GENERALIZED RANKINE-HUGONIOT CONDITION AND SHOCK SOLUTIONS FOR QUASILINEAR HYPERBOLIC SYSTEMS

XIAO-BIAO LIN

ABSTRACT. For a quasilinear hyperbolic system, we use the method of vanishing viscosity to construct shock solutions. The solution consists of two regular regions separated by a free boundary (shock). We use Melnikov's integral to obtain a system of differential/algebraic equations that governs the motion of the shock. For Lax shocks in conservation laws, these equations are equivalent to the Rankine-Hugoniot condition. For under compressive shocks in conservation laws, or shocks in non-conservation systems, the Melnikov type integral obtained in this paper generalizes the Rankine-Hugoniot condition. Under some generic conditions, we show that the initial value problem of shock solutions can be solved as a free boundary problem by the method of characteristics.

1. INTRODUCTION

The initial value problem of a quasilinear hyperbolic system

$$u_t + A(x,t,u)u_x + b(x,t,u) = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

appears in many areas of theoretical and applied sciences–control theory, game theory, variational calculus, fluid mechanics, nonlinear elasticity and the conservation law [2, 5, 6, 13, 14, 17].

The method of characteristics can be used to solved (1.1), (1.2) for a short time. The example of Burger's equation with a smooth initial data $u_0(x) = -\arctan(x)$ shows that the characteristics $dx/dt = \lambda = u$ may intersect with each other after a finite time $t = \tilde{t}$, creating a cusp region where three branches of solutions, $u^L(x,t) > u^C(x,t) > u^R(x,t)$ corresponding to three characteristics $\lambda^L > \lambda^C > \lambda^R$, coexist. Therefore, the classical solution exits only for $t < \tilde{t}$. See Figures 1.1 and 1.2.

To continue the solution to $t > \tilde{t}$, we will allow the solution to be discontinuous (shock). If we can determine the trajectory of a shock $x(t)$, then in the cusp region, the shock solution is defined by

$$u(x,t) = \begin{cases} u^L(x,t), & \text{if } x < x(t), \\ u^R(x,t), & \text{if } x > x(t). \end{cases}$$

For systems in the conservation form,

$$u_t + f(u)_x = 0, \tag{1.3}$$

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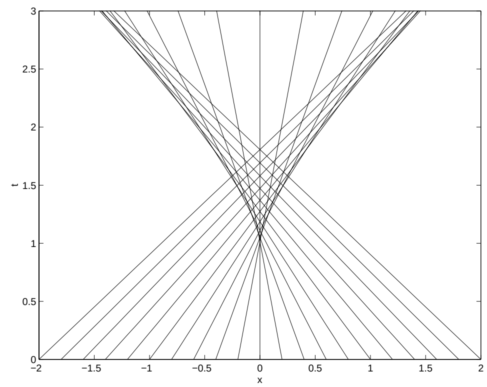

FIGURE 1.1. The intersection of characteristics in the cusp region.

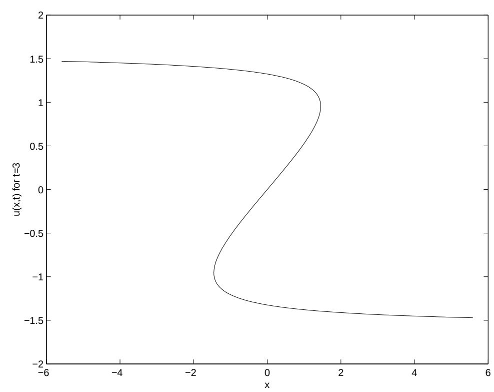

FIGURE 1.2. There are three branches of solutions for $t > \tilde{t}$.

shock solutions can be defined as weak solutions in the sense of distribution [16]. It can be shown that the shock $x(t)$ must satisfy the Rankine-Hugoniot condition (RH),

$$s[u] = [f(u)].$$

Here $s = dx/dt$ is the shock speed, $[u] = u(x(t)_+, t) - u(x(t)_-, t)$ and $[f(u)] = f(u(x(t)_+, t)) - f(u(x(t)_-, t))$ stand for the jumps of u and $f(u)$ across the shock. The (RH) is very useful for if the equation is scalar, then the shock $x(t)$ in the cusp region is determined by an ordinary differential equation derived from the (RH),

$$dx(t)/dt = \frac{f(u^R(x(t), t)) - f(u^L(x(t), t))}{u^R(x(t), t) - u^L(x(t), t)}.$$

However, for systems of conservation laws, the (RH) consists of a system of conditions. We must clarify how the initial value problem of shock solutions is determined by these conditions.

For systems in non-conservation form, it is not clear how to define shock solutions as weak solutions. The difficulty comes from defining the product of a Heaviside function with a delta function in the sense of distribution. See [14] for discussions and references on this. Even if the weak solution can be defined, from the study of

Riemann problems, it is known that there may be too many. Additional criterion must be used to single out the physically relevant weak solutions.

Due to the problem in the weak solution approach, we adopt the vanishing viscosity method to define shock solutions. A shock solution of (1.1) is a discontinuous limit of the perturbed equation

$$u_t + A(x,t,u)u_x + b(x,t,u) = \epsilon u_{xx}, \quad \epsilon \to 0+. \tag{1.4}$$

In other words, a shock solution has a viscous profile. In the outer region (not at the shock), the convergence is uniform, so the shock solution satisfies (1.1), which is from (1.4) by setting $\epsilon = 0$. In the inner region (at the shock), if we use the stretched variables

$$\xi = (x - x_0)/\epsilon, \quad \tau = (t - t_0)/\epsilon$$

near a point (x_0, t_0) on the shock, then after substituting into (1.4) and letting $\epsilon = 0$, we have the reduced system in the inner region

$$u_\tau + A(x_0, t_0, u)u_\xi = u_{\xi\xi}. \tag{1.5}$$

The basic assumption for the vanishing viscosity method is that for every (x_0, t_0) on the shock, (1.5) has a traveling wave solution $\tilde{u}$ (viscous profile) that connects $u^L(x_0, t_0)$ to $u^R(x_0, t_0)$. Assume that $s_0 = \frac{d}{dt}x(t_0)$, $\zeta = \xi - s_0\tau$ and $q(\zeta) = \tilde{u}(\xi, \tau)$. With (x_0, t_0) being parameters and $' = d/d\zeta$, q satisfies

$$-s_0 q' + A(x_0, t_0, q)q' = q''. \tag{1.6}$$

The small viscous term naturally occurs in fluid mechanics and other mechanical process. In control and game theories, the viscosity often comes from a small stochastic perturbation of the definite process [2]. In these cases, the vanishing viscosity method is extremely satisfactory. Moreover, it is known that for a system in conservation law, the shock solution determined by the vanishing viscosity method also satisfies the (RH) condition. For systems with no real dissipation mechanism, we treat the viscous term as an artificial regularization term.

For simplicity, we assume the viscous term has the form ϵu_{xx}. The method works if the viscous term has the form $\epsilon D u_{xx}$ where D is a positive definite matrix. Notice that the evolution of the shock will depend on the choice of D, unless the system is a conservation law with a Lax shock.

Authors like P. Le Floch [5], L Sainsaulieu [14] and S. Schecter [15] have studied the existence of shock solutions for the Riemann problem of non conservative systems, but they did not consider initial value problems with general initial data. The purpose of this paper is to study the evolution of a shock solution from a given initial data $u_0(x)$ at $t_0 = 0$, which has a jump at $x = x_0$ but is not a step function. We assume that the two sided limits $u_0(x_0-)$ and $u_0(x_0+)$ can be connected by a traveling wave solution like (1.6). The case where the initial data does not admit a traveling wave solution is not considered, since the solution of the initial value problem may have more complicated structure, as suggested by the Riemann problem of conservation laws. The formation of shock from a smooth initial data, as in Figures 1.1 and 1.2, will not be considered here either.

Under some generic conditions, we show that for a non-conservation quasilinear hyperbolic system, the shock solution uniquely exists for a short time $0 \le t \le \delta$. Moreover, there may exist weak discontinuity along characteristic curves issuing from (x_0, t_0). Our method can be used to further continue the shock starting at $t = \delta$, until the generic conditions are no longer satisfied. But a compatibility condition is satisfied by the initial data at $t = \delta$, so there is no more weak discontinuity issuing form the shock.

A diffusively perturbed conservation system can be integrated once to reduce the order, therefore it is not generic among quasilinear systems. If we pose generic conditions on the reduced system and assume that the shock is not over compressive, we can show that the same result hold for systems of conservation laws. Note the last condition is not required for non-conservation systems. See examples in §7.

Rigorously speaking, we still need to prove that there is a solution of (1.4) for small and positive ϵ near our shock solution. By doing so we would have proved that the shock solution is indeed a discontinuous limit of solutions of (1.4). Recent advance in singular perturbation theory has provided several geometric and analytic tools to this end, one of them is the "spatial shadowing lemma" as in [8, 9]. The idea is that by truncation, the formal matched solution provides piecewise excellent approximations to a real solution of (1.4) in inner and outer regions if ϵ is small. But between the inner and two outer regions, the approximations are not matched exactly due to the truncation. Small correction terms must be found to make the gap disappear. The idea is similar to the shadowing lemma in the dynamical systems theory, only the jump is along the spatial direction rather than the time direction. The system considered in [8] is general enough to include system (1.4).

In §2, we state assumptions and main results of the paper. In §3, we state definitions and lemmas related to exponential dichotomies and trichotomies. §4 is devoted to deriving a generalized Rankine-Hugoniot condition (GRH) that ensures the existence of shock profile for (x, t) near (x_0, t_0). The (GRH) is actually a system of bifurcation equations for the existence of a heteroclinic connection of two non hyperbolic equilibria. To derive the (GRH), we generalize the Melnikov method to the case where the equilibrium points possess large dimensional center manifolds. The number of bifurcation equations depends on the number of bounded solutions of an adjoint system, which is studied in §5. In §6, we prove our main result – the existence of shock solutions under generic conditions. Here we use the results from Li and Yu [10]. Very general results on boundary and free boundary problems of quasilinear hyperbolic systems have been obtained in [10]. The result we used is in Chapter 4 of [10], called "the free boundary problems in functional forms in a fan-shaped domain". Some short examples are presented in §7.

