu_2 = \mu_2^*(\omega)$ which has a nondegenerate quadratic turning point. It is possible to show that as μ_2 increases, $\omega_3 - \omega_4 \to 0$ and $x_3(t)$ also meets that one parameter family of simple periodic solutions and disappears on it. In terms of Poincaré maps, the situation is much like a periodic doubling bifurcation when moving along the curve $\mu_2 = \mu_2^*(\omega)$ on the (ω, μ_2)-plane.

Theorem 9.3. *i) There exists $\bar{c} > 0$ such that system (2.1) has a double periodic solution $x_3(t)$ with $\mathcal{T}x_3(\cdot) = \{\omega_3, \omega_4\}_p$ where $p(-\omega_3 + 2\tau, \mu) \in \widetilde{P_iP_{-i}}$ and $P(-\omega_4 + 2\tau) \in \widetilde{P_iP_{-i-1}}$, provided that $0 \leq \mu_2 \leq z_1(i) - 2\bar{c}\delta^{\perp}(|\mu| + R)$.*

ii) There is a one parameter family of curve in (ω_3, ω_4)-plane

$$\omega_3 = \omega_3(s), \qquad \omega_4 = \omega_4(s), \qquad -s_1 < s < s_2, \qquad s_1 > 0,\ s_2 > 0,$$

with $\omega_3(0) = \omega_4(0)$, $\omega_3(s) = \omega_4(-s)$ and $\omega_3'(0) = 1$. On that curve we have a double periodic solution $x_3(t)$ with $\mathcal{T}x_3(\cdot) = \{\omega_3(s), \omega_4(s)\}_p$ which becomes a simple periodic solution at $s = 0$. Moreover, $p(-\omega_j + 2\tau, \mu) \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}$, $j = 3, 4$.

iii) All the double periodic solutions $x_3(t)$ with $\mathcal{T}x_3(\cdot) = \{\omega_3, \omega_4\}_p$, $p(-\omega_j + 2\tau, \mu) \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}$ for $j = 3, 4$ and $0 \leq \mu_2$ are given in i) and ii).

Proof. i) The existence of solutions described in (i) can be proved like Theorems 8.3 and 8.4.

ii) The proof below is a refinement of [7] since we need an estimate on the domain of s. To this end, estimates for third order derivatives as in Lemma 9.4 are needed. Let $x_3(t)$ be a generalized double periodic solution with $\mathcal{T}x_3(\cdot) = \{t_1, t_2\}_p$ and jumps of the sizes $G_1(\{t_1, t_2\}_p, \mu)$, $G_2(\{t_1, t_2\}_p, \mu)$. By a shift of time, we obtain a solution $x_4(t)$ from $x_3(t)$ with $\mathcal{T}x_4(\cdot) = \{t_2, t_1\}_p$ and sizes of jumps $G_1(\{t_2, t_1\}_p, \mu)$, $G_2(\{t_2, t_1\}_p, \mu)$. The sizes of jumps are uniquely determined by the orbit, therefore,

$$G_2(\{t_1, t_2\}_p, \mu) = G_1(\{t_2, t_1\}_p, \mu).$$

In order to have a genuine solution $x_3(t)$, we need to solve two bifurcation equations simultaneously

$$G_1(\{t_1, t_2\}_p, \mu) = 0,$$
$$G_2(\{t_1, t_2\}_p, \mu) = 0,$$

for μ_2 near $z_1(i)$ and t_1, t_2 near t_0 where $p(-t_0 + 2\tau, \mu) = P_i$. Let $\mu_2 = \mu^*(t_1, t_2)$ be the solution of the first equation, then $\mu_2 = \mu^*(t_2, t_1)$ is a solution of the second equation. Here we have used the fact $\frac{\partial G_1}{\partial \mu_2} \neq 0$. We need to solve

$$\mu_2 = \mu^*(t_1, t_2) = \mu^*(t_2, t_1).$$

Let $R = |P_i|$ and $y(t) = p(-t + 2\tau, \mu)$. By differentiating $G_1(\{t_1, t_2\}_p, \mu^*(t_1, t_2)) = 0$, we can prove the following estimates similar to those in Theorem 8.1, 9.1 and 9.2.

