Construction of periodic orbits for a singularly perturbed system

Xiao-Biao Lin

North Carolina State university

Raleigh, North Carolina, USA

Workshop on Bifurcation Methods of Limit Cycles and Related Studies

May 20-30, 2008

Shanghai Normal University, Shanghai, China

Abstract

$\sim$ For a singularly perturbed higher dimensional system, we describe a general method to construct periodic orbits and study their asymptotic expansions.
$\sim$ Using the slow and fast variables and letting epsilon goes to zero, two kinds of limiting system can be derived. One is the slow system the other is the fast system.
$\sim$ Dynamics on the slow or fast system is of lower dimensional and usually easier to study.
$\sim$ If one can find "singular orbits" on the two limiting systems and piece them together to form a "singular" periodic orbit, then under some general transversality conditions one can show that for small, nonzero epsilon the system also has a periodic orbit.

"Heteroclinic Bifurcation and Singularly Perturbed Boundary Value Problems", JDE 1990

Why should we study singular perturbation problems?

(1) In many physical and biological systems, small and large scales of variables naturally occur.

(2) Reduction to lower dimensional systems.

(3) Study a complex system by localization.

(4) Study the solutions by linearization.

(5) Approximation of solutions by asymptotic series, using a similar linear system.

FitzHugh-Nagumo equaton with a small parameter:

$$\epsilon u_t = \epsilon^2 u_{xx} + f(u) - w,$$
$$w_t = u - \gamma w.$$

Traveling wave solution of the FitzHugh-Nagumo equation, with the waves speed $\theta > 0$, satisfies

$$\epsilon \dot{u} = v,$$
$$\epsilon \dot{v} = \theta v - [f(u) - w],$$
$$\dot{w} = \theta^{-1}(u - \gamma w).$$
$$f(u) = -u(u-b)(u-1), \quad 0 < b < \frac{1}{2}.$$

This is called the slow system where $\dot{u} = \frac{du}{dt}$.

Zoom in, $t = t_i + \epsilon\tau$ to catch the fast motion at $t = t_i$:

$$u' = v,$$
$$v' = \theta v - [f(u) - w],$$
$$w' = \epsilon\theta^{-1}(u - \gamma w).$$

This is called the fast system where $u' = \frac{du}{d\tau}$.

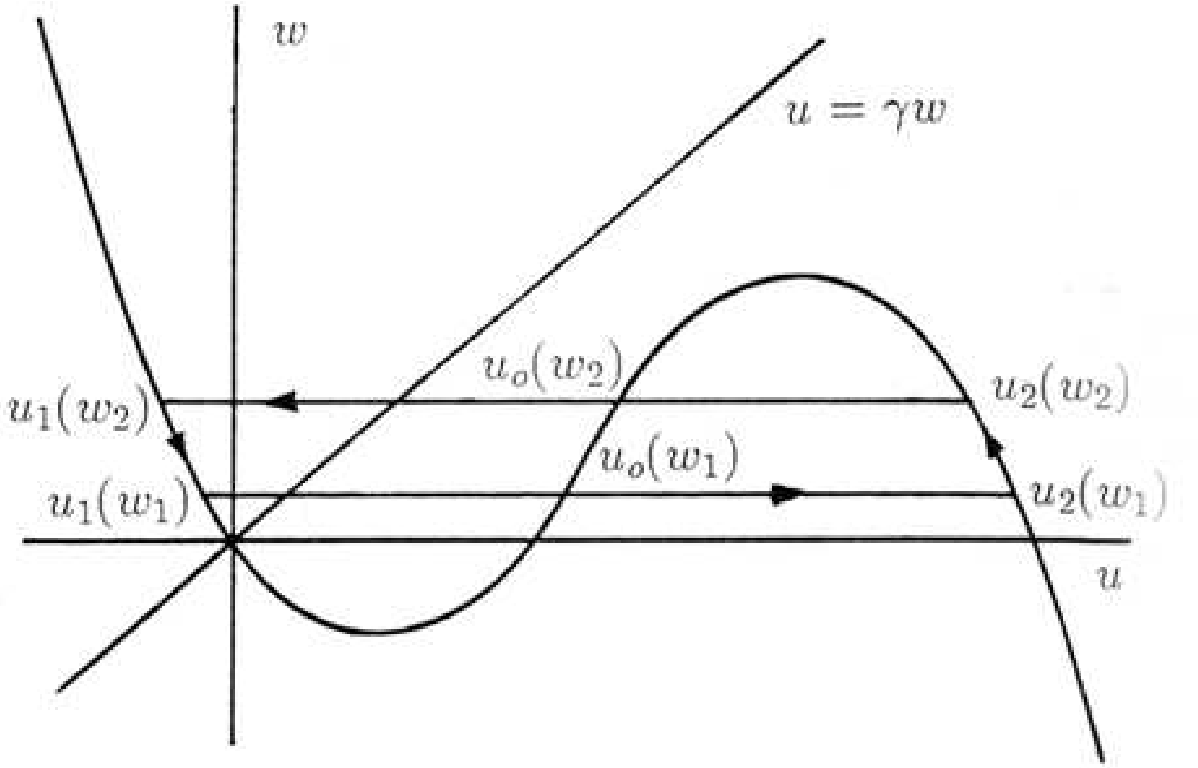


FIGURE 3

The slow flow on the cubic curve and two fast jumps that are saddle – saddle connections.
Assume that $u = \gamma w$ intersects $w = f(u)$ only once.

The slow system describe the motion in regular layers where $(\dot{u}, \dot{v}, \dot{w})$ are bounded as $\epsilon \to 0$.
The fast system describe the motion in singular layers, or internal layers where $(\dot{u}, \dot{v})$ are $O(1/\epsilon)$.

In the singular limit $\epsilon = 0$, the motion in regular layers satisfies

$$
\begin{aligned}
0 &= v, \\
0 &= \theta v - [f(u) - w], \\
\dot{w} &= \theta^{-1}(u - \gamma w).
\end{aligned}
$$

The first two equation define the slow manifold, $u = f^{-1}(w)$. It has three branches: $\mathcal{S}_1, \mathcal{S}_0, \mathcal{S}_2$.
$\mathcal{S}_1, \mathcal{S}_2$ are hyperbolic center manifolds. The flow on the slow manifold is

$$
\dot{w} = \theta^{-1}(f^{-1}(w) - \gamma w).
$$

dw/dt changes signs across $u = \gamma w$.

In the singular limit $\epsilon = 0$, th motion in singular layers satisfies

$$\begin{aligned} u' &= v, \\ v' &= \theta v - [f(u) - w], \\ w' &= 0. \end{aligned}$$

$w = w_i := w(t_i)$ appears as a constant.
Reduce to a two-dimensional system

$$u' = v, \quad v' = \theta v - [f(u) - w_i].$$

The equilibrium points are on the slow manifold $(u, v) = (f^{-1}(w), 0)$.
$\sim$ For each $\bar{w} < w < \bar{\bar{w}}$, $f(u) - w$ has three zeros,

$$u_1(w) < u_0(w) < u_2(w).$$

The u_1 and u_2 are saddle points on $\mathcal{S}_1$ and $\mathcal{S}_2$.
For each fixed θ, we look or saddle to saddle connections by adjusting w.

Lemma 1 *There is* $\theta_0 > 0$ *such that for* $0 \le \theta \le \theta_0$, *the* (u, v) *system has a unique hoteroclinic solution connecting* $(u_1(w), 0)$ *to* $(u_2(w), 0)$ *if* $w = w_1(\theta)$. *The function* $w_1(\theta)$ *satisfies* $w_1(\theta_0) = 0$, $w_1'(\theta) < 0$. $w_1(0)$ *satisfies the equal-area principle*

$$\int_{u_1}^{u_2} (f(u) - w_1(0))du = 0.$$

Also for $0 \le \theta \le \theta_0$, *the* (u, v) *system has a unique heteroclinic solution connecting* $(u_2(w), 0)$ *to* $(u_1(w), 0)$ *if* $w = w_2(\theta)$. $w_2'(\theta) > 0$ *and* $w_2(0) = w_1(0)$ *(equal area line).*
(At θ_0, w_1 *is the lowest and* w_2 *the highest.)*

proof: Three methods:
(1) Compute $w_1(\theta)$ explicitly, Casten, Cohen and Lagerstrom.
(2) It is a Hamiltonian system when $\theta = 0$.
Phase plane analysis can be used when $\theta \ne 0$.
(3) Melnikov's method can be used to compute $w_j'(\theta)$.
Then a homotopy method can be used to continue the solution to $\theta > 0$. qud