This paper is dedicated to Professor Jack K. Hale on the occasion of his 70th birthday. During the writing of this paper, I have benefited from many discussions with A. Melikyan, M. Shearer, W. Fleming, C. Dafermos and K. Zumbrun. I am especially grateful to S. Schecter who has pointed out some subtle error in the first draft of this paper.

2. Assumptions and main results

Suppose at $t_0 = 0$, a piecewise C^1 function $u_0(x)$ is given with a jump at x_0. Let $u^L = u_0(x_0-)$ and $u^R = u_0(x_0+)$. We say that u_0 has a shock profile if there exists a traveling wave solution q of (1.6) connecting u^L to u^R.

We look for the shock solution $u(x,t)$ of (1.1) with the shock trajectory $x = x(t)$ for $t > t_0$. That is, for each $t > t_0$, $u(t,x)$ has a shock profile with the shock position at $x = x(t)$.

We assume that the system is hyperbolic at (x_0, t_0) and the wave speed satisfies "entropy conditions":

H1. For $u = u^L(x_0,t_0)$ or $u^R(x_0,t_0)$, $A(x_0,t_0,u)$ is strictly hyperbolic. The eigenvalues, $\{\lambda_i^L\}$ and $\{\lambda_i^R\}$ for $u = u^L$ and u^R, and the shock speed s_0 satisfy

$$\lambda_1^L < \cdots < \lambda_k^L < s_0 < \lambda_{k+1}^L < \cdots < \lambda_n^L,$$
$$\lambda_1^R < \cdots < \lambda_j^R < s_0 < \lambda_{j+1}^R < \cdots < \lambda_n^R,$$

with $0 \le k < n$ and $0 < j \le n$.

Note that we allow $k = 0$ and/or $j = n$, that is $s_0 < \lambda_1^L$ and/or $\lambda_n^R < s_0$. However, we require that $\lambda_n^L > s_0 > \lambda_1^R$.

Write (1.6) as a first order system

$$\begin{aligned} u' &= v, \\ v' &= (A(x_0,t_0,u) - s_0)v. \end{aligned} \tag{2.1}$$

For any $u \in \mathbb{R}^n$, $(u,0)$ is an equilibrium for (2.1). The Jacobian matrix at $(u,0)$ is

$$J(u,0) = \begin{pmatrix} 0 & I \\ 0 & A(x_0,t_0,u) - s_0 \end{pmatrix}.$$

From (H1), the matrix $A(x_0,t_0,u)$ has $n-k$ unstable and k stable eigenvalues if $u = u^L$. It has $n-j$ unstable and j stable eigenvalues if $u = u^R$. Therefore, $J(u^L,0)$ has n zero, $n-k$ unstable and k stable eigenvalues while $J(u^R,0)$ has n zero, $n-j$ unstable and j stable eigenvalues.

There exist stable, unstable, center stable, center unstable, center manifolds for each of the equilibrium $(u^L,0)$ and $(u^R,0)$. Their dimensions are

$$\begin{aligned}
&\dim W^u(u^L,0) = n-k, && \dim W^s(u^L,0) = k, && \dim W^c(u^L,0) = n, \\
&\dim W^{cu}(u^L,0) = 2n-k, && \dim W^{cs}(u^R,0) = n+k, && \\
&\dim W^u(u^R,0) = n-j, && \dim W^s(u^R,0) = j, && \dim W^c(u^R,0) = n, \\
&\dim W^{cu}(u^R,0) = 2n-j, && \dim W^{cs}(u^R,0) = n+j. &&
\end{aligned}$$

The transverse intersection of two manifolds E and F shall be dented $E \pitchfork F$. Our next assumption depends on whether the system is in the conservation form or not.

First, we consider a quasilinear system in non-conservation form. Let $M_1 = W^u(u^L,0)$ or $M_1 = W^{cu}(u^L,0)$ and let $M_2 = W^s(u^R,0)$ or $M_2 = W^{cs}(u^R,0)$. In all the cases, we have $(q(t),q'(t)) \in M_1 \cap M_2$. We assume that

H2. (non-conservation law). If $\dim M_1 + \dim M_2 \le 2n+1$, then

$$M_1 \cap M_2 = \mathrm{spann}\{(q(t),q'(t))\};$$

while if $\dim M_1 + \dim M_2 \geq 2n+1$, then

$$M_1 \pitchfork M_2 \quad \text{in } \mathbb{R}^{2n}.$$

Next, we consider a system of generalized conservation law (balancing law)

$$u_t + \frac{d}{dx} f(x,t,u) + g(x,t,u) = 0, \tag{2.2}$$

where $\frac{d}{dx} f(x,t,u) = f_x(x,t,u) + f_u(x,t,u)u_x$. The viscous profile satisfies

$$-su' + f_u(x,t,u)u' = u''. \tag{2.3}$$

They can be written as (1.1) and (2.1) with $A(x,t,u) = f_u(x,t,u)$. However, the conservation law is not a generic quasilinear system. We can integrate (2.3) and obtain a first order system

$$u' = -su + f(x,t,u) + w, \tag{2.4}$$

where $w \in \mathbb{R}^n$ is a constant vector. If $w = su^L - f(x,t,u^L)$ and q is a shock profile connecting u^L and u^R, then the (RH) is satisfied and u^L and u^R are equilibria of (2.4). From (H1), u^L and u^R are hyperbolic equilibria. The homogeneous part of the linear variational equation of (2.4) is

$$\phi' = (A(x,t,u) - s)\phi. \tag{2.5}$$

Let $M_1 = W^u(u^L)$ and $M_2 = W^s(u^R)$. We impose the following generic conditions on (2.4):
H2'. (conservation law).
(1) If $\dim M_1 + \dim M_2 \leq n+1$, then $M_1 \cap M_2 = \text{spann}\{q(t)\}$;
while if $\dim M_1 + \dim M_2 \geq n+1$, then

$$M_1 \pitchfork M_2 \quad \text{in } \mathbb{R}^{n}.$$

(2) If η is a nonzero bounded solution of the adjoint equation of (2.5)

$$\eta' + (A(x,t,u) - s)^* \eta = 0, \quad A(x,t,u) = f_u(x,t,u), \tag{2.6}$$

then

$$\int_{-\infty}^{\infty} \eta(\zeta) d\zeta \neq 0.$$

Note that a bounded solution of (2.6) satisfies $|\eta(\zeta)| \leq Ce^{-\lambda\zeta}$ for some $\lambda > 0$. Thus, the integral converges. If $\eta_i, i = 1, \ldots, \ell$ is a basis for the linear space of bounded solutions of (2.6), then from (H2'), $\psi_i := \int_{-\infty}^{\infty} \eta_i(\zeta) d\zeta, i = 1, \ldots, \ell$ are linearly independent.

For a conservation law, we also assume that the shock is not over compressive:
H3. $k+1 \geq j$ for a conservation system.

We need another generic type condition (H4) that certain matrix is nonsingular. The statement of (H4) will be left to §6 where some technical terminologies have been defined.

We now state the main result of this paper:

Theorem 2.1. *For a generic non-conservation system, assume that (H1), (H2) and (H4) are satisfied, and for a generic conservation system, assume that (H1), (H2'), (H3) and (H4) are satisfied. If the initial data $u_0(x)$ admits a shock profile at $x = x_0$, then the shock solution uniquely exists on a domain $\{(x,t) : a(t) \le x \le b(t), 0 \le t \le \delta\}$. Here $a(0) < x_0 < b(0)$ and $\delta > 0$. The shock solution in that domain is completely determined by the restriction of u_0 on $[a(0), b(0)]$.*

Moreover, if $k > 0$, issuing from (x_0, t_0), there are k characteristics entering the left of the shock; and if $j < n$, there are $n - j$ characteristics entering the right of the shock. The solution may have weak discontinuities along these characteristics.

3. Basic definitions and lemmas

Let $T(t,s)$ be the principal matrix solution to the linear system of the ODE

$$x' = A(t)x, \quad x \in \mathbb{R}^{2n}, \, t \in I. \tag{3.1}$$

(3.1) is said to have an exponential trichotomy on the interval I if there exists positive constants $K, 0 \le \gamma < \alpha$ and projections $P_s(t), P_u(t), P_c(t)$, $P_s(t) + P_u(t) + P_c(t) = id$ such that for $t, s \in I$,

$$\begin{aligned}
&T(t,s)P_\nu(s) = P_\nu(t)T(t,s), \quad \nu = u, s, c, \\
&|T(t,s)P_s(s)| \le Ke^{-\alpha(t-s)}, \quad s \le t, \\
&|T(s,t)P_u(t)| \le Ke^{-\alpha(t-s)}, \quad s \le t, \\
&|T(t,s)P_c(s)| \le Ke^{\gamma|t-s|}, \quad \text{any } t, s \in I.
\end{aligned} \tag{3.2}$$

(3.1) is said to have an exponential dichotomy if $P_c(t) = 0$.

