SIAM J. APPL. MATH.
Vol. 49, No. 1, pp. 26-54, February 1989

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SHADOWING LEMMA AND SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS*

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Abstract. A complete procedure is given to determine the outer and inner expansions of a singularly perturbed boundary value problem in $\mathbb{R}^n$. The validity of such expansions is deduced from a generalized Shadowing Lemma, where the inner and outer approximations are treated like pseudo-orbits in the classical dynamical system theory.

Key words. shadowing lemma, Fredholm alternatives, exponential diehotomics, inner and outer expansions, matching principles

AMS(MOS) subject classifications. 34E15, 34B15, 34C35

1. Introduction. In the past few years, several people have attempted to bring some of the methods of dynamical systems to bear on singularly perturbed boundary value problems, e.g., for centermanifolds see Fenichel [7] and Carr and Pego (unpublished manuscript), for the Lyapunov-Schmidt method see Hale and Sakamoto [12]. The early work of Hoppensteadt [13] also used the idea of dynamical systems.

In this paper we shall use the method developed in the theory of dynamical systems to study the matched asymptotic expansion for the singularly perturbed boundary value problem

$$\begin{aligned} &\varepsilon \dot{x} = f(x, t, \varepsilon), \qquad a \leqq t \leqq b, \\ &B_1(x(a), \varepsilon) = 0, \\ &B_2(x(b), \varepsilon) = 0. \end{aligned} \tag{1.1}$$

We shall discuss the following problems. Given a candidate for zero-order asymptotic approximations of (1.1), is there an exact solution for the full problem that lies near it? How are the higher-order expansions for the exact solution computed, and how is the exact solution computed based on the asymptotic expansions, if there is such an exact solution?

Suppose that $t_0 = a < t_1 < \cdots < t_r = b$, $[a, b] = \bigcup_{i=1}^r [t_{i-1}, t_i]$ is a partition of $[a, b]$. A sequence of C^1 functions $\{x_i(t)\}_{i=1}^r$, each defined on $[t_{i-1}, t_i]$, is called a *formal approximation* subordinate to the partition if the *residuals* $f_i(t) = \varepsilon\dot{x}_i(t) - f(x_i(t), t, \varepsilon)$ in $[t_{i-1}, t_i]$, the *jumps* at common points $g_i = x_i(t_i) - x_{i+1}(t_i)$, and the *boundary errors* $B_1(x_1(a), \varepsilon)$ and $B_2(x_r(b), \varepsilon)$ are small. If (1.1) is solved by the matched asymptotic expansion method, and the *outer and inner approximations* (some authors prefer the terms regular and local approximations) are presented in the *fast variable* $\tau = t/\varepsilon$, the entire domain $[a/\varepsilon, b/\varepsilon]$ is then divided into subintervals $[\tau_{i-1}, \tau_i]$ on which either the regular or the singular approximation is defined. Truncated at a certain order of accuracy, these asymptotic solutions are, in fact, formal approximations. We shall also refer to the piecewise C^1 function $x(t)$, which is equal to $x_i(t)$ in $[t_{i-1}, t_i]$, as a formal approximation.

It is well known that a formal approximation is not necessarily close to any true solution of (1.1); that is, the *remainder* of the approximation to the exact solution does not have to be small even when the residual, jump, and boundary errors are small.

* Received by the editors August 25, 1987; accepted for publication (in revised form) February 22, 1988.
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Example 1.1. Consider

$$\varepsilon \ddot{u} = u\dot{u}, \qquad -1 \leqq t \leqq 1,$$
$$u(-1) = a,$$
$$u(1) = -a,$$

where $a > 0$ is a constant. $u_1 = a$, $-1 \leqq t \leqq \frac{1}{2}$ and $u_2 = -a$, $\frac{1}{2} \leqq t \leqq 1$ are two regular approximations. Let $t = \varepsilon\tau$ and use $'$ for $d/d\tau$; then we have $u'' = uu'$, $u' = u^2/2 + C$. The phase portrait for various C is depicted in Fig. 1.

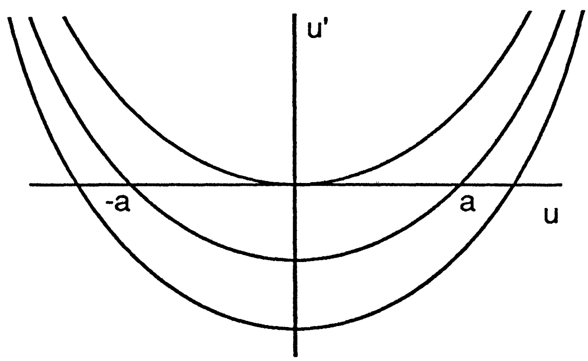

FIG. 1

It is clear from the phase portrait that there is a unique heteroclinic solution $q(\tau)$, with $q(0) = 0$, $q(\tau) \to \pm a$ as $\tau \to \mp\infty$. Define $u(\tau) = a$, $-1/\varepsilon \leqq \tau \leqq 1/2\varepsilon - \varepsilon^{\beta-1}$, $u(\tau) = q(\tau - 1/2\varepsilon)$, $1/2\varepsilon - \varepsilon^{\beta-1} \leqq \tau \leqq 1/2\varepsilon + \varepsilon^{\beta-1}$, and $u(\tau) = -a$, $1/2\varepsilon + \varepsilon^{\beta-1} \leqq \tau \leqq 1/\varepsilon$, where $0 < \beta < 1$. $u(\tau)$ is a formal approximation. However, no exact solution is close to $u(\tau)$ as $\varepsilon \to 0$, since by symmetry an exact solution $u(\tau, \varepsilon)$ must satisfy $u(0, \varepsilon) = 0$.

Many efforts have been made to give rigorous foundations for methods of matched asymptotic expansions. Here we must distinguish the work on the matching principles from the work on the existence of an exact solution and the estimates of the remainder of the exact solution to the asymptotic approximations. The matching principles are a set of auxiliary conditions that ensures the unique solvability of the inner expansions and the asymptotic matching of the outer and inner expansions so that a composite expansion can be constructed. The most general methods on this area are: (i) the method of intermediate variables originated by Kaplun and Lagerstrom; and (ii) the method of asymptotic matching principle originated by Van Dyke. It is known that in many cases, the two methods are equivalent (see Eckhaus [5]), and many simple examples, mostly in $\mathbb{R}^2$, have been treated thoroughly by both methods. However, there does not seem to exist an explicitly stated complete procedure for computing the inner expansions or boundary expansions for the system (1.1) in $\mathbb{R}^n$. The auxiliary conditions on the inner expansions given in this paper are a set of simple growth conditions that do not depend on the specific outer expansions. However, the matching of the inner and outer expansions can be proved as a consequence of the growth conditions.

The correctness of the asymptotic expansion obtained by various matching principles cannot be justified by the asymptotic analysis itself. At this stage the small parameter ε has to be fixed and the increasing of the order of truncation does not help either. Here we face the problems of the existence of an exact solution close to the formal expansion solution and how to compute the exact solution for a small but fixed ε. There are two major schools working in this direction. The first uses the maximum principle and various comparison theorems (see Chang and Howes [3], Angenent, Mallet-Paret, and Peletier [1], and Nagumo [16]). The second uses the contraction principle, the Inverse Function Theorem, Newton's method, or the like (see Eckhaus [5] and van Harten [14]). In the application of the latter methods, we must frequently investigate the inversion of certain linear operators, obtained from linearizing the whole boundary value problem (1.1). The method developed in this paper uses a modification of the classical Shadowing Lemma (see Guckenheimer, Moser, and Newhouse [11] for a proof of the classical Shadowing Lemma). The extended version used in this paper seems to be new and its application to singular perturbations is close to the Inverse Function Theorem or Newton's method. By virtue of the Shadowing Lemma, the investigation is reduced to the study of local solutions of the linear variational equations of the outer and inner approximations. This is much easier than the global inversion of the linear operators mentioned above. From a computational point of view, the justification of the validity of the asymptotic expansion automatically leads to a numerical scheme of obtaining an arbitrarily accurate solution based on that approximation. One of the major characteristics of classical singular perturbation methods is to treat the outer and inner layers separately and then use some form of matching. Therefore it seems natural to extend the application of the Shadowing Lemma to singular perturbation problems, which allows the inversion of the linear variational equations in the outer and inner regions separately.

We state our hypotheses and results in § 2, which also includes some examples that have been treated in previously published articles and that can be shown to fit our hypotheses. Basic definitions and lemmas concerning exponential dichotomies and Fredholm operators induced by the linearization around the formal approximations are given in § 3. The perturbation of angles between the stable and unstable manifolds is studied in Lemma 3.10, which is crucial in studying the interior transition layers. The Shadowing Lemmas are given in § 4. It is first proved for a system on the whole real axis (Theorem 4.3) and then applied to the boundary value problem (Theorem 4.4) and the periodic system (Theorem 4.5). In § 5, we give the complete procedure for the construction of inner and outer expansions. The major features of our expansions are given in Theorems 5.5 and 5.6. The proof of the validity of the formal solutions obtained in § 5 is given in § 6. The most unpleasant fact about exponential dichotomy on finite intervals is that the stable and unstable spaces are not uniquely defined. We have to extend the equations to the whole real axis such that it is compatible with the change of ε. This makes the proof very technical.

The general references for the singular perturbation problem are so extensive that we mention only a few that happened to catch our attention. The books of O'Malley [15], Eckhaus [5], and Wasow [18] offer comprehensive descriptions of the method and the theory, as well as many references. The work we present is closely related to the early work of Fife [8], [9]. The conditions we imposed on the boundary points are geometry-oriented and also have appeared in Hoppensteadt [13]. Hale and Sakamato [12] use the Lyapunov–Schmidt method and bifurcation theory to obtain a necessary and sufficient condition for the existence of the transition layers in a second-order problem in the neighborhood of a given approximation. The matching of the outer and inner expansions is a consequence of the method when we use the Shadowing Lemma. The method of Hale and Sakamoto [12] also gives the stability of the solution as an equilibrium point of a parabolic PDE. Exponential dichotomies are employed by Battelli and Lazzari [2] in their paper studying singular perturbation problems.

We use $\cdot$ to denote d/dt and $'$ to denote $d/d\tau$, where $\tau=(t-t_i)/\varepsilon$ is a fast variable (stretched variable). Partial derivatives shall be denoted by $D_x^{i_1}$, etc., where i_1 is a multi-index. In the notation y_j^i, both i and j are indices unless the contrary is specified. $\mathcal{R}$ and $\mathcal{K}$ are used for the range and kernel of linear operators.

2. Assumptions and main results. We assume that $f:\mathbb{R}^n\times[a,b]\times\mathbb{R}^+\to\mathbb{R}^n$ is C^∞ with bounded derivatives. $t_0=a<t_1<\cdots<t_{r-1}<t_r=b$ is a partition of $[a,b]$. Let $p_i(t)$, $i=1,\cdots,r$, be a C^∞ function on $[t_{i-1},t_i]$ such that

$$f(p_i(t),t,0)=0,\qquad t_{i-1}\leqq t\leqq t_i. \tag{2.1}_i$$

We introduce a new variable $\tau=(t-t_i)/\varepsilon$ at the neighborhood of each t_i, $0\leqq i\leqq r$. We assume that a function $q_i(\tau)$ is defined for $\tau\in\mathbb{R}$, $1\leqq i\leqq r-1$ and $\tau\in\mathbb{R}^+$, $i=0$, $\tau\in\mathbb{R}^-$, $i=r$ such that

$$q_i'(\tau)=f(q_i(\tau),t_i,0) \tag{2.2}_i$$

in the domain of definition of $q_i(\tau)$, and $q_i(\tau)\to p_i(t_i)$ as $\tau\to-\infty$, $1\leqq i\leqq r$ and $q_i(\tau)\to p_{i+1}(t_i)$ as $\tau\to+\infty$, $0\leqq i\leqq r-1$. We assume that $q_0(\tau)$ and $q_r(\tau)$ satisfy the boundary conditions $B_1(q_0(0),0)=0$ and $B_2(q_r(0),0)=0$.

We shall need some hyperbolicity assumptions on $p_i(t)$ and $q_i(\tau)$. We assume there is an $\alpha_0>0$ and an integer d^+ such that for $1\leqq i\leqq r$,

(H1) $\sigma\{f_x(p_i(t),t,0)\}\cap\{|\mathrm{Re}\,\lambda|\leqq\alpha_0\}=\varnothing$, and the dimension of the unstable space of $f_x(p_i(t),t,0)$ is d^+

for each fixed $t\in[t_{i-1},t_i]$. The dimension of the stable spaces of f_x is $d^-=n-d^+$. We assume that $B_1:\mathbb{R}^n\times\mathbb{R}^+\to\mathbb{R}^{d-}$ is C^∞, rank $B_{1x}(q_0(0),0)=d_-$ and $B_2:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{d+}$ is C^∞, rank $B_{2x}(q_r(0),0)=d_+$. The linear homogeneous equation

$$\varphi'(\tau)-f_x(q_i(\tau),t_i,0)\varphi(\tau)=0 \tag{2.3}_i$$

and the adjoint equation

$$\psi'(\tau)+f_x^*(q_i(\tau),t_i,0)\psi(\tau)=0 \tag{2.4}_i$$

are important in our investigations. Let $\varphi_1(\tau)$, $\tau\in\mathbb{R}^+$ be any nontrivial bounded solution of $(2.3)_o$ and $\varphi_2(\tau)$, $\tau\in\mathbb{R}^-$ be any nontrivial bounded solution of $(2.3)_r$; then

(H2) $B_{1x}(q_0(0),0)\cdot\varphi_1(0)\neq0,\qquad B_{2x}(q_r(0),0)\varphi_2(0)\neq0.$

It should be clear that $q_i'(\tau)$, $\tau\in\mathbb{R}$, $1\leqq i\leqq r-1$, is a nontrivial solution for $(2.3)_i$. Assume that $q_i'(\tau)$, $1\leqq i\leqq r-1$, is the only bounded solution for $(2.3)_i$ up to a scalar factor; then from the general theory of exponential dichotomy and Fredholm alternative (see Lemma 4.2 in Palmer [17] and § 3 of this paper), there exists a unique bounded solution $\psi_i(\tau)$, $\tau\in\mathbb{R}$, $1\leqq i\leqq r-1$, of $(2.4)_i$, up to a scalar factor. Moreover, $\psi_i(\tau)\to0$ exponentially as $\tau\to\pm\infty$. The following generic assumption will be crucial for our investigation:

(H3) $\displaystyle\int_{-\infty}^{\infty}\psi_i^*(\tau)f_t(q_i(\tau),t_i,0)\,d\tau\neq0,\qquad 1\leqq i\leqq r-1.$

We now state our main results as follows.

THEOREM 2.1. *Suppose* $\{p_i(t)\}$, $1 \leqq i \leqq r$, $\{q_i(\tau)\}$, $0 \leqq i \leqq r$, *satisfy* $(2.1)_i$, $(2.2)_i$, *and* (H1), (H2), (H3) *are satisfied. Then there exist formal power series*:

$$\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t), \quad X_0^i(t) = p_i(t), \quad 1 \leqq i \leqq r,$$

$$\sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau), \quad y_0^i(\tau) = q_i(\tau) \quad \text{with} \begin{cases} \tau \in R, & 1 \leqq i \leqq r-1, \\ \tau \in R^+, & i = 0, \\ \tau \in R^-, & i = r, \end{cases}$$

$$\sum_{j=0}^{\infty} \varepsilon^j \tau_j^i, \quad 0 \leqq i \leqq r, \quad \tau_j^0 = \tau_j^r = 0 \quad \text{and for all } j \geqq 0$$

with the functions X_j^i *and constants* τ_j^i *computable by a system of recursive linear algebraic equations*, y_j^i *computable by a system of recursive linear nonhomogeneous differential equations such that, for any*

$$m \geqq 0, \qquad 0 < \beta < 1,$$

the function

$$x(t, \varepsilon) = \begin{cases} \sum_{j=0}^{m} \varepsilon^j X_j^i(t), \quad t \in [t_{i-1} + \varepsilon^\beta, t_i - \varepsilon^\beta], \quad 1 \leqq i \leqq r, \\ \sum_{j=0}^{m} \varepsilon^j y_j^i\left(\tau - \sum_{j=0}^{m-1} \varepsilon^j \tau_j^i\right), \quad t \in [t_i - \varepsilon^\beta, t_i + \varepsilon^\beta] \cap [a, b], \quad \tau = (t - t_i)/\varepsilon, \quad 0 \leqq i \leqq r \end{cases}$$

is a formal approximation of (1.1) *with residuals and jumps as* $O(\varepsilon^{\beta(m+1)})$ *and boundary errors* $O(\varepsilon^{m+1})$ *uniformly for* $t \in [a, b]$ *and* $0 \leqq i \leqq r$.

Observe that $x(t, \varepsilon)$ has two values at common points $t_i \pm \varepsilon^\beta$, but that does not affect the conclusion. For the convenience, we shall tolerate that ambiguity.

Let the inner expansion of the outer approximation be

$$\sum_{j=0}^{\infty} \varepsilon^j X_j^i\left(t_i + \varepsilon\left(\tau + \sum_{j=0}^{\infty} \varepsilon^j \tau_j^i\right)\right) = \sum_{j=0}^{\infty} \varepsilon^j x_{j,1}^i(\tau),$$

$$\sum_{j=0}^{\infty} \varepsilon^j X_j^{i+1}\left(t_i + \varepsilon\left(\tau + \sum_{j=0}^{\infty} \varepsilon^j \tau_j^i\right)\right) = \sum_{j=0}^{\infty} \varepsilon_j x_{j,2}^i(\tau),$$

as in (5.24) of § 5. Define the *composite expansion* $\bar{x}(t, \varepsilon)$ as follows:

$$\begin{aligned} \bar{x}(t, \varepsilon) = {} & \sum_{j=0}^{m} \varepsilon^j X_j^i(t) + \sum_{j=0}^{m} \varepsilon^j y_j^{i-1}\left(\frac{t - t_{i-1}}{\varepsilon} - \sum_{j=0}^{m-1} \varepsilon^j \tau_j^{i-1}\right) \\ & + \sum_{j=0}^{m} \varepsilon^j y_j^i\left(\frac{t - t_i}{\varepsilon} - \sum_{j=0}^{m-1} \varepsilon^j \tau_j^i\right) - \sum_{j=0}^{m} \varepsilon^j x_{j,2}^{i-1}\left(\frac{t - t_{i-1}}{\varepsilon} - \sum_{j=0}^{m-1} \varepsilon^j \tau_j^{i-1}\right) \\ & - \sum_{j=0}^{m} \varepsilon^j x_{j,1}^i\left(\frac{t - t_i}{\varepsilon} - \sum_{j=0}^{m-1} \varepsilon^j \tau_j^i\right), \end{aligned} \tag{2.5}$$

for $t \in [t_{i-1}, t_i]$, $1 \leqq i \leqq r$.

