Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 3144–3177

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of concatenated traveling waves: Alternate approaches ☆

Xiao-Biao Lin, Stephen Schecter *

Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695, USA

Received 10 September 2014; revised 30 January 2015

Available online 29 April 2015

Abstract

We consider a reaction–diffusion equation in one space dimension whose initial condition is approximately a sequence of widely separated traveling waves with increasing velocity, each of which is asymptotically stable. As in [25,24,14], we show that the sequence of traveling waves is itself asymptotically stable: as $t \to \infty$, the solution approaches the concatenated wave pattern, with different shifts of each wave allowed. Our proof is similar to that of [14] in that it is based on spatial dynamics, Laplace transform, and exponential dichotomies, but it incorporates a number of modifications.

Keywords: Traveling wave; Stability; Spatial dynamics; Laplace transform; Exponential dichotomy

1. Introduction

Consider a system of reaction–diffusion equations in one space dimension,

$$u_t = u_{xx} + f(u), \tag{1.1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ of class C^2. A *concatenated wave pattern* for (1.1) consists of the following data:

☆ Lin was supported in part by NSF grant DMS-1211070. Schecter was supported in part by NSF grant DMS-1211707.
* Corresponding author.
E-mail addresses: xblin@ncsu.edu (X.-B. Lin), schecter@ncsu.edu (S. Schecter).

http://dx.doi.org/10.1016/j.jde.2015.04.015
0022-0396/

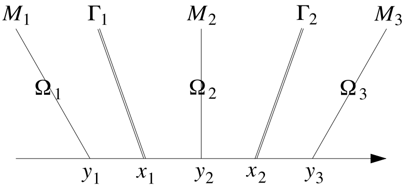

Fig. 1.1. For the case $m = 3$, the concatenated wave pattern consists of three waves separated by two lines Γ_1 and Γ_2.

- a sequence $e_0, e_1, \dots, e_m$ of equilibria of the ordinary differential equation $u_t = f(u)$, with $m \geq 2$;
- an increasing sequence $c_1 < c_2 < \cdots < c_m$ of real numbers;
- for $j = 1, \dots, m$, a traveling wave solution of (1.1) with velocity c_j, $q_j(\zeta)$, $\zeta = x - c_j t$, with $q_j(-\infty) = e_{j-1}$ and $q_j(\infty) = e_j$.

The equilibria e_j are not necessarily distinct.

We are interested in solutions of (1.1) that are close to the sequence $q_1, \dots, q_m$ of traveling waves. To discuss such solutions, let $\mathbf{y}$ denote an increasing sequence $y_1 < y_2 < \cdots < y_m$ of real numbers. Associated with $\mathbf{y}$ is a *realization* of the concatenated wave pattern defined by dividing the domain $\mathbb{R} \times \mathbb{R}^+$, with coordinates (x, t), into m regions and placing one traveling wave in each region; the jth wave is initially centered at y_j. More precisely, for $j = 1, \dots, m-1$, let $\bar{c}_j = \frac{1}{2}(c_j + c_{j+1})$, the average speed of the waves q_j and q_{j+1}, and let $x_j = \frac{1}{2}(y_j + y_{j+1})$, $j = 1, \dots, m-1$. Let $x_0 = -\infty$ and $x_m = \infty$. Define

$$\Gamma_j = \{(x,t) : x = x_j + \bar{c}_j t,\ t \geq 0\}, \quad j = 1, \dots, m-1;$$

$$\Omega_j = \{(x,t) : x_{j-1} + \bar{c}_{j-1} t < x < x_j + \bar{c}_j t,\ t \geq 0\}, \quad j = 1, \dots, m.$$

Thus $\Omega_1 = \{(x,t) : -\infty < x < x_1 + \bar{c}_1 t,\ t \geq 0\}$ and $\Omega_m = \{(x,t) : x_{m-1} + \bar{c}_{m-1} t < x < \infty, t \geq 0\}$. See Fig. 1.1; Γ_j separates Ω_j and Ω_{j+1}. The realization of the concatenated wave pattern associated with the sequence $\mathbf{y}$ is the function $u^{\mathbf{y}}(x,t)$ defined on the union of the Ω_j by

$$u^{\mathbf{y}}(x,t) = q_j(x - y_j - c_j t) \text{ for } (x,t) \in \Omega_j.$$

The center of the wave q_j in Ω_j moves on the line

$$M_j = \{(x,t) : x = y_j + c_j t,\ t \geq 0\}.$$

The lines $\Gamma_1, \dots, \Gamma_{m-1}, M_1, \dots, M_m$ spread apart as t increases. The function $u^{\mathbf{y}}(x,t)$ satisfies (1.1) in each Ω_j. It is not continuous across the Γ_j, but the jump in $u^{\mathbf{y}}(x,t)$ along Γ_j goes to 0 exponentially as $t \to \infty$.

In Ω_j it is natural to replace x with the moving coordinate $\xi_j = x - y_j - c_j t$. In $\xi_j t$-coordinates, Ω_j corresponds to

$$\tilde{\Omega}_j = \{(\xi_j, t) : x_{j-1} - y_j + (\bar{c}_{j-1} - c_j)t < \xi_j < x_j - y_j + (\bar{c}_j - c_j)t\}.$$

The line M_j in Ω_j becomes $\xi_j = 0$ in $\tilde{\Omega}_j$.

Given a function $u(x,t)$ on $\mathbb{R} \times \mathbb{R}^+$, its restriction to Ω_j corresponds to a function $\tilde{u}_j(\xi_j, t)$ on $\tilde{\Omega}_j$ given by $\tilde{u}_j(\xi_j, t) = u(\xi_j + y_j + c_j t, t)$.

Let I_j denote the interval (x_{j-1}, x_j).

Definition 1.1. A concatenated wave pattern is *exponentially stable* provided there exists $\gamma < 0$ such that for each $\epsilon > 0$ there exist $\chi > 0$ and $\delta > 0$ for which the following is true. Suppose $\min(y_{j+1} - y_j) > \chi$ and $u_0^{ex} \in H^1(\mathbb{R})$ satisfies $\|u_0^{ex}(x) - q_j(x - y_j)\|_{H^1(I_j)} < \delta$ for $j = 1, \dots, m$. Then there is a solution $u^{ex}(x,t)$ in $\mathbb{R} \times \mathbb{R}^+$ to (1.1) such that

(1) $u^{ex}(x, 0) = u_0^{ex}(x)$;
(2) in $\tilde{\Omega}_j$, $\tilde{u}_j^{ex}(\xi_j, t) = q_j(\xi_j - \beta_j(t)) + Y_j(\xi_j, t)$, and as $t \to \infty$, $\dot{\beta}_j(t)$ and $Y_j(\xi_j, t)$ are $\mathcal{O}(e^{\gamma t})$;
(3) in appropriate function spaces, $\|\dot{\beta}_j(t)\| < \epsilon$ and $\|Y_j(\xi_j, t)\| < \epsilon$.

Since $\dot{\beta}_j(t)$ is $\mathcal{O}(e^{\gamma t})$, $\lim_{t\to\infty} \beta_j(t)$ exists. Thus the definition says that if the initial condition $u^{ex}(x, 0)$ is close to the concatenated wave pattern at $t = 0$, then the solution approaches a shifted concatenated wave pattern as $t \to \infty$. Different shifts are allowed in different Ω_j.

In the coordinates $(\xi, t) = (\xi_j, t)$, (1.1) becomes

$$u_t = u_{\xi\xi} + c_j u_\xi + f(u), \quad \xi = \xi_j = x - y_j - c_j t. \tag{1.2}$$

The traveling wave $q_j(\xi)$ is a stationary solution of (1.2). The linearization of (1.2) at the traveling wave $q_j(\xi)$ is

$$U_t = U_{\xi\xi} + c_j U_\xi + Df(q_j(\xi))U, \quad \xi = \xi_j = x - y_j - c_j t. \tag{1.3}$$

Define the linear operator $\mathcal{L}_j$ on $L^2(\mathbb{R})$ by

$$\mathcal{L}_j U = U_{\xi\xi} + c_j U_\xi + Df(q_j(\xi))U.$$

We shall assume that for $j = 1, \dots, m$,

(A1) $q_j(\xi)$ approaches its end states exponentially as $\xi \to \pm\infty$.
(A2) There is a number $\eta < 0$ such that for $j = 1, \dots, m$, the half-plane $\Re(\lambda) \ge \eta$ contains only resolvent points of $\mathcal{L}_j$, except for the simple eigenvalue $\lambda = 0$, with one-dimensional eigenspace spanned by q_j'.

Assumptions (A1) and (A2) imply that the individual traveling waves are exponentially stable. From the form (1.3) of the system, the linear operators $\mathcal{L}_j$ are sectorial.

Assumption (A1) implies

(A1′) There are numbers $K > 0$ and $\mu > 0$ such that for $j = 1, \dots, m$,
  (i) $\|q_j(\xi) - e_{j-1}\| \le K e^{\mu\xi}$ for $\xi << 0$;
  (ii) $\|q_j(\xi) - e_j\| \le K e^{-\mu\xi}$ for $\xi >> 0$;
  (iii) $\|q_j'(\xi)\| \le K e^{-\mu|\xi|}$ and $\|q_j''(\xi)\| \le K e^{-\mu|\xi|}$ for $|\xi| >> 0$.

We now state the main result of this paper.

Theorem 1.1. *Assume (A1) and (A2). Then the concatenated wave pattern is exponentially stable. Moreover, let η and μ be given by (A2) and (A1′), and let*

$$\nu = \max\left(\eta, -\frac{1}{2}\mu(c_2 - c_1), \dots, -\frac{1}{2}\mu(c_k - c_{k-1})\right) < 0. \tag{1.4}$$

Then in the definition of stability, one may take $\nu < \gamma < 0$.

The result says that if the individual traveling waves are exponentially stable, then the concatenated wave pattern is exponentially stable.

Essentially the same result was proved by Doug Wright [25] and Sabrina Selle [24]. However, their approach does not use concatenated wave patterns, but instead uses a sum of traveling waves. We have found it difficult to apply this approach to problems in which the traveling waves were degenerate in some manner at their end states, because of the "smearing" of the waves inherent in using sums. We were therefore motivated to try to develop a stability approach to concatenated wave patterns in which the different waves are more clearly separated, and their interactions can perhaps be more precisely seen.

Wright's work has been generalized to two space dimensions [26] and to lattice differential equations [6]. There is also work on concatenated wave patterns for scalar problems that uses methods limited to scalar problems such as the comparison principle; see [4,3]. In addition, there is a literature on the related problem of fronts that approach each other; see for example [27,18,23].

Our approach uses "spatial dynamics," which was developed by Kirchgassner [7], Renardy [20], Mielke [17], Sandstede, Scheel, and collaborators [19,1], Lin [10,11] and others. The idea of spatial dynamics is to treat the space variable as time, and evolve functions of t to the left and right. With the aid of Laplace transform in t and exponential dichotomies in ξ, one can decompose a function of t into a part that decays to the left and a part that decays to the right. A recent paper that uses Laplace transform in this manner is [21].

Here is a brief outline of the paper. In Section 2 we reduce the problem of proving Theorem 1.1 to one of proving a linear result. In Section 3 we reduce the problem of proving the linear result to one of proving two lemmas. The proofs of these lemmas occupy the remainder of the paper. One, the Tail Lemma, deals with canceling a discontinuity in the solution along one of the lines Γ_j, where the tails of two traveling waves interact. The proof of this lemma uses the linearization of the reaction–diffusion equation at the equilibrium e_j, which has spectrum in $\Re\lambda < \nu < 0$. The other, the Interior Lemma, deals with canceling a discontinuity in the solution near one of the lines M_j, i.e., in the interior of the jth traveling wave. The proof of this lemma uses the linear operator $\mathcal{L}_j$, which has the eigenvalue 0.

In order to prove these two lemmas, in Section 4 we give some background about exponential dichotomies and Laplace transform. Much of this background comes from [10]. The two lemmas are proved using the material of Section 4 in Sections 5–6.

The organization of the paper is intended to make the structure of the argument clear in Sections 2–3. Our hope is that the structure of the argument can be used in less standard situations, for example, when the operators $\mathcal{L}_j$ are not sectorial or have essential spectrum on the imaginary axis. However, the choice of spaces used in Sections 2–3, and the proofs of the lemmas in Sections 5–6, rely on the strong assumptions we have made.

Our approach to the problem of stability of concatenated waves was first presented in [14]. The present paper differs from [14] in a number of respects, including: (1) a more standard

$\epsilon\delta$-definition of stability of the concatenated wave pattern; (2) a simpler treatment of linear implies nonlinear stability (Section 2) that in particular does not attempt to determine the ultimate phase shifts as variables in the problem, and thus differs from the approach of Sattinger in [22]; (3) greater emphasis on the structure of the argument (Sections 2–3); (4) generalization and detailed proof of a key lemma about the smoothness of the solution of a second-order linear partial differential equation that is almost time-independent (Lemma 4.6); (5) an alternate construction of the solution to a certain linear ODE satisfying an internal jump condition, near a parameter value where there is no exponential dichotomy (Section 6, Step 2).

2. Reduction to a linear problem

2.1. Spaces and jumps

Define the Banach spaces

$$H^k(\mathbb{R}^+) = W^{k,2}(\mathbb{R}^+, \mathbb{R}^n),\ k \geq 0,\ \text{the usual Sobolev space};$$
$$H^{k_1\times k_2}(\mathbb{R}^+) = H^{k_1}(\mathbb{R}^+) \times H^{k_2}(\mathbb{R}^+),\ k_1 \geq 0,\ k_2 \geq 0.$$

As usual, $H^0 = L^2$.

Let Ω be an open subset of $\mathbb{R} \times \mathbb{R}^+$, with coordinates (x, t). Define the Banach space

$$H^{2,1}(\Omega) = \{u : \Omega \to \mathbb{R}^n \mid u, u_x, u_{xx} \text{ and } u_t \in L^2(\Omega; \mathbb{R}^n\};$$
$$\|u\|_{H^{2,1}(\Omega)} = \|u\|_{L^2} + \|u_x\|_{L^2} + \|u_{xx}\|_{L^2} + \|u_t\|_{L^2}.$$

We use the subscript 0 to denote a subspace of functions that equal 0 at $t = 0$; thus $H_0^k(\mathbb{R}^+) \subseteq H^k(\mathbb{R}^+)$ consists of functions in $H^k(\mathbb{R}^+)$ that are 0 at $t = 0$.

For a constant $\gamma < 0$, define:

$$H^k(\mathbb{R}^+, \gamma) = \{u : \mathbb{R}^+ \to \mathbb{R}^n \mid e^{-\gamma t}u \in H^k(\mathbb{R}^+\};\ \|u\|_{H^k(\mathbb{R}^+,\gamma)} = \|e^{-\gamma t}u\|_{H^k(\mathbb{R}^+)}.$$
$$H^{k_1\times k_2}(\mathbb{R}^+, \gamma) = H^{k_1}(\mathbb{R}^+, \gamma) \times H^{k_2}(\mathbb{R}^+, \gamma).$$
$$L^2(\Omega, \gamma) = \{u : \Omega \to \mathbb{R}^n \mid e^{-\gamma t}u \in L^2(\Omega)\};\ \|u\|_{L^2(\Omega,\gamma)} = \|e^{-\gamma t}u\|_{L^2(\Omega)}.$$
$$H^{2,1}(\Omega, \gamma) = \{u : \Omega \to \mathbb{R}^n \mid e^{-\gamma t}u \in H^{2,1}(\Omega)\};\ \|u\|_{H^{2,1}(\Omega,\gamma)} = \|e^{-\gamma t}u\|_{H^{2,1}(\Omega)}.$$
$$X^1(\mathbb{R}^+, \gamma) = \{u : \mathbb{R}^+ \to \mathbb{R}^n \mid e^{-\gamma t}\dot{u} \in L^2(\mathbb{R}^+\};\ \|u\|_{X^1(\mathbb{R}^+,\gamma)} = \|u(0)\| + \|e^{-\gamma t}\dot{u}\|_{L^2(\mathbb{R}^+)}.$$

The change of coordinates $\xi = x - y - ct$ converts Ω to a subset $\tilde{\Omega}$ of $\mathbb{R} \times \mathbb{R}^+$, with coordinates (ξ, t), and converts a function $u(x, t)$ on Ω to a function $\tilde{u}(\xi, t) = u(\xi + y + ct, t)$ on $\tilde{\Omega}$.