Let $T^*(t,s) = -[T(s,t)]^*$ be the principal matrix solution to the adjoint equation of (3.1):

$$\psi' + A^*(t)\psi = 0, \quad t \in I. \tag{3.3}$$

If (3.1) has an exponential trichotomy on I, then (3.3) has an exponential trichotomy on I with the projections $P^*_\nu(t) = (P_\nu(t))^*$, $\nu = s, u, c$ and the same constants K, α, γ. The adjoint system should be solved backward in time when speaking of the stable or unstable subspaces. the following can be derived by taking adjoint of (3.2).

$$\begin{aligned}
&T^*(t,s)P^*_\nu(s) = P^*_\nu(t)T^*(t,s), \quad \nu = u, s, c, \\
&|T^*(t,s)P^*_s(s)| \le Ke^{-\alpha(t-s)}, \quad s \ge t, \\
&|T^*(s,t)P^*_u(t)| \le Ke^{-\alpha(t-s)}, \quad s \ge t, \\
&|T^*(t,s)P^*_c(s)| \le Ke^{\gamma|t-s|}, \quad \text{any } t, s \in I.
\end{aligned} \tag{3.4}$$

Suppose now (3.1) has exponential trichotomies on $I = \mathbb{R}^-$ and $\mathbb{R}^+$ respectively. Let the dimensions of the the nine possible subspaces defined by the intersections $\mathcal{R}P_\mu(0-) \cap \mathcal{R}P_\nu(0+)$ where $\mu, \nu = s, u, c$ be given in Table 3.1

dim	$\mathcal{R}P_s(0+)$	$\mathcal{R}P_c(0+)$	$\mathcal{R}P_u(0+)$
$\mathcal{R}P_u(0-)$	a	b	c
$\mathcal{R}P_c(0-)$	d	e	f
$\mathcal{R}P_s(0-)$	g	h	i

TABLE 3.1. Dimensions of the intersections of invariant subspaces for compatible trichotomies

We say that a pair of two trichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ are compatible if there exists a split of the space $\mathbb{R}^{2n}$ by the direct sum:

$$\sum_{\mu,\nu=u,c,s} \mathcal{R}P_\mu(0-) \cap \mathcal{R}P_\nu(0+) = \mathbb{R}^{2n}.$$

We can show that the definition is equivalent to having nine projections:

$$\sum_{\mu,\nu=u,c,s} P_{\mu,\nu} = id, \text{ with } P_{\mu,\nu} = P_\mu(0-)P_\nu(0+) = P_\nu(0+)P_\mu(0-).$$

Clearly, $P_{\mu,\nu}$ projects onto $\mathcal{R}P_\mu(0-) \cap \mathcal{R}P_\nu(0+)$.

If the trichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ are compatible then one can express dimensions of any of the subspaces of the form $M_1 \cap M_2$ by the numbers $a, b, \dots, i$ from Table 3.1, where M_1 is any of the subspaces at $0-$ of the forms $\mathcal{R}P_u(0-), \mathcal{R}P_{cu}(0-), \cdots, \mathbb{R}^{2n}$, and M_2 is any of the subspace at $0+$ of the forms $\mathcal{R}P_s(0+), \mathcal{R}P_{cs}(0+), \cdots, \mathbb{R}^{2n}$. For example, $\dim \mathcal{R}P_u(0-) = a+b+c$, $\dim \mathcal{R}P_c(0-) = d+e+f$, $\dim \mathcal{R}P_s(0-) = g+h+i$ and $\dim \mathcal{R}P_{cu}(0-) \cap \mathcal{R}P_{cs}(0+) = a+b+d+e$, etc..

Lemma 3.1. *If (3.1) admits exponential trichotomies on $\mathbb{R}^\pm$ resp., then there exits a pair of compatible exponential dichotomies on $\mathbb{R}^\pm$ with the same exponents γ, α. Moreover, although compatible exponential trichotomies are non unique, the dimensions $a, b, \dots, i$ are unique.*

Proof. Although exponential trichotomies on $\mathbb{R}^\pm$ are not unique, the invariant subspaces $\mathcal{R}P_u(0-), \mathcal{R}P_{cu}(0-), \mathcal{R}(P_s(0+)), \mathcal{R}P_{cs}(0+)$ are unique. The proof follows by examining the dimensions of the intersections of these unique subspaces and redefine the non unique subspaces $\mathcal{R}P_c(0-), \mathcal{R}P_s(0-), \mathcal{R}P_c(0+)$ and $\mathcal{R}P_u(0+)$. Once these invariant subspaces are determined, the projections that define the trichotomies are uniquely defined.

□

Lemma 3.2. *(**S**tructure Lemma for the Adjoint System) If (3.1) has a pair of compatible exponential dichotomies on $\mathbb{R}^\pm$, and if the projections are $P_\nu(t), \nu = s, c, u$, $t \in \mathbb{R}^-$ or $\mathbb{R}^+$, then the adjoint system (3.3) also has a pair of compatible exponential trichotomies on $\mathbb{R}^\pm$, with $P^*_\mu(t) = (P_\mu(t))^*$ and $P^*_{\mu,\nu} = (P_{\mu,\nu})^*$, $\mu, \nu = u, c, s$. Moreover, $\dim \mathcal{R}P^*_\mu(0-) \cap \mathcal{R}P^*_\nu(0+) = \dim \mathcal{R}P_\mu(0-) \cap \mathcal{R}P_\nu(0+)$. See Table 3.2.*

Proof. Let $\mu, \nu = u, c, s$. Consider one of the nine projections $P^*_{\mu,\nu} = (P_\mu(0-)P_\nu(0+))^* = (P_\nu(0+)P_\mu(0-))^*$. It can also be expressed as $P^*_\mu(0-)P^*_\nu(0+) = P^*_\nu(0+)P^*_\mu(0-)$. The rank of the projection $P_{\mu,\nu}$ is equal to its adjoint $P^*_{\mu,\nu}$. □

dim	$\mathcal{R}P_s^*(0+)$	$\mathcal{R}P_c^*(0+)$	$\mathcal{R}P_u^*(0+)$
$\mathcal{R}P_u^*(0-)$	a	b	c
$\mathcal{R}P_c^*(0-)$	d	e	f
$\mathcal{R}P_s^*(0-)$	g	h	i

TABLE 3.2. Dimensions of intersections of invariant subspaces for compatible trichotomies of the adjoint system

Many authors have studied the relation between exponential dichotomies (trichotomies) of (3.1) and (3.3) [12, 3, 18]. Most of their results can be re derived by using Lemma 3.2. One of such example will be presented in §5.

Since Lemma 3.2 is important in this paper, we show the growth or decay of solutions in the nine subspaces graphically in Figures 3.1, 3.2. The norms of solutions in the nine spaces, with their dimensions indicated, are plotted against the time t. The vertical axis is in the log scale so that exponential curves become straight lines.

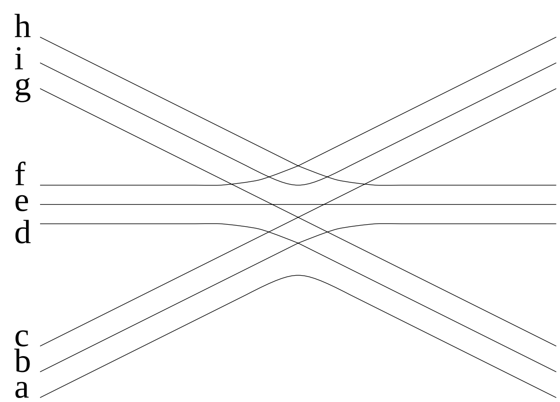

FIGURE 3.1. The norms and dimensions of solutions in the nine subspaces for compatible trichotomies of $x' = A(t)x$.

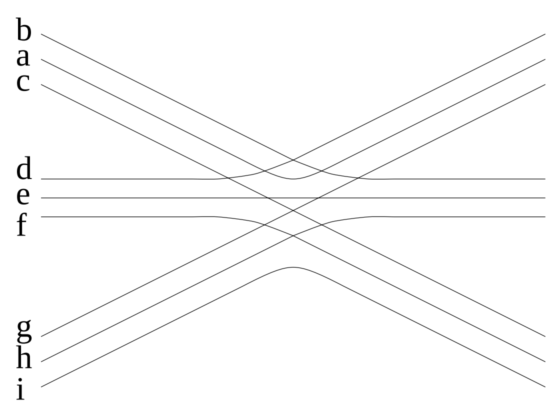

FIGURE 3.2. The norms and dimensions of solutions in the nine subspaces for compatible trichotomies of $\psi' + A^*(t)\psi = 0$.

4. Shock Profile and Rankine-Hugoniot Conditions

Assume that at $t = t_0$, to the system (1.1), $u(x, t_0)$ admits a shock profile with the shock position at $x = x_0$ and the shock speed $dx/dt = s_0$. Let q be the traveling wave connecting $u^L = u(x_0-, t_0)$ to $u^R = u(x_0+, t_0)$. For a small positive Δt, suppose that the shock can be continued to $t_0 + \Delta t$ with its new position at $x = x_0 + \Delta x$. The shock profile connects $u^L + \Delta u^L$ to $u^R + \Delta u^R$ and the speed of the shock is $s_0 + \Delta s$.

We look for relations among the parameters $(\Delta x, \Delta t, \Delta u^L, \Delta u^R, \Delta s)$ such that the shock profile exists. Let Θ be a function that is liner in Δu^L and Δu^R and satisfies

$$\Theta(\zeta, \Delta u^L, \Delta u^R) = \begin{cases} \Delta u^L, & \zeta \le -1, \\ \Delta u^R, & \zeta \ge 1, \\ 0, & -0.5 \le \zeta \le 0.5. \end{cases}$$

Such Θ can be constructed by cut off functions.

For $\beta > 0$, define

$$\begin{aligned} &\|f\|_B = \sup_\zeta \{|f(\zeta)|e^{\beta|\zeta|}\}, \\ &\|f\|_{B_1} = \|f\|_B + \|f'\|_B, \\ &\|f\|_{B_2} = \|f\|_B + \|f'\|_B + \|f''\|_B, \\ &B = \{f \in C(-\infty, \infty) : \|f\|_B < \infty\}, \\ &B_1 = \{f \in B : f' \in B\}, \\ &B_2 = \{f \in B : f',\ f'' \in B\}. \end{aligned}$$

The function space B, B_1, and B_2 are Banach spaces with the norms $\|\cdot\|_B$, $\|\cdot\|_{B_1}$, and $\|\cdot\|_{B_2}$ respectively.