THEOREM 2.2. *Let* $x(t, \varepsilon)$ *be the formal approximation as in Theorem* 2.1, *corresponding to some* $m \geqq 0$. *Then there exists* $\varepsilon_0 > 0$ *and* $\delta_0 > 0$ *such that there exists a unique exact solution* $x_{\text{exact}}(t, \varepsilon)$ *of the boundary problem* (1.1) *with* $|x(t, \varepsilon) - x_{\text{exact}}(t, \varepsilon)| \leqq \delta_0$ *if* $0 < \varepsilon \leqq \varepsilon_0$. *The remainder* $x(t, \varepsilon) - x_{\text{exact}}(t, \varepsilon)$ *is* $O(\varepsilon^{\beta(m+1)})$ *for any* β *with* $0 < \beta < 1$. *The composite expansion* $\bar{x}(t, \varepsilon)$ *is uniformly valid for* $t \in [a, b]$ *with* $\bar{x}(t, \varepsilon) - x_{\text{exact}}(t, \varepsilon)$ *being* $O(\varepsilon^{m+1})$.

Results similar to Theorem 2.1 and 2.2 are also valid for periodic systems. Consider a periodic system

$$\varepsilon\dot{x}(t) = f(x(t), t, \varepsilon) \tag{2.6}$$

where $f(x, t, \varepsilon) = f(x, t+\omega, \varepsilon)$ is $C^\infty: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$. Let $[0, \omega] = \bigcup_{i=1}^{r} [t_{i-1}, t_i]$, and let $p_i(t)$, $1 \leqq i \leqq r$, be a C^∞ function on $[t_{i-1}, t_i]$, satisfying $(2.1)_i$. Assume that (H1) is satisfied. In the fast variable $\tau = (t - t_i)/\varepsilon$, assume that a heteroclinic solution $q_i(\tau)$, $0 \leqq i \leqq r$, of $(2.2)_i$ is given, with $q_0(\tau) = q_r(\tau)$, $q_i(\tau) \to p_i(t_i)$ as $\tau \to -\infty$ for $1 \leqq i \leqq r$ and $q_i(\tau) \to p_{i+1}(t_i)$ as $\tau \to +\infty$ for $0 \leqq i \leqq r-1$.

Assume that the linear variational equation $(2.3)_i$ has the unique bounded solution $q_i'(\tau)$ and the formal adjoint equation $(2.4)_i$ has a unique bounded solution $\psi_i(\tau)$, $\tau \in \mathbb{R}$, $0 \leqq i \leqq r$, up to a scalar factor. We shall also assume the generic assumption (H3).

THEOREM 2.3. *For the periodic system* (2.6), *suppose* $\{p_i(t)\}$, $1 \leqq i \leqq r$, $\{q_i(\tau)\}$, $0 \leqq i \leqq r$, *satisfy* $(2.1)_i$, $(2.2)_i$ *with* $q_0(\tau) = q_r(\tau)$, $a = 0$, $b = \omega$, *and* (H1), (H3) *are satisfied for* $1 \leqq i \leqq r$. *Then there exist formal power series*:

$$\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t), \quad X_0^i(t) = p_i(t), \quad 1 \leqq i \leqq r,$$

$$\sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau), \quad y_0^i(\tau) = q_i(\tau), \quad \tau \in \mathbb{R}, \quad 0 \leqq i \leqq r,$$

$$\sum_{j=0}^{\infty} \varepsilon^j \tau_j^i, \quad 0 \leqq i \leqq r$$

with the functions X_j^i *and constants* τ_j^i *computable by a system of recursive linear algebraic equations,* y_j^i *computable by a system of recursive linear nonhomogeneous differential equations. Moreover, for any* $m \geqq 0$, $0 < \beta < 1$, *if* $x(t, \varepsilon)$ *and* $\bar{x}(t, \varepsilon)$ *are defined as in Theorems* 2.1 *and* 2.2 *for* $t \in [0, \omega]$ *and are periodic with period* ω *for* $t \in \mathbb{R}$, *then* $x(t, \varepsilon)$ *is a formal approximation of* (2.6) *with residuals and jumps as* $O(\varepsilon^{\beta(m+1)})$. *There exists a unique exact periodic solution* $x_{\text{exact}}(t, \varepsilon)$ *of period* ω *in a small neighborhood of* $x(t, \varepsilon)$ *provided that* $0 < \varepsilon \leqq \varepsilon_0$. *The remainder* $x(t, \varepsilon) - x_{\text{exact}}(t, \varepsilon)$ *is* $O(\varepsilon^{\beta(m+1)})$. *The composite expansion* $\bar{x}(t, \varepsilon)$ *is uniformly valid with* $\bar{x}(t, \varepsilon) - x_{\text{exact}}(t, \varepsilon)$ *being* $O(\varepsilon^{m+1})$.

Example 2.4. Consider

$$\varepsilon^2 \ddot{x} = f(x, t), \qquad a \leqq t \leqq b,$$

Boundary conditions at $t = a$ and $t = b$

where $x \in \mathbb{R}^n$ and $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. If $y = \dot{x}$, we have

$$\varepsilon\dot{x} = y, \qquad \varepsilon\dot{y} = f(x, t), \tag{2.7}$$

which is a system in $\mathbb{R}^{2n}$. Suppose $f(p(t), t) = 0$ for $t \in J \subset [a, b]$, then $(p(t), \varepsilon\dot{p}(t))$ is a regular approximation for (2.7) in J, with residual $O(\varepsilon^2)$. Suppose that $D_x f(p(t), t)$ has positive eigenvalues $\lambda_1(t), \cdots, \lambda_n(t)$, $\min_{1 \leqq j \leqq n} \lambda_j(t) \geqq \lambda_0 > 0$ for each $t \in J$. The Jacobian for the $2n$-system (2.7) is $A = \begin{pmatrix} 0 & I \\ f_x & 0 \end{pmatrix}$. It is easy to show that $\det(\lambda I - A) = \prod_{j=1}^{n} (\lambda^2 - \lambda_j)$. Thus (H1) is satisfied, with $d^- = d^+ = n$ for each $t \in J$.

Suppose $p_1(t)$, $t \in [a, t_1]$ and $p_2(t)$, $t \in [t_1, b]$ are such regular approximations. Let $(q_1(\tau), q_2(\tau))'$, $q_i \in \mathbb{R}^n$, $i = 1, 2$ be a heteroclinic solution for

$$x' = y, \qquad y' = f(x, t_1), \tag{2.8}$$

which satisfies all the conditions preceding Theorem 2.1. Let $(\psi_1(\tau), \psi_2(\tau))'$, $\psi_i \in \mathbb{R}^n$, $i = 1, 2$ be the unique nontrivial bounded solution (up to a scalar factor) for the formal adjoint system of the linear variational equation of (2.8). Condition (H3) reduces to

$$\int_{-\infty}^{\infty} \psi_2^*(\tau) f_t(q_1(\tau), t_1)\, d\tau \neq 0.$$

An often studied case is $n = 1$. It is easy to verify that $\psi_2(\tau) = q_1'(\tau)$ up to a scalar factor. We can write (H3) as

$$\int_{p_1(t_1)}^{p_2(t_1)} f_t(x, t_1)\, dx \neq 0. \tag{2.9}$$

Condition (2.9) can be found in Fife [8], [9]. Angenent, Mallet-Paret, and Peletier [1] studied the scalar boundary value problem

$$\varepsilon^2 u_{tt} + f(u, t) = 0, \qquad u_t(0) = u_t(1) = 0,$$

where $f(u, t) = u(1-u)(u-a(t))$, $a \in C^1([0, 1])$ and $0 < a(t) < 1$. If $0 < t_1 < 1$ is such that $a(t_1) = \frac{1}{2}$, $a'(t_1) \neq 0$. Then there exists a heteroclinic solution connecting $u = 0$ and $u = 1$, and (2.9) is satisfied.

Dirichlet boundary conditions can be found in Fife [9], although they have been treated in early papers. It is of interest to point out that condition (2.5) in his paper implies (H2), which can easily be justified by a phase portrait analysis.

We remark here that there are several obvious generalizations of Theorem 2.2. For example, $[a, b]$ may be replaced by $[a, +\infty)$ or $(-\infty, b]$, and in each case one of the boundary conditions is missing and a global boundedness condition is imposed. We can also consider $(-\infty, +\infty)$ with no boundary condition at all. Moreover, the number of subintervals does not have to be finite. In some sense, the boundary value problem and initial value problem need not be treated separately. However, in the semistable case, only part of the initial conditions should be specified as follows from our general theory easily.

3. Preliminaries. We first present some properties of the exponential dichotomies of the linear nonautonomous equations and the application to the linear variational equation of the heteroclinic solution $q_i(\tau)$. We refer to Coppel [4] and Palmer [17] for proofs of the following results.

Consider a linear ODE in $\mathbb{R}^n$

$$\dot{x}(t) - A(t)x(t) = h(t), \qquad t \in J \tag{3.1}$$

where $A(t)$, $t \in J$ is a continuous and uniformly bounded matrix-valued function. Let $T(t, s)$ be the solution map for the linear homogeneous equation.

DEFINITION 3.1. We say that (3.1), or $T(t, s)$, has an *exponential dichotomy* in J if there exist projections $P_s(t)$ and $P_u(t) = I - P_s(t)$, $t \in J$, such that

$$T(t, s)P_s(s) = P_s(t)T(t, s), \qquad t \geqq s \quad \text{in } J,$$

$$|T(t, s)P_s(s)| \leqq K e^{-\alpha(t-s)}, \qquad t \geqq s \quad \text{in } J,$$

$$|T(t, s)P_u(s)| \leqq K e^{-\alpha(s-t)}, \qquad s \geqq t \quad \text{in } J$$

where K and α are positive constants.

LEMMA 3.2 (roughness of exponential dichotomies). *Suppose the linear equation* (3.1) *has an expoential dichotomy on* $\mathbb{R}^+$. *If*

$$\delta = \sup_{t \in \mathbb{R}^+} |B(t)| < \alpha / 4K^2$$

then $\dot{y}(t) - [A(t) + B(t)]y(t) = 0$ *also has an exponential dichotomy in* $\mathbb{R}^+$, *with the constants* $\tilde{K}$ *and* $\tilde{\alpha}$ *determined by* K, α, *and* δ. *Let the projections be* $\tilde{P}_s(t)$ *and* $\tilde{P}_u(t)$; *then* $|\tilde{P}_s(t) - P_s(t)| = O(\delta)$ *uniformly in* $t \in \mathbb{R}^+$, $|\tilde{\alpha} - \alpha| = O(\delta)$ *and* $\tilde{K}$ *is bounded.*

LEMMA 3.3. *Assume that* $|A(t)| \leqq M$ *for all* $t \in J$, *and* $A(t)$ *has* d^-*-eigenvalues with real part* $\leqq -\alpha < 0$ *and* $d^+ = n - d^-$ *eigenvalues with real part* $\geqq \alpha > 0$ *for all* $t \in J$. *Assume that for any* $0 < \varepsilon < \alpha$, *there exists* $0 < \delta = \delta(M, \alpha, \varepsilon)$ *such that if* $|A(t_2) - A(t_1)| \leqq \delta$ *for* $|t_2 - t_1| \leqq h$, *where* $h > 0$ *is a fixed number not greater than the length of* J. *Then* (3.1) *has exponential dichotomy in* J *with constants* $K = K(M, \alpha, \varepsilon)$ *and exponent* $\alpha - \varepsilon$. *Moreover,* $P_s(t)$ *approaches the spectral projection to the stable eigenspace of* $A(t)$ *for each fixed* t, *as* $\delta \to 0$.

Proofs of Lemma 3.2 and 3.3 may be found in Coppel [4]. Suppose that $q(t)$, $t \geqq 0$ is a solution for a nonlinear autonomous ODE, which approaches a hyperbolic equilibrium $x = x_0$ as $t \to \infty$. The linear variational equation around x_0 clearly has an exponential dichotomy. Because of Lemma 3.2, we conclude that the linear variational equation around $q(t)$, $t \geqq \tilde{t}$, has exponential dichotomy for sufficiently large $\tilde{t} > 0$, with the projections close to those of the linearization around x_0. And it is easy to see that the dichotomy around $q(t)$ extends to $J = \mathbb{R}^+$. Similar results also hold for $J = \mathbb{R}^-$. These observations will be useful throughout the paper.

DEFINITION 3.4. Define a subset $E_J(\gamma, l)$ of continuous functions on J as $E_J(\gamma, l) = \{x(t) | \sup_{t \in J} (|x(t)| e^{\gamma|t|}(1 + |t|^l)^{-1}) < \infty\}$, which is a Banach space with the norm $\|x\|_{E_J(\gamma, l)} = \sup_{t \in J} \{|x(t)| e^{\gamma|t|}(1 + |t|^l)^{-1}\}$, where γ is a real constant and $l \geqq 0$ an integer. Let $E_J^k(\gamma, l) = \{x(t) | x(t), x'(t), \cdots, x^{(k)}(t) \in E_J(\gamma, l)\}$, which is a Banach space with $\|x\|_{E_J^k(\gamma, l)} = \sum_{j=0}^k \|x^{(j)}\|_{E_J(\gamma, l)}$.

DEFINITION 3.5. Let $\mathscr{F}: E_J^1(\gamma, l) \to E_J(\gamma, l)$, $x \mapsto h$, be defined as $h(t) = \dot{x}(t) - A(t)x(t)$. Let $\mathscr{F}^*: E_J^1(\gamma, l) \to E_J(\gamma, l)$, $y \mapsto g$ be defined as $g(t) = \dot{y}(t) + A(t)^* y(t)$.

Clearly $\mathscr{F}$ and $\mathscr{F}^*$ are linear bounded. Assume that (3.1) has exponential dichotomy in J with constant K and exponent α. Let γ be a constant, $|\gamma| < \alpha$.

LEMMA 3.6. (i) *If* $J = \mathbb{R}^-$, *then for any* $h \in E_{\mathbb{R}^-}(\gamma, l)$ *and* $u \in \mathscr{R}P_u(0)$, *there exists a unique solution* $x \in E^1_{\mathbb{R}^-}(\gamma, l)$ *of* (3.1) *with* $P_u(0)x(0) = u$. *Moreover* $\|x\|_{E^1_{\mathbb{R}^+}(\gamma, l)} \leqq C\{\|h\|_{E_{\mathbb{R}^-}}(\gamma, l) + \|u\|\}$.

(ii) *If* $J = \mathbb{R}^+$, *then for any* $h \in E_{\mathbb{R}^+}(\gamma, l)$ *and* $v \in \mathscr{R}P_s(0)$, *there exists a unique solution* $x \in E^1_{\mathbb{R}^+}(\gamma, l)$ *of* (3.1) *with* $P_s(0)x(0) = v$. *Moreover,* $\|x\|_{E^1_{\mathbb{R}^+}(\gamma, l)} \leqq C\{\|h\|_{E_{\mathbb{R}^+}(\gamma, l)} + \|v\|\}$.

(iii) *If* $J = \mathbb{R}$, *then for any* $h \in E_{\mathbb{R}}(\gamma, l)$, *there exists a unique solution* $x \in E^1_{\mathbb{R}}(\gamma, l)$ *of* (3.1) *with* $|x\|_{E^1_{\mathbb{R}}(\gamma, l)} \leqq C\|h\|_{E_{\mathbb{R}}(\gamma, l)}$.

Proof. (i) We can write the solution as

$$x(t) = T(t, 0)u + \int_0^t T(t, s)P_u(s)h(s)\, ds + \int_{-\infty}^t T(t, s)P_s(s)h(s)\, ds.$$

From a simple estimate using Definitions 3.1 and 3.4, we have $\|x\|_{E_{\mathbb{R}^-}(\gamma, l)} \leqq C\{\|h\|_{E_{\mathbb{R}^-}(\gamma, l)} + \|u\|\}$. The estimate for $\|x\|_{E^1_{\mathbb{R}^-}(\gamma, l)}$ comes from (3.1).

Proofs for (ii) and (iii) are similar to that of (i).

LEMMA 3.7. *If* (3.1) *has exponential dichotomies in* $\mathbb{R}^-$ *and* $\mathbb{R}^+$ *with constant* K *and exponent* α *being the same in* $\mathbb{R}^-$ *and* $\mathbb{R}^+$, $|\gamma| < \alpha$. *Let the projections, which define the exponential dichotomies, be* $P_u^-(t)$ *and* $P_s^-(t)$ *for* $t \in \mathbb{R}^-$ *and* $P_u^+(t)$ *and* $P_s^+(t)$ *for*

$t \in \mathbb{R}^-$. Then $\mathscr{F}: E^1_{\mathbb{R}}(\gamma, l) \to E_{\mathbb{R}}(\gamma, l)$ is <u>*Fredholm*</u> *with Index* $\mathscr{F} = \dim \mathscr{R}P_u^-(0) - \dim \mathscr{R}P_0^+(0)$. *$h \in \mathscr{R}\mathscr{F}$ if and only if*

$$\int_{-\infty}^{\infty} \psi^*(t)h(t) = 0$$

for all $\psi \in \mathscr{K}\mathscr{F}^*$. Indeed, $\mathscr{K}\mathscr{F}^* \subset E_{\mathbb{R}}(\alpha, 0)$.

Proof. The proof is completely similar to Lemma 4.2 in Palmer [17].

The following definition is from Gohberg and Krein [10], which also contains proof of Lemma 3.9.

DEFINITION 3.8. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two linear subspaces of $\mathbb{R}^n$. By the *minimal angle* between $\mathscr{M}_1$ and $\mathscr{M}_2$, it is meant the angle $\theta(\mathscr{M}_1, \mathscr{M}_2)(0 \leqq \theta \leqq \pi/2)$, defined by

$$\cos \theta(\mathscr{M}_1, \mathscr{M}_2) = \sup \{|(x, y)| \, x \in \mathscr{M}_1, y \in \mathscr{M}_2, |x| = |y| = 1\}.$$

Obviously $\theta(\mathscr{M}_1, \mathscr{M}_2) \neq 0$ and $\dim \mathscr{M}_1 + \dim \mathscr{M}_2 = n$ is equivalent to $\mathscr{M}_1 \oplus \mathscr{M}_2 = \mathbb{R}^n$. In this case, there is a projection P such that $\mathscr{R}P = \mathscr{M}_1$ and $\mathscr{K}P = \mathscr{M}_2$. We are interested in the case that $\theta(\mathscr{M}_1, \mathscr{M}_2)$ is small or, equivalently, $\|P\|$ is large. The following lemma is useful.