Lemma 2.1. *The map $u \to \tilde{u}$ is a linear isomorphism of $H^{2,1}(\Omega, \gamma)$ to $H^{2,1}(\tilde{\Omega}, \gamma)$. The map $u \to \tilde{u}$ and its inverse $\tilde{u} \to u$ both have norm at most $1 + |c|$.*

Proof. Let $u \in H_0^{2,1}(\Omega, \gamma)$. Then $\tilde{u}_\xi = u_x$, $\tilde{u}_{\xi\xi} = u_{xx}$, $\tilde{u}_t = u_t + c_j u_x$. Thus

$$\|\tilde{u}\| + \|\tilde{u}_\xi\| + \|\tilde{u}_{\xi\xi}\| + \|\tilde{u}_t\| \le \|u\| + \|u_x\| + \|u_{xx}\| + \|u_t\| + |c|\|u_x\|.$$

Here all the norms are in $L^2(\Omega, \gamma)$. The lemma follows easily. □

Let $\ell(x_0, c) = \{(x, t) : x = x_0 - ct,\ t \ge 0\}$. We shall sometimes denote $\ell(x_0, c)$ simply by ℓ. If $u \in H^{2,1}(\Omega, \gamma)$ and $\ell(x_0, c) \subset \Omega$ (resp. $\ell(x_0, c)$ is the right- or left-hand boundary of Ω), then u has a naturally defined restriction (resp. extension) to $\ell(x_0, c)$ called the *trace* of u on $\ell(x_0, c)$.

Lemma 2.2.

(1) *If $u \in H^{2,1}(\Omega, \gamma)$ and $\ell(x_0, c) \subset \Omega$ or $\ell(x_0, c)$ is the right- or left-hand boundary of Ω, then the trace of u on $\ell(x_0, c)$, as a function of t, belongs to $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$. The mapping $u \to u|_{\ell(x_0,c)}$ is bounded linear from $H^{2,1}(\Omega, \gamma)$ to $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$. Moreover, there is a number $K > 0$, independent of x_0 and c, such the norm of the linear map is at most $K(1 + |c|)$.*

Proof. For $c = 0$, see [15], vol. 2, Theorem 2.1. For $c \ne 0$, use Lemma 2.1 followed by restriction or extension for $c = 0$. □

Now let Ω^- and Ω^+ be open subsets of $\mathbb{R} \times \mathbb{R}^+$ such that the line $\ell = \ell(x_0, c)$ is both the right-hand boundary of Ω^- and the left-hand boundary of Ω^+. Let $u^- \in H^{2,1}(\Omega^-, \gamma)$ and $u^+ \in H^{2,1}(\Omega^+, \gamma)$. Let u be the function on $\Omega^- \cup \Omega^+$ that equals u^- on Ω^- and u^+ on Ω^+. We denote the jump in u across ℓ, a function of t, by $[u](\ell) = u_+(t) - u_-(t)$. By the previous lemma, $[u](\ell)$ is defined and belongs to $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$.

Lemma 2.3. *If $[u](\ell) = 0$ in $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$, then u extends to a function in $H^{2,1}(\Omega^- \cup \ell \cup \Omega^+, \gamma)$.*

Proof. The proof is an exercise in trace theory. Let w be the function on $\Omega^- \cup \ell \cup \Omega^+$ that equals u_x^- on Ω^- and u_x^+ on Ω^+. Integrating against a test function and using integration by parts, it is easy to show that $u_x = w$ in $L^2(\Omega^- \cup \ell \cup \Omega^+)$ if $[u](\ell) = 0$. Thus $u_x \in L^2(\Omega^- \cup \ell \cup \Omega^+)$. Similarly one shows that $u_{xx} \in L^2(\Omega^- \cup \ell \cup \Omega^+)$. It is clear that u and u_t are in $L^2(\Omega^- \cup \ell \cup \Omega^+)$. Therefore $u \in H^{2,1}(\Omega^- \cup \ell \cup \Omega^+, \gamma)$. □

Let $u(x, t)$ be any function defined on $\cup\Omega_j$. Then $u(x, t)$ corresponds to a sequence of functions $\tilde{u}_1(\xi_1, t), \dots, \tilde{u}_m(\xi_m, t)$, where $\tilde{u}_j(\xi_j, t)$ is defined on $\tilde{\Omega}_j$, with $\tilde{u}_j(\xi_j, t) = u(\xi_j + y_j + c_j t, t)$. Conversely, given $\tilde{u}_1(\xi_1, t), \dots, \tilde{u}_m(\xi_m, t)$, one can define $u(x, t)$ on $\cup\Omega_j$ by

$$u(x, t) = \tilde{u}_j(x - y_j - c_j t, t) \text{ on } \Omega_j.$$

If each $\tilde{u}_j$ is in $H^{2,1}(\tilde{\Omega}_j, \gamma)$, then the trace of $\tilde{u}_j$ is defined on the left and right boundaries of $\tilde{\Omega}_j$, so we have

$$[u](\Gamma_j) = \tilde{u}_{j+1}(x_j - y_{j+1} + (\bar{c}_j - c_{j+1})t, t) - \tilde{u}_j(x_j - y_j + (\bar{c}_j - c_j)t, t). \tag{2.1}$$

We shall also use the notation $[\tilde{u}_j](\Gamma_j)$ to denote (2.1).

2.2. Reformulation of the problem in $\tilde{\Omega}_j$

Recall from Definition 1.1 that in $\tilde{\Omega}_j$, the solution of the initial value problem of interest is

$$\tilde{u}_j^{ex}(\xi_j, t) = q_j(\xi_j + \beta_j(t)) + Y_j(\xi_j, t).$$

Then in $\tilde{\Omega}_j$, (1.1) can be written

$$\begin{aligned}&\partial_t Y_j + q_j'(\xi + \beta_j)\dot{\beta}_j = \partial_{\xi\xi} Y_j + c_j \partial_\xi Y_j + f(q_j(\xi + \beta_j) + Y_j) - f(q_j(\xi + \beta_j)),\\ &(\xi, t) = (\xi_j, t) \in \tilde{\Omega}_j.\end{aligned} \tag{2.2}$$

We now reformulate (2.2) by expanding about $(Y_j, \beta_j) = (0, 0)$. With $(\xi, t) = (\xi_j, t)$, write

$$\begin{aligned}f(q_j(\xi + \beta_j) + Y_j) - f(q_j(\xi + \beta_j)) - Df(q_j(\xi + \beta_j))Y_j &= F_{j1}(\xi, Y_j, \beta_j)(Y_j, Y_j),\\ Df(q_j(\xi + \beta_j)) - Df(q_j(\xi)) &= F_{j2}(\xi, \beta_j)\beta_j,\\ q_j'(\xi + \beta_j) - q_j'(\xi) &= F_{j3}(\xi, \beta_j)\beta_j,\end{aligned}$$

with

$$\begin{aligned}F_{j1}(\xi, Y_j, \beta_j) &= \int_0^1 (1 - s)D^2 f(q_j(\xi + \beta_j) + sY_j)\, ds,\\ F_{j2}(\xi, \beta_j) &= \int_0^1 D^2 f(q_j(\xi + s\beta_j))\, ds,\\ F_{j3}(\xi, \beta_j) &= \int_0^1 q_j''(\xi + s\beta_j)\, ds.\end{aligned}$$

Let

$$F_j(\xi, Y_j, \beta_j, \dot{\beta}_j) = F_{j1}(\xi, Y_j, \beta_j)(Y_j, Y_j) + F_{j2}(\xi, \beta_j)Y_j\beta_j - F_{j3}(\xi, \beta_j)\beta_j\dot{\beta}_j.$$

Write (2.2) as

$$\begin{aligned}&\partial_t Y_j + q_j'(\xi)\dot{\beta}_j = \partial_{\xi\xi} Y_j + c_j \partial_\xi Y_j + Df(q_j(\xi))Y_j + F_j(\xi, Y_j, \beta_j, \dot{\beta}_j),\\ &(\xi, t) = (\xi_j, t) \in \tilde{\Omega}_j.\end{aligned} \tag{2.3}$$

We also expand the initial conditions and the jump conditions across the Γ_j about $(Y_j, \beta_j) = (0, 0)$. With $(\xi, t) = (\xi_j, t)$, write

$$q_j(\xi+\beta_j)-q_j(\xi)-\beta_j q_j'(\xi)=G_j(\xi,\beta_j)\beta_j^2, \quad G_j(\xi,\beta_j)=\int_0^1 (1-s)q_j''(\xi+s\beta_j)\,ds.$$

Initial conditions for $(Y_j(\xi,0),\beta_j(0))$ for (2.3) must satisfy the equation

$$\begin{aligned} u^{ex}(\xi+y_j,0) &= \tilde{u}_j^{ex}(\xi,0)=q_j(\xi+\beta_j(0))+Y_j(\xi,0) \\ &= q_j(\xi)+\beta_j(0)q_j'(\xi)+G_j(\xi,\beta_j(0))\beta_j^2(0)+Y_j(\xi,0), \quad \xi=\xi_j\in I_j. \end{aligned} \tag{2.4}$$

Across Γ_j we have

$$\begin{aligned} 0=[u^{ex}](\Gamma_j) &= [\tilde{u}_j^{ex}](\Gamma_j)=[q_j(\xi_j+\beta_j)](\Gamma_j)+[Y_j](\Gamma_j) \\ &= [q_j(\xi_j)](\Gamma_j)+[\beta_j q_j'(\xi_j)](\Gamma_j)+[G_j(\xi_j,\beta_j)\beta_j^2](\Gamma_j)+[Y_j](\Gamma_j). \end{aligned} \tag{2.5}$$

2.3. Approach

Given $\gamma<0$, to solve (1.1) with the initial condition $u^{ex}(x,0)=u_0^{ex}(x)\in H^1(\mathbb{R})$, we need to find pairs $(Y_j(\xi_j,t),\beta_j(t))\in H^{2,1}(\tilde{\Omega}_j,\gamma)\times X^1(\mathbb{R}^+,\gamma)$, $j=1,\dots,m$, that satisfy the following conditions, which come from (2.3), (2.4), and (2.5):

(N1) In $\tilde{\Omega}_j$, (Y_j,β_j) satisfies

$$\begin{aligned} &\partial_t Y_j+q_j'(\xi)\dot{\beta}_j=\partial_{\xi\xi}Y_j+c_j\partial_\xi Y_j+Df(q_j(\xi)Y_j+F_j(\xi,Y_j,\beta_j,\dot{\beta}_j), \\ &\quad (\xi,t)=(\xi_j,t)\in\tilde{\Omega}_j. \\ &Y_j(\xi,0)+\beta_j(0)q_j'(\xi)=u^{ex}(\xi+y_j,0)-q_j(\xi)-G_j(\xi,\beta_j(0))\beta_j^2(0), \quad \xi=\xi_j\in I_j. \end{aligned}$$

(N2) Along Γ_j,

$$\begin{aligned} &[Y_j](\Gamma_j)+[\beta_j q_j'(\xi_j)](\Gamma_j)=-[q_j(\xi_j)](\Gamma_j)-[G_j(\xi_j,\beta_j)\beta_j^2](\Gamma_j) \\ &[Y_{j\xi}](\Gamma_j)+[\beta_j q_j''(\xi_j)](\Gamma_j)=-[q_j'(\xi_j)](\Gamma_j)-[G_{j\xi}(\xi_j,\beta_j)\beta_j^2](\Gamma_j). \end{aligned}$$

Note that the requirement that $\beta_j\in X^1(\mathbb{R}^+,\gamma)$ implies that $\beta_j(\infty)=\beta_j(0)+\int_0^\infty \dot{\beta}_j(t)\,dt$ is finite, because

$$\begin{aligned} \int_0^\infty |\dot{\beta}_j(t)|dt &= \int_0^\infty e^{\gamma t}e^{-\gamma t}|\dot{\beta}_j(t)|dt \le \|e^{\gamma t}\|_{L^2(\mathbb{R}^+)}\|e^{-\gamma t}|\dot{\beta}_j(t)|\|_{L^2(\mathbb{R}^+)} \\ &= (2|\gamma|)^{-\frac{1}{2}}\|\dot{\beta}_j(t)\|_{L^2(\mathbb{R}^+,\gamma)}. \end{aligned}$$

In order to solve the nonlinear system (N1)–(N2) we shall first consider the following nonhomogeneous linear problem. Let $\eta\le\gamma<0$. Let $h_j\in L^2(\tilde{\Omega}_j,\gamma)$, $j=1,\dots m$; $w_j\in H^1(I_j)$, $j=1,\dots m$; and $J_j\in H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$, $j=1,\dots m-1$. Assume

$$[(w_j, w_{j\xi})](x_j) = J_j(0). \tag{2.6}$$

Find pairs $(Y_j, \beta_j) \in H^{2,1}(\tilde{\Omega}_j, \gamma) \times X^1(\mathbb{R}^+, \gamma)$, $j = 1, \ldots, m$, that satisfy the following conditions.

(L1) In $\tilde{\Omega}_j$, (Y_j, β_j) satisfies

$$\partial_t Y_j + q_j'(\xi)\dot{\beta}_j = \partial_{\xi\xi} Y_j + c_j \partial_\xi Y_j + Df(q_j(\xi)Y_j + h_j(\xi, t), \quad (\xi, t) = (\xi_j, t) \in \tilde{\Omega}_j,$$
$$Y_j(\xi, 0) + \beta_j(0)q_j'(\xi) = w_j(\xi), \quad \xi = \xi_j \in I_j.$$

(L2) Along Γ_j,

$$[(Y_j, Y_{j\xi})](\Gamma_j) + [(\beta_j q_j'(\xi_j), \beta_j q_j''(\xi_j))] = J_j.$$

Let $\mathcal{Y} = \prod_1^m (H^{2,1}(\tilde{\Omega}_j, \gamma) \times X^1(\mathbb{R}^+, \gamma))$, and let $\mathcal{Z}$ denote the subspace of $\prod_1^m L^2(\Omega_j, \gamma) \times \prod_1^m H^1(I_j) \times \prod_1^{m-1} H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$ for which (2.6) is satisfied.

Theorem 2.4 *(Linear Theorem). Assume (A1) and (A2). Fix γ, $\eta \le \gamma < 0$. If $\min(y_{j+1} - y_j)$ is sufficiently large, then the linear problem (L1)–(L2) has a solution $((Y_1, \beta_1), \ldots, (Y_m, \beta_m)) \in \mathcal{Y}$ given by a bounded linear mapping*

$$\mathcal{K} : \mathcal{Z} \to \mathcal{Y}, \quad \mathcal{K}(h_1, \ldots, h_m, w_1, \ldots, w_m, J_1, \ldots, J_{m-1}) = ((Y_1, \beta_1), \ldots, (Y_m, \beta_m)).$$

The bound is independent of $y_1, \ldots, y_m$.

We emphasize that the linear problem (L1)–(L2) does not have a unique solution, since we have not specified subspaces in which the Y_j are to lie. The assertion of the Linear Theorem is that there is a precisely defined linear map $\mathcal{K}$ that picks out one solution of (L1)–(L2) for each value of $(h_1, \ldots, h_m, w_1, \ldots, w_m, J_1, \ldots, J_{m-1})$.

Actually, for the linear problem (L1)–(L2), the sum $U_j(\xi, t) = Y_j(\xi, t) + \beta_j(t)q_j'(\xi)$ is uniquely defined, although the pair (Y_j, β_j) is not. Moreover, each $\beta_j(\infty)$ is uniquely defined.

We outline the proof of the Linear Theorem in Section 3, with the proofs of two key lemmas deferred to the remainder of the paper. Given the linear result, one proves the following more precise formulation of Theorem 1.1.

Theorem 2.5 *(Nonlinear Theorem). Assume (A1) and (A2). Let ν be given by (1.4). Fix γ, $\nu < \gamma < 0$, and let $\epsilon > 0$. Then there exist $\chi > 0$ and $\delta > 0$ such that the following is true. If $\min(y_{j+1} - y_j) > \chi$ and $u_0^{ex} \in H^1(\mathbb{R})$ satisfies $\|u_0^{ex}(x) - q_j(x - y_j)\|_{H^1(I_j)} < \delta$ for $j = 1, \ldots, m$, then the nonlinear problem (N1)–(N2) has a solution $((Y_1, \beta_1), \ldots, (Y_m, \beta_m)) \in \mathcal{Y}$. Moreover, for each j, $\|Y_j\|_{H^{2,1}(\tilde{\Omega}_j, \gamma)} < \epsilon$ and $\|\beta_j\|_{X^1(\mathbb{R}^+, \gamma)} < \epsilon$.*

Proof. The argument follows a standard template. A solution of the nonlinear problem (N1)–N(2) is given by the fixed point of the mapping $\mathcal{F} = \mathcal{K} \circ \mathcal{N}$, where $\mathcal{N} : \mathcal{Y} \to \mathcal{Z}$ is given by

$$(Y_j, \beta_j)|_{j=1}^m \to (F_j(\xi_j, Y_j, \beta_j, \dot{\beta}_j)|_{j=1}^m, u^{ex}(\xi_j + y_j, 0) - q_j(\xi_j) - G_j(\xi_j, \beta_j(0))\beta_j^2(0)|_{j=1}^m,$$
$$-[q_j(\xi_j), q_j'(\xi_j)](\Gamma_j) - [G_j(\xi_j, \beta_j)\beta_j^2, G_{j\xi_j}(\xi_j, \beta_j)\beta_j^2](\Gamma_j)|_{j=1}^{m-1}).$$

Note that

$$\begin{aligned}\|[q_j(\xi_j)]\| &= \|q_{j+1}(x_j - y_{j+1} + (\bar{c}_j - c_{j+1})t) - q_j(x_j - y_j + (\bar{c}_j - c_j)t)\| \\ &\le \|q_{j+1}(x_j - y_{j+1} + (\bar{c}_j - c_{j+1})t) - e_j\| + \|e_j - q_j(x_j - y_j + (\bar{c}_j - c_j)t)\| \\ &\le K e^{\mu(x_j - y_{j+1} + (\bar{c}_j - c_{j+1})t)} + K e^{-\mu(x_j - y_j + (\bar{c}_j - c_j)t)} \le \tilde{K} e^{-\frac{1}{2}\mu(c_{j+1} - c_j)t}.\end{aligned}$$

Similar estimates apply to $[q_j'(\xi_j)]$, $[G_j(\xi_j, \beta_j)]$, and $[G_{j\xi_j}(\xi_j, \beta_j)]$. This is the reason for the requirement that $\nu < \gamma$.