Since the shock profile u approaches the end limits exponentially fast, we assume that

$$\begin{aligned} &u = q + \Delta u + \Theta(\zeta, \Delta u^L, \Delta u^R), \\ &< \Delta u(0), q'(0) >= 0. \end{aligned}$$

where $\Delta u \in B$. The second is a phase condition so that the wave position and speed is well defined.

For the convenience, let $K(x, t, u, v) = A(x, t, u)v$. Consider a linear equation and its adjoint system:

$$\phi'' - K_u\phi - (K_v - s_0)\phi' = 0, \tag{4.1}$$

$$\psi'' - K_u^*\psi + [(K_v^* - s_0)\psi]' = 0, \tag{4.2}$$

where K and its partial derivatives are evaluated at (x_0, t_0, q, q'). Define $\mathcal{L}\phi$ by the left hand side of (4.1). Then (4.2) can be written as $\mathcal{L}^*\psi = 0$.

We remark that $K_v = A$, but it is not easy to define $K_u\phi$ without using tensor notations.

Let $(U, U') = (\Delta u, \Delta u')$. It is clear that q satisfies $\mathcal{L}q' = 0$. from (1.6), the unknown function U satisfies the equation

$$\mathcal{L}U = -\Delta s\ q' + \mathcal{N}, \quad < U(0), q'(0) >= 0, \tag{4.3}$$

and $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3$, with

$$\begin{aligned}
\mathcal{N}_1 &= -\mathcal{L}\Theta, \\
\mathcal{N}_2 &= K_x \Delta x + K_t \Delta t, \\
\mathcal{N}_3 &= K(x_0 + \Delta x, t_0 + \Delta t, q + U + \Theta, q' + U' + \Theta') \\
&\quad - K(x_0, t_0, q, q') - \Delta s(U' + \Theta') \\
&\quad - K_x \Delta x - K_t \Delta t - K_u \cdot (U + \Theta) - K_v \cdot (U' + \Theta').
\end{aligned}$$

Observe that $\mathcal{N}_1$ is a linear function of $(\Delta u^L, \Delta u^R)$ through the function Θ, and $\mathcal{N}_2$ is linear a function of $(\Delta x, \Delta t)$. It is tedious, but straight forward to verify that $\mathcal{N}_3 \in B$ and

$$\|\mathcal{N}_3\|_B = O(\|U\|_B + \|U'\|_B + |\Delta x| + |\Delta t| + |\Delta u^L| + |\Delta u^R|)^2.$$

Before solving (4.3), let us consider a linear problem

$$\mathcal{L}U = h.$$

We have the following lemma.

Lemma 4.1. *The operator $\mathcal{L} : B_2 \to B$ is Fredholm. Let $\{\psi_i\}_1^m$ be a basis of the space of bounded solutions of the adjoint equation (4.2). Then the range of $\mathcal{L}$ is*

$$\mathcal{RL} = \{h \in B : \int_{-\infty}^{\infty} < \psi_i(\zeta), h(\zeta) > d\zeta = 0, \quad i = 1, \ldots, m\}.$$

Proof. Convert $\mathcal{L}U = h$ and $\mathcal{L}^* V = 0$ into first order systems in $\mathbb{R}^{2n}$,

$$\begin{aligned}
U' &= U_1, \\
U_1' &= K_u U + (K_v - s_0) U_1 + h;
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
V_1' + K_u^* V &= 0, \\
V' + V_1 + (K_v^* - s_0) V &= 0.
\end{aligned} \tag{4.5}$$

(4.5) is the adjoint system of (4.4).

As $\zeta \to \pm\infty$, $K_u \to 0$ and $K_v \to A(x, t, u^L)$ or $A(x, t, u^R)$ exponentially fast. Thus as $\zeta \to \pm\infty$ the limiting autonomous systems of (4.4) and (4.5) have exponential trichotomies on $\mathbb{R}^{\pm}$, so do the systems (4.4) and (4.5). This is based on the "roughness of exponential trichotomies", and the proof of which is similar to the "roughness of exponential dichotomies". Let the exponents of the trichotomies be $0 \le \gamma < \alpha$. Let the exponent β defining the spaces B, B_1, B_2 be $\gamma < \beta < \alpha$. By a theorem of Hale and Lin [4], (4.4) defines a Fredholm $\mathcal{F} : B_1 \times B_1 \to B \times B$. The function $(0, h)$ is in the range of $\mathcal{F}$ if and only if

$$\int_{-\infty}^{\infty} < (0, h), (V_1, V) >_{\mathbb{R}^{2n}} d\zeta = 0,$$

for any solution (V_1, V) of (4.5) that satisfies $|V_1| + |V| \le Ce^{\beta|\zeta|}$. The condition simplifies to $\int_{-\infty}^{\infty} < V(\zeta), h(\zeta) > d\zeta = 0$ for any solution V of (4.2) that satisfies $|V| \le Ce^{\beta|\zeta|}$.

We now prove that such V is bounded as $\zeta \to \pm\infty$. In fact, as $\zeta \to \pm\infty$, the limit of (4.2) is the autonomous equation

$$V'' + ((A^*(x, t, u^\pm) - s_0)V)' = 0, \tag{4.6}$$

where $u^- = u^L$ and $u^+ = u^R$. From a theorem of Hartman, any solution V of (4.2) that satisfies $|V| \le Ce^{\beta|\zeta|}$ approaches a solution $\tilde{V}$ of (4.6) exponentially fast. Since the matrix $A^*(x, t, u^\pm) - s_0$ is hyperbolic, it is easy to see that a solution $\tilde{V}$ of (4.6) is bounded if it satisfies $|\tilde{V}| \le Ce^{\beta|\zeta|}$. The desired result follows. □

As seen from the above proof, we can choose $\gamma = 0$ in the definition of exponential trichotomies for systems (4.4) and (4.5).

Let $Q : B \to \mathcal{RL}$ be a projection from B to the range of $\mathcal{L}$. (4.3) can be written as

$$\mathcal{L}U = Q(-\Delta s\, q' + \mathcal{N}), \tag{4.7}$$

$$(I - Q)(-\Delta s\, q' + \mathcal{N}) = 0. \tag{4.8}$$

Assume that the kernel $\mathcal{KL}$ is one dimensional, spanned by q'. In the next section, we will show that this is the only case needed in this paper. Let $\mathcal{K}^\perp = \{U \in B :< U(0), q'(0) >= 0$, a complementary subspace to $\mathcal{KL}$. Let $\tilde{L}^{-1}$ be the inverse of $\mathcal{L} : \mathcal{K}^\perp \to \mathcal{RL}$. Then the solution of (4.7) can be expressed as

$$U = \tilde{L}^{-1}Q(-\Delta s\, q' + \mathcal{N}). \tag{4.9}$$

Since $\mathcal{N}$ contains only higher order terms of (U, U'), (4.9) can be solved for U by the contraction mapping principle. Denote the solution

$$U = \tilde{U}(\Delta t, \Delta x, \Delta s, \Delta u^L, \Delta u^R). \tag{4.10}$$

Substituting (4.10) into (4.8), and using Lemma 4.1 to express the range of $\mathcal{L}$, we have

$$\int_{-\infty}^{\infty} < \psi_i, -\Delta s\, q' + \mathcal{N} > d\zeta = 0, \quad i = 1, \dots, m. \tag{4.11}$$

Using integration by parts, we have

$$\begin{aligned} &< \psi_i, \mathcal{N}_1 >=< \psi_i, \mathcal{L}(-\Theta) > \\ &= < \mathcal{L}^*\psi_i, -\Theta > + < \psi_i, (K_v - s_0)\Theta > |_{-\infty}^{\infty}. \end{aligned}$$

Observe that $\mathcal{N} = \sum_{i=1}^{3} \mathcal{N}_i$ and $\mathcal{N}_3$ contains only higher order terms. Also $K_v = A(x_0, t_0, q)$ and $\Theta(\zeta) \to u^R$ and u^L as $\zeta \to \pm\infty$ resp. If we only show linear terms in

(4.11) explicitly, we have

$$
\begin{aligned}
(4.12)\qquad & \text{for } i = 1, \ldots, m, \\
& \Delta s \int_{-\infty}^{\infty} < \psi_i, q' > d\zeta =< \psi_i(+\infty), (A(x_0, t_0, u^R) - s_0)\Delta u^R > \\
& - < \psi_i(-\infty), (A(x_0, t_0, u^L) - s_0)\Delta u^L > \\
& + \int_{-\infty}^{\infty} < \psi_i, A_x(x_0, t_0, q(\zeta))q'(\zeta)\Delta x + A_t(x_0, t_0, q(\zeta))q'(\zeta)\Delta t > d\zeta \\
& + \mathcal{H}(\Delta x, \Delta t, \Delta s, \Delta u^L, \Delta u^R),
\end{aligned}
$$

where $\mathcal{H}(\Delta x, \Delta t, \Delta s, \Delta u^L, \Delta u^R)$ represents higher order terms.

(4.12) is a necessary and sufficient condition on $(\Delta t, \Delta x, \Delta s, \Delta u^L, \Delta u^R)$ for the existence of a shock profile. We shall call it the generalized (RH) condition (GRH). However, it is rather complicated. If we assume that $(\Delta x, \Delta s, \Delta u^L, \Delta u^R)$ can be expressed as C^1 functions of Δt, we can obtain a simple necessary condition for the existence of shock profile.