LEMMA 3.9. *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 \theta(\mathscr{M}_1, \mathscr{M}_2)^{-1} \leqq \|P\| \leqq C_2 \theta(\mathscr{M}_1, \mathscr{M}_2)^{-1}.$$

We shall be interested in the ε-dependent systems

$$\dot{x}(t) - A(t, \varepsilon)x(t) = 0 \tag{3.2$_\varepsilon$}$$

where $A(t, \varepsilon)$ is C^1 in ε and uniformly bounded in t, $\partial A(t, \varepsilon)/\partial \varepsilon \in E_{\mathbb{R}}(0, 1)$. Suppose for $\varepsilon = 0$, $(3.2)_o$ has exponential dichotomies in $\mathbb{R}^-$ and $\mathbb{R}^+$; then for ε small, $(3.2)_\varepsilon$ still possesses exponential dichotomies in $\mathbb{R}^-$ and $\mathbb{R}^+$. Let the projections be $P_s^-(t, \varepsilon) + P_u^-(t, \varepsilon) = I$ and $P_s^+(t, \varepsilon) + P_u^+(t, \varepsilon) = I$, respectively, in $\mathbb{R}^-$ and $\mathbb{R}^+$. We are interested in the following situation: (i) $\dim \mathscr{R}P_u^-(t, \varepsilon) = \dim P_u^+(t, \varepsilon) = d^+$, $(d^- = n - d^+)$; (ii) $\mathscr{R}P_u^-(0, 0) \cap \mathscr{R}P_s^+(0, 0)$ is one-dimensional. Let $\varphi(t)$ be the only bounded solution of $(3.2)_o$, up to a scalar factor. From Lemma 3.7, Ind $\mathscr{F} = 0$ and there is a unique bounded solution $\psi(t)$ for $\dot{\psi}(t) + A(t, 0)^* \psi(t) = 0$. Indeed, both φ and $\psi \in E(\alpha, 0)$. Since at $\varepsilon = 0$, $\mathscr{R}P_u^-(0, 0) \cap \mathscr{R}P_s^+(0, 0) \neq \{0\}$, we find that $\theta(\mathscr{R}P_u^-(0, 0), \mathscr{R}P_s^+(0, 0)) = 0$. We are interested in $\theta(\mathscr{R}P_u^-(0, \varepsilon), \mathscr{R}P_s^+(0, \varepsilon))$ for small and nonzero ε.

Lemma 3.10. *Under our assumptions on $(3.2)_\varepsilon$, if $\int_{-\infty}^{\infty} \psi^*(t)(\partial/\partial\varepsilon)A(t, 0)\varphi(t) \neq 0$, then there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\theta(\mathscr{R}P_u^-(0, \varepsilon), \mathscr{R}P_s^+(0, \varepsilon) \geqq C|\varepsilon|,$$

for $|\varepsilon| \leqq \varepsilon_0$.

Proof. We shall choose an orthonormal basis in $\mathscr{R}P_u^-(0, 0)$: $\{u_1, \cdots, u_{d^+}\}$ and an orthonormal basis in $\mathscr{R}P_s^+(0, 0)$: $\{v_1, \cdots, v_{d^-}\}$. Without loss of generality, assume that $|\varphi(0)| = 1$ and $u_1 = v_1 = \varphi(0)$. We claim that for $\varepsilon \neq 0$ and small, we have a basis $\{u_1(\varepsilon), \cdots, u_{d^+}(\varepsilon)\} = \{u_1 + \bar{u}_1(\varepsilon), \cdots, u_{d^+} + \bar{u}_{d^+}(\varepsilon)\}$ in $\mathscr{R}P_u^-(0, \varepsilon)$, with $\bar{u}_i(\varepsilon) \in \mathscr{R}P_s^-(0, 0)$, $i = 1, \cdots, d^+$. We also have a basis $\{v_1(\varepsilon), \cdots, v_{d^-}(\varepsilon)\} = \{v_1 + \bar{v}_1(\varepsilon), \cdots, v_{d^-} + \bar{v}_{d^-}(\varepsilon)\}$ in $\mathscr{R}P_s^+(0, \varepsilon)$, with $\bar{v}_i(\varepsilon) \in \mathscr{R}P_u^+(0, 0)$. To show these, let $x(t, \varepsilon)$ be a solution of $(3.2)_\varepsilon$, $x(0, \varepsilon) = u_i(\varepsilon)$ $1 \leqq i \leqq d^+$, and $x(t, \varepsilon) \to 0$ as $t \to -\infty$. Clearly, $x(t, \varepsilon)$ satisfies the following integral equation

$$\begin{aligned} x(t, \varepsilon) &= x(t, 0) + \int_0^t T(t, s)P_u^-(s, 0)[A(s, \varepsilon) - A(s, 0)]x(s, \varepsilon)\, ds \\ &\quad + \int_{-\infty}^t T(t, s)P_s^-(s, 0)[A(s, \varepsilon) - A(s, 0)]x(s, \varepsilon)\, ds, \qquad t \in \mathbb{R}. \end{aligned} \tag{3.3}$$

By the contraction mapping principle, it is easy to prove that there exists a unique solution $x(t, \varepsilon)$, $t \in \mathbb{R}^-$, $|\varepsilon| \leqq \varepsilon_0$. Moreover, $x(t, \varepsilon)$ and $\partial x(t, \varepsilon)/\partial \varepsilon$ is continuous and uniformly bounded for $t \in \mathbb{R}^-$, $|\varepsilon| \leqq \varepsilon_0$. And we have

$$\frac{\partial \bar{u}_i(\varepsilon)}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon} x(0, \varepsilon) = \int_{-\infty}^{0} T(0, s) P_s^-(s, 0)[A(s, \varepsilon) - A(s, 0)] \frac{\partial}{\partial \varepsilon} x(s, \varepsilon)\, ds + \int_{-\infty}^{0} T(0, s) P_s^-(s, 0) \frac{\partial}{\partial \varepsilon} A(s, \varepsilon) x(s, \varepsilon)\, ds. \tag{3.4}$$

In particular, $\bar{u}_i(\varepsilon) = O(\varepsilon)$, $1 \leqq i \leqq d^+$. This proves the assertion for the basis in $\mathscr{R}P_u^-(0, \varepsilon)$. Similar results are also valid for $\bar{v}_i(\varepsilon)$, $1 \leqq i \leqq d^-$. We infer that

$$\frac{\partial}{\partial \varepsilon}\{\bar{u}_1(0) - \bar{v}_1(0)\} = \int_{-\infty}^{0} T(0, s) P_s^-(s, 0) \frac{\partial}{\partial \varepsilon} A(s, 0) \varphi(s)\, ds + \int_{0}^{\infty} T(0, s) P_u^+(s, 0) \frac{\partial}{\partial \varepsilon} A(s, 0) \varphi(s)\, ds. \tag{3.5}$$

Define a projection $Q(\varepsilon)$, which is from $\mathbb{R}^n \to \mathscr{R}P_u^-(0, \varepsilon)$ and parallel to $\mathscr{R}P_s^+(0, \varepsilon)$. Consider a linear algebra equation

$$w = (u_1(\varepsilon), \cdots, u_{d^+}(\varepsilon), v_1(\varepsilon), \cdots, v_{d^-}(\varepsilon)) \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix}, \quad \text{with } |w| = 1.$$

Clearly, $Q(\varepsilon) w = \sum_{i=1}^{d^+} \zeta_i u_i(\varepsilon)$ and $|Q(\varepsilon)| \leqq C \sum_{i=1}^{d^+} |\zeta_i|$. Observe that $\sum_{i=1}^{d^+} |\zeta_i| \leqq C\{\det(u_1(\varepsilon), \cdots, u_{d^+}(\varepsilon), v_1(\varepsilon), \cdots, v_{d^-}(\varepsilon))\}^{-1}$, as can be seen from the inversion formula of matrices. Observe that

$$\begin{aligned} &\det(u_1(\varepsilon), \cdots, u_{d^+}(\varepsilon), v_1(\varepsilon), \cdots, v_{d^-}(\varepsilon)) \\ &= \det(\bar{u}_1(\varepsilon) - \bar{v}_1(\varepsilon), \cdots, v_{d^-}(\varepsilon)) \\ &= \det(\bar{u}_1(\varepsilon) - \bar{v}_1(\varepsilon), u_2, \cdots, u_{d^+}, v_1, \cdots, v_{d^-}) + O(\varepsilon^2). \end{aligned} \tag{3.6}$$

From our definition of $\psi(t)$, $\psi(0) \in \mathscr{R}P_s^{-*}(0, 0) \cap \mathscr{R}P_u^{+*}(0, 0)$, where $*$ denotes the adjoint of an operator. Thus, $\psi(0) \perp \mathscr{R}P_u^-(0, 0)$ and $\psi(0) \perp \mathscr{R}P_s^+(0, 0)$. Without loss of generality, let $|\psi(0)| = 1$. Then (3.6) is equal to

$$\begin{aligned} &\det(\psi(0), u_2, \cdots, u_{d^+}, v_1, \cdots, v_{d^-})\{\psi^*(0)(\bar{u}_1(\varepsilon) - \bar{v}_1(\varepsilon))\} + O(\varepsilon^2) \\ &= C\{\psi^*(0)(\bar{u}_1(\varepsilon) - \bar{v}_1(\varepsilon))\} + O(\varepsilon^2). \end{aligned}$$

From (3.5), we have

$$\frac{\partial}{\partial \varepsilon}\{\psi^*(0)(\bar{u}_1(0) - \bar{v}_1(0))\} = \int_{-\infty}^{\infty} \psi^*(s) \frac{\partial A(s, 0)}{\partial \varepsilon} \varphi(s)\, ds \neq 0.$$

From (3.4), we observe that

$$\frac{\partial \bar{u}_1(\varepsilon)}{\partial \varepsilon} - \frac{\partial \bar{u}_1(0)}{\partial \varepsilon} = o(1).$$

It is obvious that if $\varepsilon_0 > 0$ is sufficiently small and $0 < |\varepsilon| \leqq \varepsilon_0$, $|\det(u_1(\varepsilon), \cdots, v_{d^-}(\varepsilon))| \geqq C|\varepsilon|$, $C > 0$ and $|Q(\varepsilon)| \leqq C|\varepsilon|^{-1}$. From Lemma 3.9, $C_1 \theta(\mathscr{R}P_u^-(0, \varepsilon), \mathscr{R}P_s^+(0, \varepsilon))^{-1} \leqq C|\varepsilon|^{-1}$. Whence the desired result follows.

4. Variants of shadowing lemmas. As mentioned previously, the basic tool of our investigation is a generalized *Shadowing Lemma.* We state our theorem first for bounded continuous functions defined on the whole real axis. Then we indicate how the general theory applies to boundary value problems.

If $\mathbb{R}=\bigcup_{m_0<i<m_1}[t_{i-1}, t_i]$ is a partition of the real axis into finitely or infinitely many subintervals. We shall allow $m_0=-\infty$ and/or $m_1=+\infty$. We shall agree that if m_0 and/or m_1 are finite, the first interval $(-\infty, t_{m_0+1}]$ and/or the last interval $[t_{m_1-2}, +\infty)$ are still denoted by $[t_{m_0}, t_{m_0+1}]$ and/or $[t_{m_1-2}, t_{m_1-1}]$ for simplification of notation. Let $A_i(t)$, $m_0<i<m_1$ be a continuous matrix valued function in $[t_{i-1}, t_i]$. Let $T^i(t, s)$ be the solution map for

$$\dot{x}(t)=A_i(t)x(t)=0, \qquad t\in[t_{i-1}, t_i].$$

We assume the following:

(i) $T^i(t, s)$ has exponential dichotomy in $[t_{i-1}, t_i]$, with projections $P_u^i(t)$ and $P_s^i(t)$, the constants K, $\alpha>0$ do not depend on i.

(ii) $\mathscr{R}P_u^i(t_i)\oplus\mathscr{R}P_s^{i+1}(t_i)=\mathbb{R}^n$, $m_0<i<m_1-1$. Let $Q^i:\mathbb{R}^n\to\mathscr{R}P_u^i(t_i)$ be a projection with $\mathscr{K}Q^i=\mathscr{R}P_s^{i+1}(t_i)$. Then $|Q^i|\leqq M$ and $\|I-Q^i|\leqq M$, M does not depend on i.

(iii) $t_i-t_{i-1}\geqq\nu$, $\nu>0$ is independent of i, $m_0<i<m_1$. Moreover, $4KMe^{-\alpha\nu}<1$ and $\sqrt{2(1+K)}\,KMe^{-\alpha\nu}<1$.

Let $\mathscr{X}$ be the Banach space of sequences of continuous and bounded functions $\{u_i(t)\}_{m_0<i<m_1}$, each defined in $[t_{i-1}, t_i]$, and on each sequence the following norm is finite:

$$\|\{u_i\}\|_{\mathscr{X}}=\sup\{|u_i(t)|: t\in[t_{i-1}, t_i], m_0<i<m_1\}.$$

Let Z be the Banach space of bounded sequences of real numbers $\{g_i\}_{m_0<i<m_1-1}$, with the norm $\|\{g_i\}\|_Z=\sup\{|g_i|: m_0<i<m_1-1\}$. Let $Y=\mathscr{X}\times Z$ be a Banach space with the norm

$$\|\{f_i\}\times\{g_i\}\|_Y=\|\{g_i\}\|_Z+\frac{2K}{\alpha}\|\{f_i\}\|_{\mathscr{X}}.$$

Let $\mathscr{A}:\mathscr{D}(\mathscr{A})\subset\mathscr{X}\to Y$ be defined as $\mathscr{A}:\{u_i\}\to\{f_i\}\times\{g_i\}$:

$$\begin{aligned}&\dot{u}_i(t)-A_i(t)u_i(t)=f_i(t), \qquad t\in[t_{i-1}, t_i], \quad m_0<i<m_1,\\ &u_i(t_i)-u_{i+1}(t_i)=g_i, \qquad m_0<i<m_1-1,\end{aligned} \tag{4.1}$$

where the domain $\mathscr{D}(\mathscr{A})$ is a subset of $\mathscr{X}$ on which the right-hand side of (4.1) is well defined and is in Y.

LEMMA 4.1. *Under the assumptions of* (i), (ii), *and* (iii), $\mathscr{A}:\mathscr{D}(\mathscr{A})\to Y$ *is one-to-one and onto.* $\mathscr{A}^{-1}: Y\to\mathscr{X}$ *is bounded with*

$$\|\mathscr{A}^{-1}\|\leqq 4MK+1.$$

Proof. Let $v_i(t)=\int_{t_{i-1}}^t T^i(t, s)P_s^i(s)f_i(s)\,ds+\int_{t_i}^t T^i(t, s)P_u^i(s)f_i(s)\,ds$, $t\in[t_{i-1}, t_i]$. Clearly, $\mathscr{A}\{v_i\}=\{f_i\}\times\{\bar{g}_i\}$, where $\bar{g}_i=v_i(t_i)-v_{i+1}(t_i)$:

$$\begin{aligned}|v_i(t)|&\leqq\int_{t_{i-1}}^t Ke^{-\alpha(t-s)}|f_i|\,ds+\int_t^{t_i}Ke^{-\alpha(s-t)}|f_i|\,ds\\ &\leqq\frac{2K}{\alpha}\|\{f_i\}\|_{\mathscr{X}}, \qquad t\in[t_{i-1}, t_i],\end{aligned} \tag{4.2}$$

$$|\bar{g}_i|\leqq\frac{2K}{\alpha}\|\{f_i\}\|_{\mathscr{X}}. \tag{4.3}$$

Let $g_i^1 = g_i - \bar{g}_i$. If g_i^k, $1 \leqq k$, has been computed, define

$$v_i^k(t) = T(t, t_{i-1})P_s^i(t_{i-1})(Q^{i-1} - I)g_{i-1}^k + T(t, t_i)P_u^i(t_i)Q^i g_i^k.$$

Then

$$(4.4)\qquad \begin{aligned} g_i^{k+1} &\overset{\text{def}}{=} g_i^k - [v_i^k(t_i) - v_{i+1}^k(t_i)] \\ &= T(t_i, t_{i-1})P_s^i(t_{i-1})(I - Q^{i-1})g_{i-1}^k + T(t_i, t_{i+1})P_u^{i+1}(t_{i+1})Q^{i+1}g_{i+1}^k. \end{aligned}$$

Obviously, from (4.3),

$$\|\{g_i^1\}\|_Z \leqq \|\{f_i\} \times \{g_i\}\|_Y.$$

From (4.4),

$$\|\{g_i^{k+1}\}\|_Z \leqq 2KM e^{-\alpha\nu}\|\{g_i^k\}\|_Z, \qquad k \geqq 1,$$

$$\|\{v_i^k\}\|_{\mathscr{X}} \leqq 2MK\|\{g_i^k\}\|_Z, \qquad k \geqq 1.$$

Since $2KM e^{-\alpha\nu} < \frac{1}{2}$, therefore $\sum_{k=1}^\infty \|\{g_i^k\}\|_Z < \infty$, and $\sum_{k=1}^\infty \|\{v_i^k\}\|_{\mathscr{X}} < \infty$. Let $u_i(t) = \sum_{k=1}^\infty v_i^k(t) + v_i(t)$, $t \in [t_{i-1}, t_i]$, $m_0 < i < m_1$; then $u_i(t)$ is continuous in its domain of definition and by adding (4.4) through $k \geqq 1$, $u_i(t_i) - u_{i+1}(t_i) = g_i^1 + \bar{g}_i = g_i$. Observe that $\mathscr{A}$ is a closed operator; thus

$$\mathscr{A}\{u_i\} = \{f_i\} \times \{g_i\}.$$

We have the estimate

$$\begin{aligned} \|\{u_i\}\|_{\mathscr{X}} &\leqq 2MK\left(\sum_{k=1}^\infty (2KM e^{-\alpha\nu})^k\right)\|\{g_i^1\}\|_Z + \|\{v_i\}\|_{\mathscr{X}} \\ &\leqq (4MK+1)\|\{f_i\} \times \{g_i\}\|_Y. \end{aligned}$$