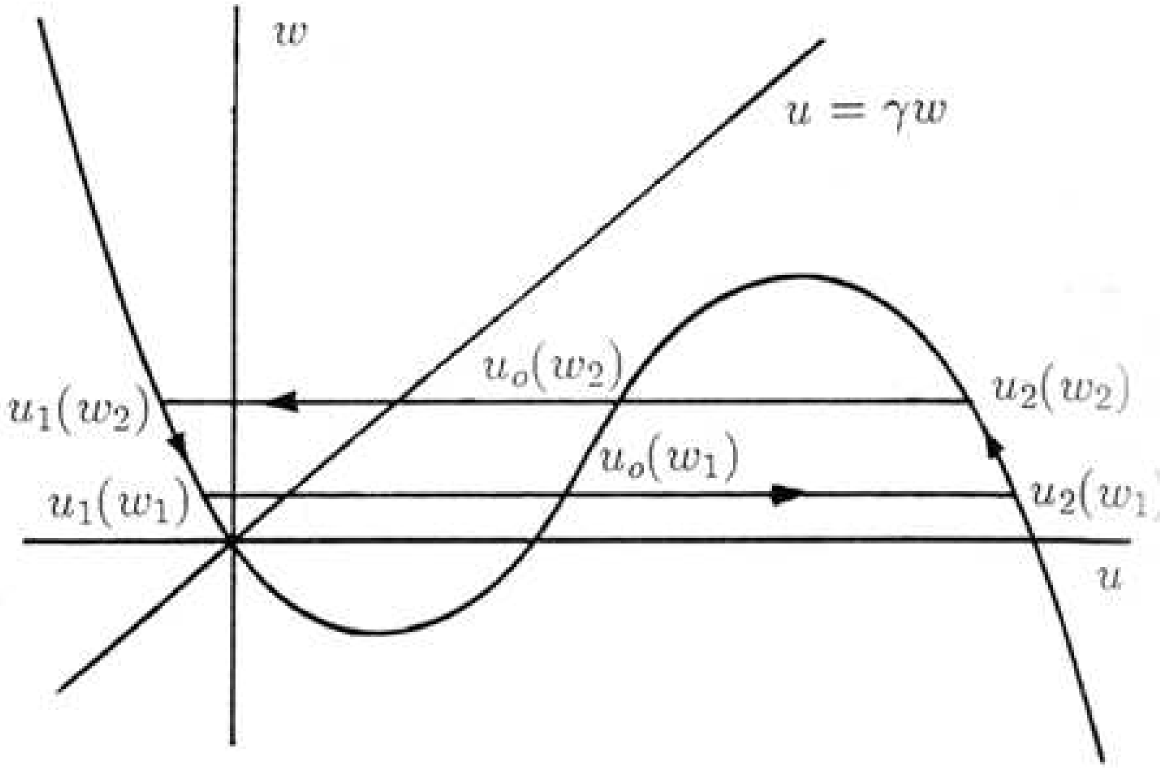


FIGURE 3

For each θ, $w_1(\theta), w_2(\theta)$ are take-off points on the slow manifold for the heteroclinic connections from one branch to another.
Fast and slow singular orbits form a closed loop.
We can show that when ϵ is small, there is a periodic orbit near the union of the singular orbits.

For a given $(\tilde{\theta}, \tilde{w})$, let $(\tilde{u}(\tau), \tilde{v}(\tau))$ be the unique heteroclinic orbit. For any (θ, w) near $(\tilde{\theta}, \tilde{w})$, the heterolinic orbit may break. The gap between the unstable and stable manifolds is $G(\theta, w)$. To have a heterolinic orbit, we need

$$G(\theta, w) = 0.$$

Implicit function theorem can be used to locally solve $G = 0$ as $w = w^*(\theta)$. Let the vector field be F. Then

$$F = \begin{pmatrix} u'(\tau) \\ v'(\tau) \end{pmatrix}, \quad \frac{\partial F}{\partial \theta} = \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix}, \quad \frac{\partial F}{\partial w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Melnikov's integrals, cf. Guckenheimer and Holmes, $\theta = tr(A)$, generalized to $\mathbb{R}^n$ by Ken Palmer, 1984:

$$\frac{\partial G}{\partial \theta} = \int_{-\infty}^{\infty} e^{-\theta\tau}(F \wedge \frac{\partial F}{\partial \theta})d\tau = \int_{-\infty}^{\infty} e^{-\theta\tau}(\tilde{v})^2 d\tau > 0,$$
$$\frac{\partial G}{\partial w} = \int_{-\infty}^{\infty} e^{-\theta\tau}(F \wedge \frac{\partial F}{\partial w})d\tau = \int_{-\infty}^{\infty} e^{-\theta\tau}\tilde{u}'(\tau)d\tau$$
$$> 0 \text{ if } u(-\infty) < u(\infty), \quad < 0 \text{ if } u(-\infty) > u(\infty).$$

$\frac{\partial w^*}{\partial \theta} > 0$ if jumps to the left, $\frac{\partial w^*}{\partial \theta} < 0$ if jumps to the right.

Conditions that ensure the existence of an exact periodic orbit for $\epsilon \neq 0$ near the singular orbits:

(H1) The slow manifolds $\mathcal{S}_1, \mathcal{S}_2$ are hyperbolic with the same dimension of unstable index.

$\sim$ This can be verified by computing eigenvalues on slow manifolds.

(H2) $\partial_w G(\theta, w) \neq 0$ at the take-off points $w_1(\theta), w_2(\theta)$.

(H3) The flow on the slow manifold cross the take-off points transversely.

$\sim$ This can be verified by $\frac{dw}{dt} \neq 0$ at the $w = w_j(\theta)$.

Formulation of pseudo-orbits

In regular layers, the ith solution between take-off points w_i and w_{i+1}:

$Z_i = (U, V, W)$ that satisfies

$U = f^{-1}(W), \quad V = 0, \quad \dot{W} = \theta^{-1}(f^{-1}(W) - \gamma W),$

solution: $Z_i(t), t_i \le t \le t_{i+1}, W(t_i) = w_i, W(t_{i+1}) = w_{i+1}.$

In singular layers, the ith solution between $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$:

$z_i = (u, v, w)$ that satisfies

$u' = v, \quad v' = \theta v - [f(u) - w_i], \quad w = w_i,$

solution: $z_i(\tau, w_i), \quad -\infty < \tau < \infty,$

$z_i(-\infty) = (u_-(w_i), 0, w_i), \quad z_i(\infty) = (u_+(w_i), 0, w_i),$

$z_i(0+, w_i) - z_i(0_-, w_i) = G_i(\theta, w_i).$

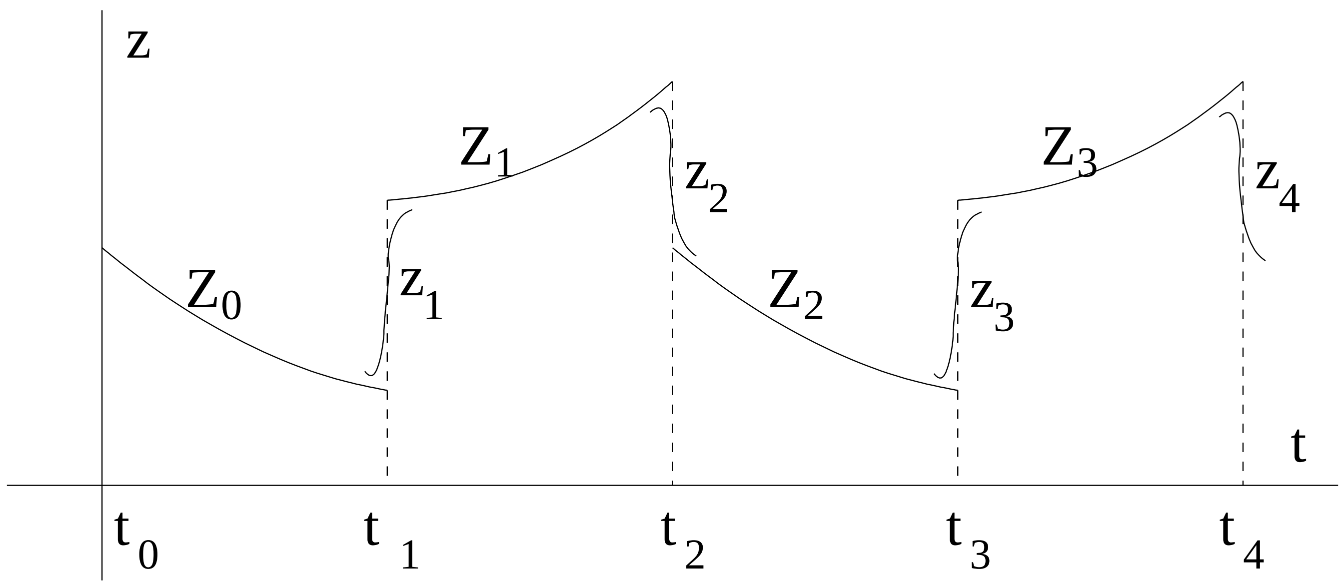
z_i is a pseudo-heteroclinic solution. To have a true heteroclinic solution, need $G_i(\theta, w_i) = 0$.