As $\Delta t \to 0$, let

$$
\frac{\Delta x}{\Delta t} \to s_0, \quad \frac{\Delta s}{\Delta t} \to \frac{ds}{dt}, \quad \frac{\Delta u^L}{\Delta t} \to \frac{du^L}{dt}, \quad \frac{\Delta u^R}{\Delta t} \to \frac{du^R}{dt}.
$$

Then, we have a linearized (GRH):

$$
\begin{aligned}
(4.13)\qquad & \text{For } i = 1, 2, \ldots, m, \\
& \frac{ds}{dt} \int_{-\infty}^{\infty} < \psi_i, q' > d\zeta =< \psi_i(+\infty), (A(x_0, t_0, u^R) - s_0)\frac{du^R}{dt} > \\
& - < \psi_i(-\infty), (A(x_0, t_0, u^L) - s_0)\frac{du^L}{dt} > \\
& + \int_{-\infty}^{\infty} < \psi_i, A_x(x_0, t_0, q(\zeta))q'(\zeta)s_0 + A_t(x_0, t_0, q(\zeta))q'(\zeta) > d\zeta.
\end{aligned}
$$

We now compare (4.13) with the usual (RH) for conservation laws. It is shown in the next section that for the conservation law, $\mathcal{L}U = 0$ simplifies to (5.1) and the adjoint system becomes $\mathcal{L}^*\psi = \psi'' + (A - s_0)^*\psi' = 0$. Therefore, every constant vector $\psi \in \mathbb{R}^n$ is a solution to the adjoint equation (4.2). If (4.13) is valid for every constant ψ, we obtain an identity in $\mathbb{R}^n$.

$$
\begin{aligned}
(4.14)\qquad \frac{ds}{dt}(u^R - u^L) =& (A(x_0, t_0, u^R) - s_0)\frac{du^R}{dt} - (A(x_0, t_0, u^L) - s_0)\frac{du^L}{dt} \\
& + \int_{-\infty}^{\infty} (A_x(x_0, t_0, q(\zeta))q'(\zeta)s_0 + A_t(x_0, t_0, q(\zeta))q'(\zeta))d\zeta.
\end{aligned}
$$

Let $\frac{d}{d\tau} = s_0\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ be the directional derivative along the shock. Observe that

$$\begin{aligned}\int_{-\infty}^{\infty}(K_x s_0 + K_t)d\zeta &= \int_{-\infty}^{\infty}(s_0 A_x(x_0,t_0,q(\zeta)) + A_t(x_0,t_0,q(\zeta)))q'(\zeta)d\zeta \\ &= (s_0\frac{\partial}{\partial x} + \frac{\partial}{\partial t})(f(x_0,t_0,u^R) - f(x_0,t_0,u^L)).\end{aligned}$$

It is now obvious that (4.14) can be obtained by differentiating the (RH) condition

$$s_0(u^R - u^L) = f(x_0,t_0,u^R) - f(x_0,t_0,u^L).$$

along the shock.

However, for under compressive shocks of the conservation law, there may exist non-constant bounded solutions to (4.2) and the number of conditions offered by (4.12) and (4.13) may exceed n. Thus our (GRH) may contain more than n conditions. The extra conditions for the under compressive shocks can also be obtained by the Melnikov method on the existence of saddle connections of (2.4). Our approach unifies two kind of conditions.

At this point, it is appropriate to give some intuitive idea as how (4.12) can be used to extend the shock for small $\Delta t > 0$. To this end, assume that (1.1) is linear and is in the characteristic form,

$$u_t + \Lambda^{\pm}(x,t)u_x + b(x,t) = 0,$$

where $\Lambda^- = \mathrm{diag}(\lambda_1^L, \dots, \lambda_n^L)$ and $\Lambda^+ = \mathrm{diag}(\lambda_1^R, \dots, \lambda_n^R)$.

Let

$$\Gamma_i^- = \{(x,t) : \frac{dx}{dt} = \lambda_i^L(x,t)\}, \quad \Gamma_i^+ = \{(x,t) : \frac{dx}{dt} = \lambda_i^R(x,t)\}.$$

For (x,t) in the left of Γ_1^- and in the right of Γ_n^+, the solution $u(x,t)$ of (1.1), is completely determined by the initial date $u_0(x)$. To determine $u(x,t)$ between Γ_1^- and Γ_n^+, we need to know the shock trajectory Γ and the data $u_i^L, 1 \le i \le k$ and $u_i^R, j+1 \le i \le n$ on Γ, since they correspond to characteristics leaving Γ to the left and right respectively. Thus, there are $k + (n-j) + 1$ unknown data on Γ. These conditions must come from the m-(GRH) conditions in (4.12).

Observe that $u_i^L, k+1 \le i \le n$ and $u_i^R, 1 \le i \le j$ are known functions of $(\Delta x, \Delta t)$, computable from the initial data $u_0(x)$, by the method of characteristics. Assume that $m = k + (n-j) + 1$, which will be proved under general conditions in the next section, and also assume that certain rank conditions are satisfied. Then from the implicit function theorem, we can solve $\Delta x, \Delta u_i^L, 1 \le i \le k, \Delta u_i^R, j+1 \le i \le n$ from (4.12), as functions of $(\Delta x, \Delta t)$. In particular, along the shock Γ,

$$\frac{dx}{dt} = s_0 + \Delta s = s_0 + \mathcal{S}^*(\Delta x, \Delta t) = s_0 + \mathcal{S}^*(x - x_0, t - t_0),$$

where $\mathcal{S}^*$ is a C^1 function. This allows us to solve the shock $x(t)$ for t near t_0 and x near x_0. Then, the data $\Delta u_i^L, 1 \le i \le k, \Delta u_i^R, j+1 \le i \le n$ are completely determined on Γ and the shock solution $u(x,t)$ in the region between between Γ_1^- and Γ_n^+ can be determined.

We should mention that the shock solution can also be constructed using (4.13). This condition can be used to formulate an Euler's method to approximate (x, u^L, v^R, s)

and q along the shock trajectory. Again, the characteristic method can be used to the left and right of the shock. However, the proof of convergence may be complicated. Our current approach allows us to use results from [10] directly.

In the next section, we will discuss the dimension m of the space of bounded solutions to the adjoint equation (4.2). Using the information from the inner solution to find a matched solution is presented in §6.

5. Linear equation and its the adjoint

In this section, we study the linear equation (4.1) and its adjoint system (4.2). Let m be the dimension of the space of bounded solutions of (4.2).

Lemma 5.1. *For a generic quasilinear hyperbolic system, assume that (H1) and (H2) are satisfied. Then for a pair of compatible exponential trichotomies on $\mathbb{R}^{\pm}$, we have (1) if $k+1 \geq j$, then $a=1, b=0, c=n-k-1, d=0, e=n+j-k-1, f=k+1-j, g=j-1, h=k+1-j, i=0$;*
(2) if $k+1<j$, then $a=1, b=j-k-1, c=n-j, d=j-k-1, e=n-j+k+1, f=0, g=k, h=i=0$.

Proof. Notice from (H1), we have $\dim W^u(u^L,0)=n-k$ and $\dim W^s(u^R,0)=j$. Since $\dim W^u(u^L,0)+\dim W^s(u^R,0)<2n+1$, from (H2), we have that $W^u(u^L,0)\cap W^s(u^R,0)$ is one dimensional. Therefore, $\mathcal{R}P_u(0-)\cap\mathcal{R}P_s(0+)$ is one dimensional, spanned by $(q'(0),q''(0))$. For the Tables 3.1, 3.2, we have $a=1$.

(1) First, assume that $k+1\geq j$. Then from (H2), $\mathcal{R}P_{cu}(0-)\cap\mathcal{R}P_s(0+)$ and $\mathcal{R}P_u(0-)\cap\mathcal{R}P_{cs}(0+)$ are both one dimensional. Since $a=1$, we have $b=d=0$. By (H2) again, $\mathcal{R}P_{cu}(0-)\cap\mathcal{R}P_{cs}(0+)$ is $n-k+j$ dimensional, thus, $a+b+d+e=n-k+j$. This implies $e=n+j-k-1$. Using the compatibility of trichotomies, we obtain the other dimensions on the Tables 3.1, 3.2: $c=n-k-1, g=j-1, f=h=k-j+1$ and $i=0$.

(2) Next, assume that $k+1<j$. From (H2), we have $\mathcal{R}P_{cu}(0-)\cap\mathcal{R}P_s(0+)$ and $\mathcal{R}P_u(0-)\cap\mathcal{R}P_{cs}(0+)$ are both $j-k$ dimensional. Since $a=1$, we have $b=d=j-k-1$. By (H2) again, $\dim\mathcal{R}P_{cu}(0-)\cap\mathcal{R}P_{cs}(0+)=n+j-k$, thus, $a+b+d+e=n+j-k$. From this, $e=n+k-j+1$. By the definition of compatibility of the trichotomies, we have $c=n-j, g=k, f=h=i=0$.

□

The result of Lemma 5.1 is depicted in Figures 5.1 and 5.2.

Lemma 5.2. *For a generic quasilinear system, assume that (H1) and (H2) are satisfied. Then $m=k+(n-j)+1$ (m as in Lemma 4.1). Moreover, there does not exist nonzero bounded solution to the adjoint system that approaches zero as $\zeta\to\pm\infty$.*

Proof. By Lemma 3.2, the dimension of the space of bounded solutions to the adjoint system is equal to $\dim\mathcal{R}P^*_{cu}(0+)\cap\mathcal{R}P^*_{cs}(0-)=e+f+h+i$ and the dimension of the space of solutions that approach zero as $\zeta\to\pm\infty$ is $\dim\mathcal{R}P^*_u(0+)\cap\mathcal{R}P^*_s(0-)=i=0$. From Lemma 5.1, $e+f+h+i=k+(n-j)+1$ and $i=0$. □

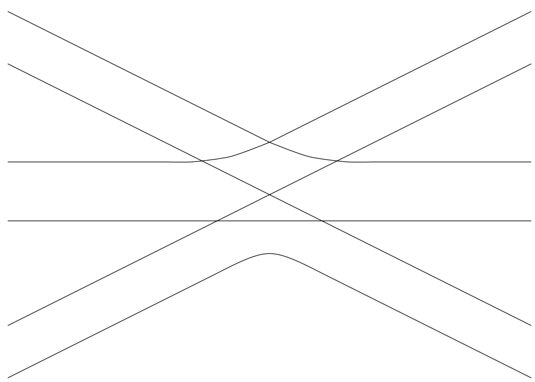

FIGURE 5.1. The dimensions of solutions of a pair of compatible trichotomies of (4.1), for the generic conservation or non-conservation system with $k+1 \geq j$.