For notational simplicity, let (Y, β) denote $(Y_j, \beta_j)|_{j=1}^m$. Then

$$\mathcal{N}(Y, \beta) = (0|_{j=1}^m, u^{ex}(\xi_j + y_j, 0) - q_j(\xi_j)|_{j=1}^m, -[q_j(\xi_j), q_j'(\xi_j)](\Gamma_j)|_{j=1}^{m-1}) + \mathcal{O}(\|(Y, \beta)\|^2).$$

In fact, there are numbers $K > 1$ and $\epsilon > 0$ such that if $\min(y_{j+1} - y_j)$ is sufficiently large and $\max \|u_0^{ex}(x) - q_j(x - y_j)\|_{H^1(I_j)}$ is sufficiently small, then,

(1) $\|\mathcal{N}(Y, \beta) - \mathcal{N}(0, 0)\| \le K\|(Y, \beta)\|^2$ for $\|(Y, \beta)\| \le \epsilon$;
(2) $\|D\mathcal{N}(Y, \beta)\| \le K\|(Y, \beta)\|$ for $\|(Y, \beta)\| \le \epsilon$;
(3) $\|\mathcal{K}\| \le K$;
(4) $\|\mathcal{N}(0, 0)\| \le \frac{\epsilon}{2K}$.

We may assume $\epsilon \le 1/2K^2$. Then $\|\mathcal{F}(0, 0)\| = \|\mathcal{K}\mathcal{N}(0, 0)\| \le \frac{\epsilon}{2}$ and

$$\|\mathcal{F}(Y, \beta) - \mathcal{F}(0, 0)\| \le K\|\mathcal{N}(Y, \beta) - \mathcal{N}(0, 0)\| \le K^2\epsilon^2 \le \frac{\epsilon}{2}.$$

Therefore $\mathcal{F}$ maps the ϵ-ball in $\prod_1^m (H^{2,1}(\tilde{\Omega}_j, \gamma) \times X^1(\mathbb{R}^+, \gamma))$ into itself. For $\|(Y, \beta)\| \le \epsilon$, $\|D\mathcal{F}(Y, \beta)\| \le K\|D\mathcal{N}(Y, \beta)\| \le K^2\epsilon \le \frac{1}{2}$. Therefore $\mathcal{F}$ is a contraction of that ball. The fixed point is a solution of (N1)–(N2). □

3. Proof of the Linear Theorem 2.4

In Section 3.1 we review some facts about a single traveling wave. In Section 3.2, we solve the initial value problem (L1) in each $\tilde{\Omega}_j$, ignoring the jump condition (L2). In Section 3.3, we state a result, the Jump Theorem 3.2, about the solution of the full linear system (L1)–(L2) with $h_j = 0$, $w_j = 0$, and $J_j(0) = 0$, i.e., zero forcing, initial condition zero, and jump at $t = 0$ equal to zero to match the initial condition. Then we combine the two solutions to prove the Linear Theorem 2.4. In Section 3.4 we state two key lemmas, and in Section 3.5 we use these two lemmas to prove the Jump Theorem. The proofs of the two lemmas are given in the remainder of the paper.

3.1. A single traveling wave

The traveling wave $q_j(\xi)$, $\xi = \xi_j = x - y_j - c_j t$, satisfies the ODE

$$q_j'' + c_j q_j' + f(q_j) = 0.$$

The function $(u(\xi), v(\xi)) = (q_j(\xi), q_j'(\xi))$, $-\infty < \xi < \infty$, is a heteroclinic solution of the associated first-order system

$$u_\xi = v, \quad v_\xi = -c_j v - f(q_j) \tag{3.1}$$

that connects the equilibria $(e_{j-1}, 0)$ and $(e_j, 0)$. Assumption (A2) implies that these equilibria are hyperbolic saddles with n eigenvalues having positive real part and n having negative real part.

Let $\mathcal{L}_j^*$ be the adjoint operator for $\mathcal{L}_j$ on $L^2(\mathbb{R})$,

$$\mathcal{L}_j^* z = z_{\xi\xi} - c_j z_\xi + Df(q_j(\xi))^* z. \tag{3.2}$$

The kernel of $\mathcal{L}_j^*$ is spanned by a function z_j. Since q_j' is not in the range of $\mathcal{L}_j$, $\int_{-\infty}^{\infty} \langle z_j, q_j' \rangle d\xi \neq 0$. We choose z_j so that

$$\int_{-\infty}^{\infty} \langle z_j, q_j' \rangle \, d\xi = 1. \tag{3.3}$$

The spectral projection to $\ker(\mathcal{L}_j)$ is

$$\mathcal{P}_j U = \left(\int_{-\infty}^{\infty} \langle z_j, U \rangle d\xi \right) q_j'.$$

We have the spectral decomposition $U = Y + \beta q_j'$ with

$$Y \in R(I - \mathcal{P}_j) = R\mathcal{L}_j = \{Y : \int_{-\infty}^{\infty} \langle z_j, Y \rangle d\xi = 0\}.$$

3.2. Solution of (L1)

In this subsection we consider each $\tilde{\Omega}_j$ separately. Given a forcing function $h_j \in L^2(\tilde{\Omega}_j, \gamma)$ and an initial condition $w_j \in H^1(I_j)$, we look for a solution (Y_j, β_j) of (L1) in $H^{2,1}(\tilde{\Omega}_j, \gamma) \times X^1(\mathbb{R}^+, \gamma)$.

Proposition 3.1. *Assume (A1) and (A2), and assume* $\min(y_{j+1} - y_j)$ *is bounded below by a positive number. Fix* γ, $\eta \le \gamma < 0$. *Then the linear problem (L1) has a solution* $((Y_1^{(1)}, \beta_1^{(1)}), \dots, (Y_m^{(1)}, \beta_m^{(1)})) \in \mathcal{Y}$ *that is given by a bounded linear mapping*

$$\mathcal{K}^{(1)} : \prod_1^m L^2(\Omega_j, \gamma) \times \prod_1^m H^1(I_j) \to \mathcal{Y},$$

$$\mathcal{K}^{(1)}(h_1, \dots, h_m, w_1, \dots, w_m) = ((Y_1^{(1)}, \beta_1^{(1)}), \dots, (Y_m^{(1)}, \beta_m^{(1)})).$$

The bound is independent of $y_1, \dots, y_m$.

We remark that (L1) does not have a unique solution. The assertion of Theorem 3.1 is that there is a precisely defined linear map $\mathcal{K}^{(1)}$ that picks out one solution of (L1) for each value of $(h_1, \dots, h_m, w_1, \dots, w_m)$.

Proof. We shall need linear extension operators from $L^2(\tilde{\Omega}_j, \gamma)$ to $L^2(\mathbb{R} \times \mathbb{R}^+, \gamma)$ and from $H^1(I_j) \to H^1(\mathbb{R})$. Functions in $L^2(\tilde{\Omega}_j, \gamma)$ can be extended to $L^2(\mathbb{R} \times \mathbb{R}^+, \gamma)$ by taking them to be 0 outside $\tilde{\Omega}_j$. For a method of extending functions in $H^1(I_j)$ to $H^1(\mathbb{R})$, see [5], Theorem 6.44. If $\min(y_{j+1} - y_j)$ is bounded below by a positive number, then these extension operators can be chosen to have a uniform bound K independent of the choice of $y_1, \dots, y_m$. We shall denote all such extension operators by $\mathcal{E}$.

Let $\bar{h}_j = \mathcal{E} h_j \in L^2(\mathbb{R} \times \mathbb{R}^+, \gamma)$, and let $\bar{w}_j = \mathcal{E} w_j \in H^1(\mathbb{R})$. Then consider the initial value problem

$$U_t = U_{\xi\xi} + c_j U_\xi + Df(q_j(\xi))U + \bar{h}_j(\xi, t), \ (\xi, t) \in \mathbb{R} \times \mathbb{R}^+; \quad U(\xi, 0) = \bar{w}_j(\xi), \ \xi \in \mathbb{R}. \tag{3.4}$$

The solution of (3.4) can be written as $U(\xi, t) = Y(\xi, t) + \beta(t) q_j'$ with $Y(\cdot, t) \in R(I - \mathcal{P}_j)$. Applying the operators $I - \mathcal{P}_j$ and $\mathcal{P}_j$ to (3.4), we obtain

$$Y_t = Y_{\xi\xi} + c_j Y_\xi + Df(q_j(\xi))Y + (I - \mathcal{P}_j)\bar{h}_j(\xi, t), \qquad Y(\xi, 0) = (I - \mathcal{P}_j)\bar{w}_j, \tag{3.5}$$

$$\dot{\beta} = \mathcal{P}_j \bar{h}_j, \qquad \beta(0) = \mathcal{P}_j \bar{w}_j. \tag{3.6}$$

Now $R(I - \mathcal{P}_j) = R\mathcal{L}_j$ is invariant under $\mathcal{L}_j$, and $\mathcal{L}_j | R(I - \mathcal{P}_j)$ is sectorial and generates an analytic semigroup $e^{\mathcal{L}_j t}$. We have

$$Y(\xi, t) = e^{\mathcal{L}_j t}(I - \mathcal{P}_j)\bar{w}_j(\xi) + \int_0^t e^{\mathcal{L}_j (t-\tau)}(I - \mathcal{P}_j)\bar{h}_j(\xi, \tau)\, d\tau,$$

$$\beta(t) = \mathcal{P}_j \bar{w}_j + \int_0^t \mathcal{P}_j \bar{h}_j(\xi, \tau)\, d\tau.$$

From Lemma 3.11 of [10] it is easy to show that $Y \in H^{2,1}(\mathbb{R} \times \mathbb{R}^+, \gamma)$ and satisfies

$$\|Y\|_{H^{2,1}(\mathbb{R}\times\mathbb{R}^+,\gamma)} \le C_1(\|\bar{w}_j\|_{H^1(\mathbb{R})} + \|\bar{h}_j\|_{L^2(\mathbb{R}\times\mathbb{R}^+,\gamma)}) \le C_1 K(\|w_j\|_{H^1(I_j)} + \|h_j\|_{L^2(\tilde{\Omega}_j,\gamma)}).$$

Also, we clearly have

$$\begin{aligned}\|\beta\|_{X^1(\mathbb{R})} = |\beta(0)| + \|\dot{\beta}\|_{L^2(\mathbb{R}^+,\gamma)} &\le C_2(\|\bar{w}_j\|_{H^1(\mathbb{R})} + \|\bar{h}_j\|_{L^2(\mathbb{R}\times\mathbb{R}^+,\gamma)}) \\ &\le C_2 K(\|w_j\|_{H^1(I_j)} + \|h_j\|_{L^2(\tilde{\Omega}_j,\gamma)}).\end{aligned}$$

Finally, to solve (L1) in $\tilde{\Omega}_j$, we let

$$Y_j^{(1)} = Y|\tilde{\Omega}_j, \quad \beta_j^{(1)} = \beta. \qquad \square$$

3.3. The Jump Theorem and the proof of the Linear Theorem

Consider the following linear problem: given jumps $\tilde{J}_j \in H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$, $j=1,\dots,m-1$, find pairs $(Y_j^{(2)},\beta_j^{(2)}) \in H_0^{2,1}(\tilde{\Omega}_j,\gamma)\times X_0^1(\mathbb{R}^+,\gamma)$, $j=1,\dots,m$, such that the functions $\tilde{U}_j^{(2)}(\xi_j,t)=Y_j^{(2)}(\xi_j,t)+\beta_j^{(2)}q_j'(\xi_j)$ satisfy

$$U_t = U_{\xi\xi}+c_jU_\xi+Df(q_j(\xi))U, \quad (\xi,t)=(\xi_j,t)\in\tilde{\Omega}_j, \tag{3.7}$$

$$U(\xi,0)=0, \quad \xi=\xi_j\in I_j, \tag{3.8}$$

$$[U_j,U_{j\xi})](\Gamma_j)=\tilde{J}_j(\Gamma_j). \tag{3.9}$$

Note that this linear problem has zero forcing and initial condition zero.

Theorem 3.2 *(Jump Theorem). Assume (A1) and (A2). Fix γ, $\eta\le\gamma<0$. If $\min(y_{j+1}-y_j)$ is sufficiently large, then the linear problem (3.7)–(3.9) has a solution $((Y_1^{(2)},\beta_1^{(2)}),\dots,(Y_m^{(2)},\beta_m^{(2)}))\in\mathcal{Y}_0$ that is given by a bounded linear mapping*

$$\mathcal{K}^{(2)}:\prod_1^{m-1}H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)\to\mathcal{Y}_0,\quad \mathcal{K}^{(2)}(\tilde{J}_1,\dots,\tilde{J}_{m-1})=((Y_1^{(2)},\beta_1^{(2)}),\dots,(Y_m^{(2)},\beta_m^{(2)})).$$

The bound is independent of $y_1,\dots,y_m$.

As usual the solution is not unique, but one solution is given by a precisely defined linear map.

Given Proposition 3.1 and Theorem 3.2, the Linear Theorem 2.4 is proved as follows. Given h_j, w_j, and J_j, let $((Y_1^{(1)},\beta_1^{(1)}),\dots,(Y_m^{(1)},\beta_m^{(1)}))$ be the solution of (L1) found in Theorem 3.1. For $j=1,\dots m$, define the function $\tilde{U}_j^{(1)}(\xi_j,t)$ on $\tilde{\Omega}_j$ by $\tilde{U}_j^{(1)}(\xi_j,t)=Y_j^{(1)}(\xi_j,t)+\beta_j^{(1)}q_j'(\xi_j)$. Then define

$$\tilde{J}_j(\Gamma_j)=J_j(\Gamma_j)-[(\tilde{U}_j^{(1)},\tilde{U}_{j\xi}^{(1)})](\Gamma_j).$$

Note that $\tilde{J}_j\in H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$ and

$$\tilde{J}_j(0)=[J_j(0)]-[w](0)=0.$$

Use Theorem 3.2 to find the functions $\tilde{U}_j^{(2)}=Y_j^{(2)}(\xi_j,t)+\beta_j^{(2)}q_j'(\xi_j)$. Set

$$Y_j=Y_j^{(1)}+Y_j^{(2)},\quad \beta_j=\beta_j^{(1)}+\beta_j^{(2)},\quad j=1,\dots,m.$$

3.4. Two lemmas

The proof of the Jump Theorem 3.2 relies on two lemmas. In this subsection we state the two lemmas and show how they yield Theorem 3.2.

Let $N>0$. Assume that $\min(y_{j+1}-y_j)>2N$. In xt-coordinates, let

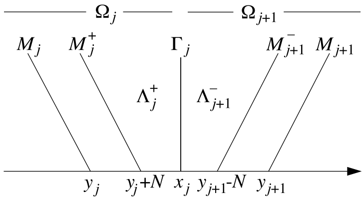

Fig. 3.1. Sets defined in Subsection 3.4.

$$\begin{aligned}
M_j^+ &= \{(x,t) : x = y_j + N + c_j t,\ t \ge 0\}, \quad j = 1,\dots,m-1,\\
M_{j+1}^- &= \{(x,t) : x = y_{j+1} - N + c_{j+1} t,\ t \ge 0\}, \quad j = 1,\dots,m-1,\\
M &= \text{the union of the } M_j^+ \text{ and } M_{j+1}^-, \quad j = 1,\dots,m-1,\\
\Lambda_j^+ &= \text{the open subset of } \Omega_j \text{ between } M_j^+ \text{ and } \Gamma_j, \quad j = 1,\dots,m-1,\\
\Lambda_{j+1}^- &= \text{the open subset of } \Omega_{j+1} \text{ between } \Gamma_j \text{ and } M_{j+1}^-, \quad j = 1,\dots,m-1,\\
\Lambda &= \text{the union of the } \Lambda_j^+ \text{ and } \Lambda_{j+1}^-, \quad j = 1,\dots,m-1.
\end{aligned}$$

See Fig. 3.1.