Let $0 < \beta < 1$. . For $\epsilon > 0$, small, defined the approximations:
In regular layers, the intermediate variable $\epsilon^\beta \to 0$, $\epsilon \to 0$.

$$Z_i(t), \quad t_i + \epsilon^\beta < t < t_{i+1} - \epsilon^\beta,$$
$$W_i(t_i) = w_i, \; W_i(t_{i+1}) = w_{i+1}.$$

In singular layers, $\epsilon^{\beta-1} \to \infty$, $\epsilon \to 0$.

$$z_i(\tau, w_i), \quad -\epsilon^{\beta-1} \le \tau \le \epsilon^{\beta-1},$$
$$\tau = (t - t_i)/\epsilon, \quad t_i - \epsilon^\beta \le t \le t_i + \epsilon^\beta.$$



The union of fast and slow orbits

$$(z_1, Z_1, z_2, Z_2)$$

are approximations of a closed orbit in the sense: check!

(1) The residual errors are small in regular and singular layers.

$$H_i(\epsilon) = \begin{pmatrix} \epsilon\dot{U} - V \\ \epsilon\dot{V} - \theta V + f(U) - W \\ \epsilon\dot{W} - \epsilon\theta^{-1}(U - \gamma W) \end{pmatrix}.$$

$$h_i(\epsilon) = \begin{pmatrix} u' - v \\ v' - \theta v + f(u) - w \\ w' - \epsilon\theta^{-1}(u - \gamma w) \end{pmatrix}.$$

(2) The jump errors are small between regular and singular layers.

$$A_i = z_i(\epsilon^{\beta-1}, w_i) - Z_i(t_i + \epsilon^{\beta}),$$
$$B_i = Z_i(t_{i+1} - \epsilon^{\beta}) - z_{i+1}(-\epsilon^{\beta-1}, w_{i+1}).$$
$$H_i(\epsilon),\ h_i(\epsilon),\ A_i,\ B_i \to 0 \text{ as } \epsilon \to 0. \text{ Why?}$$

Add correction terms to eliminate errors:

$$z_i + \Delta z_i,\ Z_i + \Delta Z_i,\ t_i + \Delta t_i.$$

Let $Z = (Y, W)$, $z = (y, w)$ where $Y, y \in \mathbb{R}^2$. The corrections satisfies:

$$\begin{aligned}
\Delta Y_i(\tau)' &= \mathcal{F}(Y_i + \Delta Y_i, W_i + \Delta W_i) - \mathcal{F}(Y_i, W_i) - H_{i1},\\
\Delta W_i(\tau)' &= \epsilon\mathcal{G}(Y_i + \Delta Y_i, W_i + \Delta W_i) - \epsilon\mathcal{G}(Y_i, W_i) - H_{i2},\\
\Delta y_i(\tau)' &= \mathcal{F}(y_i + \Delta y_i, w_i + \Delta w_i) - \mathcal{F}(y_i, w_i) - h_{i1},\\
\Delta w_i(\tau)' &= \epsilon\mathcal{G}(y_i + \Delta y_i, w_i + \Delta w_i) - \epsilon\mathcal{G}(y_i, w_i) - h_{i2}.
\end{aligned}$$

Try to simplify:

$\sim$ In singular layers

$$\begin{aligned}
\Delta y_i(\tau)' - \mathcal{F}_y^i(\tau)\Delta y_i(\tau) - \mathcal{F}^i(\tau)\Delta w_i(\tau) &= -h_{i1} + h.o.t.\\
\Delta w_i(\tau)' &= -h_{i2} + h.o.t.
\end{aligned}$$

Can solve Δw_i then solve for Δy_i.

$\sim$ In regular layers, cannot drop $\epsilon\mathcal{G}$ terms. WHY?

Using $W = \bar{W} + \epsilon\omega$ as time:

$$\frac{dU}{d\omega} = \frac{V}{\theta^{-1}(U - \gamma(\bar{W} + \epsilon\omega))}, \qquad \frac{dV}{d\omega} = \frac{\theta V - f(U) + \bar{W} + \epsilon\omega}{\theta^{-1}(U - \gamma(\bar{W} + \epsilon\omega))}.$$

This is a slow varying system. The linear variational system at each fixed $\bar{W}$ and at $V = 0, U = f^{-1}(\bar{W})$ is hyperbolic. For small ϵ it has an exponential dichotomy on each regular layer.

$$\frac{d\Delta U_i}{d\omega} = \mathcal{F}^i_{1u}\Delta U_i + \mathcal{F}^i_{1v}\Delta V_i = -h_{iu} + h.o.t.$$
$$\frac{d\Delta V_i}{d\omega} = \mathcal{F}^i_{2u}\Delta U_i + \mathcal{F}^i_{2v}\Delta V_i = -h_{iv} + h.o.t.$$

Integral formuals can be used to write solutions $(\Delta U(\omega), \Delta V(\omega))$, then solve for $\Delta W(t)$.

To cancel out the jump errors, we adjust the interval (t_i, t_{i+1}):

$$\begin{aligned}
&A_i = z_i(\epsilon^{\beta-1}, w_i) - Z_i(t_i + \epsilon^{\beta}),\\
&B_i = Z_i(t_{i+1} - \epsilon^{\beta}) - z_{i+1}(-\epsilon^{\beta-1}, w_{i+1}),\\
&\Delta Z_i(t_i + \Delta t_i + \epsilon^{\beta}) - \Delta z_i(\epsilon^{\beta-1}) = A_i,\\
&\Delta z_{i+1}(-\epsilon^{\beta-1}) - \Delta Z_i(t_{i+1} + \Delta t_{i+1} - \epsilon^{\beta}) = B_i.
\end{aligned}$$

Linearization:

$$\begin{aligned}
&\Delta Z_i(t_i + \epsilon^{\beta}) + \Delta t_i F_i(t_i) - \Delta z_i(\epsilon^{\beta-1}) = A_i + h.o.t.\\
&\Delta z_{i+1}(-\epsilon^{\beta-1}) - \Delta Z_i(t_{i+1} - \epsilon^{\beta}) - \Delta t_{i+1} F_i(t_{i+1}) = B_i + h.o.t.
\end{aligned}$$

The vector field $F_i(t_i)$, $F_i(t_{i+1})$ has nonzero W components, can choose Δt_i to eliminate jumps in W, leaving only jumps in (U, V).

Jack Hale, "Ordinary Differential Equations", page 145:
Basic to any discussion of problems concerned with perturbed linear systems is a complete understanding of the nonhomogeneous linear system:

$$\dot{x} = A(t)x + f(t).$$

Drop the higher order terms in differential equations
and drop the Δt_i in jump conditions,
the linear system where $\mathbf{k} = (0, 1)^\tau$:

$$\begin{aligned}
&\Delta y_i' - L(\tau)\Delta y_i - \mathbf{k}\Delta w_i = -h_i(\tau, \epsilon), \quad -\epsilon^{\beta-1} \le \tau \le \epsilon^{\beta-1}, \\
&\Delta Y_i' - M(\omega)\Delta Y_i = -H_i(\omega, \epsilon), \quad t_i + \epsilon^\beta \le t \le t_{i+1} - \epsilon^\beta, \\
&\Delta Y_i(t_i + \epsilon^\beta) - \Delta y_i(\epsilon^{\beta-1}) = A_i, \\
&\Delta y_{i+1}(-\epsilon^{\beta-1}) - \Delta Y_i(t_{i+1} - \epsilon^\beta) = B_i.
\end{aligned}$$

(1) Study a linearized system around the heteroclinic solution. Related Fredholm properties and the Liapunov-Schmidt reduction. Generalized heteroclinnic solution.

(2) A version of shadowing lemma that can handle a sequence of differential equations with jump conditions at junction points $\{t_i\}_{i=-\infty}^{\infty}$.

Let $T(t,s)$ be the principle matrix solution for the ODE in $\mathbb{R}^n$.

$$\dot{x} - A(t)x = h(t), \quad t \in J = (a,b).$$

Definition: We say $T(t,s)$ has an exponential dichotomy in J if ther exists projections $P_s(t) + P_u(t) = I$ such that

$$T(t,s)P_s(s) = P_s(t)T(t,s), \quad t,s \in J,$$
$$|T(t,s)P_s(s)| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,$$
$$|T(t,s)P_u(s)| \leq Ke^{-\alpha(s-t)}, \quad s \geq t.$$

Lemma(Frozen coefficients)Assume that $\|A(t)\| \leq M, A(t)$ has d_- eigenvalues with negative real parts and $d_+ = n - d_-$ eigenvalues with positive real parts and $\|A'(t)\| \leq \delta$. If δ is sufficiently small, then $\dot{x} = A(t)x$ has an exponential dichotomy in J.