To prove uniqueness, we show that the only bounded and continuous solution of (4.1), with $\{f_i\} \times \{g_i\} = 0$, is $\{u_i\} \equiv 0$. Assume the contrary; suppose at certain t_i, we have $0 \neq v = Q^i u(t_i)$, and $|(I - Q^i)u(t_i)| \leqq |Q^i u(t_i)|$, as the case $|(I - Q^i)u(t_i)| \geqq |Q^i u(t_i)|$ can be studied similarly. First, from $Q^i u(t_i) = P_u^{i+1}(t_i)Q^i u(t_i) + P_s^{i+1}(t_i)Q^i u(t_i)$, applying Q^i to the equation, we have

$$|Q^i u(t_i)| = |Q^i P_u^{i+1}(t_i)Q^i u(t_i)| \leqq M|P_u^{i+1}(t_i)Q^i u(t_i)|;$$

thus

$$(4.5)\qquad |P_u^{i+1}(t_i)v| \geqq \frac{1}{M}|v|,$$

$$(4.6)\qquad |P_s^{i+1}(t_i)v| \leqq K|v|,$$

$$(4.7)\qquad |P_s^{i+1}(t_i)(I - Q^i)u(t_i)| = |(I - Q^i)u(t_i)| \leqq |v|,$$

$$(4.8)\qquad |P_u^{i+1}(t_i)(I - Q^i)u(t_i)| = 0.$$

From (4.6) and (4.7),

$$|P_s^{i+1}(t_i)u(t_i)| \leqq (1+K)|v|.$$

From (4.5) and (4.8),

$$|P_u^{i+1}(t_i)u(t_i)| \geqq \frac{1}{M}|v|.$$

Using the relation $T(t_{i+1}, t_i): \mathcal{R}P_s^{i+1}(t_i) \to \mathcal{R}P_s^{i+1}(t_{i+1})$ and $T(t_{i+1}, t_i): \mathcal{R}P_u^{i+1}(t_i) \to \mathcal{R}P_u^{i+1}(t_{i+1})$, and the exponential estimates, we have

$$|P_s^{i+1}(t_{i+1})u(t_{i+1})| \leqq (1+K)K\,e^{-\alpha\nu}|v|,$$

$$|P_u^{i+1}(t_{i+1})u(t_{i+1})| \geqq \frac{1}{KM}\,e^{\alpha\nu}|\nu|,$$

$$|Q^{i+1}P_u^{i+1}(t_{i+1})u(t_{i+1})| = |P_u^{i+1}(t_{i+1})u(t_{i+1})| \geqq \frac{1}{KM}\,e^{\alpha\nu}|v|, \tag{4.9}$$

$$|Q^{i+1}P_s^{i+1}(t_{i+1})u(t_{i+1})| \leqq M|P_s^{i+1}(t_{i+1})u(t_{i+1})| \leqq (1+K)KM\,e^{-\alpha\nu}|v|, \tag{4.10}$$

$$(I-Q^{i+1})P_u^{i+1}(t_{i+1})u(t_{i+1}) = 0, \tag{4.11}$$

$$|(I-Q^{i+1})P_s^{i+1}(t_{i+1})u(t_{i+1})| \leqq M|P_s^{i+1}(t_{i+1})u(t_{i+1})| \leqq (1+K)KM\,e^{-\alpha\nu}|\nu|,$$

From (4.9) and (4.10),

$$|Q^{i+1}u(t_{i+1})| \geqq \left(\frac{1}{KM}\,e^{\alpha\nu} - (1+K)KM\,e^{-\alpha\nu}\right)|v|. \tag{4.12}$$

From (4.11) and (4.12),

$$|(I-Q^{i+1})u(t_{i+1})| \leqq (1+K)KM\,e^{-\alpha\nu}|v|. \tag{4.13}$$

Since $\sqrt{2(K+1)}\,KM\,e^{-\alpha\nu} < 1$, $(1+K)KM\,e^{-\alpha\nu} < (1/2KM)\,e^{\alpha\nu}$. Therefore $\bar{\lambda} = (1/KM)\,e^{\alpha\nu} - (1+K)MK\,e^{-\alpha\nu} > (1/2KM)\,e^{\alpha\nu} > 1$. We have obtained that $|Q^{i+1}u(t_{i+1})| \geqq \bar{\lambda}|Q^i u(t_i)|$ with $\bar{\lambda} > 1$. Now that $|Q^{i+1}u(t_{i+1})| - |(I-Q^{i+1})u(t_{i+1})| \geqq ((1/KM)\,e^{\alpha\nu} - 2(1+K)KM\,e^{-\alpha\nu})|v| \geqq 0$, we can repeat the whole argument. Therefore, $|Q^j u(t_j)| \to \infty$ as $j \to \infty$, contradicting the boundedness of $|Q^j|$ and $|u(t_j)|$. This completes the proof of $\{u_i(t)\} = 0$, and whence the whole lemma.

Before we prove the nonlinear Shadowing Lemma, we need the following lemma.

LEMMA 4.2 (Inverse Function Theorem). *Let X and Y be two Banach spaces. If $A: X \to Y$ is linear, $A^{-1}: Y \to X$ exists and is bounded with the norm $|A^{-1}|$. Let $f: X \to Y$ be C^1, $f(0) = 0$, $f'(0) = 0$. Also $|f'(x)| \leqq M_1|x|$, $M_1 > 0$ is a constant. Then, for each $y \in Y$ with $|y| \leqq \frac{1}{4}M_1^{-1}|A^{-1}|^{-2}$, there exists a unique $x \in X$ such that $y = Ax + f(x)$, with $|x| \leqq \frac{1}{2}M_1^{-1}|A^{-1}|^{-1}$. Moreover, $|x| \leqq 2|A^{-1}| \cdot |y|$.*

Proof. Consider $x_1 = A^{-1}y - A^{-1}f(x)$, the right-hand side maps $|x| \leqq \frac{1}{2}M_1^{-1}|A^{-1}|^{-1}$ into $|x_1| \leqq \frac{1}{2}M_1^{-1}|A^{-1}|^{-1}$ provided that $|y| \leqq \frac{1}{4}M_1^{-1}|A^{-1}|^{-2}$. And it is easy to verify that $x \mapsto x_1$ is a contraction of rate $\frac{1}{2}$. Therefore, there exists a unique fixed element $x = x_1$. Moreover, $|x| \leqq |A^{-1}||y| + |A^{-1}|M_1 \cdot |x|^2 \leqq |A^{-1}||y| + \frac{1}{2}|x|$. Thus, $|x| \leqq 2|A^{-1}||y|$.

Consider a nonlinear equation in $\mathbb{R}^n$

$$\dot{u}(t) = f(u(t), t), \tag{4.14}$$

where $f \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, $|f|_{C^2} \leqq \bar{M}_1$. Assume that $\mathbb{R} = \bigcup_{m_0 < i < m_1} [t_{i-1}, t_i]$ is a partition of $\mathbb{R}$ as in the beginning of this section and $\{u_i(t)\}$ is a formal approximation of (4.14), subordinate to the given partition, with

$$\dot{u}_i(t) - f(u_i(t), t) = h_i(t), \qquad t \in [t_{i-1}, t_i]$$

$$u_i(t_i) - u_{i+1}(t_i) = k_i.$$

Let $A_i(t) = f_x(u_i(t), t)$, $t \in [t_{i-1}, t_i]$ and $T^i(t, s)$ be the solutions map of the linear homogeneous equation

$$\dot{u}(t) - A_i(t)u(t) = 0. \tag{4.15}$$

THEOREM 4.3. *For the partition* $\mathbb{R} = \bigcup_{m_0<i<m_1} [t_{i-1}, t_i]$ *and the solution map* $T^i(t, s)$ *of* (4.15), *assume that the hypotheses* (i), (ii), *and* (iii), *as in the beginning of this section, are satisfied. Suppose that* $\delta, \eta, \varepsilon_1$ *are positive constants with* $\delta + (2K/\alpha)\eta \leqq \frac{1}{4}M_1^{-1}(4KM+1)^{-2}$ *and* $\varepsilon_1 = \frac{1}{2}M_1^{-1}(4KM+1)^{-1}$, *and suppose that* $\|\{h_i\}\|_{\mathscr{X}} \leqq \eta$ *and* $\|\{k_i\}\|_Z \leqq \delta$. *Then* (4.14) *possesses a unique bounded continuous solution* $u(t)$ *with* $|u(t) - u_i(t)| \leqq \varepsilon_1$. *Moreover*

$$\|\{u - u_i\}\|_{\mathscr{X}} \leqq 2(4KM+1)\|\{h_i\} \times \{k_i\}\|_Y.$$

Proof. Let $u(t) = u_i(t) + z_i(t)$, $t \in [t_{i-1}, t_i]$. We have to solve

$$\dot{z}_i(t) - A_i(t)z_i(t) = N_i(z_i)(t) - h_i(t), \qquad t \in [t_{i-1}, t_i],$$

$$z_i(t_i) - z_{i+1}(t_i) = -k_i,$$

where $N_i(z_i)(t) = f(u_i(t) + z_i(t), t) - f(u_i(t), t) - A_i(t)z_i(t)$. Equivalently,

$$\mathscr{A}\{z_i\} = \{N_i(z_i) - h_i\} \times \{-k_i\}.$$

Applying Lemma 4.1, we have that $\mathscr{A}: \mathscr{X} \to Y$ is one-to-one and onto with $\|\mathscr{A}^{-1}\| \leqq (4MK+1)$. Observe that $\{N_i(0)\} = 0$, $D\{N_i(0)\} = 0$, and $\|D\{N_i(z_i)\}\|_{\mathscr{L}(\mathscr{X})} \leqq \bar{M}_1 \|\{z_i\}\|_{\mathscr{X}}$. From Lemma 4.2, for each $\{h_i\} \times \{k_i\} \in Y$ with $\|\{h_i\} \times \{k_i\}\|_Y \leqq \frac{1}{4}M_1^{-1}(4MK+1)^{-2}$, $M_1 = (2K/\alpha)\bar{M}_1$, there exists a unique $\{z_i\} \in \mathscr{X}$ with $\|\{z_i\}\|_{\mathscr{X}} \leqq \frac{1}{2}M_1^{-1}(4MK+1)^{-1}$. Moreover,

$$\|\{z_i\}\|_{\mathscr{X}} \leqq 2(4KM+1)\|\{h_i\} \times \{k_i\}\|_Y.$$

We now indicate how Theorem 4.3 can be adapted to boundary value problems in finite interval or periodic systems in $\mathbb{R}$.

First, consider the boundary value problem

$$\begin{aligned} &\dot{u}(t) = f(u, t), \qquad a \leqq t \leqq b, \\ &B_1(u(a)) = 0, \\ &B_2(u(b)) = 0 \end{aligned} \tag{4.16}$$

where $f: \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$, $B_1: \mathbb{R}^n \to \mathbb{R}^{d^-}$, and $B_2: \mathbb{R}^n \to \mathbb{R}^{d^+}$, $d^- + d^+ = n$, are C^2, bounded functions with $\sup\{|f|_{C^2}, |B_1|_{C^2}, |B_2|_{C^2}\} \leqq \bar{M}_1$. Let $M_1 = \sup\{\bar{M}_1, (2K/\alpha)\bar{M}_1\}$.

Suppose $[a, b] = \bigcup_{1 \leqq i \leqq r} [t_{i-1}, t_i]$ is a partition of $[a, b]$ and $\{u_i(t)\}$ is a formal approximation subordinate to this partition with

$$\begin{aligned} &\dot{u}_i(t) - f(u_i(t), t) = h_i(t), \qquad t \in [t_{i-1}, t_i], \quad 1 \leqq i \leqq r, \\ &u_i(t_i) - u_{i+1}(t_i) = k_i, \qquad 1 \leqq i \leqq r-1, \\ &B_1(u_1(a)) = b_1, \\ &B_2(u_m(b)) = b_2. \end{aligned} \tag{4.17}$$

$\{h_i(t)\}$, $\{k_i\}$, b_1, and b_2 are residual, jump, and boundary errors, respectively.

Assume the $B_{1X}(u_1(a)): \mathbb{R}^n \to \mathbb{R}^{d^-}$ is of rank d^-. That is, $\mathscr{K}B_{1X}(u_1(a))$ is of dimension d^+ and $B_{1x}(u_1(a)): \{\mathscr{K}B_{1x}(u_1(a))\}^\perp \to \mathbb{R}^{d^-}$ is invertible, with the inverse denoted by $B_{1x}(u_1(a))^{-1}$. Let $\xi_0 \in \mathscr{K}B_{1x}(u_1(a))$ and $\xi_1 \in \{\mathscr{K}B_{1x}(u_1(a))\}^\perp$. Consider

$$B_1(u_1(a) + \xi_0 + \xi_1) = 0, \tag{4.18}$$

$$\begin{aligned} &B_{1x}(u_1(a))\xi_1 + \{B_1(u_1(a) + \xi_1) - B_1(u_1(a)) - B_{1x}(u_1(a))\xi_1\} \\ &\quad = -b_1 + B_1(u_1(a) + \xi_1) - B_1(u_1(a) + \xi_0 + \xi_1). \end{aligned} \tag{4.19}$$

Clearly, the term in { } is a nonlinear function in ξ_1, denoted by $N_1(\xi_1)$, $N_1(0)=0$, $DN_1(0)=0$, $D^2N_1(\xi_1)\leqq M_1$. And the right-hand side of (4.19) is bounded by $|b_1|+M_1|\xi_0|$. It follows from Lemma 4.2 that there is a C^2 function $\xi_1=\xi_1^*(\xi_0)$, $|\xi_0|\leqq \frac{1}{8}M_1^{-2}\|B_{1X}(u_1(a))^{-1}\|^{-2}$, which solves (4.18), provided that $|b_1|\leqq\frac{1}{8}M_1^{-1}\|B_{1x}(u_1(a))^{-1}\|^{-2}$. Let $\bar{\xi}_1=\bar{\xi}_1^*(0)$; then $|\bar{\xi}_1|\leqq 2\|B_{1x}(u_1(a))^{-1}\|\,|b_1|$.

We can construct $f_1(x)$ such that $f_1(u_1(a)+\bar{\xi}_1)=0$, $u_1(a)+\bar{\xi}_1$ is a hyperbolic equilibrium of the equation $\dot{u}(t)=f_1(u(t))$, and the image of $u=u_1(a)+\xi_0+\xi_1^*(\xi_0)$ with $\xi_0\in\mathcal{K}B_{1x}(u_1(a))$, $|\xi_0|\leqq\frac{1}{8}M_1^{-2}\|B_{1x}(u_1(a))^{-1}\|^{-2}$ is the local unstable manifold. Such a construction does exist and we shall not render it here.

Similarly, assume that $B_{2x}(u_r(b)):\mathbb{R}^n\to\mathbb{R}^{d^+}$ is of rank d^+. Let $\zeta_0\in\mathcal{K}B_{2x}(u_r(b))$ and $\zeta_1\in\{\mathcal{K}B_{2x}(u_r(b))\}^\perp$; then

$$B_2(u_r(b)+\zeta_0+\zeta_1)=0,$$

is uniquely solvable by a C^2 function $\zeta_1=\zeta_1^*(\zeta_0)$, $|\zeta_0|\leqq\frac{1}{8}M_1^{-2}\|B_{2x}(u_r(b))^{-1}\|^{-2}$, provided that $|b_2|\leqq\frac{1}{8}M_1^{-1}\|B_{2x}(u_r(b))^{-1}\|^{-2}$. Here $B_{2x}(u_r(b))^{-1}:\mathbb{R}^{d^+}\to\{\mathcal{K}B_{2x}(u_r(b))\}^\perp$ is a right inverse of $B_{2x}(u_r(b))$. Let $\bar{\zeta}_1=\zeta_1^*(0)$, $|\bar{\zeta}_1|\leqq 2\|B_{2x}(u_r(b))^{-1}\|\,|b_2|$. We can construct an autonomous ODE $\dot{u}(t)=f_2(u(t))$ such that $u_r(b)+\bar{\zeta}_1$ is a hyperbolic equilibrium and the image of $u=u_r(b)+\zeta_0+\zeta_1^*(\zeta_0)$ with $\zeta_0\in\mathcal{K}B_{1x}(U_r(b))$, $|\zeta_0|\leqq\frac{1}{8}M_1^{-2}\|B_{1x}(u_r(b))^{-1}\|^{-2}$ is a the local stable manifold.

Consider a partition $\mathbb{R}=(-\infty,a]\cup(b,+\infty)\cup\bigcup_{1\leqq i\leqq r}[t_{i-1},t_i]\}$; and an extended system in $\mathbb{R}$:

$$\dot{u}(t)=\tilde{f}(u(t),t), \tag{4.20}$$

where $\tilde{f}(x,t)=f_1(x)$, $t\in(-\infty,a)$, $\tilde{f}(x,t)=f(x,t)$, $t\in[a,b]$ and $\tilde{f}(x,t)=f_2(x)$, $t\in(b,+\infty)$. Let $\{u_i(t)\}$, $0\leqq i\leqq r+1$ be a formal approximation of (4.20), with $u_0(t)\equiv u_1(a)+\bar{\xi}_1$, $u_{r+1}(t)\equiv u_r(b)+\bar{\zeta}_1$ and $u_i(t)$, $1\leqq i\leqq r$ being the previous formal approximation of the boundary value problem (4.16). We remark here that $\tilde{f}(x,t)$ is only piecewise C^2 in $(-\infty,a)\cup[a,b]\cup(b,+\infty)$. However, the proof of Theorem 4.3 shows that we do not need the vector field to be C^2 in the entire domain. We also observe the jump errors $u_0(a)-u_1(a)=\bar{\xi}_1=O(b_1)$ and $u_r(b)-u_{r+1}(b)=O(b_2)$. Let $T^i(t,s)$ be the solution map of

$$\dot{u}(t)-A_i(t)u(t)=0,\qquad t\in[t_{i-1},t_i],$$

where $A_i(t)=f_x(u_i(t),t)$. Theorem 4.4 follows from an application of Theorem 4.3 to (4.20).

THEOREM 4.4. *Assume that* (i), (ii), *and* (iii) *are satisfied with* $m_0=0$ *and* $m_1=r+1$, $B_{1x}(u_1(a))$ *is of rank* d^-, *and* $B_{2x}(u_2(b))$ *is of rank* d^+; $\mathcal{K}B_{1x}(u_1(a))\oplus\mathcal{R}P_s(a)=\mathcal{K}B_{2x}(u_r(b))\oplus\mathcal{R}P_u(b)=\mathbb{R}^n$; $Q^0:\mathbb{R}^n\to\mathcal{K}B_{1x}(u_1(a)+\bar{\xi}_1)$ *with* $\mathcal{K}Q^0=\mathcal{R}P_s^1(a)$, *and* $Q^r:\mathbb{R}^n\to\mathcal{R}P_u^r(b)$ *with* $\mathcal{K}Q^r=\mathcal{K}B_{2x}(u_r(b)+\bar{\zeta}_1)$ *are two projections with* $\sup\{|Q^0|,|I-Q^0|,|Q^r|,|I-Q^r|\}\leqq M$. *Suppose that* δ, η, *and* ε_1 *are positive constants with* $\delta+(2K/\alpha)\eta\leqq\frac{1}{4}M_1^{-1}(4KM+1)^{-2}$ *and* $\varepsilon_1=\frac{1}{2}M_1^{-1}(4KM+1)^{-1}$, *and suppose* $\sup\{|k_i|\}$, $1\leqq i\leqq r-1\}\leqq\delta$, $\sup\{|h_i(t)|,\ 1\leqq i\leqq r,\ t\in[t_{i-1},t_i]\}\leqq\eta$ *and* $|b_i|\leqq\delta/\bar{C}$, $i=1,2$, *where* $\bar{C}=2\sup\{\|B_{1x}(u_1(a))^{-1}\|,\ \|B_{2x}(u_r(b))^{-1}\|\}$. *Then* (4.16) *possesses a unique solution* $u(t)$ *with* $|u(t)-u_i(t)|\leqq\varepsilon_1$. *Moreover,*

$$\begin{aligned}\sup\{&|u(t)-u_i(t)|:1\leqq i\leqq r,\ t\in[t_{i-1},t_i]\}\\ &\leqq 2(4KM+1)(\sup\{\bar{C}|b_i|,|k_j|:i=1,2,1\leqq j\leqq r-1\}\\ &\quad+2K/\alpha\sup\{|h_i(t)|:1\leqq i\leqq r,\ t\in[t_{i-1},t_i]\}).\end{aligned}$$

Next consider a periodic system of period ω,

$$\dot{u}(t)=f(u(t),t), \tag{4.21}$$

where $f(x, t) = f(x, t+\omega)$ is a C^2 bounded function with $|f|_{C^2} \leqq M_1$. Let $[0, \omega] = \bigcup_{1 \leqq i \leqq r} [t_{i-1}, t_i]$ be a partition of $[0, \omega]$ and $\{u_i(t)\}$ a formal approximation subordinate to this partition, with

$$\dot{u}(t) - f(u_i(t), t) = h_i(t), \qquad t \in [t_{i-1}, t_i], \quad 1 \leqq i \leqq r,$$

$$u_i(t_i) - u_{i+1}(t_i) = k_i, \qquad 1 \leqq i \leqq r-1,$$

$$u_r(\omega) - u_1(0) = k_r.$$

Let $T^i(t, s)$ be the solution map of $\dot{u}(t) - A_i(t)u(t) = 0$, $t \in [t_{i-1}, t_i]$, where $A_i(t) = f_x(u_i(t), t)$. We have the following theorem for the formal approximation $\{u_i(t)\}$ for the periodic system (4.21).