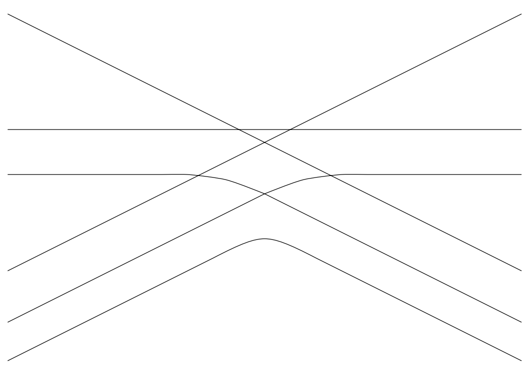

FIGURE 5.2. The dimensions of solutions of a pair of compatible trichotomies of (4.1), for the generic non-conservation system with $k+1 < j$,.

A conservation law (2.2) can be written as a quasilinear system (1.1), with

$$A(x,t,u) = f_u(x,t,u),$$
$$b(s,t,u) = g(x,t,u) + f_x(x,t,u).$$

Its viscous profile satisfies

$$[f(x,t,u) - su]' = u''.$$

The linear variational system of the above is

$$U'' = [(A(x,t,u) - s)U]', \tag{5.1}$$

which is much simpler than the linear variational system of a generic system,

$$U'' = \frac{\partial}{\partial u}(Av)U + (A - s)U', \quad v = u' \tag{5.2}$$

To understand the special feature of a conservation system, we show how to derive (5.1) from (5.2). Using the tensor notation, (5.2) can be written as

$$U_i'' = \sum_k \frac{\partial}{\partial u_k}(\sum_j A_{ij}v_j)U_k + (\sum_j A_{ij} - s)U_j'.$$

Since $A_{ij} = \frac{\partial}{\partial u_j} f_i$, we have

$$\frac{\partial A_{ij}}{\partial u_k} = \frac{\partial A_{ik}}{\partial u_j}. \tag{5.3}$$

Thus,

$$\begin{aligned} U_i'' &= \sum_k (\sum_j \frac{\partial A_{ik}}{\partial u_j} v_j)U_k + (\sum_j A_{ij} - s)U_j' \\ &= \sum_k (\frac{d}{d\zeta} A_{ik})U_k + (\sum_j A_{ij} - s)U_j'. \end{aligned}$$

This is equivalent to (5.1). Notice that (5.3) is only valid if the matrix A is from a conservation law.

We now determine the nine numbers in the Tables 3.1, 3.2 for a pair of compatible exponential trichotomies of (5.1).

Integrating (5.1) once, we have an equivalent system

$$U' = (A(x, t, u) - s)U + W, \qquad W \in \mathbb{R}^n. \tag{5.4}$$

From (H1), the homogeneous part of (5.4) has exponential dichotomies on $\mathbb{R}^\pm$ with $\dim\mathcal{R}P_u(0-) = n - k$ and $\dim\mathcal{R}P_s(0+) = j$. Also $\text{spann}\{\dot{q}(0)\} \subset \mathcal{R}P_u(0-) \cap \mathcal{R}P_s(0+)$. Moreover, from (H2'), $\mathcal{R}P_u(0-)$ and $\mathcal{R}P_s(0+)$ intersect generically. That is, (1), $\text{spann}\{\dot{q}(0)\} = \mathcal{R}P_u(0-) \cap \mathcal{R}P_s(0+)$ if $\dim\mathcal{R}P_u(0-) + \dim\mathcal{R}P_s(0+) \leq n + 1$; (2), $\mathcal{R}P_u(0-) + \mathcal{R}P_s(0+) = \mathbb{R}^n$ if $\dim\mathcal{R}P_u(0-) + \dim\mathcal{R}P_s(0+) \geq n + 1$.

The adjoint equation of (5.4) is (2.6):

$$\eta' + (A^*(x, t, q) - s)\eta = 0.$$

Let $\eta_i, i = 1, \ldots, \ell$ be a basis of the linear space of bounded solutions of (2.6). We can proof the following:

Lemma 5.3. *(1) If $k + 1 \geq j$ then the dimension of the space of bounded solutions of the adjoint equation (2.6) is $\ell = k + 1 - j$. This happens in the Lax or under compressive shock.*
(2) If $k + 1 < j$ then $\ell = 0$. This happens in the over compressive shock.

Proof. The lemma can be proved by using the index theory of Fredholm operators as in [12]. However, we will use Lemma 3.2. Since we have exponential dichotomies on $\mathbb{R}^\pm$, in Table 3.1, $b = d = e = f = h = 0$.

(1) Assume that $k+1 \geq j$. Since $\dim\mathcal{R}P_u(0-)+\dim\mathcal{R}P_s(0+) = n-k+j \leq n+1$, from (H2'), we have $a = 1$. Thus, $c = \dim\mathcal{R}P_u(0-) - a = n - k - 1$ and $i = \dim\mathcal{R}P_u(0+) - c = n - j - (n - k - 1) = k + 1 - j$. This shows that $\dim\mathcal{R}P_s^*(0-) \cap \mathcal{R}P_u^*(0+) = \ell = i = k + 1 - j$.

(2) Assume that $k+1<j$. Since $\dim\mathcal{R}P_u(0-)+\dim\mathcal{R}P_s(0+)=n-k+j\geq n+1$, from (H2'), we have $a=j-k$. Thus $c=\dim\mathcal{R}P_u(0-)-a=n-k-a=n-j$ and $i=\dim\mathcal{R}P_u(0+)-c=n-j-c=0$. Therefore, $\ell=0$.

$\square$

Lemma 5.4. *For a conservation law, assume that (H1) and (H2') are satisfied. Then for a pair of compatible exponential trichotomy of (5.1),*
(1) if $k+1\geq j$, then $a=1, b=0, c=n-k-1, d=0, e=n-k+j-1, f=k+1-j, g=j-1, h=k+1-j, i=0$;
(2) if $k+1<j$, then $a=j-k, b=0, c=n-j, d=0, e=n, f=0, g=k, h=i=0$.

Proof. Observe that $q\to u^\pm$ as $\zeta\to\pm\infty$, where $u^-=u^L$ and $u^+=u^R$. From a theorem of Hartman,if U is a bounded solution of (5.4), then $U\to-(A(x,t,u^\pm)-s)^{-1}W$ as $\zeta\to\pm\infty$. In particular,

$$U(\zeta)\to 0,\ \zeta\to-\infty \Longleftrightarrow U(\zeta)\to 0,\ \zeta\to+\infty. \tag{5.5}$$

(1) Consider the case $k+1\geq j$. Let $W=0$ first. Since (5.4) has exponential dichotomies on $\mathbb{R}^\pm$, and $\dim\mathcal{R}P_u(0-)+\dim\mathcal{R}P_s(0+)\leq n+1$, from (H2'), $\mathcal{R}P_u(0-)\cap\mathcal{R}P_s(0+)=\text{spann}\{(q'(0),q''(0))\}$. Thus, $a=1$. Also, due to (5.5), $b=d=0$.

Now let $W\neq 0$. If (5.4) admits a bounded solution, then for each η_i from the basis of the space of bounded solutions of (2.6), W must satisfy $\int_{-\infty}^{\infty}<\eta_i,W>d\zeta=0,\ i=1,\dots,\ell$. That is

$$<\psi_i,W>=0,\quad i=1,\dots,\ell,\quad \text{where } \psi_i=\int_{-\infty}^{\infty}\eta_i d\zeta.$$

Due to (H2'), $\{\psi\}_1^\ell$ are linearly independent. Therefore, such W forms a $n-\ell$ dimensional subspace $E_1\subset\mathbb{R}^n$. There exist $n-\ell$ linearly independent solutions of U of (5.1) that approach nonzero limits on both ends. Thus $e=n-\ell$. Let E_2 be a complementary ℓ dimensional subspace of E_1, For $W\in E_2$, the corresponding solution U only approaches a finite limit at one end and blows up at the other end. Thus $f=h=\ell$.

From Lemma 5.3, $\ell=k+1-j$. Therefore, we have $e=n+j-k-1, f=h=k+1-j$. Finally, from the compatibility of trichotomies, $c=n-k-1, g=j-1$ and $i=0$.

(2) Consider the case $k+1<j$ now. Let $W=0$ first. Since $\dim\mathcal{R}P_u(0-)+\dim\mathcal{R}P_s(0+)>n+1$, thus $\dim\mathcal{R}P_u(0)\cap\mathcal{R}P_s(0+)=j-k$. We have $a=j-k$. Also, due to (5.5), $b=d=0$.

Consider $W\neq 0$. From Lemma 5.3 again, $\ell=0$. Therefore, for every $W\neq 0$, there is a bounded solution U that approaches non zero limits as $\zeta\to\pm\infty$. Thus $e=n$. Finally, from the compatibility of the dichotomies at $\mathbb{R}^\pm$, $c=n-j, g=k$ and $f=h=i=0$.

$\square$

The result of Lemma 5.4 is depicted in Figures 5.1 and 5.3. Note that for $k+1\geq j$, the result is the same as for the non-conservation systems.