Recall the slow system:

$$\frac{du}{d\omega} = \frac{v}{\theta^{-1}(u - \gamma(\bar{w} + \epsilon\omega))}, \quad \frac{dv}{d\omega} = \frac{\theta v - f(u) + \bar{w} + \epsilon\omega}{\theta^{-1}(u - \gamma(\bar{w} + \epsilon\omega))}.$$

Proposition (1) The linearized slow system for $Y = (U, V)$:

$$\Delta Y_i' - M(\omega)\Delta Y_i = -H_i(\omega)$$

has an exponential dichotomy in $\tau \in (t_i/\epsilon, t_{i+1}/\epsilon)$, $\omega \in (w_i/\epsilon, w_{i+1}/\epsilon)$.
(2) The linearized fast system for $y = (u, v)$:

$$\Delta y_i' - L(\tau)\Delta y_i = \mathbf{k}\Delta w_i - h_i(\tau),$$

has exponential dichotomies in $(-\infty, 0]$ and $[0, \infty)$. The dimensions of unstable spaces are the same, but the projections

$$P_s(0-) \neq P_s(0+).$$

A solution in regular layers as a two point boundary value problem:
For any $\phi_s \in \Re P_s(a_i), \phi_u \in \Re P_u(b_i)$,

$$\Delta Y_i(\omega) = T(\omega, a_i)\phi_s + \int_{a_i}^{\omega} T(\omega, \eta)P_s(\eta)(-H_i(\eta))d\eta$$
$$+T(\omega, b_i)\phi_u + \int_{b_i}^{\omega} T(\omega, \eta)P_u(\eta)(-H_i(\eta))d\eta.$$

Here $a_i = w_i/\epsilon,\ b_i = w_{i+1}/\epsilon$.

$$|\Delta Y_i| \leq C(|\phi_s| + |\phi_u| + |H_i|).$$

Comment: C is independent of ϵ as $\epsilon \to 0$.
$(\phi_s,\ \phi_u)$ to be determined.

Solutions in singular layers is not so easy to write down.

Fredholm property, Ken. Palmer (JDE 1984):
(H) Assume $\dot{x} - A(t)x = 0$ has exponential dichotomies on $\mathbb{R}^{\pm}$. Then

$$\mathcal{F}x = \dot{x} - A(t)x$$

is Fredholm in the space of bounded solutions, with

$$\text{Index}\mathcal{F} = \dim\Re P_u(0-) - \dim\Re P_u(0+).$$

If $\dim\Re P_u(0+) = \dim\Re P_u(0-)$, and if there is a unique bounded solution $\phi(t)$ to the system. Then the adjoint system

$$\dot{y} + A(t)^* y = 0$$

has a unique bounded solution $\psi(t)$. For any bounded $f(t)$, the nonhomogeneous system

$$\dot{x} - A(t)x = f(t), \quad t \in \mathbb{R},$$

has a bounded solution iff

$$\int_{-\infty}^{\infty} < \psi(t), f(t) > dt = 0.$$

K. Palmer (1985): (H) is necessary for $\mathcal{F}$ to be Fredholm.

Generalization of Palmer's result, Lin 1989

$$\dot{x} - A(t)x = f(t), \quad a \le t \le b,\ a < 0 < b,$$
$$P_s(a)x(a) = \phi_s, \quad P_u(b)x(b) = \phi_u.$$

The system has a unique solution $x(t), x(0) \perp \phi(0)$ iff

$$\int_a^b < \psi(t), f(t) > dt + < \psi(a), \phi_s > - < \psi(b), \phi_u > = 0.$$

If the condition does not hold, then let the left hand side be G. There exists a unique solution $x(t), x(0) \perp \phi(0)$ such that

$$x(0+) - x(0-) = G\,\psi(0). \quad (\text{Assume } |\psi(0)| = 1).$$

The generalized solution has a jump in prescribed the direction.

$$|G| \le C(\|f\| + e^{-\alpha|a|}|\phi_s| + e^{-\alpha|b|}|\phi_u|).$$

Comment: (1) C is independent of the length of interval $b-a$.
(2) The jump is in the direction of $\psi(0) \perp \Re P_u(0-) + \Re P_s(0+)$.
(3) The jump size G_i depends weakly on the boundary data $(\phi_s,\ \phi_u)$ if $|a_i|, |b_i| >> 1$.

For $a \le t \le 0,\ x_u(0) \in \Re P_u(0-)$,

$$x(t) = T(t,a)\phi_s + \int_a^t T(t,s)P_s(s)f(s)ds + T(t,0)x_u(0) + \int_0^t T(t,s)P_u(s)f(s)ds.$$

For $0 \le t \le b,\ x_s(0) \in \Re P_s(0+)$,

$$x(t) = T(t,0)x_s(0) + \int_0^t T(t,s)P_s(s)f(s)ds + T(t,b)\phi_u + \int_0^t T(t,s)P_u(s)f(s)ds.$$

Since $\Re P_u(0-)$, $\Re P_s(0+), \psi(0)$ span the space $\mathbb{R}^n$, there exist unique $(x_u(0-),\ x_s(0+),\ G)$ such that

$$x_s(0+) - x_u(0-) + G\,\psi(0) = T(0,a)\phi_s + \int_a^0 T(0,s)P_s(s)f(s)ds - T(0,b)\phi_u + \int_0^b T(0,s)P_u(s)f(s)ds.$$

Observe that

$$< \psi(0), x_s(0+) >= 0, \ < \psi(0), x_u(0-) >= 0, \ |\psi(0)| = 1,$$
$$T^*(0,a)\psi(0) = \psi(a), \ T^*(0,b)\psi(0) = \psi(b), \ T^*(0,s)\psi(0) = \psi(s).$$

Apply $\psi(0)$ to the last equation, we have

$$G = \int_a^b < \psi(t), f(t) > dt + < \psi(a), \phi_s > - < \psi(b), \phi_u > .$$

Comment: (1) Let $a \to -\infty$, $b \to \infty$, we have

$$G = \int_{-\infty}^{\infty} < \psi(t), f(t) > dt.$$

(2) In a two dimensional system, the gap formulas reduce to the original Melnikov's integral.
(3) Solutions with a jump in the specified direction will be called pseudo-solutions.
(4) In applications f comes from differentiating with respect to parameters in linear variational equations.

Reference for the classical shadowing lemma:

J. Guckenhermer, J. Moser & S. Newhouse, dynamical systems, Birkhauser, 1980.

Generalization to system of differential eqautions:

Lin, Shadowing lemma and singularly perturbed boundary value problems, SIAM J. Appl. Math, 1989

Shadowing lemma:

$$\dot{x}_i(t) - A_i(t)x_i(t) = f_i(t), \quad t_{i-1} \le t \le t_i,$$
$$x_i(t_i) - x_{i+1}(t_i) = g_i.$$

Lemma (1)Assume that the system has exponential dichotomies in $[t_{i-1}, t_i]$ with the same constants and exponentials.
(2) $\Re P_u^i(t_i) \oplus \Re P_s^{i+1}(t_i) = \mathbb{R}^n$.
Let Q^i be the projection to the unstable space. $\|Q^i\| \le M$.
Then for each sequences of uniformly bounded continuous functions $\{f_i\}$, and sequence $\{g_i\}$, there exists a unique solution $\{x_i\}$ such that

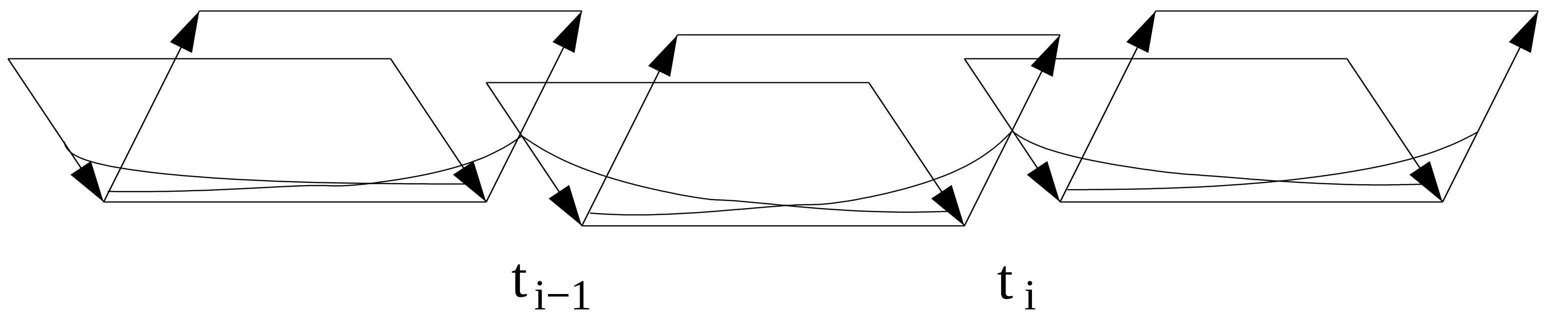
$$\|\{x_i\}\| \le C(\|\{f_i\}\| + \|\{g_i\}\|).$$