THEOREM 4.5. *Assume that* (i), (ii), *and* (iii) *are satisfied with* $m_0 = 0$ *and* $m_1 = r+1$. *Assume that* $\mathcal{R}P_u^r(\omega) \oplus \mathcal{R}P_s^1(0) = \mathbb{R}^n$. *Let* $Q^r : \mathbb{R}^n \to \mathcal{R}P_u^r(\omega)$ *be a projection with* $\mathcal{K}Q^r = \mathcal{R}P_s^1(0)$. *Assume that* $|Q^r| \leqq M$ *and* $|I - Q^r| \leqq M$. *Suppose that* δ, η, *and* ε_1 *are positive constants with* $\delta + (2K/\alpha)\eta \leqq \frac{1}{4}M_1^{-1} \cdot (4KM+1)^{-2}$, $\varepsilon_1 = \frac{1}{2}M_1^{-1}(4KM+1)^{-1}$ *and suppose that* $\sup\{|k_i|, 1 \leqq i \leqq r\} \leqq \delta$ *and* $\sup\{|h_i(t)|, 1 \leqq i \leqq r, t \in [t_{i-1}, t_i]\} \leqq \eta$. *Then* (4.21) *possesses a unique periodic solution* $u(t) = u(t) + \omega)$, *with* $|u(t) - u_i(t)| \leqq \varepsilon_1$. *Moreover,*

$$\sup\{|u(t) - u_i(t)| : 1 \leqq i \leqq r, t \in [t_{i-1}, t_i]\}$$
$$\leqq 2(4KM+1)(\sup\{|k_i| : 1 \leqq i \leqq r\} + 2K/\alpha \sup\{|h_i(t)| : 1 \leqq i \leqq r, t \in [t_{i-1}, t_i]\}).$$

Proof. Extend the formal approximation periodically to $t \in \mathbb{R}$ and apply Theorem 4.3. The bounded solution thus obtained is periodic, following from a uniqueness argument.

5. Formal power series solutions and matching principles. We start with some definitions that are slightly different from those in the standard literature.

Let $f(t, \varepsilon)$ be continuous and defined on $t \in J$ and $|\varepsilon| \leqq \varepsilon_0$, J is an interval in $\mathbb{R}$, bounded or unbounded, open or closed, and $\varepsilon_0 > 0$ is a constant. We say $f(t, \varepsilon) = O(\varepsilon^m)$ if for any compact subinterval $J_1 \subset J$, there exists a constant $C(J_1)$ such that $|f(t, \varepsilon)| \leqq C(J_1)|\varepsilon^m|$, $t \in J_1$. We say that $f(t, \varepsilon) = o(\varepsilon^m)$ if $f(t, \varepsilon)/\varepsilon^m \to 0$ uniformly in any compact subinterval J_1, as $\varepsilon \to 0$.

A *formal power series* (in ε) is a formal sum $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$; $\varphi_j(t)$ is defined and continuous in J. A formal power series $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$ is an *asymptotic expansion* of a continuous function $f(t, \varepsilon)$ in $t \in J$, $|\varepsilon| \leqq \varepsilon_0$, if

$$f(t, \varepsilon) - \sum_{j=0}^{m} \varepsilon^j \varphi_j(t) = O(\varepsilon^{m+1}) \quad \text{for all} \quad m \geqq 0.$$

This relation is denoted by $E\{f(t, \varepsilon)\} = \sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$, E is called the *expansion operator*, and f is said to be in the domain of E, and shall be called an *asymptotic sum* of $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$.

It is immediately obvious from Taylor's formula that if each $(\partial^i/\partial\varepsilon^i)f(t, \varepsilon)$ exists and is a continuous function of $t \times \varepsilon$, then $f(t, \varepsilon)$ is in the domain of E. The following lemma shows that E has a right inverse and is a generalization of a lemma of Borel and Ritt. The proof shall be omitted.

LEMMA 5.1. *There exists an (nonunique) asymptotic sum* $f(t, \varepsilon)$, $t \in J$, $\varepsilon \in \mathbb{R}$ *for each formal power series* $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$. *Moreover, for each* $i \geqq 0$, $(\partial^i/\partial\varepsilon^i)f(t, \varepsilon)$ exists and is continuous in t and ε.

Two functions, $f(t, \varepsilon)$ and $g(t, \varepsilon)$, both in the domain of E, are said to be *asymptotically equivalent*, denoted by $f(t, \varepsilon) \sim g(t, \varepsilon)$, if and only if $E(f(t, \varepsilon)) = E(g(t, \varepsilon))$. For any formal power series $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$, $E^{-1}(\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t))$ forms a nonempty equivalence class and shall be denoted by $[\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)]$, or $[f(t, \varepsilon)]$ with $f \in [\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)]$.

Let $F(u_1, u_2, \cdots, u_k, t, \varepsilon)$ be C^∞ in all the variables. Assume that $f_i(t, \varepsilon) \sim g_i(t, \varepsilon) \in [\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^i(t)]$, $i = 1, \cdots, k$. Then

$$F(f_1(t, \varepsilon), \cdots, f_k(t, \varepsilon), t, \varepsilon) \sim F(g_1(t, \varepsilon), \cdots, g_k(t, \varepsilon), t, \varepsilon).$$

Therefore,

$$F\left(\left[\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^1(t)\right], \cdots, \left[\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^k(t)\right], t, \varepsilon\right) \stackrel{\text{def.}}{=} [F(f_1(t, \varepsilon), \cdots, f_k(t, \varepsilon), t, \varepsilon)]$$

is well defined.

DEFINITION 5.2. Define $F(\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^1(t), \cdots, \sum_{j=0}^{\infty} \varepsilon^j \varphi_j^k(t), t, \varepsilon)$ as a formal power series $\sum_{j=0}^{\infty} \varepsilon^j \psi_j(t)$, which is the asymptotic expansion of $F([\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^1(t)], \cdots, [\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^k(t)], t, \varepsilon)$. The relation shall be denoted by

$$F\left(\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^1(t), \cdots, \sum_{j=0}^{\infty} \varepsilon^j \varphi_j^k(t), t, \varepsilon\right) = \sum_{j=0}^{\infty} \varepsilon^j \psi_j(t). \tag{5.1}$$

It is now clear that termwise summation and multiplication by scalar functions of formal power series, as well as multiplication of two formal power series, can be defined by using Definition 5.2. Moreover, if each $\varphi_j(t) \in C^1$, we define $d/dt \sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \varepsilon^j \dot{\varphi}_j(t)$. We remark here that this definition is not merely formal, i.e., there exists at least one function $f(t, \varepsilon) \in [\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)]$ such that $(d/dt) f(t, \varepsilon) \in [\sum_{j=0}^{\infty} \varepsilon^j \dot{\varphi}_j(t)]$.

LEMMA 5.3. *Each term $\psi_j(t)$ of (5.1) can be computed recursively by the following equation*:

$$\sum_{j=0}^{m} \varepsilon^j \psi_j(t) = F\left(\sum_{j=0}^{m_1} \varepsilon^j \varphi_j(t), \cdots, \sum_{j=0}^{m_k} \varepsilon^j \varphi_j^k(t), t, \varepsilon\right) + O(\varepsilon^{m+1}) \tag{5.2}$$

for any $m_1, \cdots, m_k$ with $m_i \geqq m$, $1 \leqq i \leqq k$.

Proof. By virtue of Lemma 5.1, we can choose $f_i(t, \varepsilon) \in [\sum_{j=0}^{\infty} \varepsilon^j \varphi_j^i(t)]$, $1 \leqq i \leqq k$. Since $\sum_{j=0}^{m_i} \varepsilon^j \varphi_j^i(t) = f_i(t, \varepsilon) + O(\varepsilon^{m+1})$, we have the right-hand side of (5.2) $= F(f_1(t, \varepsilon) + O(\varepsilon^{m+1}), \cdots, f_k(t, \varepsilon) + O(\varepsilon^{m+1}), t, \varepsilon) + O(\varepsilon^{m+1})$ whence (5.2) follows by the Taylor expansion.

We say that $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$ formally satisfies an algebraic or differential equation $L(u, t, \varepsilon) = 0$ if $L(\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t), t, \varepsilon) = 0$ in the sense defined as above. In this case we call $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$ a formal solution of the algebraic or differential equation.

LEMMA 5.4. *If $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$ is a formal solution of an algebraic or differential equation $L(u, t, \varepsilon) = 0$, then $\sum_{j=0}^{m} \varepsilon^j \varphi_j(t)$ is a formal approximation of the solution of the equation with the residual*

$$L\left(\sum_{j=0}^{m} \varepsilon^j \varphi_j(t), t, \varepsilon\right) = O(\varepsilon^{m+1}).$$

Proof. This is a direct corollary of Lemma 5.3.

Finally, for any $\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)$, we can define a change of variable $t = t_0 + \varepsilon\tau$ as another formal power series denoted by

$$\sum_{j=0}^{\infty} \varepsilon^j \psi_j(\tau) = \sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t_0 + \varepsilon\tau),$$

which is obtained by choosing an arbitrary $f(t, \varepsilon) \in [\sum_{j=0}^{\infty} \varepsilon^j \varphi_j(t)]$ and expanding

$$E\{f(t_0+\varepsilon\tau, \varepsilon)\} = \sum_{j=0}^{\infty} \varepsilon^j \psi_j(\tau).$$

Clearly the definition does not depend on the choice of $f(t, \varepsilon)$.

Throughout this section and § 6, γ is a constant with $0<\gamma<\alpha<\alpha_0$.

5.1. Formal power series solutions in $[t_{i-1}, t_i]$. Let $\{X_j\}_{j=0}^{\infty}$ be an arbitrary sequence of real vectors. Consider the formal asymptotic expansion defined in Definition 5.2:

$$\begin{aligned} f\left(\sum_{j=0}^{\infty} \varepsilon^j X_j, t, \varepsilon\right) = f(X_0, t, 0) + \sum_{j=1}^{\infty} \varepsilon^j \{ f_x(X_0, t, 0) X_j + F_j(X_1, \cdots, X_{j-1}, \cdots, \\ \cdot D_x^{i_1} D_\varepsilon^{i_2} f(X_0, t, 0), \cdots)\}, \end{aligned} \tag{5.3}$$

where $F_j(\ )$ is a sum of multilinear functionals on $X_1, \cdots, X_{j-1}$, each term has the form

$$D_x^{i_1} D_\varepsilon^{i_2} f(X_0, t, 0) X_1^{k_1} \cdots X_{j-1}^{k_{j-1}}$$

where $i_1 \geqq 0$ is an integer, and $k_1, \cdots, k_{j-1}$ and i_1 are multi-indices, satisfying $k_1+\cdots+k_{j-1}=i_1$, $|k_1|+2|k_2|+\cdots+(j-1)|k_{j-1}|+i_2=j$. Consider the recursive equations

$$0 = f(X_0(t), t, 0), \tag{5.4$_o$}$$

$$\begin{aligned} \dot{X}_{j-1}(t) = f_x(X_0(t), t, 0) X_j(t) + F_j(X_1(t), \cdots, X_{j-1}(t), \cdots, \\ \cdot D_x^{i_1} D_\varepsilon^{i_2} f(X_0(t), t, 0), \cdots). \end{aligned} \tag{5.4$_j$}$$

We shall solve the recursive system in $[t_{i-1}, t_i]$, $1 \leqq i \leqq r$. Let $X_0^i(t) = p_i(t)$, which by our assumption satisfies $(5.4)_o$. Assume that $X_0^i(t), \cdots, X_{j-1}^i(t)$ have been computed, and our assumption (H1) implies that there is a unique $X_j^i(t)$ satisfying $(5.4)_j$ for $t \in [t_{-1}, t_i]$. We shall show that $\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t)$ is a formal solution of the equation

$$\varepsilon \dot{X}(t) - f(X(t), t, \varepsilon) = 0.$$

Consider the formal power series

$$\begin{aligned} &\varepsilon \frac{d}{dt}\left(\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t)\right) - f\left(\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t), t, \varepsilon\right) \\ &\quad = -f(X_0^i(t), t, 0) + \sum_{j=1}^{\infty} \varepsilon^j \{\dot{X}_{j-1}^i(t) - f_x(X_0^i(t), t, 0) X_j^i(t) \\ &\quad\quad - F_j(X_1^i(t), \cdots, X_{j-1}^i(t), \cdots, D_x^{i_1} D_\varepsilon^{i_2} f(X_0^i(t), t, 0), \cdots)\}, \end{aligned}$$

which from our recursive equations is identically zero for $t \in [t_{i-1}, t_i]$.

5.2. Formal solutions for the boundary layers. The equation for the boundary layer near $t=a$ may be obtained by setting $t=a+\varepsilon\tau$. In the fast variable $\tau \in \mathbb{R}^+$, (1.1) becomes

$$\begin{aligned} &y'(\tau, \varepsilon) = f(y(\tau, \varepsilon), a+\varepsilon\tau, \varepsilon), \qquad 0 \leqq \tau < +\infty, \\ &B_1(y(0, \varepsilon), \varepsilon) = 0. \end{aligned} \tag{5.5}$$

Here we may assume that $f(x, t, \varepsilon)$ has been extended and defined for all $t \geqq a$.

Consider the formal asymptotic expansion

$$\begin{aligned} f\left(\sum_{j=0}^{\infty} \varepsilon^j y_j, a+\varepsilon\tau, \varepsilon\right) = f(y_0, a, 0) + \sum_{j=1}^{\infty} \varepsilon^j \{ f_x(y_0, a, 0) y_j \\ + G_j(y_1, \cdots, y_{j-1}, \cdots, D_x^{i_1} D_t^{i_2} D_\varepsilon^{i_3} f(y_0, a, 0), \cdots)\} \end{aligned} \tag{5.6}$$

where $G_j(y_1, \cdots, y_{j-1}, \cdots, D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0, a, 0), \cdots)$ is a sum of multilinear functionals on $y_1, \cdots, y_{j-1}$, each term has the form

$$D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0, a, 0)y_1^{k_1}\cdots y_{j-1}^{k_{j-1}}\tau^{i_2}, \tag{5.7}$$

where $i_2 \geqq 0$, $i_3 \geqq 0$ are integers, and $i_1, k_1, \cdots$ and k_{j-1} are vector indices, with $k_1 + \cdots + k_{j-1} = i_1$, $|k_1| + 2|k_2| + \cdots + (j-1)|k_{j-1}| + i_2 + i_3 = j$. Consider the recursive equations

$$y_0'(\tau) = f(y_0(\tau), a, 0), \tag*{(5.8)$_o$}$$

$$\begin{aligned} y_j'(\tau) &= f_x(y_0(\tau), a, 0)y_j(\tau) \\ &\quad + G_j(y_1(\tau), \cdots, y_{j-1}(\tau), \cdots, D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0(\tau), a, 0), \cdots). \end{aligned} \tag*{(5.8)$_j$}$$

Equation $(5.8)_j$ is a linear inhomogeneous equation in $y_j(\tau)$ provided that $y_0(\tau), \cdots, y_{j-1}(\tau)$ have been computed. We shall need boundary conditions for $y_j(\tau)$ at $\tau = 0$ and $\tau = +\infty$.

Consider the formal asymptotic expansion

$$\begin{aligned} B_1\left(\sum_{j=0}^{\infty} \varepsilon^j \bar{y}_j, \varepsilon\right) = B_1(\bar{y}_0, 0) + \sum_{j=1}^{\infty} \varepsilon^j\{&B_{1x}(\bar{y}_0, 0)\bar{y}_j + H_j(\bar{y}_1, \cdots, \bar{y}_{j-1}, \cdots, \\ &\cdot D_x^{i_1}D_\varepsilon^{i_2}B_1(\bar{y}_0, 0), \cdots)\} \end{aligned} \tag{5.9}$$

where $H_j(\bar{y}_1, \cdots, \bar{y}_{j-1}, \cdots, D_x^{i_1}D_\varepsilon^{i_2}B_1(\bar{y}_0, 0), \cdots)$ is a sum of multilinear functionals on $\bar{y}_1, \cdots, \bar{y}_{j-1}$, each term has the form

$$D_x^{i_1}D_\varepsilon^{i_2}B_1(\bar{y}_0, 0)\cdot \bar{y}_1^{k_1}\cdots \bar{y}_{j-1}^{k_{j-1}}$$

where $i_2 \geqq 0$ is an integer, and $k_1, \cdots, k_{j-1}$ and i_1 are indices, with $k_1 + \cdots + k_{j-1} = i_1$, $|k_1| + 2|k_2| + \cdots + (j-1)|k_{j-1}| + i_2 = j$. Consider the recursive equations

$$B_1(\bar{y}_0, 0) = 0, \tag*{(5.10)$_o$}$$

$$B_{1x}(\bar{y}_0, 0)\bar{y}_j + H_j(\bar{y}_1, \cdots, \bar{y}_{j-1}, \cdots, D_x^{i_1}D_\varepsilon^{i_2}B_1(\bar{y}_0, 0), \cdots) = 0. \tag*{(5.10)$_j$}$$

The solution $\bar{y}_i$ of $(5.10)_j$ shall be the initial condition for $(5.8)_j$, $y_j(\tau)|_{\tau=0} = \bar{y}_j$.

Finally, we assume that

$$y_j(\tau) \in E_{\mathbb{R}^+}(0, j), \tag*{(5.11)$_j$}$$

which is a *growth condition* for $(5.8)_j$ at $\tau = +\infty$.