Lemma 9.4.

$$\left|\frac{\partial \mu^*(t_1, t_2)}{\partial t_1}\right| \leq C \left|\frac{\partial}{\partial t_1} d(y(t_1), \mu)\right| + CR(|\mu| + R),$$

$$\left|\frac{\partial \mu^*(t_1, t_2)}{\partial t_2}\right| + \left|\frac{\partial^2 \mu^*(t_1, t_2)}{\partial t_1 \partial t_2}\right| + \left|\frac{\partial^2 \mu^*(t_1, t_2)}{\partial t_2^2}\right| \leq CR(|\mu| + R),$$

$$C^{-1}R \leq \left|\frac{\partial^2 \mu^*(t_1, t_2)}{\partial t_1^2}\right| \leq CR,$$

$$|D^\alpha \mu^*(t_1, t_2)| \leq CR, \ |\alpha| = 3 \quad \textit{where} \quad \alpha = (\alpha_1, \alpha_2) \quad \textit{is a multi-index.}$$

The estimate of the lower bound for $|\frac{\partial^2 \mu^*}{\partial t_1^2}|$ *is valid if* $|\varphi| \leq \frac{\pi}{3}$.

Remark. It is useful to observe that the derivatives involving t_2 are bounded by an extra factor $|\mu| + R$. This comes from (8.2) and $\frac{\partial}{\partial t_2} d(y(t_1), \mu) = 0$.

Proof of Theorem 9.3, continued. Let $t_1 = u + v + t_0$, $t_2 = u - v + t_0$, where $|u| \leq \frac{\pi}{6}$ and $|v| \leq \frac{\pi}{6}$. Denote $\mu^*(u + v + t_0, u - v + t_0)$ by $\mu(u + v, u - v)$. The equation

$$\frac{1}{2}[\mu(u + v, u - v) - \mu(u - v, u + v)] = 0$$

has a trivial branch of solution $v = 0$, which corresponds to $t_1 = t_2$. On that branch we have a simple periodic solution parameterized by $t_1 = t_2 = \omega$. We are interested in solutions with $v \neq 0$. Therefore, we define

$$H(u, v) = \frac{1}{2v}[\mu(u+v,\, u-v) - \mu(u-v,\, u+v)].$$

Using ∂_1 and ∂_2 to denote $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$, simple calculations show that

$$\begin{aligned} H(u, v) &= \frac{1}{2}\int_{-1}^{1}[\partial_1\mu(u+\theta v,\, u-\theta v) - \partial_2\mu(u+\theta v,\, u-\theta v)]d\theta \\ H(u, 0) &= \partial_1\mu(u, u) - \partial_2\mu(u, u). \end{aligned} \tag{9.33}$$

$$H_u(u, v) = \frac{1}{2}\int_{-1}^{1}[\partial_{11}\mu(u+\theta v,\, u-\theta v) - \partial_{22}\mu(u+\theta v,\, u-\theta v)]d\theta. \tag{9.34}$$

$$\begin{aligned} H_v(u, v) &= \frac{1}{2}\int_{-1}^{1} l(u+\theta v,\, u-\theta v)\cdot\theta d\theta \\ &= \frac{1}{2}\int_{0}^{1}[l(u+\theta v,\, u-\theta v) - l(u-\theta v,\, u+\theta v)]\theta d\theta, \end{aligned} \tag{9.35}$$

where $l = (\partial_{11} - 2\partial_{12} + \partial_{22})\mu$.

From Lemma 9.4, $|D^\alpha\mu^*| \le CR$, for $|\alpha| = 3$, thus by (9.35)

$$|H_v(u, v)| \le (|\partial_1 l| + |\partial_2 l|)|v| \le CR|v|. \tag{9.36}$$

From Lemma 9.4, $|\partial_2\mu(0, 0)| \le C(|\mu| + R)\cdot R$ and $|\partial_1\mu(0, 0)| \le \left|\frac{\partial}{\partial t}d(y(t), \mu)\right| + C(|\mu| + R)\cdot R$. Since at $u = v = 0$, $t_1 = t_0$, $\frac{\partial}{\partial t}d(y(t), \mu) = 0$, we have by (9.33),

$$|H(0, 0)| \le C(|\mu| + R)\cdot R \tag{9.37}$$

Since $C^{-1}R \gg CR(|\mu| + R)$, from Lemma 9.4 and (9.34),

$$|H_u(u, v)| \ge CR, \tag{9.38}$$

with possibly a smaller constant C. From (9.36) and (9.37),

$$\begin{aligned} |H(0, v)| &\le |H(0, v) - H(0, 0)| + |H(0, 0)| \\ &\le CR|v|^2 + C(|\mu| + R)R \end{aligned} \tag{9.39}$$