Proof (1) Each equation in $[t_{i-1}, t_i]$ has a solution that is uniformly bounded by $|f_i|$:

$$x_i(t) = \int_{t_{i-1}}^{t} T(t,s)P_s(s)f_i(s)ds + \int_{t_i}^{t} T(t,s)P_u(s)f_i(s)ds.$$

(2) Solve the system

$$\dot{x}_i(t) - A_i(t)x_i(t) = 0, \quad t_{i-1} \le t \le t_i,$$
$$x_i(t_i) - x_{i+1}(t_i) = \tilde{g}_i.$$



Let

$$x_i(t) = T(t, t_{i-1})P_s^i(t_{i-1})(Q^{i-1} - I)g_{i-1} + T(t, t_i)P_u^i(t_i)Q^i g_i.$$

With $x_i(t)$, the jump condition is not satisfied, but gets smaller. The proof is completed by the method of iterations. qed

Consider the linearized “correction problem”: Look for

$$\Delta Z_i,\ \Delta z_i,\ \Delta t_i.$$

$$\Delta z_i' - L(\tau)\Delta z_i - \mathbf{k}\Delta w_i = -h_i(\tau,\epsilon), \quad -\epsilon^{\beta-1} \le \tau \le \epsilon^{\beta-1},$$
$$\Delta Y_i' - M(\omega)\Delta Y_i = -H_i(\omega,\epsilon), \quad t_i + \epsilon^{\beta} \le t \le t_{i+1} - \epsilon^{\beta},$$
$$\Delta Y_i(t_i + \epsilon^{\beta}) - \Delta y_i(\epsilon^{\beta-1}) = A_i,$$
$$\Delta y_{i+1}(-\epsilon^{\beta-1}) - \Delta Y_i(t_{i+1} - \epsilon^{\beta}) = B_i.$$

Solve the nonhomogeneous system:

$$\Delta Y_i(\omega) = \int_{a_i}^{\omega} T(\omega,\eta)P_s(\eta)(-H_i(\eta))d\eta + \int_{b_i}^{\omega} T(\omega,\eta)P_u(\eta)(-H_i(\eta))d\eta.$$

Here $a_i = w_i/\epsilon,\ b_i = w_{i+1}/\epsilon$. Then find Δw_i by

$$\Delta w_i \int_{-\epsilon^{\beta-1}}^{\epsilon^{\beta-1}} < \psi(\tau), \mathbf{k}_i > d\tau = \int_{-\epsilon^{\beta-1}}^{\epsilon^{\beta-1}} < \psi(\tau), h_i(\tau,\epsilon) > d\tau,$$
$$\Delta t_i = \Delta w_i/\dot{w}(t_i).$$
$$\Delta y_i(\tau) = \int_{-\epsilon^{\beta-1}}^{\tau} T(\tau,\eta)P_s(\eta)(\mathbf{k}w_i - h_i(\eta))d\eta$$
$$+ \int_{\epsilon^{\beta-1}}^{\tau} T(\tau,\eta)P_u(\eta)(\mathbf{k}w_i - h_i(\eta))d\eta.$$

Error! Do positive and negative τ separately. Introduce $\Delta y(0\pm)$.

This causes new jump errors $(\tilde{A}, \tilde{B})$.

An iteration process to eliminate jump errors:

(LABEL I) Find $(\phi_s^i,\ \phi_u^i)$ as in the shadowing lemma. Find Δw_i,

$$\Delta w_i \int_{-\epsilon^{\beta-1}}^{\epsilon^{\beta-1}} < \psi(\tau), \mathbf{k}_i > d\tau = < \psi(-\epsilon^{\beta-1}), \phi_s^i > - < \psi(\epsilon^{\beta-1}), \phi_u^{i+1} > .$$

$$\begin{aligned} Y_i(\tau) &= T(\tau, a_i)\phi_s^i + T(\tau, b_i)\phi_u^{i+1}, \\ y_i(\tau) &= T(\tau, -\epsilon^{\beta-1})\phi_s^{i-1} + T(\tau, \epsilon^{\beta-1})\phi_u^i \\ &+ \int_{-\epsilon^{\beta-1}}^{\tau} T(\tau, \eta) P_s(\eta) \mathbf{k} \Delta w_i d\eta + \int_{\epsilon^{\beta-1}}^{\tau} T(\tau, \eta) P_u(\eta) \mathbf{k} \Delta w_i d\eta. \end{aligned}$$

GOTO I

Even if the iteration converges, the jump errors still exist due to Δt_i ($O(\Delta w_i)$), but will be smaller by an exponential factor:

$$\begin{aligned} &\Delta Z_i(t_i + \epsilon^\beta) + \Delta t_i F_i(t_i) - \Delta z_i(\epsilon^{\beta-1}) = A_i, \\ &\Delta z_{i+1}(-\epsilon^{\beta-1}) - \Delta Z_i(t_{i+1} - \epsilon^\beta) - \Delta t_{i+1} F_i(t_{i+1}) = B_i. \end{aligned}$$

Another layer of iteration will eliminate the new jump error.

PART II

A general $(m+n)$-dimensional system, $x \in \mathbb{R}^m, y \in \mathbb{R}^n$.
In regular layers:

$$\dot{x} = f(x, y, \epsilon),$$
$$\epsilon\dot{y} = g(x, y, \epsilon).$$

Singular layers:

$$x' = \epsilon f(x, y, \epsilon),$$
$$y' = g(x, y, \epsilon).$$

Let $\epsilon = 0$ in regular layers. It has d branches $y = \eta^i(x)$, where

$$|\Re\sigma\{g_y(x, y, 0)\}| \geq \alpha_0 > 0.$$

The slow manifolds are hyperbolic. Assume that the unstable indices are the same for all i.

Study the singular limit system.
An algebraic-differential system in regular layers:

$$\dot{X} = f(X, Y, 0),$$
$$0 = g(X, Y, 0).$$

Flow on the slow manifold $\mathcal{S}_i = \{Y = \eta_i(x)\}$:

$$\dot{X} = f(X, \eta^i(X), 0), \quad Y = \eta^i(X), \; t_i \le t \le t_{i+1}.$$

A heteroclinic connection problem in singular layer layers:

$$x' = 0, \; x \equiv \xi^i = \text{constant},$$
$$y' = g(\xi^i, y, 0).$$

Equilibrium points are on the slow manifolds $\mathcal{S}_i$.

$$y = y^i(\tau, \xi^i), \; -\infty < \tau < \infty$$
$$y^i(-\infty) \in \mathcal{S}_i, \; y^i(\infty) \in \mathcal{S}_{i+1},$$
$$y^i(0+) - y^i(0-) = G_i(\xi^i).$$

Look for a codimension-1 submanifold $M_i \subset \mathcal{S}_i$ such that if $\xi \in M_i$, then the heteroclinic exits (take-off lines).
Assume that $D_\xi G_i(\xi^i) \neq 0$.