3.4.1. Tail Lemma

The Tail Lemma, which deals with canceling discontinuities along the Γ_j, is most easily stated in xt-coordinates.

Let $\phi_j \in H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$, $j = 1,\dots,m$. On Λ, i.e., near the union of the Γ_j, we look for a function $U(x,t)$ that satisfies

$$U_t = U_{xx} + Df(q_j(x - y_j - c_j t))U, \quad (x,t) \text{ in } \Lambda_j^+, \tag{3.10}$$

$$U_t = U_{xx} + Df(q_{j+1}(x - y_{j+1} - c_{j+1} t))U, \quad (x,t) \text{ in } \Lambda_{j+1}^-, \tag{3.11}$$

$$U(x,0) = 0, \qquad [(U,U_x)](\Gamma_j) = \phi_j. \tag{3.12}$$

The solution should decay exponentially in t as $t \to \infty$ and in x as (x,t) moves away from Γ_j.

Lemma 3.3 *(Tail Lemma). Assume (A1) and (A2). Fix γ, $\eta \le \gamma < 0$. Then there is a number $N > 0$ such that if $\min(y_{j+1} - y_j) > 2N$, then the linear problem (3.10)–(3.12) has a solution $U(x,t)$ in $H_0^{2,1}(\Lambda,\gamma)$ that is given by a bounded linear mapping*

$$\mathcal{K}^{(3)} : \prod_1^{m-1} H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma) \to H_0^{2,1}(\Lambda,\gamma).$$

The bound is independent of $y_1,\dots,y_m$. There are numbers $\tilde{C} > 0$ and $\tilde{\alpha} > 0$, independent of $y_1,\dots,y_m$, such that the solution satisfies the estimates

$$\|U|_{M_j^+}\| \le \tilde{C} e^{-\tilde{\alpha}(x_j - y_j - N)} \|\phi_j\|, \quad \|U|_{M_{j+1}^-}\| \le \tilde{C} e^{-\tilde{\alpha}(y_{j+1} - N - x_j)} \|\phi_j\|, \tag{3.13}$$

where all the norms are in $H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$.

Note that the assumption $\min(y_{j+1} - y_j) > 2N$ implies that $y_j + N < x_j < y_{j+1} - N$, so $x_j - y_j - N > 0$ and $y_{j+1} - N - x_j > 0$.

3.4.2. Interior Lemma

The Tail Lemma produces discontinuities along the lines in M; the Interior Lemma deals with canceling them. To state the Interior Lemma, we will work in a single moving coordinate $\xi = \xi_j = x - y_j - c_j t$. In ξt-coordinates, let M_a denote the line $\xi = a$. Let $\phi \in H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$. We consider the problem

$$U_t = U_{\xi\xi} + c_j U_\xi + Df(q_j(\xi))U, \quad (\xi, t) \in \mathbb{R} \times \mathbb{R}^+ \setminus M_a), \tag{3.14}$$

$$U(\xi, 0) = 0, \quad [(U, U_\xi)](M_a) = \phi. \tag{3.15}$$

Lemma 3.4 *(Interior Lemma). Assume (A1) and (A2). With N given by Lemma 3.3 and $\min(y_{j+1} - y_j) > 2N$, let $a = -N$ or N. Fix j, and fix γ, $\eta \le \gamma < 0$. Then the linear problem (3.14)–(3.15) has a solution $\tilde{U} = Y_j(\xi, t) + \beta_j(t) q_j'(\xi)$ with*

(1) *$Y_j \in H_0^{2,1}(\mathbb{R} \times \mathbb{R}^+ \setminus M_a, \gamma)$ and $\mathcal{P}_j Y_j(\cdot, t) = 0$ for each t;*
(2) *$\beta_j \in X_0^1(\mathbb{R}^+, \gamma)$.*

The solution is given by a bounded linear mapping

$$\mathcal{K}^{(4)} : H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma) \to H_0^{2,1}(\mathbb{R} \times \mathbb{R}^+ \setminus M_a, \gamma) \times X_0^1(\mathbb{R}^+, \gamma), \quad \mathcal{K}^{(4)}(\phi) = (Y_j, \beta_j).$$

The bound is independent of j and $y_1, \dots, y_m$. There are numbers $C > 0$ and $\alpha > 0$, independent of $y_1, \dots, y_m$, such that the solution satisfies the estimates

$$\|\tilde{U}|_{\Gamma_{j-1}}\| \le Ce^{-\alpha(y_j + a - x_{j-1})}\|\phi\|, \quad \|\tilde{U}|_{\Gamma_j}\| \le Ce^{-\alpha(x_j - y_j - a)}\|\phi\|, \tag{3.16}$$

where all the norms are in $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$.

Note that the assumption $\min(y_{j+1} - y_j) > 2N$ implies that $x_{j-1} < y_j + a < x_j$, so $y_j + a - x_{j-1} > 0$ and $x_j - y_j - a > 0$.

3.5. Proof of the Jump Theorem

We use the Tail Lemma 3.3 and the Interior Lemma 3.4 to prove the Jump Theorem 3.2 by an iterative procedure. We work in xt-coordinates, in which the system (3.7)–(3.9) becomes

$$U_t = U_{xx} + Df(q_j)U, \quad (x, t) \in \Omega_j, \quad j = 1, \dots, m, \tag{3.17}$$

$$U(x, 0) = 0, \tag{3.18}$$

$$[(U, U_x)](\Gamma_j) = \tilde{J}_j(\Gamma_j). \tag{3.19}$$

Let

$$\Delta = \max_{j=1,\dots,m} \|\tilde{J}_j(\Gamma_j)\|_{H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)}.$$

Choose N so that the Tail Lemma 3.3 is true, then let $\min(y_{j+1} - y_j)$ be sufficiently large so that for $j = 1, \dots, m$, the coefficients $\tilde{C}e^{-\tilde{\alpha}(x_j - y_j - N)}$ and $\tilde{C}e^{-\tilde{\alpha}(y_{j+1} - N - x_j)}$ in (3.13), and $Ce^{-\alpha(y_j + a - x_{j-1})}$ and $Ce^{-\alpha(x_j - y_j - a)}$ in (3.16), with $a = \pm N$, are at most $\kappa < \frac{1}{2}$, so that $4\kappa^2 < 1$. Let K be the larger of the bounds for the linear maps $\mathcal{K}^{(3)}$ of the Tail Lemma and $\mathcal{K}^{(4)}$ of the Interior Lemma. K is independent of j and $y_1, \dots, y_m$.

For each $j = 1, \dots, m$, given the jump $\tilde{J}_j \in H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$, we set $\tilde{J}_j^{[0]} = \tilde{J}_j$ and use the Tail Lemma with $\phi_j = \tilde{J}_j^{[0]}$. For each j, the Tail Lemma produces a function $U_j^{\text{tail}+\,[0]}(x,t)$ defined on Λ_j^+, and a function $U_{j+1}^{\text{tail}-\,[0]}(x,t)$ defined on Λ_{j+1}^-; see Fig. 3.1. We extend each to be 0 elsewhere, and we set $U^{\text{tail}\,[0]} = \sum_{j=1}^m \left(U_j^{\text{tail}+[0]} + U_{j+1}^{\text{tail}-[0]}\right)$. $U^{\text{tail}\,[0]}(x,t)$ is a solution of the homogeneous problem (3.17)–(3.18) on the complement of the M_j^+, Γ_j, and M_{j+1}^-. For each $j = 1, \dots, m$, it has the correct jump $\tilde{J}_j^{[0]}$ along Γ_j, but has undesired jumps along M_j^+ and M_{j+1}^-, which we denote $-\phi_j^{+[0]}$ and $-\phi_{j+1}^{-[0]}$ respectively. In $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$, we have

$$\max(\|\phi_j^{+[0]}\|, \|\phi_{j+1}^{-[0]}\|) \le \kappa \|\tilde{J}_j^{[0]}\| \le \kappa\Delta.$$

To eliminate the undesired jump along M_j^+ (resp. M_{j+1}^-), we use the Interior Lemma 3.4 with $a = N$ (resp. with j replaced by $j+1$ and $a = -N$) and $\phi = \phi_j^{+[0]}$ (resp. $\phi = \phi_{j+1}^{-[0]}$). We denote by $U_j^{\text{int}+[0]}$ (resp. $U_{j+1}^{\text{int}-[0]}$) the solution produced by Lemma 3.4, translated into xt-coordinates, then restricted to Ω_j (resp. Ω_{j+1}) and extended to be 0 outside Ω_j (resp. Ω_{j+1}). We set $U^{\text{int}\,[0]} = \sum_{j=1}^m \left(U_j^{\text{int}+[0]} + U_{j+1}^{\text{int}-[0]}\right)$. $U^{\text{int}\,[0]}(x,t)$ is a solution of the homogeneous problem (3.17)–(3.18) on the complement of the M_j^+, Γ_j, and M_{j+1}^-. For $j = 1, \dots, m$, its jumps along M_j^+ and M_{j+1}^- are minus those of $U_j^{\text{tail}\,[0]}(x,t)$, but it has a small undesired jump along Γ_j, which we denote $-\tilde{J}_j^{[1]}$. In $H^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$, we have

$$\|\tilde{J}_j^{[1]}\| \le \kappa(\|\phi_j^{-[0]}\| + \|\phi_j^{+[0]}\| + \|\phi_{j+1}^{-[0]}\| + \|\phi_{j+1}^{+[0]}\|) \le 4\kappa^2\Delta.$$

For $j = 1$, the first of the four summands is missing, and for $j = m$, the last of the four summands is missing.

The function $U^{[0]} = U^{\text{tail}\,[0]} + U^{\text{int}\,[0]}$ is a solution of the homogeneous problem (3.17)–(3.18) in the complement of the Γ_j. For each j, its jump along Γ_j is $\tilde{J}_j^{[0]} - \tilde{J}_j^{[1]}$.

We now iterate the procedure, beginning by using the Tail Lemma for each j with $\phi_j = \tilde{J}_j^{[1]}$. For each $j = 1, \dots, m$ we obtain

- sequences of functions $U_j^{\text{tail}+\,[k]}(x,t)$ defined on Λ_j^+ and $U_{j+1}^{\text{tail}-\,[k]}(x,t)$ defined on Λ_{j+1}^-, $k = 0, 1, 2, \dots$;
- sequences of jumps $\phi_j^{+\,[k]}$ defined on M_j^+ and $\phi_{j+1}^{-\,[k]}$ defined on M_{j+1}^-, $k = 0, 1, 2, \dots$;
- sequences of functions $U_j^{\text{int}+[k]}$ defined on $\Omega_j \setminus M_j^+$ and $U_{j+1}^{\text{int}-[0]}$ defined on $\Omega_{j+1} \setminus M_{j+1}^-$, $k = 0, 1, 2, \dots$.

We set

$$U^{\mathrm{tail}\,[k]} = \sum_{j=1}^{m} \left(U_j^{\mathrm{tail}+[k]} + U_{j+1}^{\mathrm{tail}-[k]} \right), \quad U^{\mathrm{int}\,[k]} = \sum_{j=1}^{m} \left(U_j^{\mathrm{int}+[k]} + U_{j+1}^{\mathrm{int}-[k]} \right),$$

$U^{[k]} = U^{\mathrm{tail}\,[k]} + U^{\mathrm{int}\,[k]}$. $U^{[k]}$ is a solution of the homogeneous problem (3.17)–(3.18) in the complement of the Γ_j. Its jump along Γ_j is $\tilde{J}_j^{[k]} - \tilde{J}_j^{[k+1]}$, where

$$\|\tilde{J}_j^{[k]}\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)} \le (4\kappa^2)^k \Delta. \tag{3.20}$$

We wish to obtain the solution whose existence is asserted in the Jump Theorem by summing the $U^{[k]}$. The summation will be done in each $\tilde{\Omega}_j$. The restriction of $U^{[k]}$ to Ω_j corresponds, in $\xi_j t$-coordinates, to

$$\tilde{U}_j^{[k]} = \tilde{U}_j^{\mathrm{tail}-\,[k]} + \tilde{U}_j^{\mathrm{tail}+\,[k]} + \tilde{U}_j^{\mathrm{int}-\,[k]} + \tilde{U}_j^{\mathrm{int}+\,[k]}.$$

Given a subset S of Ω_j, we use $\tilde{S}$ to denote the corresponding subset of $\tilde{\Omega}_j$. With this notation, the first (resp. second) summand is nonzero only on $\tilde{\Lambda}_j^-$ (resp. $\tilde{\Lambda}_j^+$), and the third (resp. fourth) summand has a jump along $\tilde{M}_j^-$ (resp. $\tilde{M}_j^+$). Nevertheless the sum has no jumps along $\tilde{M}_j^{\pm}$. For $j = 1$, the first and third summands are missing; for $j = m$, the second and fourth summands are missing.

The functions $\tilde{U}_j^{\mathrm{int}\pm\,[k]}$ produced by the Interior Lemma are in the form $\tilde{U}_j^{\mathrm{int}\pm\,[k]} = Y_j^{\pm\,[k]}(\xi_j, t) + \beta_j^{\pm\,[k]} q_j'(\xi_j)$, with $Y_j^{\pm\,[k]} \in H_0^{2,1}(\tilde{\Omega}_j \setminus M_j^{\pm}, \gamma)$ and $\beta_j^{\pm\,[k]} \in X_0^1(\mathbb{R}^+, \gamma)$. We set

$$Y_j^{[k]} = \tilde{U}_j^{\mathrm{tail}-\,[k]} + \tilde{U}_j^{\mathrm{tail}+\,[k]} + Y_j^{-\,[k]} + Y_j^{+\,[k]}, \quad \beta_j^{[k]} q_j' = \beta_j^{-\,[k]} q_j' + \beta_j^{+\,[k]} q_j'.$$

Using $\|\cdot\|$ to denote $\|\cdot\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}$, from the Tail Lemma, the Interior Lemma, and (3.20), we have

$$\max\left(\|\tilde{U}_j^{\mathrm{tail}+\,[k]}\|_{H^{2,1}(\tilde{\Lambda}_j^+,\gamma)}, \|\tilde{U}_{j+1}^{\mathrm{tail}-\,[k]}\|_{H^{2,1}(\tilde{\Lambda}_{j+1}^-,\gamma)} \right) \le K \|\tilde{J}_j^{[k]}\| \le K (4\kappa^2)^k \Delta, \tag{3.21}$$

$$\max\left(\|Y_j^{\pm\,[k]}\|_{H^{2,1}(\tilde{\Omega}_j \setminus M_j^{\pm},\gamma)}, \|\beta_j^{\pm\,[k]}\|_{X^1(\mathbb{R}^+,\gamma)} \right) \le K \|\phi_j^{\pm[k]}\| \le K\kappa \|\tilde{J}_j^{[k]}\| \le K\kappa (4\kappa^2)^k \Delta. \tag{3.22}$$

Finally we set

$$Y_j = \sum_{k=1}^{\infty} Y_j^{[k]}, \quad \beta_j q_j' = \sum_{k=1}^{\infty} \beta_j^{[k]} q_j', \quad \tilde{U}_j = Y_j + \beta_j q_j'.$$

Now each summand of $Y_j^{[k]}$ is in $H_0^{2,1}(\tilde{S},\gamma)$ for an open subset $\tilde{S}$ of $\tilde{\Omega}_j$ bounded by a vertical line, and by construction the sum $Y_j^{[k]}$ has no jumps along those lines. Hence by Lemma 2.3 the sum is in $H_0^{2,1}(\tilde{\Omega}_j,\gamma)$. From estimates (3.21)–(3.22) and $\kappa < \frac{1}{2}$,

$$\|Y_j^{[k]}\|_{H^{2,1}(\tilde{\Omega}_j,\gamma)} \leq 4K(4\kappa^2)^k\Delta$$

Since $4\kappa^2 < 1$, Y_j is in $H_0^{2,1}(\tilde{\Omega}_j,\gamma)$. Similarly, using (3.22), β_j is in $X_0^1(\mathbb{R}^+,\gamma)$. The jump in the $\tilde{U}_j$ across Γ_j is

$$\sum_{k=1}^{\infty}\left(\tilde{J}_j^{[k]} - \tilde{J}_j^{[k+1]}\right) = \tilde{J}_j^{0]} = \tilde{J}_j$$

as desired. (The series converges by (3.20).) Moreover, independent of j,

$$\max\left(\|Y_j\|_{H^{2,1}(\tilde{\Omega}_j,\gamma)}, \beta_j\|_{X_0^1(\mathbb{R}^+,\gamma)}\right) \leq \frac{4K}{1-4\kappa^2}\Delta.$$

4. Exponential dichotomies and Laplace transform

In this section we gather material we will need to prove the Tail Lemma and Interior Lemma. The first four subsections are general; the last gives a lemma about the Laplace transform of the linear differential equation (1.2).