From (9.38),

$$|H(u, v) - H(0, v)| \ge CR\cdot\frac{\pi}{6}, \quad \text{if} \quad u = \pm\frac{\pi}{6}. \tag{9.40}$$

Comparing (9.39) and (9.40), we know that $H(u, v)$ changes sign for $u \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ provided $|v| \le \epsilon$, $\epsilon > 0$ being a small constant. Thus, there is a unique C^1 solution $u = u(v)$, $|v| \le \epsilon$ to $H(u, v) = 0$. Also $u'(0) = 0$ since $H_v(u, 0) = 0$.

From $H(u(0), 0) = 0$ and (9.37), there exists $0 < \theta < 1$ such that

$$|H(u(0), 0) - H(0, 0)| = |H_u(\theta u(0), 0)|\cdot|u(0)| \le C(|\mu| + R)\cdot R.$$

By (9.38), $|u(0)| \le cR(|\mu| + R)/R \le C(|\mu| + R)$. Also, by (9.36) and (9.38),

$$|u'(v)| \le \frac{|H_v|}{|H_u|} \le \frac{CR|v|}{CR} \le C|v|.$$

For $|v| \le \epsilon$,

$$|u(v)| \le |u(0)| + |u'|\,|v| \le C(|\mu| + R) + C|v|^2.$$

Let $s = v$, $\omega_3(s) = t_1 = u(s) + s + t_0$, and $\omega_4(s) = t_2 = u(s) - s + t_0$. Based on its definition, $H(u, -v) = H(u, v)$. Thus, $u(-v) = u(v)$. Therefore $\omega_3(-s) = \omega_4(s)$. Clearly we have $\omega_3'(0) = 1$. Also, $\omega_3'(s) \ge 1 - |u'(s)| \ge 1 - c\epsilon > 0$, if $|s| \le \epsilon$, and ϵ is small. If $|\mu| + R + \epsilon$ is small, then $|t_j - t_0| \le \frac{\pi}{3}$, $j = 1, 2$. Recall that $p(-t_0 + 2\tau) = P_i$ and the time to travel from P_i to P_{-i} or P_{-i-1} is almost π. Thus,

$$p(-\omega_j + 2\tau, \mu) \in \widetilde{P_iP_{-i}} \cup \widetilde{P_iP_{-i-1}}, \qquad j = 3, 4.$$

Assertion ii) has been proved.
iii)

$$\begin{aligned}\omega_3(\pm\epsilon) &= \pm\epsilon + 0(|u(0)| + |u'|\epsilon) + t_0 \\ &= \pm\epsilon + 0(|\mu| + R + \epsilon^2) + t_0\end{aligned}$$

If $|\mu|$ and R are sufficiently small, $|\omega_3(\pm\epsilon) - t_0| \ge \frac{\epsilon}{2}$. Let $\varphi(t)$ be the angle between $\dot{y}(t)$ and the vertical axis. Then since $\frac{d\varphi}{dt} = 1 + 0(R)$, $\varphi(\omega_3(-\epsilon)) \le -\frac{\epsilon}{3}$ and $\varphi(\omega_3(+\epsilon)) \ge \frac{\epsilon}{3}$. (Recall that $\varphi(t_0) = 0$). Thus from (8.12)

$$|z_1 - z_1(i)| \ge CR\epsilon^2 \ge 3\bar{c}\delta^{\perp} \cdot (|\mu| + R).$$

where (z_1, z_2) are the coordinates for $p(-\omega_3(\pm\epsilon) + 2\tau, \mu)$. The last inequality can be achieved by having μ and R small, not ϵ small. Therefore, at $s = \pm\epsilon$, we must have $\mu_2 \le z_1(i) - 2\bar{c}\delta^{\perp} \cdot (|\mu| + R)$. The double periodic solutions described by (ii) overlap with those described by (i) at $s = \pm\epsilon$. □

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