An alternation of regular and singular orbits.

$$X(t),\ t_i \le t \le t_{i+1}, \quad X(t_i) = \xi_i,\ X(t_{i+1}) = \xi^{i+1},$$
$$y(\tau, \xi^i), \quad y^i(-\infty) \in \mathcal{S}_i,\ y^i(\infty) \in \mathcal{S}_{i+1}.$$

HETEROCLINIC BIFURCATION

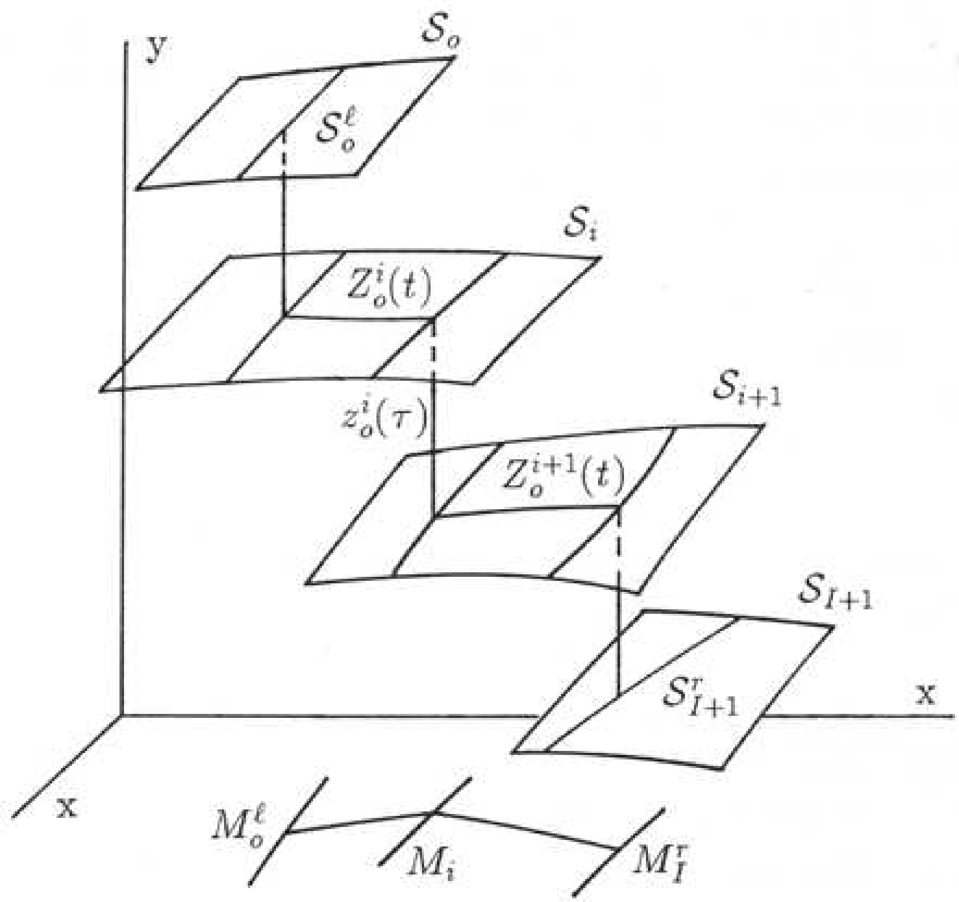


FIGURE 1

Assumptions:

(H1) $|\Re\sigma\{g_y(x,y,0)\}| \geq \alpha_0 > 0$. The dimensions of stable and unstable spaces $d^- + d^+ = n$ is independent of i.

(H2) The gap function $G_i(\xi)$ is nonsingular.

(H3) The flow on the slow manifolds are transverse to the talk-off lines.

(H4) If the slow manifolds are more than one dimensional (not needed in the previous example), then the induced piece-wise smooth periodic orbit on slow manifolds is non-degenerate:

$$M_0 \to M_1 \to \cdots \to M_I, \qquad \frac{dX}{dt} = f(X, \eta^i(X), 0).$$

(H4') The induced piecewise smooth periodic orbit on the slow manifolds is hyperbolic.

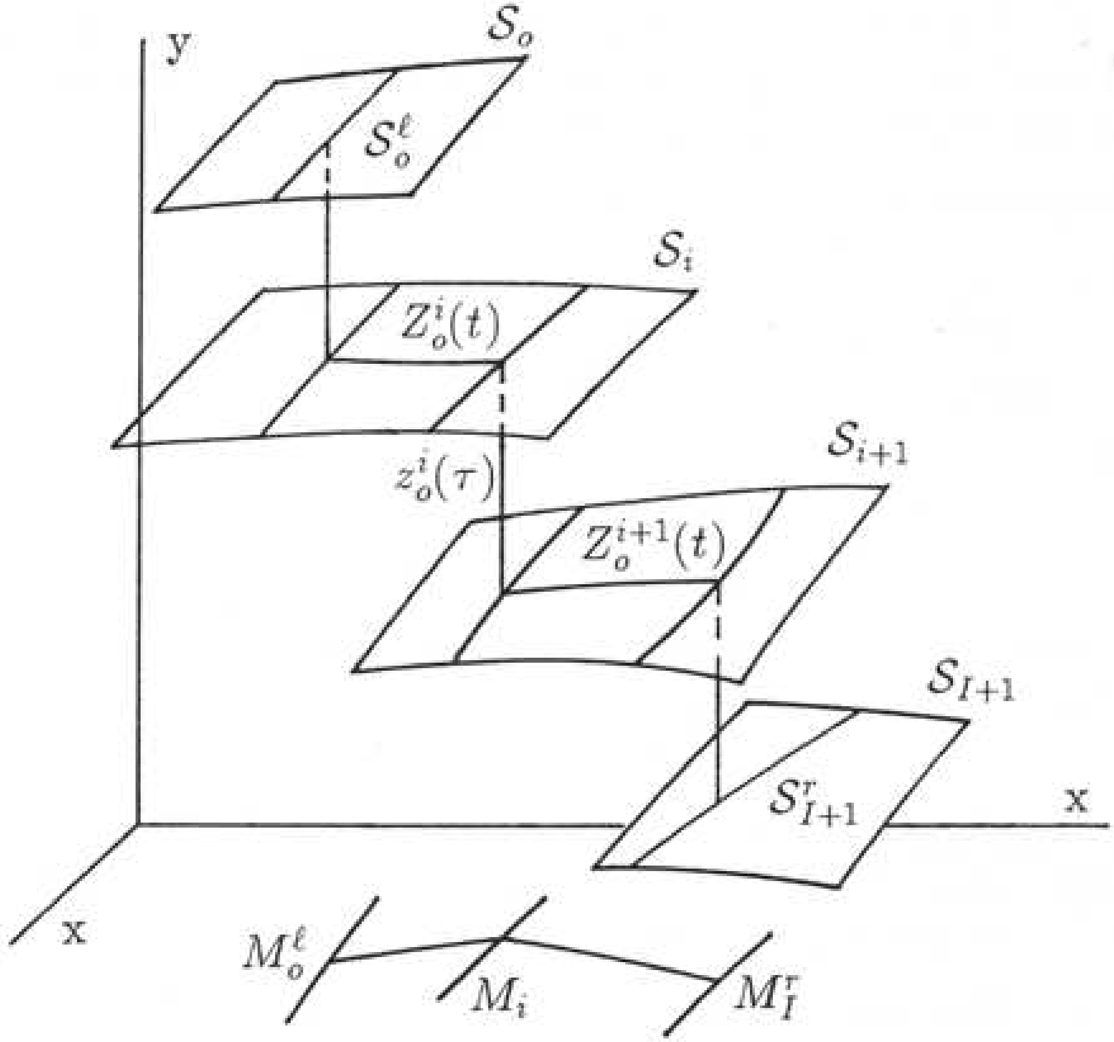

$\mathcal{S}_o$
$\mathcal{S}_o^\ell$
$\mathcal{S}_i$
$Z_o^i(t)$
$z_o^i(\tau)$
$\mathcal{S}_{i+1}$
$Z_o^{i+1}(t)$
$\mathcal{S}_{I+1}$
$\mathcal{S}_{I+1}^r$
y
x
x
M_o^ℓ
M_i
M_I^r

FIGURE 1

$\sim$ (H1) ensues that $g(x, y, 0)$ can be solved as $Y = \eta_i(X)$. The graph of which is a hyperbolic center manifold for the fast system. At $\epsilon = 0$ the center manifold consists of equilibrium points.
$\sim$ (H2): Consider $y' = g(\xi, y, 0)$ and a heteroclinic solution $y(\tau, \xi)$. When moving ξ, the heteroclinic usually breaks. A pseudo-heteroclinic solution exists with a jump in the prescribed direction of $\psi(0)$.

$$y'_\xi - g_y(\xi_i, y, 0) y_\xi = g_x(\xi_i, y, 0),$$
$$y_\xi(0+) - y_\xi(0-) = \nabla G(\xi_i) := \int_{-\infty}^{\infty} \psi_i^*(\tau) g_x(\xi_i, y(\tau), 0) d\tau.$$

$\sim$ (H3): $\nabla G_i(\xi_i) \cdot f(X(t_i), Y(t_i), 0) \neq 0$.
$\sim$ (H4): can be checked by checking that $\lambda = 1$ is not an eigenvalue of the composition of Poincare mappings between take-off surfaces (cross sections of the slow flow).

Exercise: Show that the conditions (H2) and (H3) are satisfied by the FitzHugh-Nagumo equation. Show that $\nabla G_i(\xi)$ is the same obtained by the Melnikov's integral (JDE 1990, page 379).

$$f(t,\epsilon) = \sum_0^\infty \epsilon^j f_j(t)$$

means $f(t,\epsilon) = \sum_0^m \epsilon^j f_j(t) + o(\epsilon^m)$.