We now construct our formal solution $\sum_{j=0}^{\infty} \varepsilon^j y_j^0(\tau)$ of (5.5) as follows. First let $y_0^0(\tau) = q_0(\tau)$, which satisfies $(5.8)_o$, $(5.10)_o$, and $(5.11)_o$ by our hypotheses on $q_0(\tau)$. Assume that $y_0^0(\tau), \cdots, y_{j-1}^0(\tau)$ have been computed, which satisfy $(5.8)_o$-$(5.8)_{j-1}$, $(5.10)_o$-$(5.10)_{j-1}$, and $(5.11)_o$-$(5.11)_{j-1}$. We show how $(5.8)_j$, $(5.10)_j$, and $(5.11)_j$ uniquely determine $y_j^0(\tau)$.

To begin with, it is easy to verify that

$$G_j(y_1^0(\tau), \cdots, y_{j-1}^0(\tau), \cdots, D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0^0(\tau), a, 0), \cdots) \in E_{\mathbb{R}^+}(0, j),$$

by virtue of (5.7) and the assumptions on $y_1^0(\tau), \cdots, y_{j-1}^0(\tau)$. Observe that $q_0(\tau) \to p_1(t_0)$, as $\tau \to +\infty$, which is a hyperbolic equilibrium for $(5.8)_o$. From the remark made after Lemma 3.3, the solution mapping $T^0(t, s)$ for the homogeneous part of $(5.8)_j$, $j \geqq 1$, possesses an exponential dichotomy in $\mathbb{R}^+$, with the projections being $P_s^0(s)$ and $P_u^0(s)$. Any solution of $(5.8)_j$ in $E_{\mathbb{R}^+}(0,j)$ can be written as follows:

$$(5.12)_j \qquad \begin{aligned} y_j^0(\tau) = T^0(\tau,0)P_s^0(0)y_j^0(0) + \int_0^\tau T^0(\tau,s)P_s^0(s)G_j(\cdots)(s)\,ds \\ + \int_\infty^\tau T^0(\tau,s)P_u^0(s)G_j(\cdots)(s)\,ds. \end{aligned}$$

We refer the verification of $(5.12)_j$ to Lemma 3.6(ii). From $(5.12)_j$, we have $P_u^0(0)y_j^0(0) = \int_\infty^0 T^0(0,s)P_u^0(s)G_j(\cdots)(s)\,ds$. Now substituting $y_j^0(0) = P_u^0(0)y_j^0(0) + P_s^0(0)y_j^0(0)$ into $(5.10)_j$, we have

$$(5.13)_j \qquad B_{1x}(y_0^0(0),0)P_s^0(0)y_j^0(0) + B_{1x}(y_0^0(0),0)P_u^0(0)y_j^0(0) + H_j(y_1^0(0),\cdots) = 0.$$

Notice that $\mathscr{R}P_s^0(0)\oplus\mathscr{K}B_{1x}(y_0^0(0),0) = \mathbb{R}^n$, by virtue of (H2). Thus, $P_s^0(0)y_j^0(0)$ is uniquely solvable from $(5.13)_j$. By substituting into $(5.12)_j$, $y_j^0(\tau)$ has been completely determined. And $y_j^0(\tau)\in E_{\mathbb{R}^+}(0,j)$ by Lemma 3.6(ii).

It is straightforward to verify that the formal power series $\sum_{j=0}^\infty \varepsilon^j y_j^0(\tau)$ thus obtained is a formal solution for (5.5). We shall not render the details here.

We remark that $y_j^0(\tau)$ is determined by the growth condition at $\tau = +\infty$ rather than matching principles as commonly used. However, there is a matching of $y_j^0(\tau)$ with outer layer that can be proved as the consequence of our construction and that is useful in the sequel. Consider the inner expansion of the outer formal solution $\sum_{j=0}^\infty \varepsilon^j X_j^1(t)$,

$$(5.14) \qquad \sum_{j=0}^\infty \varepsilon^j x_j^0(\tau) = \sum_{j=0}^\infty \varepsilon^j X_j^1(a+\varepsilon\tau).$$

It is easy to show that $x_j^0(\tau)$ is a polynomial of degree $\leqq j$. We can now state the following result.

THEOREM 5.5. *The formal solution* $\sum_{j=0}^\infty \varepsilon^j y_j^0(\tau)$ *of* (5.5), *with* $y_0^0(\tau) = q_0(\tau)$ *is uniquely computable from* $(5.8)_j$, $(5.10)_j$, *and* $(5.11)_j$ *recursively. Moreover, we have* $y_j^0(\tau) - x_j^0(\tau)\in E_{\mathbb{R}^+}(\gamma,j)$.

Proof. From (5.14) and the fact that $\sum_{j=0}^\infty \varepsilon^j X_j^1(t)$ formally satisfies (1.1), without boundary conditions, we easily derive that $\sum_{j=0}^\infty \varepsilon^j x_j^0(\tau)$ formally satisfies

$$y'(\tau) = f(y(\tau), a+\varepsilon\tau, \varepsilon).$$

Thus, $x_0^0(\tau)$ satisfies $(5.8)_o$ (in fact, $x_0^0(\tau) = p_1(a) = \text{const.}$), and $x_j^0(\tau)$ satisfies $(5.8)_j$ with $(y_1(\tau),\cdots,y_{j-1}(\tau)) = (x_1^0(\tau),\cdots,x_{j-1}^0(\tau))$. Assuming that $y_k^0(\tau)-x_k^0(\tau)\in E_{\mathbb{R}^+}(\gamma,\kappa)$, $0\leqq k\leqq j-1$, we consider the inhomogeneous equation for $y_j^0(\tau)-x_j^0(\tau)$:

$$(5.15) \qquad \begin{aligned} &(y_j^0(\tau)-x_j^0(\tau))' + f_x(y_0^0(\tau),a,0)(y_j^0(\tau)-x_j^0(\tau)) \\ &= \{(f_x(y_0^0(\tau),a,0)-f_x(x_0^0(\tau),a,0))x_j^0(\tau)\} \\ &\quad + \{G_j(y_1^0(\tau),\cdots,y_{j-1}^0(\tau),\cdots,D_x^{i_1}D_\tau^{i_2}D_\varepsilon^{i_3}f(y_0^0(\tau),a,0),\cdots) \\ &\quad - G_j(x_1^0(\tau),\cdots,x_{j-1}^0(\tau),\cdots,D_x^{i_1}D_\tau^{i_2}D_\varepsilon^{i_3}f(x_0^0(\tau),a,0),\cdots)\}. \end{aligned}$$

It is easy to verify from the specific forms of $G_j(\cdots)$ that the two bracketed terms are all in $E_{\mathbb{R}^+}(\gamma,j)\subset E_{\mathbb{R}^+}(0,j)$. Obviously, $y_j^0(\tau)-x_j^0(\tau)\in E_{\mathbb{R}^+}(0,j)$ and the inhomogeneous equation (5.15) has a unique solution $\bar z(\tau)\in E_{\mathbb{R}^+}(0,j)$, uniquely determined by setting $P_s^0(0)\bar z(0) = P_s^0(0)(y_j^0(0)-x_j^0(0))$; in fact, $\bar z(\tau) = x_j^0(\tau)-x_j^0(\tau)$. However, the inhomogeneous equation (5.15) also has a unique solution $z(\tau)\in E_{\mathbb{R}^+}(\gamma,j)$ if $P_s^0(0)z(0) = P_s^0(0)(y_j^0(0)-x_j^0(0))$ is given. But $z(\tau)\in E_{\mathbb{R}^+}(0,j)$, from the uniqueness,

$z(\tau)=\bar{z}(\tau)=y_j^0(\tau)-x_j^0(\tau)$. We refer the justification to Lemma 3.6(ii). Therefore, $y_j^0(\tau)-x_j^0(\tau)\in E_{\mathbb{R}^+}(\gamma,j)$. □

Similarly, we can derive the recursive equations for the formal series $\sum_{j=0}^{\infty}\varepsilon^j y_j^r(\tau)$, which formally satisfies the boundary layer equation at $t=b$,

$$y'(\tau,\varepsilon)=f(y(\tau,\varepsilon),b+\varepsilon\tau,\varepsilon),\qquad -\infty<\tau\leqq 0,$$
$$B_2(y(0,\varepsilon),\varepsilon)=0,$$

which also satisfies a *growth condition* at $\tau=-\infty$, i.e., $y_0^r(\tau)=q_r(\tau)\in E_{\mathbb{R}^-}(0,0)$, $y_j^r(\tau)\in E_{\mathbb{R}^-}(0,j)$, $j\geqq 1$. Moreover, if $\sum_{j=0}^{\infty}\varepsilon^j x_j^r(\tau)=\sum_{j=0}^{\infty}\varepsilon^j X_j^r(b+\varepsilon\tau)$, we have $y_j^r(\tau)-x_j^r(\tau)\in E_{\mathbb{R}^-}(\gamma,j)$.

5.3. Formal solutions for the interior layers. The equation for the interior layer at $t=t_i$, $1\leqq i\leqq r-1$, after setting $t=t_i+\varepsilon\tau$, becomes

$$y'(\tau,\varepsilon)=f(y(\tau,\varepsilon),t_i+\varepsilon\tau,\varepsilon),\qquad -\infty<\tau<\infty, \tag{5.16}$$

where we may assume that f has been extended to all $t\in\mathbb{R}$.

Suppose that $\sum_{j=0}^{\infty}\varepsilon^i y_j^i(\tau)$ is a formal solution of (5.16), we have $y_0^i(\tau)'=f(y_0^i(\tau),t_i,0)$. We also assume that $y_0^i(\tau)\to p_i(t_i)$ as $\tau\to-\infty$ and $y_0^i(\tau)\to p_{i+1}(t_i)$ as $t\to+\infty$. Therefore, from our assumptions on $q_i(\tau)$, we may set $y_0^i(\tau)=q_i(\tau+\tilde{\tau})$, where $\tilde{\tau}$ is a parameter to be determined. Equivalently and more conveniently we shall assume that $\sum_{j=0}^{\infty}\varepsilon^i y_j^i(\tau)$ is a formal solution of

$$y'(\tau,\varepsilon)=f(y(\tau,\varepsilon),t_i+\varepsilon(\tau+\tilde{\tau}),\varepsilon), \tag{5.17}$$

with $y_0^i(\tau)=q_i(\tau)$. We assume that $E(\tilde{\tau}(\varepsilon))=\sum_{j=0}^{\infty}\varepsilon^j\tau_j$. Consider the formal asymptotic expansion

$$f\left(\sum_{j=0}^{\infty}\varepsilon^j y_j,\ t_i+\varepsilon\left(\tau+\sum_{j=0}^{\infty}\varepsilon^j\tau_j\right),\ \varepsilon\right)$$
$$\begin{aligned}=f(y_0,t_i,0)+\sum_{j=1}^{\infty}\varepsilon^j\{&f_x(y_0,t_i,0)y_j+f_t(y_0,t_i,0)\tau_{j-1}\\&+L_j(y_1,\cdots,y_{j-1},\tau_0,\cdots,\tau_{j-2},\cdots,D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0,t_i,0),\cdots)\}\end{aligned}\tag{5.18}$$

where $L_j(\cdots)$ is a sum of multilinear functionals on $y_1,\cdots,y_{j-1},\tau_0,\cdots,\tau_{j-2}$, and each term has the form

$$D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0,t_i,0)y_1^{k_1}\cdots y_{j-1}^{k_{j-1}}\tau^{l_0}\tau_0^{l_1}\cdots\tau_{j-2}^{l_{j-1}} \tag{5.19}$$

where $i_2,i_3,l_0,\cdots,l_{j-1}$ are nonnegative integers; $i_1,k_1,\cdots,k_{j-1}$ are multi-indices with $k_1+\cdots+k_{j-1}=i_1$, $l_0+\cdots+l_{j-1}=i_2$, $|k_1|+2|k_2|+\cdots+(j-1)|k_{j-1}|+l_0+l_1+2l_2+\cdots+(j-1)l_{j-1}+i_3=j$.

Consider the recursive equation

$$\begin{aligned}y_j'(\tau)=&f_x(y_0(\tau),t_i,0)y_j(\tau)+f_t(y_0(\tau),t_i,0)\tau_{j-1}\\&+L_j(y_1(\tau),\cdots,y_{j-1}(\tau),\tau_0,\cdots,\tau_{j-2},\cdots,D_x^{i_1}D_t^{i_2}D_\varepsilon^{i_3}f(y_0(\tau),t_i,0),\cdots)\end{aligned}\tag{5.20$_j$}$$

for $j\geqq 1$ with the *growth condition* at $\tau=\pm\infty$

$$y_j(\tau)\in E_{\mathbb{R}}(0,j). \tag{5.21$_j$}$$

We also require that

$$y_j^i(0)\perp q_i'(0), \tag{5.22$_j$}$$

since the perturbation in the tangential direction of the orbit of $q_i(\tau)$ has been taken into account by $\sum_{j=0}^{\infty}\varepsilon^j\tau_j^i$. We claim that $y_0(\tau)=y_0^i(\tau)=q_i(\tau)$ and $(5.20)_j$, $(5.21)_j$, $(5.22)_j$ uniquely determine $\{\tau_j^i\}_{j=0}^{\infty}$ and $\{y_j^i(\tau)\}_{j=1}^{\infty}$. Suppose $y_k^i(\tau)$, $0\leqq k\leqq j-1$ and τ_k^i, $0\leqq k\leqq j-2$ have been computed and satisfy $(5.20)_k$, $(5.21)_k$, $1\leqq k\leqq j-1$. Then from (5.19) we easily conclude that

$$L_j(y_1^i(\tau), \cdots, y_{j-1}^i(\tau), \tau_0^i, \cdots, \tau_{j-2}^i, \cdots, D_x^{i_1} D_t^{i_2} D_\varepsilon^{i_3} f(y_0^i(\tau), t_i, 0), \cdots) \in E_{\mathbb{R}}(0, j).$$

Consider $(5.20)_j$ as an inhomogeneous equation in $y_j^i(\tau)$. Since $q_i(\tau) \to p_i(t_i)$ as $\tau \to -\infty$ and $q_i(\tau) \to p_{i+1}(t_i)$ as $\tau \to +\infty$, from Lemma 3.2, the linear homogeneous equation corresponding to $(5.20)_j$ has exponential dichotomies in $\mathbb{R}^-$ and $\mathbb{R}^+$. Our assumptions imply that $\mathcal{F}: E_{\mathbb{R}}^1(0,j) \to E_{\mathbb{R}}(0,j)$, defined as $(\mathcal{F}y)(\tau) = y'(\tau) - f_x(q_i(\tau), t_i, 0)y(\tau)$, is Fredholm with index $\mathcal{F} = 0$. Since $\dim \mathcal{K}\mathcal{F} = 1$, we have $\dim \mathcal{K}\mathcal{F}^* = 1$, and $\mathcal{K}\mathcal{F}^*$ is spanned by $\psi_i(\tau)$ by our assumptions. Now Lemma 3.7 implies that $(5.20)_j$ is uniquely solvable if and only if

$$(5.23)_j \qquad \left\{\int_{-\infty}^{\infty} \psi_i^*(\tau) f_t(q_i(\tau), t_i, 0)\, d\tau\right\} \cdot \tau_{j-1}^i + \int_{-\infty}^{\infty} \psi_i^*(\tau) L_j(y_1^i(\tau), \cdots, y_{j-1}^i(\tau), \tau_0^i, \cdots, \tau_{j-2}^i, \cdots)\, d\tau = 0$$

for $j \geqq 1$ and the solution $y_j^i(\tau)$ is in the codimension one subspace $\mathbb{Z}_i = \{y(\tau) | y(0) \perp q_i'(0)\}$, which is complementary to $\mathcal{K}\mathcal{F}$. From hypothesis (H3), τ_{j-1}^i is uniquely solvable from $(5.23)_j$. Once $(5.23)_j$ is satisfied, from standard property of Fredholm operators, $y_j^i(\tau) \in \mathbb{Z}_i$ is uniquely determined by $(5.20)_j$ and $(5.21)_j$. See Lemma 3.7.

The proof that $\sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau)$ and $\sum_{j=0}^{\infty} \varepsilon^j \tau_j^i$ formally satisfy (5.17) is straightforward.

Similar to the boundary layers, $y_j^i(\tau)$ is determined by the growth condition $(5.21)_j$ rather than by matching principles. However, the match of $y_j^i(\tau)$ with the outer expansion can be proved as a consequence of our construction. Consider the inner expansion of the outer formal solutions

$$(5.24) \qquad \begin{aligned} \sum_{j=0}^{\infty} \varepsilon^j x_{j,1}^i(\tau) &= \sum_{j=0}^{\infty} \varepsilon^j X_j^i\left(t_i + \epsilon\left(\tau + \sum_{j=0}^{\infty} \varepsilon^j \tau_j^i\right)\right), \\ \sum_{j=0}^{\infty} \varepsilon^i x_{j,2}^i(\tau) &= \sum_{j=0}^{\infty} \varepsilon^i X_j^{i+1}\left(t_i + \varepsilon\left(\tau + \sum_{j=0}^{\infty} \varepsilon^j \tau_j^i\right)\right). \end{aligned}$$

We assume that each $X_j^i(t), X_j^{i+1}(t)$ has been extended C^∞ to $t \in \mathbb{R}$. However, $\{x_{j,1}^i(\tau)\}_{j=0}^{\infty}$ and $\{x_{j,2}^i(\tau)\}_{j=0}^{\infty}$ do not depend on the extension. $x_{j,1}^i(\tau)$ and $X_{j,2}^i(\tau)$ are, in fact, polynomials of degree $\leqq j$. It is easy to see that $\sum_{j=0}^{\infty} \varepsilon^j x_{j,1}^i(\tau)$ or $\sum_{j=0}^{\infty} \varepsilon^j x_{j,2}^i(\tau)$, with $\tilde{\tau} = \sum_{j=0}^{\infty} \varepsilon^j \tau_j^i$, formally satisfies (5.17). We can now state the following result:

THEOREM 5.6. *The formal solution* $\sum_{j=0}^{\infty} \varepsilon^i y_j^i(\tau)$ *and* $\sum_{j=0}^{\infty} \varepsilon^j \tau_j^i$ *of* (5.17) *with* $y_0^i(\tau) = q_i(\tau)$ *is uniquely computable from* $(5.20)_j$*-*$(5.22)_j$ *recursively. Moreover,* $y_j^i(\tau) - x_{j,1}^i(\tau) \in E_{\mathbb{R}^-}(\gamma, j)$ *and* $y_j^i(\tau) - x_{j,2}^i(\tau) \in E_{\mathbb{R}^+}(\gamma, j)$.