4.1. Exponential dichotomies

Let us consider a linear differential equation $U_\xi = L(\xi)U$, $\xi \in I$, on a Banach space E. Here $I \subset \mathbb{R}$ is an interval, and $L(\xi) : E \to E$ is a linear operator for each $\xi \in I$, but it may be unbounded, and its domain may be a proper subspace of E. The solution operator $U(\xi) = T(\xi,\zeta)U(\zeta)$ may have a domain that depends on the pair (ξ,ζ). Of course, when E is finite dimensional, $L(\xi)$ is bounded for each ξ, and $T(\xi,\zeta)$ has domain E for each (ξ,ζ).

Definition 4.1. We say $U_\xi = L(\xi)U$ has an exponential dichotomy on E for $\xi \in I$ if there exist bounded projections $P_s(\xi) + P_u(\xi) = I$ in E, continuous in $\xi \in I$, and constants K, $\rho > 0$, such that the solution operator $T(\xi,\zeta)$ satisfies

$$T(\xi,\zeta) : RP_s(\zeta) \to RP_s(\xi) \text{ is defined for } \xi \geq \zeta; \tag{4.1a}$$

$$T(\xi,\zeta) : RP_u(\zeta) \to RP_u(\xi) \text{ is defined for } \xi \leq \zeta; \tag{4.1b}$$

$$\|T(\xi,\zeta)P_s(\zeta)\| \leq Ke^{-\rho|\xi-\zeta|}, \quad \xi \geq \zeta; \tag{4.1c}$$

$$\|T(\xi,\zeta)P_u(\zeta)\| \leq Ke^{-\rho|\xi-\zeta|}, \quad \xi \leq \zeta. \tag{4.1d}$$

We use the notation $E_s(\xi) = RP_s(\xi)$, $E_u(\xi) = RP_u(\xi)$. In general exponential dichotomies are not unique. However, if $I = (-\infty,\xi_0]$, then the unstable subspace $E_u(\xi)$ is independent of the dichotomy chosen, and if $I = [\xi_0,\infty)$, then the stable subspace $E_s(\xi)$ is independent of the dichotomy chosen.

The following result gives the basic facts about persistence of exponential dichotomies under perturbation.

Theorem 4.1 *(Roughness of exponential dichotomies). Let I be an interval, and let $U_\xi = L(\xi)U$, $\xi \in I$, be a linear differential equation on a Banach space E. Assume that $U_\xi = L(\xi)U$ has an exponential dichotomy on I with projections $P_s^0(\xi) + P_u^0(\xi) = I$ and constants K_0, $\rho_0 > 0$. Let $B(\xi) : E \to E$ be a bounded linear operator for each $\xi \in I$, with $B \in L^\infty(I)$. Let $\delta = \sup_{\xi \in I} \|B(\xi)\| < \infty$.*

Consider the perturbed linear equation

$$U_\xi = (L(\xi) + B(\xi))U. \tag{4.2}$$

Let $0 < \tilde{\rho} < \rho_0$, and assume that δ is sufficiently small so that

$$C_1\delta < 1 \text{ and } C_2\delta < 1, \text{ where } C_1 = \frac{2K_0}{\rho_0 - \tilde{\rho}},\ C_2 = \frac{2K_0^2}{(\rho_0 - \tilde{\rho})(1 - C_1\delta)}. \tag{4.3}$$

Then (4.2) also has an exponential dichotomy on I with projections $\tilde{P}_s(\xi) + \tilde{P}_u(\xi) = I$ and the exponent $\tilde{\rho}$. The multiplicative constant is $\tilde{K} = K_0(1 - C_1\delta)^{-1}(1 - C_2\delta)^{-1}$, and

$$\|\tilde{P}_s(\xi) - P_s^0(\xi)\| \le \frac{C_2\delta}{1 - C_2\delta}.$$

If E is finite-dimensional, then the proof of Theorem 4.1 is well-known [2]. If E is infinite-dimensional, the proof must be adjusted because the solution operator is usually not invertible on E; see [8,9]. For a shorter proof in the infinite-dimensional case, see [12] (simply replace the rate function $a(x)$ by e^x and the decay rate $(a(x)/a(y))^{-\rho}$ by $e^{-\rho(x-y)}$).

4.2. Second-order linear PDEs

Let I be an interval. Consider a second-order linear partial differential equation on $\mathbb{C}^n$ with zero initial conditions, of the form

$$U_t = U_{\xi\xi} + cU_\xi + A(\xi, t)U, \quad (\xi, t) \in I \times \mathbb{R}^+; \quad (U, U_\xi)(\xi, 0) = 0. \tag{4.4}$$

We assume that $A(\xi, t)$ defines a bounded, piecewise continuous mapping from $\xi \in \mathbb{R}$ to $C^1(\mathbb{R}_+)$.

Applying Laplace transform $\mathcal{L}$ in t and writing $\hat{U}(\xi, s) = \mathcal{L}U(\xi, t)$, we obtain

$$\hat{U}_{\xi\xi} = s\hat{U} - c\hat{U}_\xi - \left(\hat{A}(\xi, \cdot) \overset{s}{*} \hat{U}(\xi, \cdot)\right)(s). \tag{4.5}$$

The convolution is performed along a vertical line $\Re(p) = \sigma$ in $\mathbb{C}$ where the following integral converges:

$$\left(\hat{A}(\xi, \cdot) \overset{s}{*} \hat{U}(\xi, \cdot)\right)(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{A}(\xi, p)\hat{U}(\xi, s - p)\, dp.$$

We have $\left(\hat{A}(\xi, \cdot) \overset{s}{*} \hat{U}(\xi, \cdot)\right)(s) = \mathcal{L}(A(\xi, t)\mathcal{L}^{-1}\hat{U}(\xi, s))$.

Converting (4.4) and (4.5) to the equivalent first-order systems, we obtain

$$U_\xi = V, \qquad V_\xi = U_t - cV - A(\xi,t)U, \quad (U,V)(\xi,0) = (0,0); \tag{4.6}$$

$$\hat{U}_\xi = \hat{V}, \qquad \hat{V}_\xi = s\hat{U} - c\hat{V} - \left(\hat{A}(\xi,\cdot) \overset{s}{*} \hat{U}(\xi,\cdot)\right)(s). \tag{4.7}$$

We shall regard (4.6) as a linear differential equation in ξ on the Banach space $H_0^{(k+0.5)\times k}(\mathbb{R}^+,\gamma)$, $0 \le k \le 0.25$, which is a space of functions of t. Because of Lemma 2.2, the choice $k = 0.25$ is most natural, but some of our arguments will require this greater generality.

We wish to regard (4.7) as a linear differential equation in ξ on a space of functions of s.

We recall that a function $f(s)$ is in the Hardy–Lebesgue class $\mathcal{H}(\gamma)$, $\gamma \in \mathbb{R}$, if

(1) $f(s)$ is analytic in $\Re(s) > \gamma$;
(2) $\sup_{\sigma>\gamma}(\int_{-\infty}^{\infty} \|f(\sigma + i\omega)\|^2\, d\omega)^{1/2} < \infty$.

$\mathcal{H}(\gamma)$ is a Banach space with norm defined by the left side of (2).

According to the Paley–Wiener Theorem [28], $u(t) \in L^2(\mathbb{R}^+,\gamma)$ if and only if its Laplace transform $\hat{u}(s) \in \mathcal{H}(\gamma)$, and the mapping $u \to \hat{u}$ is a Banach space isomorphism.

For $k, k_1, k_2 \ge 0$ and $\gamma \in \mathbb{R}$, let

$$\mathcal{H}^k(\gamma) = \{u(s) : u(s) \text{ and } (s-\gamma)^k u(s) \in \mathcal{H}(\gamma)\},$$

$$\|u\|_{\mathcal{H}^k(\gamma)} = \|u\|_{\mathcal{H}(\gamma)} + \|(s-\gamma)^k u\|_{\mathcal{H}(\gamma)},$$

$$\mathcal{H}^{k_1\times k_2}(\gamma) = \mathcal{H}^{k_1}(\gamma) \times \mathcal{H}^{k_2}(\gamma).$$

An equivalent norm on $\mathcal{H}^k(\gamma)$ is

$$\|u\|_{\mathcal{H}^k(\gamma)} = \sup_{\sigma>\gamma}\left(\int_{-\infty}^{\infty} \|u(\sigma+i\omega)\|^2(1+|\sigma+i\omega|^{2k})\, d\omega\right)^{1/2}.$$

It can be shown that $u(t) \in H_0^k(\mathbb{R}^+,\gamma)$ if and only if $\hat{u}(s) \in \mathcal{H}^k(\gamma)$, and the mapping $u \to \hat{u}$ is a Banach space isomorphism. It follows that $(u,v) \in H_0^{k_1\times k_2}(\mathbb{R}^+,\gamma)$, $k_1, k_2 \ge 0$, if and only if $(\hat{u},\hat{v}) \in \mathcal{H}^{k_1\times k_2}(\gamma)$, and the mapping $(u,v) \to (\hat{u},\hat{v})$ is a Banach space isomorphism.

We shall regard (4.7) as a linear differential equation in ξ on the Banach space $\mathcal{H}^{(k+0.5)\times k}(\gamma)$, $0 \le k \le 0.25$.

The following lemma is a consequence of the fact that $\mathcal{L}$ is an isomorphism from $H_0^{(k+0.5)\times k}(\mathbb{R}^+,\gamma)$ to $\mathcal{H}^{(k+0.5)\times k}(\gamma)$.

Lemma 4.2. *The following are equivalent.*

(1) (4.7) *has an exponential dichotomy on* $\mathcal{H}^{(k+0.5)\times k}(\gamma)$ *for* $\xi \in I$*, with projections* $P_s(\xi) + P_u(\xi) = I$ *and solution operator* $T(\xi,\zeta)$.
(2) (4.6) *has an exponential dichotomy on* $H_0^{(k+0.5)\times k}(\mathbb{R}^+,\gamma)$ *for* $\xi \in I$*, with projections* $\check{P}_s(\xi) + \check{P}_u(\xi) = I$ *and solution operator* $\check{T}(\xi,\zeta)$.

Moreover, $\check{P}_j(\xi) = \mathcal{L}^{-1} P_j(\xi)\mathcal{L}$, $j = s, u$; $\check{T}(\xi, \zeta) = \mathcal{L}^{-1} T(\xi, \zeta)\mathcal{L}$ *for* $\xi, \zeta \in I$; *and the constants* K, ρ *for the two dichotomies are equal.*

An important question is whether the solution operator $\check{T}(\xi, \zeta)$ can be used to define a solution to (4.4) that is in $H^{2,1}$. For now we deal with this question by making a definition.

Definition 4.2. (1) We say that (4.4) has property (S) on $[a, \infty)$ if (4.6) has an exponential dichotomy on $H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$ for $\xi \in [a, \infty)$, with projections $\check{P}_s(\xi) + \check{P}_u(\xi) = I$ and solution operator $\check{T}(\xi, \zeta)$, and there is a number $C > 0$ such that the following is true. Let $\xi_0 \geq a$ and let $\phi \in R\check{P}_s(\xi_0)$. For $\xi > \xi_0$, define $(U, V)(\xi, t) = \check{T}(\xi, \xi_0)\phi$. Then $U \in H_0^{2,1}((\xi_0, \infty) \times \mathbb{R}^+, \gamma)$ and is a solution to (4.4). Moreover,

$$\|U\|_{H^{2,1}((\xi_0,\infty)\times\mathbb{R}^+,\gamma)} \leq C\|\phi\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}.$$

(2) Similarly, we say that (4.4) has property (S) on $(-\infty, a]$ if (4.6) has an exponential dichotomy on $H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$ for $\xi \in (-\infty, a]$, with projections $\check{P}_s(\xi) + \check{P}_u(\xi) = I$ and solution operator $\check{T}(\xi, \zeta)$, and there is a number $C > 0$ such that the following is true. Let $\xi_0 \leq a$ and let $\phi \in R\check{P}_u(\xi_0)$. For $\xi < \xi_0$, define $(U, V)(\xi, t) = \check{T}(\xi, \xi_0)\phi$. Then $U \in H_0^{2,1}((-\infty, \xi_0) \times \mathbb{R}^+, \gamma)$ and is a solution to (4.4). Moreover,

$$\|U\|_{H^{2,1}((-\infty,\xi_0)\times\mathbb{R}^+,\gamma)} \leq C\|\phi\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}.$$

Lemma 4.3. *(1) Assume that (4.4) has property (S) on* $[a, \infty)$. *Let* $x_0 \geq a$ *and* $c \leq 0$. *Then there is a number* $\tilde{C} > 0$, *which depends only on the constants* K, α *of the dichotomy, such that*

$$\|U|_{\ell(x_0,c)}\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)} \leq \tilde{C}(1 + |c|)e^{-\alpha(x_0-a)}\|\phi\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}.$$

(2) Assume that (4.4) has property (S) on $(-\infty, a]$. *Let* $x_0 \leq a$ *and* $c \geq 0$. *Then there is a number* $\tilde{C} > 0$, *which depends only on the constants* K, α *of the dichotomy, such that*

$$\|U|_{\ell(x_0,c)}\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)} \leq \tilde{C}(1 + |c|)e^{-\alpha(a-x_0)}\|\phi\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}.$$

Proof. We just prove (1). The mapping that takes $\phi \in H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$, thought of as a space of functions on the line $x = a$, to $U|_{\ell(x_0,c)} \in H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$, thought of as a space of functions on the line $\ell(x_0, c)$, is a composition of three mappings:

$$\phi \to U|_{x=x_0} \text{ from } H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma) \text{ to } H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma);$$

$$U|_{x=x_0} \to U|_{(x_0,\infty)\times\mathbb{R}^+} \text{ from } H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma) \text{ to } H_0^{2,1}((x_0, \infty) \times \mathbb{R}^+, \gamma);$$

$$U|_{(x_0,\infty)\times\mathbb{R}^+} \to U|_{\ell(x_0,c)} \text{ from } H_0^{2,1}((x_0, \infty) \times \mathbb{R}^+, \gamma) \text{ to } H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma).$$

The norm of the first map is at most $Ke^{-\alpha(b-a)}$, the norm of the second is given by Definition 4.2(1), and the norm of the third is given by Lemma 2.2. □

4.3. $A(\xi)$ independent of t

Following [10], we introduce the following families of norms on $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$.

Definition 4.3. Let $\|u\|$ denote the usual norm on $\mathbb{C}^n$. For $s \in \mathbb{C}$ and $k_1 \geq 0$, let $E^{k_1}(s)$ denote $\mathbb{C}^n$ with the norm

$$\|u\|_{E^{k_1}(s)} = (1 + |s|^{k_1})\|u\|,$$

and let $E^{k_1 \times k_2}(s)$ denote $\mathbb{C}^n \times \mathbb{C}^n$ with the norm

$$\|(u, v)\|_{E^{k_1 \times k_2}(s)} = (1 + |s|^{k_1})\|u\| + (1 + |s|^{k_2})\|v\|.$$

We can use these norms to define equivalent norms on $\mathcal{H}^{k_1}(\gamma)$ and $\mathcal{H}^{k_1 \times k_2}(\gamma)$:

$$\|u\|_{\mathcal{H}^{k_1}(\gamma)} = \sup_{\sigma > \gamma} \left(\int_{-\infty}^{\infty} \|u(\sigma + i\omega)\|^2_{E^{k_1}(\sigma + i\omega)} \, d\omega \right)^{1/2}, \tag{4.8}$$

$$\|(u, v)\|_{\mathcal{H}^{k_1 \times k_2}(\gamma)} = \sup_{\sigma > \gamma} \left(\int_{-\infty}^{\infty} \|(u, v)(\sigma + i\omega)\|^2_{E^{k_1 \times k_2}(\sigma + i\omega)} \, d\omega \right)^{1/2}. \tag{4.9}$$

If $A(\xi, t) = A(\xi)$ is independent of time t, then (4.4), (4.6), and (4.7) simplify to

$$U_t = U_{\xi\xi} + cU_\xi + A(\xi)U, \quad (U, U_\xi)(\xi, 0) = 0; \tag{4.10}$$

$$U_\xi = V, \quad V_\xi = U_t - cV - A(\xi)U, \quad (U, V)(\xi, 0) = (0, 0); \tag{4.11}$$

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c\hat{V} - A(\xi)\hat{U}. \tag{4.12}$$

We can regard (4.12) either as an ordinary differential equation on a space of functions of s, or as a family of ordinary differential equations in $\xi \in I$ on $\mathbb{C}^n$, with parameter s in a set $\mathcal{S} \subset \mathbb{C}$, having solution operator $T(\xi, \zeta, s)$.

Definition 4.4. We say that (4.12) has an *s-dependent exponential dichotomy* for $s \in \mathcal{S}$ and $\xi \in I$ if for each $s \in \mathcal{S}$, (4.12) has an exponential dichotomy on $\mathbb{C}^n \times \mathbb{C}^n$ for $\xi \in I$, and in addition the projections $P_j(\xi, s)$, $j = s, u$, are analytic in s for fixed ξ. In the dichotomy, the constants $K(s)$ and $\rho(s)$ depend on s.