Power series solutions in regular and singular regions:

$$(\sum_0^\infty \epsilon^j X_j^i(t), \sum_0^\infty \epsilon^j Y_j^i(t)),$$

$$(\sum_0^\infty \epsilon^j x_j^i(\tau), \sum_0^\infty \epsilon^j y_j^i(\tau)).$$

Taylor expansion:

$$f(\sum_0^\infty \epsilon^j X_j^i(t), \sum_0^\infty \epsilon^j Y_j^i(t), \epsilon)$$

$$= f(X_0, Y_0, 0) + \sum_1^\infty \epsilon^j \{f_x(X_0, Y_0, 0)X_j + f_y(X_0, Y_0, 0)Y_j$$

$$+ F_j(X_1, Y_1, X_2, Y_2, \ldots, X_{j-1}, Y_{j-1})\}.$$

In regular layers:

$$\begin{aligned}
\dot{X}_0^i(t) &= F(X_0^i(t), Y_0^i(t), 0), \\
0 &= g(X_0^i(t), Y_0^i(t), 0), \\
\dot{X}_j^i(t) &= f_x(X_0^i(t), Y_0^i(t), 0)X_j^i(t) + f_y(X_0^i(t), Y_0^i(t), 0)Y_j^i(t) \\
&\quad + F_j(X_1^i, Y_1^i, \ldots, X_{j-1}^i, Y_{j-1}^i), \\
\dot{Y}_{j-1}^i(t) &= g_x(X_0^i(t), Y_0^i(t), 0)X_j^i(t) + g_y(X_0^i(t), Y_0^i(t), 0)Y_j^i(t) \\
&\quad + G_j(X_1^i, Y_1^i, \ldots, X_{j-1}^i, Y_{j-1}^i).
\end{aligned}$$

The solution stays near the ith slow manifold for

$$a_i + \sum_1^\infty \epsilon^j \tau_j^i(a) \le t \le b_i + \sum_1^\infty \epsilon^j \tau_j^i(b).$$

To the lowest order, $(X_0^i(t), Y_0^i(t))$ is the singular orbits that forms part of the loop.
Method of induction: After obtaining $(X_k^i(t), Y_k^i(t)), k = 0, \ldots, j-1$, solve an algebraic-differential equation:

$$\begin{aligned}
Y_j^i(t) &= -g_y^{-1}(t)g_x(t)X_j^i(t) + \ell.o.t. \\
\dot{X}_j^i(t) &= (f_x(t) - f_y(t)g_y(t)^{-1}g_x(t))X_j^i(t) + \ell.o.t.
\end{aligned}$$

In singular layers:

$$
\begin{aligned}
x'(\tau) &= \epsilon f(x(\tau), y(\tau), \epsilon), \\
y'(\tau) &= g(x(\tau), y(\tau), \epsilon). \\
x_0^i(\tau)' &= 0, \\
y_0^i(\tau)' &= g(x_0^i(\tau), y_0^i(\tau), 0). \\
x_1^i(\tau)' &= f(x_0^i(\tau), y_0^i(\tau), 0), \\
y_1^i(\tau)' &= g_x(\tau)x_1^i(\tau) + g_y(\tau)y_1^i(\tau) + g_\epsilon(\tau). \\
x_j^i(\tau)' &= f_x(\tau)x_{j-1}^i(\tau) + f_y(\tau)y_{j-1}^i(\tau) \\
&\quad + F_{j-1}(x_1^i, y_1^i, \ldots, x_{j-2}^i, y_{j-2}^i), \\
y_j^i(\tau)' &= g_x(\tau)x_j^i(\tau) + g_y(\tau)y_j^i(\tau) \\
&\quad + G_j(x_1^i, y_1^i, \ldots, x_{j-1}^i, y_{j-1}^i).
\end{aligned}
$$

To the lowest order, (x_0^i, y_0^i) is satisfied by the heteroclinic solution

$$
x_0^i(\tau) \equiv x_0^i, \qquad \lim_{\tau\to\pm\infty} y_0^i(\tau) = G^i(x_0^i) \text{ or } G^{i+1}(x_0^i).
$$

Define $E(\gamma, m) := \{x(\cdot) | \sup(|x(t)|e^{\gamma|t|}(1 + |t|^m)^{-1}) < \infty\}$.
We require that

$$y^i_j \in E(0, j), \quad y^i_j(0) \perp y^i_j(0)'.$$

Assume $x^i_0, y^i_0, \ldots, x^i_{j-1}, y^i_{j-1}$ has been obtained, then

$$x^i_j(\tau) = x^i_0(0) + (\text{ a function in } E(0, j)),$$
$$y^i_j(\tau)' = g^i_y(\tau)y^i_j(\tau) + g^i_x(\tau)x^i_j(0) + (\text{ a function in } E(0, j)).$$

Fredholm's property to polynomial growth functions:
To have a solution $y^i_j \in E(0, j)$, need to choose $x^i_j(0)$ such that

$$\int_{-\infty}^{\infty} \psi^*_i(\tau)\{g^i_x(\tau)x^i_j(0) + (\text{ a function in } E(0, j))\}d\tau = 0.$$

The undetermined $x^i_j(0)$ lies on a codimension one surface.

$$\nabla G_i \cdot x^i_j(0) = d^i_j.$$

Matching conditions between the regular and singular layers must be satisfied. It is like the change of coordinates between two overlapping local coordinate charts representing the same differential manifolds.

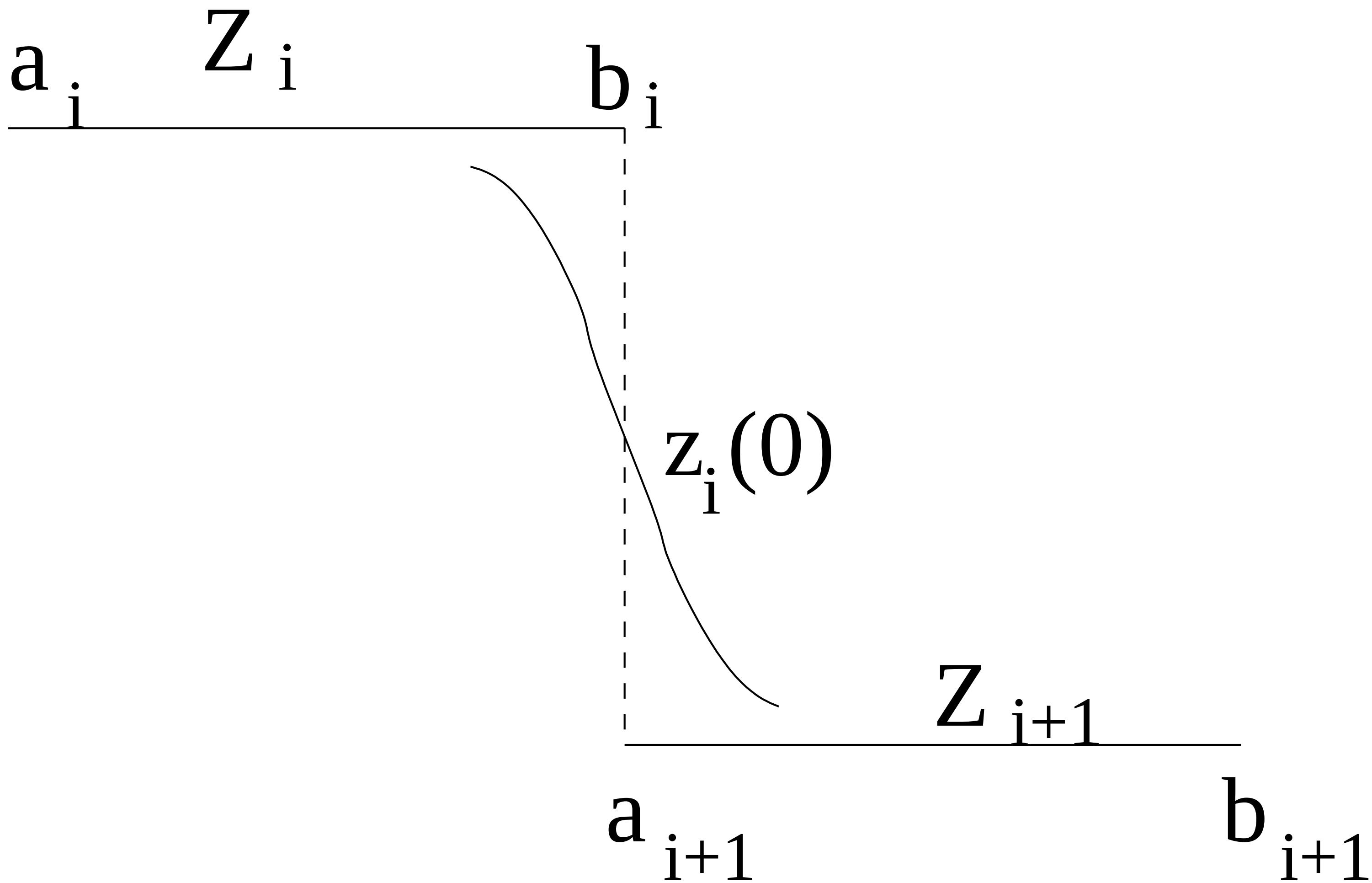
Matching condition of Van Dyke, see Eckhaus 1977, Lecture notes in Mathematics, V594.