Lemma 5.5. *For a conservation law, assume that (H1) and (H2') are satisfied, then $m=\max\{n,k+n-j+1\}$ (m as in Lemma 4.1). Moreover, there does not exist nonzero bounded solution to the adjoint system that approaches zero as $\zeta\to\pm\infty$.*

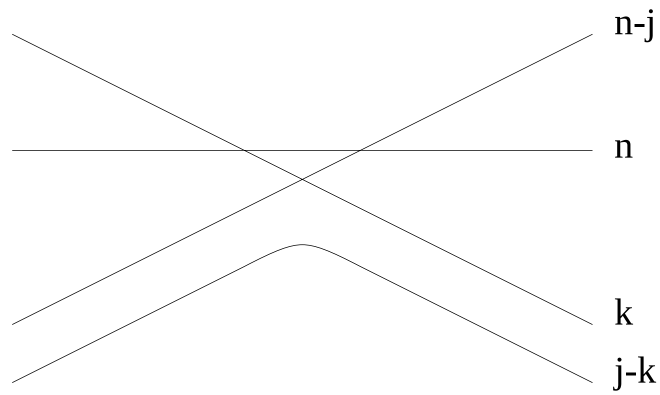

FIGURE 5.3. The dimensions of solutions of a pair of compatible trichotomies of (4.1), for the generic conservation system with $k+1 < j$.

Proof. By Lemma 3.2, the dimension of the space of bounded solution to the adjoint system is equal to $\dim \mathcal{R}P^*_{cu}(0+) \cap \mathcal{R}P^*_{cs}(0-) = e+f+h+i$. As shown in Lemma 5.2, in the case $k+1 \geq j$, we have $e+f+h+i = k+n-j+1$, while in the case $k+1 < j$, we have $e+f+h+i = n$.

Also in both cases, $\dim \mathcal{R}P^*_{u}(0+) \cap \mathcal{R}P^*_{s}(0-) = i = 0$. Therefore, there does not exist solution of (4.2) that approaches zero as $\zeta \to \pm\infty$. □

Remark . For a conservation law, the adjoint system

$$\psi'' + (A^*(x,t,q) - s)\psi' = 0 \tag{5.6}$$

can be reduced to (2.6)

$$\eta' + (A^*(x,t,q) - s)\eta = 0,$$

by setting $\eta = \psi'$. Thus the bounded solutions of (5.6) can be obtained through the bounded solutions of (2.6) using $\psi = \int \eta + C$. We can also prove Lemma 5.5 by exploring this connection.

6. THE EXISTENCE OF SHOCK SOLUTION

For a generic non-conservation system, assume that (H1) and (H2) are satisfied. For a conservation system, assume that (H1), (H2') and (H3) are satisfied.

For the convenience, we will solve (1.1) in the characteristic form. Let $\eta_\ell(x,t,u)$, $\ell = 1, \dots, n$ be the left eigenvectors of $A(x,t,u)$ corresponding to the eigenvalue $\lambda_\ell(x,t,u)$. Let $\eta_\ell = (\eta_{\ell 1} \dots, \eta_{\ell n})$, and let $L = (\eta_{\ell j}(x,t,u))$ be a $n \times n$ matrix. Then

$$LA = \Lambda L = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}.$$

Let $R = L^{-1}$. $R = (r_1, \dots, r_n)$ consists of right eigenvectors of $A(x,t,u)$.

Let $v = Lu$. The characteristic form of (1.1) is

$$v_t + \Lambda(x,t,v)v_x = \mu, \tag{6.1}$$

$$v(x,t_0) = v_0(x), \quad t_0 = 0, \tag{6.2}$$

where $v_0(x) = L(x, 0, u_0(x))u_0(x)$ and

$$\mu(x, t, v) = -Lb + (L_t + \Lambda L_x)Rv.$$

With slightly abuse of notations, the arguments for L, Λ, R and b are (x, t, v) through the substitution $u = Rv$. Also notice that $L(x, t, u)$ depends on the unknown u, thus v_t and v_x terms are involved in the term μ. However, this does not affect the iteration method used to solve the system as in [10].

We also convert (GRH) by the characteristic variables. Let $\Delta v = L\Delta u$. Using $A = R\Lambda L$, from (4.12), we have

$$\begin{aligned}
&\text{for } i = 1, \dots, m,\\
&\Delta s \int_{-\infty}^{\infty} < \psi_i, q' > d\zeta = < \psi_i(+\infty), R^R(\Lambda^R - s_0)\Delta v^R >\\
&- < \psi_i(-\infty), R^L(\Lambda^L - s_0)\Delta v^L >\\
&+ \int_{-\infty}^{\infty} < \psi_i, A_x(x_0, t_0, q(\zeta))q'(\zeta)\Delta x + A_t(x_0, t_0, q(\zeta))q'(\zeta)\Delta t > d\zeta\\
&+ \mathcal{H}(\Delta x, \Delta t, \Delta s, R^L\Delta v^L, R^R\Delta v^R).
\end{aligned} \tag{6.3}$$

Here super script L or R means that the functions are evaluated at $(x_0, t_0, v_0(x_0-))$ or $(x_0, t_0, v_0(x_0+))$ respectively.

Let $\Psi = (\psi_1 \dots, \psi_m)$. Let Ψ^τ denote the transpose of Ψ. Let $\mathcal{M}_1 = \int \Psi^\tau(\zeta)q'(\zeta)d\zeta$ be a m vector, $\mathcal{M}_2 = \Psi^\tau(-\infty)((\lambda_1^L - s_0)r_1^L, \dots, (\lambda_k^L - s_0)r_k^L)$ be a $m \times k$ matrix and $\mathcal{M}_3 = \Psi^\tau(\infty)((\lambda_{j+1}^R - s_0)r_{j+1}^R, \dots, (\lambda_n^R - s_0)r_n^R)$ be a $m \times (n - j)$ matrix. Here r_j^L or r_j^R is a right eigenvector at (x_0, t_0, v^L) or (x_0, t_0, v^R) respectively.

From Lemma 5.2, for a non-conservation system, if (H1) and (H2) are satisfied, then $m = n - j + k + 1$. From Lemma 5.5, for a conservation system, if (H1), (H2') and (H3) are satisfied, then we also have $m = n - j + k + 1$.

H4. The $m \times m$ matrix $(\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3)$ is nonsingular.

We remark that (H4) cannot be satisfied if there exist $1 \le i_1 < i_2 \le m$ such that $\psi_{i_1}(\pm\infty) = 0$ and $\psi_{i_2}(\pm\infty) = 0$. Otherwise, there are two zero rows of $\mathcal{M}_2$ and $\mathcal{M}_3$, so that (H4) cannot be valid. Notice that it is proved in Lemma 5.2 and Lemma 5.5 that $i|\psi(-\infty)| + |\psi(+\infty)| \ne 0, 1 \le i \le m$.

From (H4), using the implicit function theorem on (6.3), we can solve Δs, $\Delta v_1^L, \dots, \Delta v_k^L$, $\Delta v_{j+1}^R, \dots, v_n^R$ as functions of the input arguments Δx, Δt, $\Delta v_{k+1}^L, \dots, \Delta v_n^L$, $\Delta v_1^R, \dots, \Delta v_j^R$, provided that the inputs are sufficiently small. Therefore there is a shock profile for (x, t) near (x_0, t_0) and (v^L, v^R) near $(v_0(x_0-), v_0(x_0+))$ if and only if

$$\begin{aligned}
s &= \tilde{s}(x, t, v_{k+1}^L, \dots, v_n^L, v_1^R, \dots, v_j^R),\\
v_i^L &= \tilde{v}_i^L(x, t, v_{k+1}^L, \dots, v_n^L, v_1^R, \dots, v_j^R), \quad 1 \le i \le k,\\
v_i^R &= \tilde{v}_i^R(x, t, v_{k+1}^L, \dots, v_n^L, v_1^R, \dots, v_j^R), \quad j + 1 \le i \le n.
\end{aligned} \tag{6.4}$$

Here $v_i^L = v_0(x_0-) + \Delta v_i^L, v_i^R = v_0(x_0+) + \Delta v_i^R$. The functions $\tilde{s}, \tilde{v}_i^R, \tilde{v}_i^L$ are C^1 with respect to their arguments.

Consider an interval $[a,b]$ with $a < x_0 < b$. Assume that $[a,b]$ is sufficiently small so that Hypotheses (H1)-(H3) are satisfied for $x \in [a,b]$ and $t = 0$. For $\delta > 0$ being sufficiently small, the initial value $v_0(x)$, when restricted to $[a, x_0]$ or $[x_0, b]$, uniquely determine a classical solution of (6.1) on the domain $\{(x,t) : a(t) \le x \le x_1^L(t), 0 \le t \le \delta\}$ or $\{(x,t) : x_n^R(t) \le x \le b(t), 0 \le t \le \delta\}$ respectively. Here $a(t)$ is the nth characteristic curve passing through $(a,0)$, and $b(t)$ is the first characteristic curve passing through $(b,0)$, $x_1^L(t)$ and $x_n^R(t)$ are characteristic curves passing through $(x_0, 0)$, corresponding to λ_1^L and λ_n^R respectively.

At the shock position (x_0, t_0), comparing the characteristic curves x_1^L and x_n^R to s_0, in terms of Hypothesis (H1) we have four possibilities:
(1) $x_1^L(t_0) < s_0 < x_n^R(t_0)$, that is $k > 0,\ j < n$,
(2) $x_n^R(t_0) < s_0 < x_1^L(t_0)$, that is $k = 0,\ j = n$,
(3) $x_1^L(t_0) < s_0$ and $x_n^R(t_0) < s_0$, that is $k > 0,\ j = n$,
(4) $x_1^L(t_0) > s_0$ and $x_n^R(t_0) > s_0$, that is $k = 0,\ j < n$.

For definiteness, we will consider the case (1) only. The other cases can be handled similarly and will be left to the readers.

In the rest of the section, we will solve (6.1) in the domain $\mathcal{D} = \{(x,t) : x_1^L(t) < x < x_n^R(t), 0 \le t \le \delta\}$. The domain $\mathcal{D}$ is divided by the shock $\Gamma := \{x = x(t)\}$ into two parts, $\mathcal{D}^L$ and $\mathcal{D}^R$. Moreover, issuing from (x_0, t_0), there are characteristics $x_2^L, \dots, x_k^L$ in $\mathcal{D}^L$ and $x_{j+1}^R, \dots, x_n^R$ in $\mathcal{D}^R$ where $v(x,t)$ is weakly discontinuous.