We say that (4.12) has a *uniform exponential dichotomy* on the spaces $E^{(k+0.5)\times k}(s)$ for $s \in \mathcal{S}$ and $\xi \in I$ if it has an s-dependent exponential dichotomy, and there are constants $K, \alpha > 0$ such that, when norms in the spaces $E^{(k+0.5)\times k}(s)$ are used,

(1) each $K(s) \leq K$, and
(2) $\rho(s) = \alpha(1 + |s|^{0.5})$.

The following lemma is proved in [10] (Lemma 3.1).

Lemma 4.4. *Let $0 \le k \le 0.25$. Suppose (4.12) has a uniform exponential dichotomy on the spaces $E^{(k+0.5)\times k}(s)$ for $\Re(s) \ge \gamma$ and $\xi \in I$. Then (4.12) has an exponential dichotomy on $\mathcal{H}^{(k+0.5)\times k}(\gamma)$ for $\xi \in I$ with projections derived from those in $E^{(k+0.5)\times k}(s)$, multiplicative constant K, and exponent α. Moreover, (4.10) has property (S) on both $[a, \infty)$ and $(-\infty, a]$.*

4.4. $A(\xi, t)$ has small dependence on t

If $A(\xi, t) = A(\xi) + B(\xi, t)$, then (4.4), (4.6), and (4.7) become

$$U_t = U_{\xi\xi} + cU_\xi + A(\xi)U + B(\xi, t)U, \quad (U, U_\xi)(\xi, 0) = 0; \tag{4.13}$$

$$U_\xi = V, \quad V_\xi = U_t - cV - A(\xi)U - B(\xi, t)U, \quad (U, V)(\xi, 0) = (0, 0), \tag{4.14}$$

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c\hat{V} - A(\xi)\hat{U} - \left(\hat{B}(\xi, \cdot) \overset{s}{*} \hat{U}(\xi, \cdot)\right)(s). \tag{4.15}$$

Lemma 4.5. *Let $0 \le k \le 0.25$. Suppose the t-independent system (4.11) has an exponential dichotomy on $H_0^{(k+0.5)\times k}(\mathbb{R}^+, \gamma)$ for $\Re(s) \ge \gamma$ and $\xi \in I$. Let $K > 0$ and $\alpha > 0$ be the constant and exponential for the dichotomy. Let $\tilde{K} > K$ and $0 < \tilde{\alpha} < \alpha$. Then there is a constant $\delta > 0$ such that the following is true. If $B(\xi, t)$ defines a bounded, piecewise continuous mapping from $\xi \in \mathbb{R}$ to $C^1(\mathbb{R}_+)$, with $\sup \|B(\xi, t)\| < \delta$ and $\sup \|B_t(\xi, t)\| < \delta$, then the system (4.14) also has an exponential dichotomy on $H_0^{(k+0.5)\times k}(\mathbb{R}^+, \gamma)$ for $\xi \in I$, with constant $\tilde{K}$ and exponent $\tilde{\alpha}$.*

Proof. It is straightforward to show that for every ξ, the mapping $U(\cdot) \to B(\xi, \cdot)U(\cdot)$ satisfies

$$\|B(\xi, \cdot)U(\cdot)\|_{L^2(\mathbb{R}^+,\gamma)} \le \delta\|U\|_{L^2(\mathbb{R}^+,\gamma)}, \quad \|B(\xi, \cdot)U(\cdot)\|_{H_0^1(\mathbb{R}^+,\gamma)} \le 2\delta\|U\|_{H_0^1(\mathbb{R}^+,\gamma)}. \tag{4.16}$$

Expressed as the interpolation of two spaces, $H_0^k(\mathbb{R}^+, \gamma) = [L^2(\mathbb{R}^+, \gamma), H_0^1(\mathbb{R}^+, \gamma)]_k$. Therefore we can use the interpolation inequality [15,16] to show that

$$\|B(\xi, \cdot)\|_{H_0^k(\mathbb{R}^+,\gamma)} \le (\|B(\xi, \cdot)\|_{L^2(\mathbb{R}^+,\gamma)})^{1-k} \cdot (\|B(\xi, \cdot)\|_{H_0^1(\mathbb{R}^+,\gamma)})^k \le 2\delta.$$

If δ is sufficiently small, all the conditions in Theorem 4.1 are satisfied. Hence the perturbed system (4.14) has an exponential dichotomy on $H_0^{(k+0.5)\times k}(\mathbb{R}^+, \gamma)$ for $\xi \in I$. $\square$

Suppose that for both $k = 0$ and $k = 0.25$, the t-independent system (4.12) has a uniform exponential dichotomy on the spaces $E^{0.5\times 0}(s)$ for $\Re(s) \ge \gamma$ and $\xi \in \mathbb{R}$. We may assume that these dichotomies have the same constant $K > 0$ and exponent $\alpha > 0$. By Lemmas 4.4 and 4.2, for both $k = 0$ and $k = 0.25$, the system (4.12) has an exponential dichotomy on $H_0^{(k+0.5)\times k}(\mathbb{R}^+, \gamma)$ for $\xi \in \mathbb{R}$ with constant K and exponent α. Therefore the hypotheses of Lemma 4.5 are satisfied for $k = 0$ and $k = 0.25$ with $I = \mathbb{R}$. Let $\tilde{K} > K$ and $0 < \tilde{\alpha} < \alpha$, and let $\delta > 0$ be a number given by Lemma 4.5 that works for both $k = 0$ and $k = 0.25$.

Lemma 4.6. *In the above situation, assume $\sup \|B(\xi, t)\| < \delta$ and $\sup \|B_t(\xi, t)\| < \delta$. Let $a \in \mathbb{R}$. Then (4.13) has property (S) on both $[a, \infty)$ and $(-\infty, a]$.*

Proof. We shall only consider the interval $[a, \infty)$.

By Lemmas 4.5 and 4.2, the perturbed system (4.15) has an exponential dichotomy on $\mathcal{H}^{0.75\times 0.25}(\gamma)$ for $\xi \in \mathbb{R}$ with constant $\tilde{K}$ and exponent $\tilde{\alpha}$. Let $P_s(\xi)$ and $P_u(\xi)$ denote the projections for this dichotomy. Let $\xi_0 \geq a$; let $\phi \in R\check{P}_s(\xi_0)$, so that $\hat{\phi} \in RP_s(\xi_0)$; and let $(\hat{U}, \hat{V})(\xi) = T(\xi, \xi_0)\hat{\phi}$ for $\xi \geq \xi_0$. Then

$$\|(\hat{U}, \hat{V})(\xi)\|_{\mathcal{H}^{0.5\times 0}(\gamma)} \leq \|(\hat{U}, \hat{V})(\xi)\|_{\mathcal{H}^{0.75\times 0.25}(\gamma)} \leq \tilde{K}e^{-\tilde{\alpha}(\xi-\xi_0)}\|\hat{\phi}\|_{\mathcal{H}^{0.75\times 0.25}(\gamma)}. \tag{4.17}$$

For $\Re(s) \geq \gamma$, let

$$g(\xi, s) = \begin{cases} \left(\hat{B}(\xi, \cdot) \overset{s}{*} \hat{U}(\xi, \cdot)\right)(s), & \xi \geq \xi_0, \\ 0, & \xi < \xi_0. \end{cases}$$

Rewrite (4.15) as the first-order system

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}_\xi = \begin{pmatrix} 0 & I \\ sI - A(\xi) & -c \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} + \begin{pmatrix} 0 \\ g(\xi, s) \end{pmatrix} \tag{4.18}$$

For fixed s, we can regard (4.18) as a differential equation on $E^{0.5\times 0}(s)$, in which case we regard $(0, g(\xi, s))$ as an element of $E^{0.5\times 0}(s)$. We denote the solution operator for (4.12), with parameter s, by $T(\xi, \zeta, s)$, and we denote the projections for the dichotomy on $E^{0.5\times 0}(s)$ by $P_s(\xi, s)$ and $P_u(\xi, s)$.

Alternatively, we can regard (4.18) as a differential equation on the function space $\mathcal{H}^{0.5\times 0}(\gamma)$. Using (4.16), (4.17), and the isomorphisms provided by Laplace transform, for $\sigma > \gamma$ we have

$$\begin{aligned} \|g(\xi, \sigma + i\cdot)\|_{L^2(\mathbb{R})} &\leq \|g(\xi, \cdot)\|_{\mathcal{H}(\gamma)} = \|B(\xi, \cdot)U(\xi, \cdot)\|_{L^2(\mathbb{R}^+,\gamma)} \leq \delta\|U(\xi, \cdot)\|_{L^2(\mathbb{R}^+,\gamma)} \\ &\leq \|(U, V)(\xi, \cdot)\|_{H^{0.5\times 0}(\mathbb{R}^+,\gamma)} \leq \|(U, V)(\xi, \cdot)\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)} \\ &\leq \delta\tilde{K}e^{-\tilde{\alpha}(\xi-\xi_0)}\|\phi\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}. \end{aligned} \tag{4.19}$$

Now (4.12) has an exponential dichotomy on $\mathcal{H}^{0.5\times 0}(\gamma)$ for $\xi \in \mathbb{R}$, with projections derived from those on $E^{0.5\times 0}(s)$. Since (4.19) implies that $(0, g(\xi, \cdot))$ is bounded in $\mathcal{H}^{0.5\times 0}(\gamma)$, if we regard g as given in (4.18), then the unique solution of (4.18) that is bounded in $\mathcal{H}^{0.5\times 0}(\gamma)$ for $\xi \geq \xi_0$ is given by

$$\begin{aligned} (\hat{U}, \hat{V})(\xi, s) &= T(\xi, \xi_0, s)P_s(\xi_0, s)\hat{\phi}(s) + \int_{\xi_0}^{\xi} T(\xi, \zeta, s)P_s(\zeta, s)(0, g(\zeta, s))\, d\zeta \\ &\quad + \int_{\infty}^{\xi} T(\xi, \zeta, s)P_u(\zeta, s)(0, g(\zeta, s))\, d\zeta. \end{aligned} \tag{4.20}$$

Therefore the previously defined function $(\hat{U}, \hat{V})$ is given by this formula. Then $(U, V) = \mathcal{L}^{-1}(\hat{U}, \hat{V})$ can be expressed as $(U^{(1)}, V^{(1)}) + (U^{(2)}, V^{(2)}) + (U^{(3)}, V^{(3)})$, where $(U^{(j)}, V^{(j)})$ is the inverse Laplace transforms of the jth summand of (4.20).

By Lemma 4.4, $U^{(1)} \in H_0^{2,1}((\xi_0,\infty)\times\mathbb{R}^+,\gamma)$ and has norm bounded by a constant times $\|\phi\|_{H_0^{0.5\times 0}(\mathbb{R}^+,\gamma)}$, which is in turn bounded by a constant times $\|\phi\|_{H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}$.

To show that $U^{(j)} \in H_0^{2,1}((\xi_0,\infty)\times\mathbb{R}^+,\gamma)$, $j=2,3$, we shall show that $e^{-\gamma t}U^{(j)}$, $e^{-\gamma t}U_t^{(j)}$, $e^{-\gamma t}U_\xi^{(j)}$, and $e^{-\gamma t}U_{\xi\xi}^{(j)}$ are in $L^2((\xi_0,\infty)\times\mathbb{R}^+,\gamma)$.

We shall treat only $U^{(2)}$. Motivated by the proof of Lemma 3.8 in [10], we use the uniform exponential dichotomy of (4.12) on the spaces $E^{0.5\times 0}(s)$ for $\xi\in\mathbb{R}$ to estimate

$$\begin{aligned}\|T(\xi,\zeta,s)P_s(\zeta,s)(0,g(\zeta,s))\|_{E^{0.5\times 0}(s)} &\le Ke^{-\alpha(1+|s|^{0.5})(\xi-\zeta)}\|(0,g(\zeta,s))\|_{E^{0.5\times 0}(s)}\\ &= Ke^{-\alpha(1+|s|^{0.5})(\xi-\zeta)}\|g(\zeta,s)\|, \quad \xi\ge\zeta, \end{aligned} \tag{4.21}$$

where $\|\cdot\|$ is the usual norm on $\mathbb{C}^n$. Let

$$h(\xi)=\begin{cases} e^{-\alpha(1+|s|^{0.5})\xi}, & \xi\ge 0,\\ 0, & \xi<0.\end{cases}$$

Then from (4.21),

$$\begin{aligned}\|\hat U^{(2)}(\xi,s)\| &\le (1+|s|^{0.5})^{-1}\|(\hat U^{(2)},\hat V^{(2)})(\xi,s)\|_{E^{0.5\times 0}(s)}\\ &\le (1+|s|^{0.5})^{-1}\int_{\xi_0}^{\xi} Ke^{-\alpha(1+|s|^{0.5})(\xi-\zeta)}\|g(\zeta,s)\|\,d\zeta\\ &= K(1+|s|^{0.5})^{-1}(h*\|g(\cdot,s)\|)(\xi).\end{aligned}$$

From Young's inequality for convolutions,

$$\begin{aligned}\|\hat U^{(2)}(\cdot,s)\|_{L^2} &\le K(1+|s|^{0.5})^{-1}\|h\|_{L^1}\|g(\cdot,s)\|_{L^2} \le K(1+|s|^{0.5})^{-2}\alpha^{-1}\|g(\cdot,s)\|_{L^2}\\ &\le K(1+|s|)^{-1}\alpha^{-1}\|g(\cdot,s)\|_{L^2}.\end{aligned} \tag{4.22}$$

Next we fix $\sigma>\gamma$ and show that $\hat U^{(2)}$ and, $s\hat U^{(2)}$, as functions of (ξ,s) with $s=\sigma+i\omega$, $\omega\in\mathbb{R}$, are in $L^2((\xi_0,\infty)\times\mathbb{R})$. Using (4.22) and (4.19), we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\xi_0}^{\infty}(1+|\sigma+i\omega|)^2\|\hat U^{(2)}(\xi,\sigma+i\omega)\|^2\,d\xi d\omega \le K^2\alpha^{-2}\int_{-\infty}^{\infty}\|g(\cdot,\sigma+i\omega)\|_{L^2}^2\,d\omega\\ &= K^2\alpha^{-2}\int_{\xi_0}^{\infty}\int_{-\infty}^{\infty}\|g(\xi,\sigma+i\omega)\|^2\,d\omega d\xi \le K^2\alpha^{-2}\int_{\xi_0}^{\infty}\|g(\xi,\cdot)\|_{\mathcal{H}(\gamma)}^2\,d\omega\\ &\le K^2\alpha^{-2}\delta^2\tilde K^2\int_{\xi_0}^{\infty}e^{-2\tilde\alpha(\xi-\xi_0)}\|\phi\|^2_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}\,d\omega.\end{aligned}$$

Therefore

$$
\begin{aligned}
&(1+|\sigma+i\omega|)\|\hat{U}^{(2)}(\cdot,\sigma+i\cdot)\|_{L^2((\xi_0,\infty)\times\mathbb{R})}\\
&\quad\le C_2\|\phi\|_{H^{0.75\times0.25}(\mathbb{R}^+,\gamma)},\qquad C_2=K\alpha^{-1}\delta\tilde{K}(2\tilde{\alpha})^{-0.5}.
\end{aligned}
$$

Taking the inverse Laplace transform, we have

$$
\|e^{-\sigma t}U^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}+\|e^{-\sigma t}U_t^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}\le C_2\|\phi\|_{H^{0.75\times0.25}(\mathbb{R}^+,\gamma)}.
$$

Letting $\sigma\to\gamma$, we obtain the same estimate with σ replaced by γ.

A similar argument yields

$$
\|e^{-\gamma t}U_\xi^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}=\|e^{-\gamma t}V^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}\le C_2\|\phi\|_{H^{0.75\times0.25}(\mathbb{R}^+,\gamma)}.
$$

Finally, a similar estimate for $\|e^{-\gamma t}U_{\xi\xi}^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}=\|e^{-\gamma t}V_\xi^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}$ follows from the estimates for $U^{(2)}$, $U_t^{(2)}$, and $V^{(2)}$ by noting that $(U^{(2)},V^{(2)})$ is a solution of (4.14), and using the second differential equation to estimate $\|e^{-\gamma t}V_\xi^{(2)}\|_{L^2((\xi_0,\infty)\times\mathbb{R})}$.

The proof is completed by noting that all constants are independent of ξ_0. □

4.5. *Exponential dichotomies for the Laplace transform of (1.3)*

In this subsection we consider the Laplace transform of the linear differential equation (1.3), which we write as a system:

$$
\hat{U}_\xi=\hat{V},\quad \hat{V}_\xi=(sI-Df(q_j(\xi)))\hat{U}-c_j\hat{V},\quad \xi\in\mathbb{R}. \tag{4.23}
$$

For each fixed s we regard (4.23) as defining a differential equation on $\mathbb{C}^n\times\mathbb{C}^n$. Hypothesis (A2) implies that (4.23) has an s-dependent exponential dichotomy for $\Re(s)\ge\eta$, $s\ne0$.