Proof. We shall show that $y_j^i(\tau) - x_{j,2}^i(\tau) \in E_{\mathbb{R}^+}(\gamma, j)$. Clearly, for $j = 0$, $y_0^i(\tau) = q_i(\tau) \to p_{i+1}(t_i) = x_{0,2}^i(\tau)$ as $\tau \to +\infty$ exponentially fast as does $e^{-\gamma\tau}$. Assume that $y_k^i(\tau) - x_{k,2}^i(\tau) \in E_{\mathbb{R}^+}(\gamma, k)$, $0 \leqq k \leqq j-1$. Since both $y_j^i(\tau)$ and $x_{j,2}^i(\tau)$ satisfy $(5.20)_j$, it follows that the inhomogeneous equation for $y_j^i(\tau) - x_{j,2}^i(\tau)$ is

$$(5.25) \qquad \begin{aligned} &(y_j^i(\tau) - x_{j,2}^i(\tau)' - f_x(y_0^i(\tau), t_i, 0)(y_j^i(\tau) - x_{j,2}^i(\tau)) \\ &\quad = \{f_x(y_0^i(\tau), t_i, 0) - f_x(x_{0,2}^i(\tau), t_i, 0)\} x_{j,2}^i(\tau) \\ &\qquad + \{f_t(y_0^i(\tau), t_i, 0) - f_t(x_{0,2}^i(\tau), t_i, 0)\} \tau_{j-1}^i \\ &\qquad + \{L_j(y_1^i(\tau), \cdots, y_{j-1}^i(\tau), \tau_0^i, \cdots, \tau_{j-2}^i, \cdots, D_x^{i_1} D_t^{i_2} d_\varepsilon^{i_3} f(y_0^i(\tau), t_i, 0), \cdots) \\ &\qquad\quad - L_j(x_{1,2}^i(\tau), \cdots, x_{j-1,2}^i(\tau), \tau_0^i, \cdots, \tau_{j-2}^i, \cdots, \\ &\qquad\qquad D_x^{i_1} D_t^{i_2} D_\varepsilon^{i_3} f(x_{0,2}^i(\tau), t_i, 0), \cdots)\}. \end{aligned}$$

From the specific form of $L_j(\cdots)$ and the assumptions on $y_k^i(\tau)-x_{k,2}^i(\tau)$, $0\leqq k\leqq j-1$, it is clear that the right-hand side of (5.25) is in $E_{\mathbb{R}^+}(\gamma,j)\subset E_{\mathbb{R}^+}(0,j)$. Moreover, we have that $y_j^i(\tau)-x_{j-2}^i(\tau)\in E_{\mathbb{R}^+}(0,j)$. Arguing as in the proof of Theorem 5.5, we infer that $y_j^i(\tau)-x_{j,2}^i(\tau)\in E_{\mathbb{R}^+}(\gamma,j)$. □

Remark 5.7. It is of interest to compare our method with the classical matching principles (see Eckhaus [5]). For all the boundary layers and interior layers, we have merely imposed growth conditions on $y_j^i(\tau)$, $0\leqq i\leqq r$. The limiting behavior $y_j^i(\tau)-x_j^i(\tau)\in E(\gamma,j)$ is proved as a consequence but not a constraint.

However, there exist overlap regions and the *intermediate variable* can be $(t-t_i)/\varepsilon^\beta$, for any $0<\beta<1$, in a neighborhood of t_i in $[a,b]$. Also, the *asymptotic matching principle*, using the notation of Eckhaus [5],

$$E_\tau^n E_t^m x(t,\varepsilon)=T_\tau E_t^m E_\tau^n x(t,\varepsilon)$$

is satisfied, for all $m\geqq 0$, $n\geqq 0$ integers. The proof is straightforward, though tedious. The uniformly valid *composite expansion* is

$$\sum_{j=0}^m \varepsilon^j X_j^0(t,\varepsilon)+\sum_{j=0}^m \varepsilon^j y_j^0(\tau)-\sum_{j=0}^m \varepsilon^j x_j^0(\tau),\qquad t=a+\varepsilon\tau$$

in a neighborhood of a in $[a,b]$.

6. Proof of the main results. In this section, γ is a constant with $0<\gamma<\alpha<\alpha_0$.

Proof of Theorem 2.1. The construction of the formal solutions $\sum_{j=0}^\infty \varepsilon^j X_j^i(\tau)$, $1\leqq i\leqq r$ and $\sum_{j=0}^\infty \varepsilon^j y_j^i(\tau)$, $0\leqq i\leqq r$ is given in § 5. In particular, see Theorems 5.5 and 5.6. Consider the formal approximation $x(t,\varepsilon)$ obtained piecewise from the truncations of those formal solutions as in Theorem 2.1. The estimates for the residuals of the outer approximations in $[t_{i-1}+\varepsilon^\beta, t_i-\varepsilon^\beta]$, and boundary errors at $t=a$ and $t=b$, follow from Lemma 5.4. For the residual in the boundary or interior layer, Lemma 5.4 does not offer a uniform estimate. Consider the boundary layer at $t=a$, in the fast variable τ. We need an estimate for $(\sum_{j=0}^m \varepsilon^j y_j^0(\tau))'-f(\sum_{j=0}^m \varepsilon^j y_j^0(\tau), a+\varepsilon\tau,\varepsilon)$, $0\leqq\tau\leqq\varepsilon^{\beta-1}$. From Taylor expansion and (5.6), (5.7), $(5.8)_j$, we find that the residual is $O(\sup_{0\leqq\tau\leqq\varepsilon^{\beta-1}}(\varepsilon\tau)^{m+1})=O(\varepsilon^{\beta(m+1)})$. Similarly, we can show that the residuals for all the interior layer approximations are $O(\varepsilon^{\beta(m+1)})$. For the jump errors at $t_i\pm\varepsilon^\beta$, assume that $T_0>1$ is a constant such that $|\tau_0^i|\leqq T_0-1$, $1\leqq i\leqq r-1$, then $|\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i|\leqq T_0$ if ε is small. Consider the jump error at $t_i-\varepsilon^\beta$ $1\leqq i\leqq r-1$. We need an estimate for $\sum_{j=0}^m \varepsilon^j X_j^i(t)-\sum_{j=0}^m \varepsilon^j y_j^i(\tau)$ at $\bar t_i=t_i-\varepsilon^\beta$ where $\tau=(t-t_i)/\varepsilon+\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i$. Let $\{x_{j,1}^i(\tau)\}_{j=0}^m$ be as in § 5. We have

$$\begin{aligned}\left|\sum_{j=0}^m \varepsilon^j y_j^i(\tau)-\sum_{j=0}^m \varepsilon^j x_{j,1}^i(\tau)\right| &\leqq C\left|\sum_{j=0}^m \varepsilon^j e^{-\gamma|\tau|}(1+|\tau|^j)\right|\\ &\leqq C e^{-(\gamma/2)|\tau|}\\ &\leqq C e^{-\gamma/2\cdot\varepsilon^{\beta-1}}=O(\varepsilon^{m+1}),\end{aligned}$$

for $|\tau|\geqq\varepsilon^{\beta-1}-T_0$. Here we have used the fact that $y_j^i(\tau)-x_{j,1}^i(\tau)\in E_{\mathbb{R}^-}(\gamma,j)$. Moreover,

$$\begin{aligned}\left|\sum_{j=0}^m \varepsilon^j X_j^i\left(t_i+\varepsilon\left(\tau+\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i\right)\right)-\sum_{j=0}^m \varepsilon^j x_{j,1}^i(\tau)\right| &\leqq C|\varepsilon\tau|^{m+1}\\ &\leqq C\varepsilon^{\beta(m+1)}\end{aligned}$$

for $|t-t_i|\leqq \varepsilon^{\beta}$ and $|\tau|\leqq \varepsilon^{\beta-1}+T_0$. Thus, the jump error is $O(\varepsilon^{\beta(m+1)})$, $0<\beta<1$. The other cases may be treated similarly.

Proof of Theorem 2.2. Our main tool is Theorem 4.4. We shall study the formal approximation $x(\varepsilon\tau, \varepsilon)$, which is defined piecewise by a change of variable from $\sum_{j=0}^{m} \varepsilon^j X_j^i(t)$, $1\leqq i\leqq r$ or $\sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau)$, $0\leqq i\leqq r$, and which is presented in the fast variable $\tau=t/\varepsilon$, $\tau\in[a/\varepsilon, b/\varepsilon]$. The proof is divided into several steps.

(1) In each subinterval, the residual error is

$$x(\varepsilon\tau, \varepsilon)'-f(x(\varepsilon\tau, \varepsilon), \varepsilon\tau, \varepsilon)=O(\varepsilon^{\beta(m+1)}).$$

The boundary errors are $B_1(x(\varepsilon\cdot a/\varepsilon, \varepsilon), \varepsilon)=O(\varepsilon^{m+1})$ and $B_2(x(\varepsilon\cdot b/\varepsilon, \varepsilon), \varepsilon)=O(\varepsilon^{m+1})$. Moreover, the jump error at $(t_i+\varepsilon^{\beta})/\varepsilon$, $0\leqq i\leqq r-1$ or $(t_i-\varepsilon^{\beta})/\varepsilon$, $1\leqq\tau\leqq r$ is $O(\varepsilon^{\beta(m+1)})$, $0<\beta<1$. These are the consequences of Theorem 2.1 since change of variable from t to τ does not affect these errors.

(2) For $t\in[t_{i-1}+\varepsilon^{\beta}, t_i-\varepsilon^{\beta}]$ or $\tau\in[(t_{i-1}+\varepsilon^{\beta})/\varepsilon, (t_i-\varepsilon^{\beta})/\varepsilon]$, $x(\varepsilon\tau, \varepsilon)=\sum_{j=0}^{\infty} \varepsilon^j X_j^i(\varepsilon\tau)$. The homogeneous linear variational equation around $x(\varepsilon\tau, \varepsilon)$ is

$$z'(\tau)-f_x\left(\sum_{j=0}^{m} \varepsilon^j X_j^i(\varepsilon\tau), \varepsilon\tau, \varepsilon\right)z(\tau)=0. \tag{6.1}$$

Comparing this with

$$z'(\tau)-f_x(p_i(\varepsilon\tau), \varepsilon\tau, 0)z(\tau)=0, \tag{6.2}$$

we find that the coefficients differ by $O(\varepsilon)$. We now apply Lemma 3.3 to (6.2). From (H1), for each fixed τ, (6.2) is hyperbolic with the dimension of the stable space being d^- and unstable space d^+. Also $(\partial/\partial\tau)f_x(p_i(\varepsilon\tau), \varepsilon\tau, 0)=O(\varepsilon)$. Therefore if ε is sufficiently small, (6.2) has exponential dichotomy in $[(t_{i-1}+\varepsilon^{\beta})/\varepsilon, (t_i-\varepsilon^{\beta})/\varepsilon]$, with the projections $\bar P_s(\tau, \varepsilon)$ and $\bar P_u(\tau, \varepsilon)$ approaching the spectral projections of the matrix $f_x(p_i(\varepsilon\tau), \varepsilon\tau, 0)$. Also the constant $\bar K$ is uniformly bounded and the exponent $\bar\alpha$ approaching α_0 as $\varepsilon\to 0$. It is clear from Lemma 3.2 that (6.1) also has exponential dichotomy in the same interval as (6.2) and with the projections $P_s(\tau, \varepsilon)$, $P_u(\tau, \varepsilon)$ approaching the spectral projections of $f_x(p_i(\varepsilon\tau), \varepsilon\tau, 0)$ and with K uniformly bounded and $\alpha\to\alpha_0$ as $\varepsilon\to 0$. We may notice that Lemma 3.2 is stated for semi-infinite intervals. However, we can extend (6.1) and (6.2) so that Lemma 3.2 applies.

(3) The extension of $f(t, x, \varepsilon)$ to $\tilde f(t, x, \varepsilon)$ seems to be essential in the sequel. For $1\leqq i\leqq r-1$, extend the definition of $f(t, x, \varepsilon)$ in a neighborhood of $t\in t_i$ to $t\in\mathbb{R}$ as follows:

$$\begin{aligned}\tilde f(x, t, \varepsilon)&=\xi_1\left(\frac{t-t_i}{\rho}\right)\cdot f(x, t, \varepsilon)+\xi_2\left(\frac{t-t_i}{\rho}\right)\cdot f(x, t_i+3\rho, \varepsilon)\\&\quad+\xi_3\left(\frac{t-t_i}{\rho}\right)f(x, t_i-3\rho, \varepsilon)\end{aligned} \tag{6.3}$$

where $\xi_i(t)$, $0\leqq\xi_i(t)\leqq 1$, $i=1, 2, 3$ is in $C^{\infty}(\mathbb{R})$, with $\xi_1(t)=1$ for $|t|\leqq 2$, $\xi_1(t)=0$ for $|t|\geqq 3$; $\xi_2(t)=1$ for $t\geqq 3$, $\xi_2(t)=0$ for $t\leqq 2$; and $\xi_3(t)=\xi_2(-t)$. Here $\tilde f$ also depends on i and ρ; for simplicity, we drop these dependencies. For $i=0$, we define $\tilde f(x, t, \varepsilon)$ for $t\geqq a$ only and for $i=r$, we define $\tilde f(x, t, \varepsilon)$ for $t\leqq b$ only. Both are similar to (6.3) with obvious changes.

We shall also modify $y_j^i(\tau)$ to $\tilde y_j^i(\tau)$, $j\geqq 1$. For $1\leqq i\leqq r-1$, and $j\geqq 1$, let

$$\tilde y_j^i(\tau)=\xi_1\left(\frac{\varepsilon\tau}{\rho}\right)y_j^i(\tau)+\xi_2\left(\frac{\varepsilon\tau}{\rho}\right)y_j^i\left(\frac{3\rho}{\epsilon}\right)+\xi_3\left(\frac{\varepsilon\tau}{\rho}\right)y_j^i\left(-\frac{3\rho}{\varepsilon}\right). \tag{6.4}$$

For $i=0$, $j\geqq 1$, $\tilde{y}_j^0(\tau)$ is defined for $\tau\geqq 0$ only and for $i=r$, $\tilde{y}_j^r(\tau)$ is defined for $\tau\leqq 0$ only. Both are similar to (6.4) with obvious changes.

Finally, let $\tilde{y}_0^i(\tau)=y_0^i(\tau)=q_i(\tau)$, $0\leqq i\leqq r$.

(4) For $t\in[a, a+\varepsilon^\beta]$ or $\tau\in[a/\varepsilon, (a+\varepsilon^\beta)/\varepsilon]$, in which $x(\varepsilon\tau, \varepsilon)=\sum_{j=0}^m \varepsilon^j y_j^0(\tau-a/\varepsilon)$ is a perturbation of $q_0(\tau-a/\varepsilon)$. We know that

$$z'(\tau)-f_x(q_0(\tau-a/\varepsilon), a, 0)z(\tau)=0 \tag{6.5}$$

has exponential dichtotomy in $[a/\varepsilon, +\infty)$. Comparing it with

$$z'(\tau)-\tilde{f}_x\left(\sum_{j=0}^m \varepsilon^j\tilde{y}_j^0\left(\tau-\frac{a}{\varepsilon}\right), \varepsilon\tau, \varepsilon\right)z(\tau)=0, \tag{6.6}$$

we find that the coefficients differ by $O(\rho+\varepsilon)$. Choose $\rho+\varepsilon$ sufficiently small. Then from Lemma 3.3, (6.6) also has exponential dichotomy in $[a/\varepsilon, +\infty)$, with projections and exponent close to those of (6.5). With ρ fixed, if ε is sufficiently small, $\varepsilon^\beta<\rho$, $0<\beta<1$. Thus, in $[a/\varepsilon, (a+\varepsilon^\beta)/\varepsilon]$,

$$z'(\tau)-f_x\left(\sum_{j=0}^m \varepsilon^j y_j^0\left(\tau-\frac{a}{\varepsilon}\right), \varepsilon\tau, \varepsilon\right)z(\tau)=0$$

also has exponential dichotomy. Moreover, the projections at $(a+\varepsilon^\beta)/\varepsilon$, $P_s((a+\varepsilon^\beta)/\varepsilon, \varepsilon)$ and $P_u((a+\varepsilon^\beta)/\varepsilon, \varepsilon)$ are close to the spectral projections of $f_x(p_1(a+\varepsilon^\beta), a+\varepsilon^\beta, 0)$, if we let ε be sufficiently small. We also infer that

$$\mathcal{R}P_s(a/\varepsilon, \varepsilon)\oplus\mathcal{K}B_{1x}(x(a, \varepsilon), \varepsilon)=\mathbb{R}^n$$

with the angle of the two subspaces being bounded away from zero as $\varepsilon\to 0$. Here hypothesis (H2) is employed.

Similarly, the homogeneous linear variational equation around $x(\varepsilon\tau, \varepsilon)$, $\tau\in[(b-\varepsilon^\beta)/\varepsilon, b/\varepsilon]$ has exponential dichotomy in that interval with projections at one endpoint $P_s((b-\varepsilon^\beta)/\varepsilon, \varepsilon)$ and $P_u((b-\varepsilon^\beta)/\varepsilon, \varepsilon)$ close to those of the spectral projections of $f_x(p_r(b-\varepsilon^\beta), b-\varepsilon^\beta, 0)$, if ε is sufficiently small. Moreover,

$$\mathcal{R}P_u(b/\varepsilon, \varepsilon)\oplus\mathcal{K}B_{2x}(x, b, \varepsilon), \varepsilon)=\mathbb{R}^n,$$

with the angle of the two subspaces being bounded away from zero as $\varepsilon\to 0$, by virtue of (H2) again.

(5) For $t\in[t_i-\varepsilon^\beta, t_i+\varepsilon^\beta]$, or $\tau\in[(t_i-\varepsilon^\beta)/\varepsilon, (t_i+\varepsilon^\beta)/\varepsilon]$, in which $x(\varepsilon\tau, \varepsilon)=\sum_{j=0}^m \varepsilon^j y_j^i(\tau-t_i/\varepsilon-\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i)$ is a perturbation of $q_i(\tau-t_i/\varepsilon-\tau_0)$. We first prove that the homogeneous linear variational equation around $x(\varepsilon\tau, \varepsilon)$ has exponential dichotomy in $[(t_i-\varepsilon^\beta)/\varepsilon, t_i/\varepsilon]$ and $[t_i/\varepsilon, (t_i+\varepsilon^\beta)/\varepsilon]$, with projections $P_u(\tau, \varepsilon)$ and $P_s(\tau, \varepsilon)$ close to the spectral projections of $f_x(x(\varepsilon\tau, \varepsilon), \varepsilon\tau, 0)$ at $(t_i-\varepsilon^\beta)/\varepsilon$ and $(t_i+\varepsilon^\beta)/\varepsilon$. It is convenient to make a shift in τ. Comparing in $\tau\in\mathbb{R}$,

$$z'(\tau)-\tilde{f}_x\left(\sum_{j=0}^m \varepsilon^j\tilde{y}_j^i(\tau), t_i+\varepsilon\left(\tau+\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i\right), \varepsilon\right)z(\tau)=0, \tag{6.7}$$

with

$$z'(\tau)-f_x(q_i(\tau), t_i, 0)z(\tau)=0, \tag{6.8}$$

we find that the coefficients differ by $O(\rho+\varepsilon)$. Since (6.8) has exponential dichotomies in $\mathbb{R}^-$ and $\mathbb{R}^+$, so does (6.7), provided that $\rho+\varepsilon$ is sufficiently small. Moreover, the projections $\bar{P}_s(\tau, \varepsilon)$ and $\bar{P}_u(\tau, \varepsilon)$ corresponding to (6.7) are close to those of (6.8).