Inner expansion of outer layers: $\tau = 0$ in the ith singular layer corresponds to $t = b_i + \sum_1^\infty \epsilon^k \tau_k^i(b)$, and $t = a_{i+1} + \sum_1^\infty \epsilon^k \tau_k^{i+1}(a)$.

$$\sum_{j=0}^{\infty} Z_j^i(b_i + \sum_{k=1}^{\infty} \epsilon^k \tau_k^i(b) + \epsilon\tau) = \sum_{j=0}^{\infty} \epsilon^j z_j(\tau, b, i),$$
$$\sum_{j=0}^{\infty} Z_j^i(a_i + \sum_{k=1}^{\infty} \epsilon^k \tau_k^i(a) + \epsilon\tau) = \sum_{j=0}^{\infty} \epsilon^j z_j(\tau, a, i).$$

The matching condition:

$$z_j^i(\tau) - z_j(\tau, b, i) \in E_{R^-}(\gamma, j),$$
$$z_j^i(\tau) - z_j(\tau, a, i+1) \in E_{R^+}(\gamma, j).$$


a i
Z i
b i
z i(0)
Z i+1
a i+1
b i+1

Let $Z^i = (X^i, Y^i)$, $z^i = (x^i, y^i)$.
The matching of $y^i_j(\tau), \tau \to \pm\infty$ to the outer layers can be proved based on the growth conditions of $y^i_j(\tau)$.
The matching of $x^i_j(\tau)$ to X^i_j is complicated.

$$\begin{aligned}
x_j(0, b, i) &= X^i_j(b_i) + \dot{X}^i_0(b_i)\tau^i_j(b) + \dots \\
&= X^i_j(b_i) + f(X^i_0(b_i), Y^i_)(b_i), 0)\tau^i_j(b) + \dots
\end{aligned}$$

$$x^i_j(\tau) - x_j(\tau, b, i) = x^i_j(0) - x_j(0, b, i)$$

$$+ \int_0^\tau (\text{ a funciton of } \sigma \text{ which is in } E_{R^-}(\gamma, j-1))d\sigma.$$

$$x^i_j(0) - X^i_j(b_i) - f(X^i_0(b_i), Y^i_0(b_i), 0)\tau^i_j(b) = C_j(b, i),$$

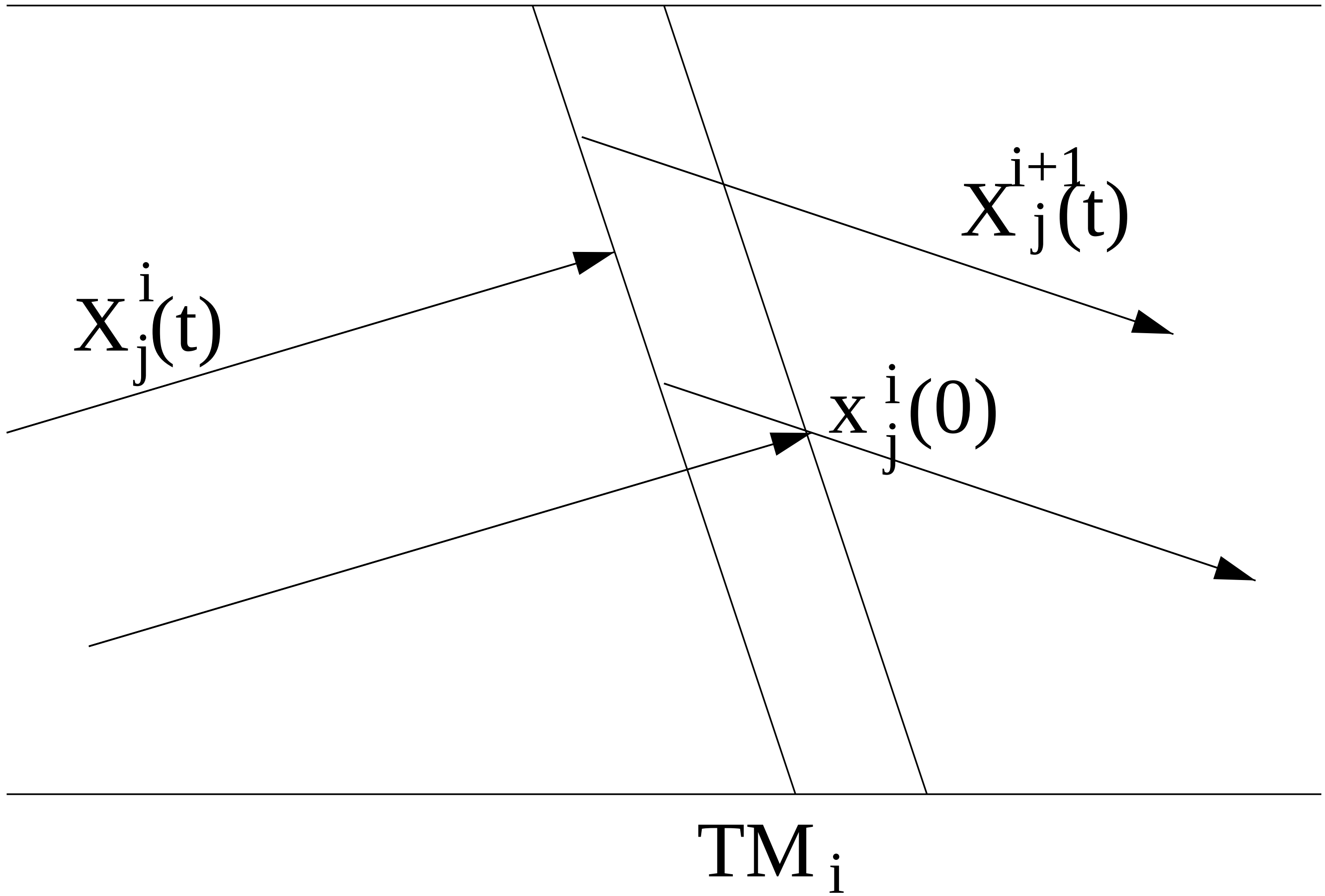
$$x^i_j(0) - X^{i+1}_j(a_{i+1}) - f(X^{i+1}_0(a_{i+1}, Y^{i+1}_0(a_{i+1}), 0)\tau^{i+1}_j(a)$$

$$= C_j(a, i+1).$$

Let $P(a, i)$, $P(b, i)$ be the projection in $\mathbb{R}^m$ with the range TM_{i-1}, TM_i and kernel $f(X^i_0(a_i), Y^i_0(a_i), 0)$, $f(X^i_0(b_i), Y^i_0(b_i), 0)$.

$$P(a, i+1)(x^i_j(0) - X^{i+1}_j(a_{i+1})) = P(a, i+1)C_j(a, i+1),$$

$$P(b, i)(x^i_j(0) - X^i_j(b_i)) = P(b, i)C_j(b, i).$$


$X_j^{i+1}(t)$
$X_j^i(t)$
$x_j^i(0)$
TM_i

One can compute $P(a,i+1)x_j^i(0)-P(b,i)x_j^i(0)$ from $\Delta_i \cdot x_j^i(0) = d_j^i$, hence

$$P(b,i)X_j^i(b_i) - P(a,i)X_j^{i+1}(a_{i+1}) = C_{ij}.$$

Let $\Pi^i : TM_{i-1} \to TM_i$ be the Poincare mapping induced by

$$\dot{X}_j^i(t) = (f_x^i(t) - f_y^i(t)g_y^i(t)^{-1}g_x^i(t))X_j^i(t) + \ell.o.t.$$

Then $\Pi_i : P(a,i-1)X_j^i(a_i) \to P(b,i)X_j^i(b_i)$.
Assume $\lambda = 1$ is not an eigenvalue of $\Pi_N \circ \Pi_{N-1} \circ \cdots \circ \Pi_2 \circ \Pi_1$.

$$X = \Pi_N(\Pi_{N-1}(\cdots(\Pi_2(\Pi_1 X + C_{1j}) + C_{2j}) + \cdots)C_{N-1,j}) + C_{Nj}$$

has a fixed point $P(a,1)X_j^i(a_1)$.
$\sim$ If we specify some phase conditions, $X_j^i(t)$ and $\tau_j^i(a), \tau_j^i(b)$ can be determined.