Lemma 4.7. *Assume (A2). Fix γ, $\eta\le\gamma<0$, let $\epsilon>0$, and let $\mathcal{S}_\epsilon=\{s:\Re(s)\ge\gamma$ and $|s|\ge\epsilon\}$. Then (4.23) has a uniform exponential dichotomy on the spaces $E^{0.75\times0.25}(s)$ for $s\in\mathcal{S}_\epsilon$ and $\xi\in\mathbb{R}$. The multiplicative constant $K(\epsilon)$ depends on ϵ and approaches infinity as $\epsilon\to0$, but for some $\alpha>0$, the exponent is $\alpha(1+|s|^{0.5})$ independent of ϵ.*

Moreover, let $P_s(e_{j-1},s)$ and $P_u(e_j,s)$ be the spectral projections at the two limiting points $(e_{j-1},0)$ and $(e_j,0)$. Then there are constants $M>0$, $N>0$, $\delta_1>0$, and $\delta_2>0$ such that for $s\in\mathcal{S}_\epsilon$, $0<\epsilon<M$,

$$
\begin{aligned}
&\|P_s(\xi,s)-P_s(e_{j-1},s)\|\le\frac{16K^2(\epsilon)\delta_k}{\alpha(1+|s|^{0.5})},\quad \xi\le-N,\\
&\|P_u(\xi,s)-P_u(e_j,s)\|\le\frac{16K^2(\epsilon)\delta_k}{\alpha(1+|s|^{0.5})},\quad \xi\ge N,
\end{aligned}\tag{4.24}
$$

where $k=1$ for $|s|\ge M$, and $k=2$ for $\epsilon\le|s|<M$.

Proof. The proof is adapted from that of [10], see also [13].

Step 1: exponential dichotomy for $|s| \geq M$. Let $M > 0$. For $s \in \mathcal{S}_M$, we treat (4.23) as a perturbation to the system

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c_j\hat{V}. \tag{4.25}$$

From [10], (4.25) has a uniform exponential dichotomy on the spaces $E^{0.75\times 0.25}(s)$ for $\Re(s) \geq \gamma$ and $\xi \in \mathbb{R}$, with multiplicative constant K_0 and exponent $\rho_0 = \alpha_0(1 + |s|^{0.5})$.

Let $s \in \mathcal{S}_M$. Let $\delta_1 = \sup_\xi |Df(q_j(\xi))|$. Although δ_1 is not small, the conditions $C_1\delta_1 < 1$ and $C_2\delta_1 < 1$ in Theorem 4.1 can be satisfied if we choose $\tilde{\rho} = \frac{\alpha_0}{2}(1 + |s|^{0.5})$. Then $\rho_0 - \tilde{\rho} = \frac{\alpha_0}{2}(1 + |s|^{0.5})$ is large for $s \in \mathcal{S}_M$ with M is sufficiently large. Hence for $s \in \mathcal{S}_M$ with M sufficiently large, (4.3) in Theorem 4.1 is satisfied, so (4.23) has an s-dependent exponential dichotomy for $\xi \in \mathbb{R}$. The multiplicative constant $\tilde{K}$ is independent of s, and the exponent is $\tilde{\rho} = \frac{\alpha_0}{2}(1 + |s|^{0.5})$. The projections satisfy (4.24) with $k = 1$. Thus we in fact have a uniform exponential dichotomy on the spaces $E^{0.75\times 0.25}(s)$ for $s \in \mathcal{S}_M$ and $\xi \in \mathbb{R}$.

Step 2: exponential dichotomy for $0 < |s| \leq M$. Using M found in Step 1, we consider the spectral equation (4.23) with s in the compact set $\{s : \Re(s) \geq \gamma \text{ and } |s| \leq M\}$.

Consider the constant-coefficient systems

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = (sI - Df(e_k))\hat{U} - c_j\hat{V}, \quad k = j-1, j, \tag{4.26}$$

in which s is a parameter. Hypothesis (A2) implies that these systems have n eigenvalues with positive real part and n eigenvalues with negative real part. For s in the compact set $\{s : \Re(s) \geq \gamma \text{ and } |s| \leq M\}$, the systems (4.26) have exponential dichotomies for $\xi \in \mathbb{R}$ with the common exponent $\rho_1 > 0$ and the common multiplicative constant $K > 0$.

Let $N > 0$, and let

$$\begin{aligned} \delta_2 = \max(&\sup\{\|Df(q_j(\xi)) - Df(e_{j-1})\| : \xi \leq -N\}, \\ &\sup\{\|Df(q_j(\xi)) - Df(e_j)\| : \xi \geq N\}). \end{aligned} \tag{4.27}$$

As $N \to \infty$, $\delta_2 \to 0$, so for N sufficiently large, (4.3) in Theorem 4.1 is satisfied. Therefore, for $\Re(s) \geq \gamma$ and $|s| \leq M$, (4.23) has exponential dichotomies in $\xi \leq -N$ and in $N \leq \xi$. The dichotomies are not unique, but the unstable subspace $E_u(\xi, s)$, $\xi \leq -N$, and the stable subspace $E_s(\xi, s)$, $\xi \geq N$, are unique and depend analytically on s. We shall use them to construct an exponential dichotomy on $\mathbb{R}$.

We extend $E_u(\xi, s)$, $\xi \leq -N$, and $E_s(\xi, s)$, $\xi \geq N$, to $\xi \in \mathbb{R}$ by defining

$$E_u(\xi, s) = T(\xi, -N, s)E_u(-N, s) \quad \text{for } -N \leq \xi \leq \infty,$$
$$E_s(\xi, s) = T(\xi, N, s)E_s(N, s) \quad \text{for } -\infty \leq \xi \leq N.$$

From (A2), if $\Re(s) \geq \gamma$ and $0 < |s| \leq M$, $T(N, -N, s)E_u(-N, s)$ is transverse to $E_s(N, s)$. The dichotomy has been extended to $\xi \in \mathbb{R}$ for $\Re(s) \geq \gamma$ and $0 < |s| \leq M$. The exponent of the dichotomy can be taken to be $\alpha_1(1 + |s|^{0.5})$ with α_1 independent of s.

For $0 < \epsilon < M$, in the compact set $\{s : \Re(s) \geq \gamma \text{ and } \epsilon \leq |s| \leq M\}$, the angle between $E_u(\pm N_-, s)$ and $E_s(\pm N_+, s)$ is bounded below by a constant that approaches 0 as $\epsilon \to 0$.

Thus, the multiplicative constant $K_1(\epsilon)$ for the dichotomy on $\{s : \Re(s) \geq \gamma \text{ and } \epsilon \leq |s| \leq M\}$ approaches infinity as $\epsilon \to 0$.

Step 3: completion of proof. We combine the two cases and select $\alpha = \min\{\frac{\alpha_0}{2}, \alpha_1\}$. Then (4.23) has a uniform exponential dichotomy on the spaces $E^{0.75\times 0.25}(s)$ for $s \in \mathcal{S}_\epsilon$ and $\xi \in \mathbb{R}$. The exponent is $\alpha(1 + |s|^{0.5})$. The multiplicative constant is $K(\epsilon) = \max(\tilde{K}, K_1(\epsilon))$.

The fact that the exponential dichotomy is analytic in s follows from a simple perturbation argument. Assume that for a given $s_0 \in \mathbb{C}$, the system (4.23) has an exponential dichotomy on $\mathbb{R}$. Then the contraction mapping principle can be used to find the stable and unstable subspaces of (4.23) for $|s - s_0| < \epsilon$. Since the equation depends analytically on s, so do the stable and unstable subspaces. □

5. Proof of the Tail Lemma 3.3

The proof of the Tail Lemma uses the material in Subsection 4.4.

Fix j. We use coordinates (ξ, t), $\xi = x - x_j - \bar{c}_j t$. Notice that

$$\xi_j = x - y_j - c_j t = \xi + x_j - y_j + (\bar{c}_j - c_j)t = \xi + x_j - y_j + \frac{1}{2}(c_{j+1} - c_j)t,$$

$$\begin{aligned}\xi_{j+1} &= x - y_{j+1} - c_{j+1}t = \xi + x_{j+1} - y_{j+1} + (\bar{c}_{j+1} - c_{j+1})t \\ &= \xi + x_{j+1} - y_{j+1} - \frac{1}{2}(c_{j+1} - c_j)t.\end{aligned}$$

The line Γ_j becomes $\xi = 0$. The lines M_j^+ and M_{j+1}^- become

$$\begin{aligned}\tilde{M}_j^+ &= \{(\xi, t) : \xi = y_j - x_j + N - \frac{1}{2}(c_{j+1} - c_j)t,\ t \geq 0\}, \\ \tilde{M}_{j+1}^- &= \{(\xi, t) : \xi = y_{j+1} - x_j - N + \frac{1}{2}(c_{j+1} - c_j)t,\ t \geq 0\}.\end{aligned}$$

Between $\tilde{M}_j^+$ and $\xi = 0$, (3.10) becomes

$$U_t = U_{\xi\xi} + \bar{c}_j U_\xi + A(\xi, t)U, \quad A(\xi, t) = Df(q_j(\xi + x_j - y_j + \frac{1}{2}(c_{j+1} - c_j)t).$$

Between $\xi = 0$ and $\tilde{M}_{j+1}^-$, (3.11) becomes

$$U_t = U_{\xi\xi} + \bar{c}_{j+1} U_\xi + A(\xi, t)U, \quad A(\xi, t) = Df(q_{j+1}(\xi + x_{j+1} - y_{j+1} - \frac{1}{2}(c_{j+1} - c_j)t).$$

Let K and μ be the constants from (A1′). We claim that for (ξ, t) between $\tilde{M}_j^+$ and $\xi = 0$, and between $\xi = 0$ and $\tilde{M}_{j+1}^-$, $\|A(\xi, t) - Df(e_j)\| \leq K e^{-\mu N}$ and $A_t(\xi, t) \leq K e^{-\mu N}$.

To see this, first note that for $y_j - x_j + N \leq \xi < 0$ and $t \geq 0$,

$$\|A(\xi, t) - Df(e_j)\| \leq K e^{-\mu(\xi + x_j - y_j + \frac{1}{2}(c_{j+1} - c_j)t)} \leq K e^{-\mu(\xi + x_j - y_j)} \leq K e^{-\mu N}.$$

Then note that for $\xi < y_j - x_j + N$ and (ξ, t) above $\tilde{M}_j^+$, $\xi + x_j - y_j + \frac{1}{2}(c_{j+1} - c_j)t \geq N$, so

$$\|A(\xi,t) - Df(e_j)\| \leq Ke^{-\mu(\xi + x_j - y_j + \frac{1}{2}(c_{j+1}-c_j)t)} \leq Ke^{-\mu N}.$$

Similarly, $A_t(\xi,t) \leq Ke^{-\mu N}$. Analogous arguments apply to the right of $\xi = 0$.

Using smooth cut-off function, we can extend $A(\xi,t)$ to all of $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^+$, so that for all (ξ,t), $\|A(\xi,t) - Df(e_j)\| \leq Ke^{-\mu N}$ and $A_t(\xi,t) \leq Ke^{-\mu N}$. ($K$ may have to be increased slightly independent of N.)

It is shown in [10] that the system

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - \bar{c}_j\hat{V} - Df(e_j)\hat{U}$$

has a uniform exponential dichotomy on the spaces $E^{(k+0.5)\times k}(s)$, $0 \leq k \leq 0.25$, for $\Re(s) \geq \gamma$ and $\xi \in \mathbb{R}$. By Lemmas 4.4, 4.2, and 4.5, for N sufficiently large, the linear system

$$U_\xi = V,\ V_\xi = U_t - \bar{c}_j V - A(\xi,t)U$$

has an exponential dichotomy on $H_0^{(k+0.5)\times k}(\gamma)$ for $\xi \in \mathbb{R}$, with, for each k, constants $\tilde{K}$ and $\tilde{\alpha}$ that are independent of y_j, y_{j+1}. The estimate (3.13) follows from Lemma 4.3. The remainder of the Tail Lemma follows from Lemma 4.6.

6. Proof of the Interior Lemma 3.4

The proof of the Interior Lemma uses the material in Subsection 4.5.

After Laplace transform, the system (3.14)–(3.15) becomes

$$0 = \hat{U}_{\xi\xi} + c_j\hat{U}_\xi - s\hat{U} + Df(q_j(\xi))\hat{U}, \quad [(\hat{U}, \hat{U}_\xi)](M_a) = \hat{\phi}(s). \tag{6.1}$$

Writing (6.1) as a first-order system, we obtain

$$\hat{U}_\xi = \hat{V},\ \hat{V}_\xi = (sI - Df(q_j(\xi)))\hat{U} - c_j\hat{V}, \quad [(\hat{U}, \hat{V})](M_a) = \hat{\phi}(s). \tag{6.2}$$

Since the jump $\phi(t)$ is in $H_0^{0.75\times 0.25}(\gamma)$, $\hat{\phi}(s)$ is in $\mathcal{H}^{0.75\times 0.25}(\gamma)$. We look for solutions of (6.2) that decay to zero as ξ moves away from M_a.

Equivalently, we can rewrite $U(\xi,t)$ in (3.14) as $U(\xi,t) = Y(\xi,t) + \beta(t)q'(\xi)$ with $\mathcal{P}_j Y(\cdot,t) = 0$ for each t. The system (3.14)–(3.15) becomes

$$Y_t = Y_{\xi\xi} + c_jY_\xi + Df(q_j(\xi))Y - \dot{\beta}(t)q_j'(\xi),$$
$$Y(\xi,0) = 0,\ \beta(0) = 0, \quad [(Y, Y_\xi)](M_a) = \hat{\phi}(s).$$

Write $h(t) = \dot{\beta}(t)$. Then taking the Laplace transform, we obtain

$$\hat{Y}_{\xi\xi} + c_j\hat{Y}_\xi + Df(q_j(\xi))\hat{Y} - s\hat{Y} = \hat{h}(s)q'(\xi), \quad [(\hat{Y}, \hat{Y}_\xi)](a) = \hat{\phi}(s), \tag{6.3}$$

with $\mathcal{P}_j\hat{Y}(\cdot,s)=0$ for each t. Written as a first order system, (6.3) becomes

$$(\hat{Y},\hat{Z})_\xi=(\hat{Z},(sI-Df(q_j(\xi)))\hat{Y}-c_j\hat{Z})+(0,\hat{h}(s)q_j'(\xi)),\quad [(\hat{Y},\hat{Z})](a)=\hat{\phi}(s). \tag{6.4}$$

Step 1: $|s|\geq\epsilon$. Let $\epsilon>0$. We shall show that *for $\Re(s)\geq\eta$ and $|s|\geq\epsilon$, system (6.2) has a unique solution $(\hat{U},\hat{V})(\xi,s)$ that decays exponentially as $\xi\to\pm\infty$. Moreover, the solution depends analytically on s, and there are constants $C_1(\epsilon)>0$ and $\alpha_1(\epsilon)>0$ such that for $\Re(s)\geq\eta$, $|s|\geq\epsilon$, and $\rho_1(\epsilon)=\alpha_1(\epsilon)(1+|s|^{0.5})$, the solution satisfies*

$$\|(\hat{U},\hat{V})(\xi,s)\|_{E^{0.72\times0.25}(s)}\leq C_1(\epsilon)e^{-\rho_1(\epsilon)|\xi-a|}\|\hat{\phi}(s)\|_{E^{0.75\times0.25}(s)}. \tag{6.5}$$

To prove this result, we note that by Lemma 4.7, for $\Re(s)\geq\eta$ and $|s|\geq\epsilon$, the system (4.23) has a uniform exponential dichotomy on the spaces $E^{0.75\times0.25}(s)$ for $\xi\in\mathbb{R}$. Let $T(\xi,\zeta,s)$ denote the solution operator, and let the projections be $P_s(\xi,s)+P_u(\xi,s)=I$. The unique solution of (6.1) that decays as $\xi\to\pm\infty$ is then

$$\begin{aligned}(\hat{U},\hat{V})(\xi,s)&=-T(\xi,a,s)P_u(a,s)\hat{\phi}(s),\quad \xi\leq a,\\ (\hat{U},\hat{V})(\xi,s)&=T(\xi,a,s)P_s(a,s)\hat{\phi}(s),\quad \xi\geq a.\end{aligned} \tag{6.6}$$

Estimate (6.5) follows from the definition of uniform exponential dichotomy.