Our next step is to use Lemma 3.10 to show that $\mathscr{R}\bar{P}_u(0^-, \varepsilon) \oplus \mathscr{R}\bar{P}_s(0^+, \varepsilon) = \mathbb{R}^n$ with $\theta(\mathscr{R}\bar{P}_u(0^-, \varepsilon), \mathscr{R}\bar{P}_s(0^+, \varepsilon)) \geqq C|\varepsilon|$, $C > 0$ is a constant, provided that $0 < \varepsilon \leqq \varepsilon_0$, $\varepsilon_0 > 0$ being a small constant.

Let $\bar{\varepsilon} > 0$ be a small and fixed constant and consider $\rho = \rho(\bar{\varepsilon}) = \bar{\varepsilon}^\beta$, $0 < \beta < 1$ and $0 \leqq \varepsilon \leqq \bar{\varepsilon}$. Let $A_i(\tau, \varepsilon) = \tilde{f}_x(\sum_{j=0}^m \varepsilon^j \tilde{y}_j^i(\tau), t_i + \varepsilon(\tau + \sum_{j=0}^{m-1} \varepsilon^j \tau_j^i), \varepsilon)$. We observe that $A_i(\tau, \varepsilon)$ is $C^1/1$ in ε and uniformly bounded in τ, $(\partial/\partial\varepsilon)A_i(\tau, \varepsilon) \in E_{\mathbb{R}}(0, 1)$. Moreover, $A_i(\tau, 0) = f_x(q_i(\tau), t_i, 0)$. From our assumptions on $q_i(\tau)$, the only bounded solution of (6.8) is $\varphi_i(\tau) = q_i'(\tau)$, up to a scalar factor. Let $\psi_i(\tau)$ be the only bounded solution, up to a scalar factor, of the formal adjoint equation of (6.8). To apply Lemma 3.10, we have to show that

$$I \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi_i^*(\tau) \frac{\partial}{\partial \varepsilon} A_i(\tau, 0) \varphi_i(\tau)\, d\tau \neq 0,$$

$$\frac{\partial}{\partial \varepsilon} A_i(\tau, 0) = \tilde{f}_{xx}(q_i(\tau), t_i, 0)\tilde{y}_1^i(\tau) + \tilde{f}_{xt}(q_i(\tau), t_i, 0)(\tau + \tau_0^i) + \tilde{f}_{x\varepsilon}(q_i(\tau), t_i, 0).$$

Let $\bar{\tau}$ be an arbitrary parameter, we have

$$\begin{aligned} I = \int_{-\infty}^{\infty} & \psi_i^*(\tau + \bar{\tau})[\,\tilde{f}_{xx}(q_i(\tau + \bar{\tau}), t_i, 0)\tilde{y}_1^i(\tau + \bar{\tau}) \\ & \qquad + f_{xt}(q_i(\tau + \bar{\tau}), t_i, 0)(\tau + \tau + \tau_0^i) + \tilde{f}_{x\varepsilon}(q_i(\tau + \bar{\tau}), t_i, 0)] \\ & \cdot q_i'(\tau + \bar{\tau})\, d\tau = I_1 + I_2, \end{aligned}$$

$$\begin{aligned} I_1 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} & \psi_i^*(\tau + \bar{\tau}) \Big\{ \frac{\partial}{\partial \bar{\tau}} [\tilde{f}_x(q_i(\tau + \bar{\tau}), t_i, 0) y_1^i(\tau + \bar{\tau}) \\ & \qquad + \tilde{f}_t(q_i(\tau + \bar{\tau}), t_i, 0)(\tau + \bar{\tau} + \tau_0^i) + \tilde{f}_\varepsilon(q_i(\tau + \bar{\tau}), t_i, 0)] \\ & \qquad - \tilde{f}_x(q_i(\tau + \bar{\tau}), t_i, 0) y_1^i(\tau + \bar{\tau})' - \tilde{f}_t(q_i(\tau + \bar{\tau}), t_i, 0) \Big\}\, d\tau. \end{aligned}$$

From $(5.20)_1$, $y_1^i(\tau)' = \tilde{f}_x(q_i(\tau), t_i, 0) y_1^i(\tau) + \tilde{f}_t(q_i(\tau), t_i, 0)(\tau + \tau_0^i) + \tilde{f}_\varepsilon(q_i(\tau), t_i, 0)$, we have that

$$I_1 = \int_{-\infty}^{\infty} \psi_i^*(\tau + \bar{\tau}) \Big\{ \frac{\partial}{\partial \bar{\tau}} y_1^i(\tau + \bar{\tau})' - \tilde{f}_x(q_i(\tau + \bar{\tau}), t_i, 0) y_1^i(\tau + \bar{\tau})' - \tilde{f}_t(q_i(\tau + \bar{\tau}), t_i, 0) \Big\}\, d\tau.$$

However, $S(\tau) = (\partial/\partial\bar{\tau}) y_1^i(\tau + \bar{\tau})' - \tilde{f}_x(q_i(\tau + \bar{\tau}), t_i, 0) y_1^i(\tau + \bar{\tau})'$ is in the range of the Fredholm operator $z \to z'(\tau) - A(\tau + \bar{\tau})z(\tau)$, therefore $\int_{-\infty}^{\infty} \psi_i^*(\tau + \bar{\tau}) S(\tau)\, d\tau = 0$. And

$$I_1 = -\int_{-\infty}^{\infty} \psi_i^*(\tau) f_t(q_i(\tau), t_i, 0)\, d\tau \neq 0$$

by virtue of (H3). Consider

$$\begin{aligned} I_2 &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi_i^*(\tau + \bar{\tau}) \tilde{f}_{xx}(q_i(\tau + \bar{\tau}), t_i, 0)\{\tilde{y}_1^i(\tau + \bar{\tau}) - y_1^i(\tau + \bar{\tau}\}\, d\tau \\ &\leqq \int_{-\infty}^{-\rho(\bar{\varepsilon})/\bar{\varepsilon}} |\psi_i(\tau)|\,|f_{xx}|(1 + |\tau|)\, d\tau + \int_{\rho(\bar{\varepsilon})/\bar{\varepsilon}}^{\infty} |\psi_i(\tau)|\,|f_{xx}|(1 + |\tau|)\, d\tau \\ &\to 0 \quad \text{as } \bar{\varepsilon} \to 0, \end{aligned}$$

since $\psi_i(\tau) \leqq C e^{-\alpha|\tau|}$ and $\rho(\bar{\varepsilon})/\bar{\varepsilon} \to \infty$ as $\bar{\varepsilon} \to 0$. Therefore, for $\bar{\varepsilon}$ sufficiently small, $|I_1 + I_2| \geqq |I_1|/2 \neq 0$.

Strictly speaking, the entire equation, (6.7), depends on $\bar{\varepsilon}$, i.e., it should be written as

$$z'(\tau)-A(\tau, \varepsilon, \bar{\varepsilon})z(\tau)=0.$$

The projections $\bar{P}_s(\tau, \varepsilon, \bar{\varepsilon})$ and $\bar{P}_u(\tau, \varepsilon, \bar{\varepsilon})$ also depend on $\bar{\varepsilon}$. From Lemma 3.10,

$$\theta(\bar{P}_s(0^+, \varepsilon, \bar{\varepsilon}), \bar{P}_u(0^-, \varepsilon, \bar{\varepsilon})) \geqq C(\bar{\varepsilon})|\varepsilon|, \tag{6.9}$$

the constant $C(\bar{\varepsilon})>0$ also depends on $\bar{\varepsilon}$. Suppose we can prove that there exists $\varepsilon_0>0$ such that for all $0<\bar{\varepsilon}\leqq\varepsilon_0$, $C(\bar{\varepsilon})\geqq C>0$, and (6.9) is valid for all $0<\varepsilon\leqq\varepsilon_0$, the desired results can be obtained by setting $\varepsilon=\bar{\varepsilon}$ in (6.9). Here we refer back to the proof of Lemma 3.10 and make the following observations. If $\varepsilon_0>0$ is sufficiently small, and $0\leqq\bar{\varepsilon}\leqq\varepsilon_0$, then (i) the unique solvability of (3.3) for $|\varepsilon|\leqq\varepsilon_0$ does not depend on $\bar{\varepsilon}$; (ii) the estimates $\bar{u}_i(\varepsilon)=O(\varepsilon)$, $\bar{v}_j(\varepsilon)=O(\varepsilon)$ and (3.6) are uniform with respect to $\bar{\varepsilon}$; (iii) we have shown that $|\int_{-\infty}^{\infty}\psi_i(\tau)(\partial/\partial\varepsilon)A(\tau, 0, \bar{\varepsilon})\varphi_i(\tau)|\geqq|I_1|/2>0$ uniformly with respect to $\bar{\varepsilon}$; and (iv) from (3.4) $\partial\bar{u}_i(\varepsilon)/\partial\varepsilon-\partial\bar{u}_i(0)/\partial\varepsilon=O(\varepsilon)$, uniformly with respect to $\bar{\varepsilon}$. Thus, the dependence of $\bar{\varepsilon}$ of the equation (6.7) does not matter. Let us now consider the restriction of (6.7) on $[-\varepsilon^{\beta-1}-\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i, \varepsilon^{\beta-1}-\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i]$. If ε_0 is small and $0<\varepsilon=\bar{\varepsilon}\leqq\varepsilon_0$, then $|\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i|\leqq T_0\leqq\varepsilon^{\beta-1}$ and

$$\tilde{f}_x\left(\sum_{j=0}^{m}\varepsilon^j y_j^i(\tau), t_i+\varepsilon\left(\tau+\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i\right), \varepsilon\right)=f_x\left(\sum_{j=0}^{m}\varepsilon^j y_j^i(\tau), t_i+\varepsilon\left(\tau+\sum_{j=0}^{m-1}\varepsilon^j\tau_j^i\right), \varepsilon\right).$$

This completes the proof that the linearization around $x(\varepsilon\tau, \varepsilon)$ has exponential dichotomies in $[(t_i-\varepsilon^\beta)/\varepsilon, t_i/\varepsilon]$ and $[t_i/\varepsilon, (t_i+\varepsilon^\beta)/\varepsilon]$, respectively. Moreover, $\theta(\mathscr{R}P_u((t_i/\varepsilon)^-, \varepsilon), \mathscr{R}P_s((t_i/\varepsilon)^+, \varepsilon))\geqq C|\varepsilon|$, and the projections are close to the spectral projections of $f_x(x(\varepsilon\tau, \varepsilon), \varepsilon\tau, 0)$ at $\tau=(t_i\pm\varepsilon^\beta)/\varepsilon$.

(6) To complete the proof of Theorem 2.2, we shall recall the main facts concerning the formal approximation $x(\varepsilon\tau, \varepsilon)$, which is defined piecewise in $2r+1$ subintervals of $[a/\varepsilon, b/\varepsilon]$. The residual and the jump errors are $O(\varepsilon^{\beta(m+1)})$. The boundary errors are $O(\varepsilon^{m+1})$. We may further divide each interior layer into two and make the total number of subintervals into 3r, i.e.,

$$\left[\frac{a}{\varepsilon}, \frac{b}{\varepsilon}\right]=\bigcup_{i=1}^{r}\left\{\left[\frac{t_{i-1}}{\varepsilon}, \frac{t_{i-1}+\varepsilon^\beta}{\varepsilon}\right]\cup\left[\frac{t_{i-1}+\varepsilon^\beta}{\varepsilon}, \frac{t_i-\varepsilon^\beta}{\varepsilon}\right]\cup\left[\frac{t_i-\varepsilon^\beta}{\varepsilon}, \frac{t_i}{\varepsilon}\right]\right\}.$$

The homogeneous linear variational equation around $x(\varepsilon\tau, \varepsilon)$ has exponential dichotomy in each subinterval as has been proved in (2)-(5) of this proof. We also know that the constants K and α, for all the subintervals are uniform with respect to $0\leqq\varepsilon\leqq\varepsilon_0$. The projections $P_u(\zeta^-, \varepsilon)$, $P_s(\zeta^-, \varepsilon)$, $P_u(\zeta^+, \varepsilon)$ and $P_s(\zeta^+, \varepsilon)$, at the common points $\zeta=(t_{i-1}+\varepsilon^\beta)/\varepsilon$ or $\zeta=(t_i-\varepsilon^\beta)/\varepsilon$, are close to the spectral projections of $f_x(x(\varepsilon\zeta, \varepsilon), \varepsilon\zeta, \varepsilon)$ provided that ε_0 is small. Therefore $\theta(\mathscr{R}P_u(\zeta^-, \varepsilon), \mathscr{R}P_s(\zeta^+, \varepsilon))\geqq C_1$, $C_1>0$ is a constant. Our result in (5) also shows that $\theta(\mathscr{R}P_u(\zeta^-, \varepsilon), \mathscr{R}P_s(\zeta^+, \varepsilon))\geqq C_2|\varepsilon|$, $C_2>0$ is a constant for all the common points $\zeta=t_i/\varepsilon$, $i=1, \cdots, r-1$. Let $Q(\zeta)$ be the projections $\mathbb{R}^n\to P_u(\zeta^-, \varepsilon)$, parallel to $P_s(\zeta^+, \varepsilon)$, here ζ is one of the $3r-1$ common points of the $3r$ subintervals, then $|Q(\zeta)|=O(1/\varepsilon)$, $0<\varepsilon\leqq\varepsilon_0$. We also know that $\theta(\mathscr{K}B_{1x}(x(a, \varepsilon), \varepsilon), \mathscr{R}P_s(a/\varepsilon, \varepsilon))$ and $\theta(\mathscr{K}B_{2x}(x(b, \varepsilon), \varepsilon), \mathscr{R}P_u(b/\varepsilon, \varepsilon))$ are bounded away from zero, uniformly with respect to $0<\varepsilon\leqq\varepsilon_0$. Define the projections $Q(a/\varepsilon):\mathbb{R}^n\to\mathscr{K}B_{1x}(x(a, \varepsilon), \varepsilon)$, $\mathscr{K}Q(a/\varepsilon)=\mathscr{R}P_s(a/\varepsilon, \varepsilon)$ and $Q(b/\varepsilon):\mathscr{R}^n\to\mathscr{R}P_u(b/\varepsilon, \varepsilon)$, $\mathscr{K}Q(b/\varepsilon)=\mathscr{K}B_{2x}(x(b, \varepsilon), \varepsilon)$. It is clear that $Q(a/\varepsilon)$ and $Q(b/\varepsilon)$ are bounded uniformly with respect to $0<\varepsilon\leqq\varepsilon_0$. Finally, the length of each interval is no less than $\varepsilon^{\beta-1}\geqq\varepsilon_0^{\beta-1}$. For the time being, suppose $\beta(m+1)>2$. It should be clear that if ε_0 is sufficiently small, all the assumptions of Theorem 4.4 are satisfied, in particular, $\|\mathscr{A}^{-1}\| = O(1/\varepsilon)$ and $\varepsilon^{\beta(m+1)} = o(\|\mathscr{A}^{-1}\|^{-2})$. Therefore, we obtain a unique solution $x_{\text{exact}}(\varepsilon\tau, \varepsilon)$ in a neighborhood of the orbit of $x(\varepsilon\tau, \varepsilon)$. It follows from the estimate in Theorem 4.4 that

$$\sup_{\tau\in[a/\varepsilon, b/\varepsilon]} \{|x_{\text{exact}}(\varepsilon\tau, \varepsilon) - x(\varepsilon\tau, \varepsilon)|\} = O(\varepsilon^{\beta(m+1)-1}). \tag{6.10}$$

We now consider $|x(\tau, \varepsilon) - \bar{x}(\tau, \varepsilon)|$ where $\bar{x}(\tau, \varepsilon)$ is the composite expansion in (2.5). For $t \in [t_{i-1} + \varepsilon^\beta, t_i - \varepsilon^\beta]$, by virtue of the fact $y_j^{i-1} - x_{j,2}^{i-1} \in E_{\mathbb{R}^+}(\gamma, j)$ and $y_j^i - x_{j,1}^i \in E_{\mathbb{R}^-}(\gamma, j)$, $|x(\tau, \varepsilon) - \bar{x}(\tau, \varepsilon)| = O(\varepsilon^{(m+1)})$. For $t \in [t_{i-1}, t_{i-1} + \varepsilon^\beta]$ by virtue of the fact

$$\begin{aligned}\sum_{j=0}^{m} \varepsilon^j X_j^i(t) &- \sum_{j=0}^{m} \varepsilon^j x_{j,2}^{i-1}\left(\frac{t - t_{i-1}}{\varepsilon} - \sum_{j=0}^{m-1} \varepsilon^j \tau_j^{i-1}\right) \\ &= \sum_{j=0}^{m} \varepsilon^j X_j^i\left(t_{i-1} + \varepsilon\left(\tau + \sum_{j=0}^{m-1} \varepsilon^j \tau_j^{i-1}\right)\right) - \sum_{j=0}^{m} \varepsilon^j x_{j,2}^{i-1}(\tau) = O(\varepsilon^{\beta(m+1)}),\end{aligned}$$

$|x(\tau, \varepsilon) - \bar{x}(\tau, \varepsilon)| = O(\varepsilon^{\beta(m+1)})$. This is similar for $t \in [t_i - \varepsilon^\beta, t_i]$. Therefore

$$\sup_{\tau\in[a/\varepsilon, b/\varepsilon]} \{|x_{\text{exact}}(\varepsilon\tau, \varepsilon) - \bar{x}(\varepsilon\tau, \varepsilon)|\} = O(\varepsilon^{\beta(m+1)-1}). \tag{6.11}$$

Recall that our approximations $x(\tau, \varepsilon)$ and $\bar{x}(\tau, \varepsilon)$ depend on m, and should be denoted by $x(\tau, \varepsilon, m)$ and $\bar{x}(\tau, \varepsilon, m)$. For any $m \geqq 0$, we can always choose $m_1 > m$ such that $\beta(m_1 + 1) - 1 \geqq m + 1$. It is easy to see that

$$|x(t, \varepsilon, m) - x(t, \varepsilon, m_1)| = O(\varepsilon^{\beta(m+1)}),$$

$$|\bar{x}(t, \varepsilon, m) - \bar{x}(t, \varepsilon, m_1)| = O(\varepsilon^{m+1}).$$

We now apply (6.10) and (6.11) to $x(t, \varepsilon, m_1)$ and $\bar{x}(t, \varepsilon, m_1)$, and the desired estimates in Theorem 2.2 follow easily.

The proof of Theorem 2.3 uses Theorem 4.5 and is analogous to those of Theorem 2.1 and 2.2. Details shall be omitted.

Acknowledgments. I am grateful for many encouraging discussions with S.-N. Chow, J. Hale, F. Hoppensteadt, L. Kelly, and D. Yen. This work was done when I was visiting Michigan State University.

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