Step 2: $|s|\leq\epsilon$. We shall show that *there exists $\epsilon>0$ such that for $|s|\leq\epsilon$, (6.4) has a unique solution $((\hat{Y},\hat{Z})(\xi,s),h(s))$ such that $\mathcal{P}_j\hat{Y}(\cdot,s)=0$ for each s and $(\hat{Y},\hat{Z})(\xi,s)$ decays exponentially as $\xi\to\pm\infty$. Moreover, the solution depends analytically on s, and there are constants $C_2>0$ and $\alpha_2>0$ such that for $|s|\leq\epsilon$ and $\rho_2=\alpha_2(1+|s|^{0.5})$, the solution satisfies*

$$\|(\hat{Y},\hat{Z})(\xi,s)\|_{E^{0.72\times0.25}(s)}\leq C_2e^{-\rho_2|\xi-a|}\|\hat{\phi}(s)\|_{E^{0.72\times0.25}(s)}. \tag{6.7}$$

To prove this result, we note that for each small s, there exist two exponential dichotomies for (4.23) one for $\xi\leq a$, the other for $\xi\geq a$. We denote the projections by $P_s^-(\xi,s)+P_u^-(\xi,s)=I$ for $\xi\leq a$ and $P_s^+(\xi,s)+P_u^+(\xi,s)=I$ for $\xi\geq a$. The spaces $E_u^-(\xi,s)=RP_u^-(\xi,s)$ and $E_s^+(\xi,s)=RP_s^+(\xi,s)$ are uniquely defined and depend analytically on s. Complementary invariant spaces, and hence the projections, can be chosen to depend analytically on s. Assumption (A2) implies that $RP_u^-(\xi,s)\cap RP_s^+(\xi,s)=\{0\}$ for $s\neq0$, and $RP_u^-(\xi,0)\cap RP_s^+(\xi,0)$ is spanned by $(\gamma'(\xi),\gamma''(\xi))$.

Bounded solutions of (6.4) can be expressed as follows:

$$\begin{aligned}&\text{for } \xi\leq a,\quad (\hat{Y},\hat{Z})(\xi,s)=T(\xi,a,s)P_u^-(a,s)(\hat{Y},\hat{Z})(a-,s)\\ &\quad+\int_{-\infty}^{\xi}T(\xi,\zeta,s)P_s^-(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta+\int_a^{\xi}T(\xi,\zeta,s)P_u^-(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta;\\ &\text{for } \xi\geq a,\quad (\hat{Y},\hat{Z})(\xi,s)=T(\xi,a,s)P_s^+(a,s)(\hat{Y},\hat{Z})(a+,s)\\ &\quad+\int_a^{\xi}T(\xi,\zeta,s)P_s^+(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta+\int_{\infty}^{\xi}T(\xi,\zeta,s)P_u^+(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta.\end{aligned} \tag{6.8}$$

Let

$$\mu_u^-(a,s) = P_u^-(a,s)(\hat{Y},\hat{Z})(a-,s), \quad \mu_s^+(a,s) = P_s^+(a,s)(\hat{Y},\hat{Z})(a+,s),$$
$$v(s) = \int_{-\infty}^{a} T(a,\zeta,s)P_s^-(\zeta,s)(0,q_j'(\zeta))d\zeta + \int_{a}^{\infty} T(a,\zeta,s)P_u^+(\zeta,s)(0,q_j'(\zeta))d\zeta.$$

From (6.8), the jump condition at $\xi = a$ is satisfied provided

$$\mu_s^+(a,s) - \mu_u^-(a,s) - h(s)v(s) = \hat{\phi}(s). \tag{6.9}$$

We have $\mathcal{P}_j\hat{Y} = 0$ provided

$$\int_{-\infty}^{\infty} < z_j(\xi), \hat{Y}_j(\xi,s) > d\xi = 0 \text{ for each } s. \tag{6.10}$$

For each s we regard the left-hand side of (6.9) as a linear map

$$L_1(s) : E_u^-(a,s) \times E_s^+(a,s) \times \mathbb{R} \to \mathbb{R}^{2n}, \quad (\mu_u^-, \mu_s^+, h) \to \mu_s^+ - \mu_u^- - hv(s).$$

Moreover, since $\hat{Y}_j(\xi,s)$ depends linearly on $(\mu_u^-(a,s), \mu_s^+(a,s), h(s))$ through (6.8), for each s we can regard the left-hand side of (6.10) as a linear map

$$L_2(s) : E_u^-(a,s) \times E_s^+(a,s) \times \mathbb{R} \to \mathbb{R}, \quad (\mu_u^-, \mu_s^+, h) \to \int_{-\infty}^{\infty} < z_j(\xi), \hat{Y}_j(\xi,s) > d\xi.$$

Define $L(s) : E_u^-(a,s) \times E_s^+(a,s) \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}$ by $L(s) = (L_1(s), L_2(s))$. $L(s)$ depends analytically on s. Since $\dim E_u^-(a,s) + \dim E_s^+(a,s) = 2n$, $L(s)$ is a linear map from a space of dimension $2n+1$ to a space of the same dimension. The formula (6.8) gives a solution to (6.3) if and only if $(\mu_u^-(a,s), \mu_s^+(a,s), h(s))$ is a solution of

$$L(s)(\mu_u^-, \mu_s^+, h) = (\hat{\phi}(s), 0). \tag{6.11}$$

We shall show that $L(0)$ is invertible. It follows that $L(s)$ is invertible for small s, so for all small s there is a unique solution $(\mu_u^-(a,s), \mu_s^+(a,s), h(s))$ of (6.11) that depends analytically on s. The estimates in the lemma follow from the formulas (6.8) and the compactness of the set $|s| \le \epsilon$.

To prove that $L(0)$ is invertible, we will show that its kernel is trivial. Let (μ_u^-, μ_s^+, h) belong to $\ker L(0)$. The adjoint equation of (4.23) has a bounded solution $\Psi = (c_j z_j - z_{j\xi}, z_j)$. $\Psi(a)$ is orthogonal to the codimension-one space $E_u^-(a,s) + E_+(a,s)$. In addition, we claim that $\langle \Psi(a), v(0) \rangle = 1$:

$$
\begin{aligned}
\langle \Psi(a), v(0)\rangle &= \Psi(a)^\top v(0) \\
&= \Psi(a)^\top \int_{-\infty}^{a} T(a,\zeta,0)P_s^-(\zeta,0)(0,q_j'(\zeta))d\zeta \\
&\quad + \Psi(a)^\top \int_{a}^{\infty} T(a,\zeta,0)P_u^+(\zeta,0)(0,q_j'(\zeta))d\zeta \\
&= \int_{-\infty}^{a} \Psi(\zeta)^\top(0,q_j'(\zeta))d\zeta + \int_{a}^{\infty} \Psi(\zeta)^\top(0,q_j'(\zeta))d\zeta \\
&= \int_{-\infty}^{a} z_j(\zeta)^\top q_j'(\zeta)d\zeta + \int_{a}^{\infty} z_j(\zeta)^\top q_j'(\zeta)d\zeta = \int_{-\infty}^{\infty} \langle z_j(\zeta), q_j'(\zeta)\rangle d\zeta = 1.
\end{aligned}
$$

Hence we can multiply the equation $L_1(0)(\mu_u^-,\mu_s^+,h) = \mu_s^+ - \mu_u^- - hv(0) = 0$ by $\Psi(a)^\top$ and obtain $h=0$. Therefore $\mu_s^+ - \mu_u^- = 0$. Since $E_u^-(a,0)\cap E_s^+(a,0)$ is spanned by $(q_j'(a), q_j''(a))$, there is a number k such that $\mu_s^+ = \mu_u^- = k(q_j'(a), q_j''(a))$. With these values of (μ_u^-,μ_s^+,h), the function $(\hat{Y}(\xi,0), \hat{Z}(\xi,0)$ produced by (6.8) is simply $k(q_j'(\xi), q_j''(\xi))$. Then the equation $L_2(0)(\mu_u^-,\mu_s^+,h)=0$ reduces to

$$
\int_{-\infty}^{\infty} \langle z_j, kq_j'\rangle = k = 0.
$$

We conclude that $(\mu_u^-,\mu_s^+,h) = (0,0,0)$. Thus $L(0)$ has trivial kernel, so it is invertible.

Step 3: combining solutions from Steps 1 and 2. Using $\epsilon > 0$ given by Step 2, we consider $(\hat{U},\hat{V})(\xi,s)$ defined by (6.1) for $\Re s \geq \gamma$ and $|s| \geq \epsilon$. For $\Re s \geq \gamma$ and $|s|\geq\epsilon$, define

$$
\begin{aligned}
h(s) &= \int_{-\infty}^{\infty} \langle z_j(\xi), \hat{U}(\xi,s)\rangle ds, \\
\hat{Y}(\xi,s) &= (I-\mathcal{P}_j)\hat{U}(\xi,s) = \hat{U}(\xi,s) - h(s)q_j'(\xi), \\
\hat{Z}(\xi,s) &= \hat{V}(\xi,s) - h(s)q_j''(\xi).
\end{aligned}
$$

For $\Re s \geq \gamma$ and $|s|\geq\epsilon$, $((\hat{Y},\hat{Z})(\xi,s), h(s))$ is the unique solution of (6.4) such that $\mathcal{P}_j\hat{Y} = 0$ and $(\hat{Y},\hat{Z})(\xi,s)$ decays exponentially as $\xi\to\pm\infty$. $(\hat{Y},\hat{Z})(\xi,s)$ satisfies (6.7) with C_2 replaced by $C_1(\epsilon)$ and ρ_2 replaced by $\rho_1(\epsilon)$.

We have now defined $((\hat{Y},\hat{Z})(\xi,s), h(s))$ for all s with $\Re s\geq\gamma$. These functions have been defined twice for $|s|=\epsilon$, but from their uniqueness, the two solutions agree. Thus $((\hat{Y},\hat{Z})(\xi,s), h(s))$ is analytic in $\Re s \geq \gamma$.

Step 4: $\dot{\beta}(t)$. We shall show that $\dot{\beta}\in L^2(\mathbb{R}^+,\gamma)$ and that $\|\dot{\beta}\|_{L^2(\mathbb{R}^+,\gamma)}$ is at most a constant times $\|\phi\|_{H^{0.75}(\mathbb{R}^+,\gamma)}$.

From Steps 1 and 3, for $\Re s \geq \gamma$ and $|s| \geq \epsilon$,

$$|h(s)| \leq \|z\|_{L^2(\mathbb{R})} \|\hat{U}(\cdot, s)\|_{L^2(\mathbb{R})} \leq \|z\|_{L^2(\mathbb{R})} C_1(\epsilon) \|\hat{\phi}(s)\|_{E^{0.75\times 0.25}(s)}$$

It follows easily that there is a constant $C > 0$ such that for $\Re s \geq \gamma$,

$$|h(s)| \leq C\|z\|_{L^2(\mathbb{R})} \|\hat{\phi}(s)\|_{E^{0.75\times 0.25}(s)}.$$

Therefore

$$\begin{aligned}
\|\dot{\beta}\|_{L^2(\mathbb{R}^+,\gamma)} = \|h\|_{\mathcal{H}(\gamma)} &= \sup_{\sigma>\gamma} \left(\int_{-\infty}^{\infty} |h(\sigma + i\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \\
&\leq C\|z\|_{L^2(\mathbb{R})} \sup_{\sigma>\gamma} \left(\int_{-\infty}^{\infty} |\hat{\phi}(\sigma + i\omega)|^2_{E^{0.75\times 0.25}(\sigma+i\omega)} d\omega \right)^{\frac{1}{2}} \\
&= C\|z\|_{L^2(\mathbb{R})} \|\hat{\phi}\|_{\mathcal{H}^{0.75}(\gamma)} = C\|z\|_{L^2(\mathbb{R})} \|\phi\|_{H^{0.75}(\mathbb{R}^+,\gamma)}.
\end{aligned}$$

Step 5: $(Y, Z)(\xi, t)$. Let $C = \max(C_1(\epsilon), C_2)$, $\alpha = \min(\alpha_1(\epsilon), \alpha_2)$, and $\rho = \alpha(1 + |s|^{0.5})$; then from Step 3, for $\Re s \geq \gamma$,

$$\|(\hat{Y}, \hat{Z})(\xi, s)\|_{E^{0.72\times 0.25}(s)} \leq Ce^{-\rho|\xi - a|} \|\hat{\phi}(s)\|_{E^{0.72\times 0.25}(s)}. \tag{6.12}$$

Thus on $-\infty < \xi \leq a$ (resp. on $a \leq \xi < \infty$), $(\hat{Y}, \hat{Z})(\xi, s)$ satisfies the unstable subspace part (resp. the stable subspace part) of the estimate required for a uniform exponential dichotomy on the spaces $E^{0.72\times 0.25}(s)$ for $\Re s \geq \gamma$. Then as in Lemma 4.4, it follows that $Y \in H_0^{2,1}(\mathbb{R} \times \mathbb{R}^+ \setminus M_a, \gamma)$, and there is a constant $\check{C}$ such that

$$\|Y\|_{H_0^{2,1}(\mathbb{R}\times\mathbb{R}^+\setminus M_a,\gamma)} \leq \check{C}\|\phi\|_{H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)}$$

Finally, estimate (3.16) in the Interior Lemma follows from Lemma 4.3.

References

[1] M. Beck, B. Sandstede, K. Zumbrun, Nonlinear stability of time-periodic viscous shocks, Arch. Ration. Mech. Anal. 196 (2010) 1011–1076.
[2] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math., vol. 629, Springer-Verlag, New York, 1978.
[3] A. Ducrot, T. Giletti, H. Matano, Existence and convergence to a propagating terrace in one-dimensional reaction–diffusion equations, Trans. Amer. Math. Soc. 366 (2014) 5541–5566.
[4] P. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Ration. Mech. Anal. 65 (1977) 335–361.
[5] G.B. Folland, Introduction to Partial Differential Equations, Princeton University Press, Princeton, 1995.
[6] A. Hoffman, J.D. Wright, Exit manifolds for lattice differential equations, Proc. Roy. Soc. Edinburgh Sect. A 141 (2011) 77–92.
[7] K. Kirchgassner, Homoclinic bifurcation of perturbed reversible systems, in: W. Knobloch, K. Smith (Eds.), Lecture Notes in Math., vol. 1017, Springer-Verlag, New York, 1982, pp. 328–363.

[8] J.K. Hale, X.-B. Lin, Heteroclinic orbits for retarded functional differential equations, J. Differential Equations 65 (1986) 175–202.
[9] X.-B. Lin, Exponential dichotomies in intermediate spaces with applications to a diffusively perturbed predator–prey model, J. Differential Equations 108 (1994) 36–63.
[10] X.-B. Lin, Shadowing matching errors for wave-front-like solutions, J. Differential Equations 129 (1996) 403–457.
[11] X.-B. Lin, Local and global existence of multiple waves near formal approximations, Progr. Nonlinear Differential Equations Appl. 19 (1996) 385–404.
[12] X.-B. Lin, Algebraic dichotomies with an application to the stability of Riemann solutions of conservation laws, J. Differential Equations 11 (2009) 2924–2965.
[13] X.-B. Lin, Stability of standing waves for monostable reaction–convection equations in a large bounded domain with boundary conditions, J. Differential Equations 255 (2013) 58–84.
[14] X.-B. Lin, S. Schecter, Stability of concatenated traveling waves, preprint, 2014.
[15] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin, New York, 1972.
[16] A. Lunardi, Interpolation Theory, Scuola Normale Superiore di Pisa, Pisa, 2009.
[17] A. Mielke, A reduction principle for nonautonomous systems in infinite-dimensional spaces, J. Differential Equations 65 (1986) 68–88.
[18] Y. Morita, H. Ninomiya, Entire solutions with merging fronts to reaction–diffusion equations, J. Dynam. Differential Equations 18 (2006) 841–861.
[19] D. Peterhof, B. Sandstede, A. Scheel, Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders, J. Differential Equations 140 (1997) 266–308.
[20] M. Renardy, Bifurcation of singular solutions in reversible systems and applications to reaction–diffusion equations, Adv. Math. 3 (1982) 324–406.
[21] J. Rottmann-Matthes, Linear stability of traveling waves in first-order hyperbolic PDEs, J. Dynam. Differential Equations 23 (2011) 365–393.
[22] D.H. Sattinger, On the stability of waves of nonlinear parabolic systems, Adv. Math. 22 (1976) 312–355.
[23] A. Scheel, J.D. Wright, Colliding dissipative pulses—the shooting manifold, J. Differential Equations 245 (2008) 59–79.
[24] S. Selle, Decomposition and stability of multifronts and multipulses, Ph.D. thesis, University of Bielefeld, 2009.
[25] J.D. Wright, Separating dissipative pulses: the exit manifold, J. Dynam. Differential Equations 21 (2009) 315–328.
[26] J.D. Wright, Interaction manifolds for reaction diffusion equations in two dimensions, SIAM J. Appl. Dyn. Syst. 9 (2010) 734–768.
[27] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, Publ. Res. Inst. Math. Sci. 39 (2003) 117–164.
[28] K. Yosida, Functional Analysis, Grundlehren Math. Wiss., vol. 123, Springer-Verlag, New York, 1980.