We give a precise formulation of the boundary value problem on $\mathcal{D}$. Define $y_i(t), -k \le i \le n-j$ by

$$
y_i(t) = \begin{cases} x(t), & \text{if } i = 0, \\ x^L_{k+1+i}(t) & \text{if } -k \le i < 0, \\ x^R_{j+i}(t) & \text{if } 0 < i \le n-j. \end{cases} \tag{6.5}
$$

The curves $\{y_i(t)\}_{-k}^{n-j}$ divide $\mathcal{D}$ into $n-j+k$ sectors, $\mathcal{D}^{(i)} := \{(x,t) : y_i(t) \le x \le y_{i+1}(t), 0 \le t \le \delta\}$. Let the restriction of v in $\mathcal{D}^i$ be $v^{(i)}, -k \le i \le n-j-1$, which is C^1 in $\mathcal{D}^{(i)}$ and satisfies (6.1). Following Li and Yu, we call $\mathcal{D}$ the fan-shaped domain.

If we denote the values of v on the outer boundaries of $\mathcal{D}$ by $v^{(-k)}(y_{-k}(t), t) = \alpha(t)$ and $v^{(n-j-1)}(y_{n-j}(t), t) = \beta(t)$, then from the previous calculation, $(\alpha_2, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_{n-1})$ have been obtained. They correspond to characteristic curves that enter $\mathcal{D}$ from the outer boundaries. The following conditions must be satisfied:

$$
\begin{aligned}
v_s^{(-k)}(y_{-k}(t), t) &= \alpha_s(t),\ s = 2, \dots, n \\
v_r^{(n-j-1)}(y_{n-j}(t), t) &= \beta_r(t),\ r = 1, \dots, n-1 \\
\frac{d}{dt} y_{-k}(t) &= \lambda_1^L(y_{-k}(t), t, \alpha(t)), \quad y_{-k}(0) = x_0, \\
\frac{d}{dt} y_{n-j}(t) &= \lambda_n^R(y_{n-j}(t), t, \beta(t)), \quad y_{n-j}(0) = x_0.
\end{aligned} \tag{6.6}
$$

From (6.6)and the equations

$$\begin{aligned} \frac{d}{dt}\alpha_1(t) &= \mu_1(y_{-k}(t), t, \alpha(t)), \\ \frac{d}{dt}\beta_n(t) &= \mu_n(y_{n-j}(t), t, \beta(t)), \end{aligned} \tag{6.7}$$

we can uniquely solve $(\alpha_1(t), \beta_n(t), y_{-k}(t), y_{n-j}(t))$ as the solution to an initial value problem. The result should agree with the previous calculation that determines $v(x,t)$ to the left and right of $\mathcal{D}$, if the following compatibility conditions are satisfied,

$$\begin{aligned} \alpha(0) &= v_0(x_0-), \\ \beta(0) &= v_0(x_0+). \end{aligned} \tag{6.8}$$

The internal boundaries $y_i(t), -k < i < n-j$ are free boundaries which must be determined, as well as the values of $v^{(i-1)}$ and $v^{(i)}$ on them. If $i \neq 0$, $y_i(t)$ is a characteristic, and we must have

$$v^{(i-1)} = v^{(i)}, \tag{6.9}$$

$$\frac{d}{dt}y_i(t) = \lambda^L_{i+1+k}(y_i(t), t, v^{(i)}), \quad i < 0, \tag{6.10}$$

$$\frac{d}{dt}y_i(t) = \lambda^R_{i+j}(y_i(t), t, v^{(i-1)}), \quad i > 0. \tag{6.11}$$

On $y_0(t)$ there is a shock, so from (6.4) we must have

$$\frac{dy_0(t)}{dt} = \tilde{s}(y_0(t), t, v^{(-1)}_s, v^{(0)}_r), s = k+1, \dots, n; r = 1, \dots, j, \tag{6.12}$$

$$v^{(-1)}_{\hat{r}} = \tilde{v}^L_{\hat{r}}(y_0(t), t, v^{(-1)}_s, v^{(0)}_r), s = k+1, \dots, n; r = 1, \dots, j; 1 \le \hat{r} \le k, \tag{6.13}$$

$$v^{(0)}_{\hat{s}} = \tilde{v}^R_{\hat{s}}(y_0(t), t, v^{(-1)}_s, v^{(0)}_r), s = k+1, \dots, n; r = 1, \dots, j; j+1 \le \hat{s} \le n. \tag{6.14}$$

Boundary value problem (6.1), (6.6)–(6.14) has been studied by Li and Yu in their book [10]. We shall use their result in Chapter 4 on "general free boundary value problems". It is impossible to describe Li and Yu's method in details (to do so would need a book), so we will present an outline of their approach.

Following [10], the unknown vector $v^{(i)} = (v^{(i)}_1, \dots, v^{(i)}_n)$ in $\mathcal{D}^{(i)}$ is written as $(v^{(i)}_r, v^{(i)}_s)$, where $v^{(i)}_r$ corresponds to characteristics that leave y_{i+1} and enter $\mathcal{D}^{(i)}$ and $v^{(i)}_s$ corresponds to characteristics that leave y_i and enter $\mathcal{D}^{(i)}$. For $-k \le i \le -1$, i.e., $\mathcal{D}^{(i)} \subset \mathcal{D}^L$, we have $r = 1, \dots, i+k+1$, $s = i+k+2, \dots, n$; for $0 \le i \le n-j-1$, i.e., $\mathcal{D}^{(i)} \subset \mathcal{D}^R$, we have $r = 1, \dots, i+j$ and $s = j+i+1, \dots, n$. Note that in each $\mathcal{D}^{(i)}$, $\dim(v^{(i)}_r)$+$\dim(v^{(i)}_s) = n$. Also, if $y_i(t)$ is a characteristic ($i \neq 0$), and if we cross $y_i(t)$ from $\mathcal{D}^{(i-1)}$ to $\mathcal{D}^{(i)}$,then $\dim(v^{(i)}_r)$ increases by one and $\dim(v^{(i)}_s)$ decreases by one.

We write the boundary conditions for $v^{(i)}$ on $\mathcal{D}^{(i)}$ as

$$\begin{aligned} v^{(i)}_s &= G^{(i)}_s(x, t, v^{(i-1)}, v^{(i)}), \quad \text{on } x = y_i(t) \\ v^{(i)}_r &= G^{(i)}_r(x, t, v^{(i)}, v^{(i+1)}), \quad \text{on } x = y_{i+1}(t). \end{aligned} \tag{6.15}$$

The functions $G_s^{(i)}$ and $G_r^{(i)}$ are at least C^1. We remark that if $i = -k$, then $v^{(i-1)}$ should be dropped, and if $i = n - j - 1$, then $v^{(i+1)}$ should be dropped.

In our particular case, these functions are:

$$
\begin{aligned}
G_s^{(-k)} &= \alpha_s(t), \quad s = 2, \ldots, n, \\
G_r^{(n-j-1)} &= \beta_r(t), \quad r = 1, \ldots, n-1, \\
G_s^{(i)} &= v_s^{(i-1)}, \quad i = -k+1, \ldots, n-j-1, \text{ except } i = 0, \\
G_r^{(i)} &= v_r^{(i+1)}, \quad i = -k. \ldots, n-j-2, \text{ except } i = -1, \\
G_{\hat{s}}^{(0)} &= \tilde{v}_{\hat{s}}^R(x, t, v_s^{(-1)}, v_r^{(0)}), \hat{s} = j+1, \ldots, n; s = k+1, \ldots, n; r = 1, \ldots, j, \\
G_{\hat{r}}^{(-1)} &= \tilde{v}_{\hat{r}}^L(x, t, v_s^{(-1)}, v_r^{(0)}), \hat{r} = 1, \ldots, k; s = k+1, \ldots, n; r = 1, \ldots, j.
\end{aligned}
$$

When $y_i(t)$ is a characteristics curve ($i \neq 0$), the above conditions, with (6.15), are weaker than (6.9) since they do not guarantee all the components of $v^{(i-1)}$ and $v^{(i)}$ are equal on $y_i(t)$. For definiteness, assume $i < 0$. If $\ell \neq i + k + 1$, then the ℓth characteristic $x_\ell^L(t)$ passes $y_i(t)$ transversely. Thus, our condition implies that $v_\ell^{(i-1)} = v_\ell^{(i)}$. However, if $\ell = i + k + 1$, then the ℓth characteristic $x_\ell^L(t) = y_i(t)$. We need to show $u_\ell^{(i)} = u_\ell^{(i-1)}$ on $y_i(t)$ from (6.1). Since along the characteristics $y_i(t)$, both $v_\ell^{(i-1)}$ and $v_\ell^{(i)}$ satisfy the same ordinary differential equation induced from (6.1), with the same initial condition at (x_0, t_0). Thus they are equal throughout the curve $y_i(t)$.

To solve system (6.1) with boundary conditions (6.15) on the characteristics $y_i(t), i \neq 0$, and the shock $y_0(t)$, we use an iteration or fixed point argument. First, an initial guess of the value $\bar{v}^{(i)}$ in $\mathcal{D}^{(i)}, -k \leq i \leq n-j-1$ is given. From (6.15), an initial guess of the boundary values $(\bar{v}_s^{(i)}, \bar{v}_r^{(i)})$ of $\bar{v}^{(i)}$ on $y_i(t)$ and $y_{i+1}(t)$ is obtained. Next, using the method of characteristics, we compute the value $\bar{\bar{v}}^{(i)}$ in $\mathcal{D}^{(i)}$. If the process: $\{\bar{v}^{(i)}, i = -k, \ldots, n-j-1\} \to \{\bar{\bar{v}}^{(i)}, i = -k, \ldots, n-j-1\}$ is a contraction mapping, then we would have a solution $v^{(i)}, i = -k, \ldots, n-j-1$. Since the boundaries $y_i(t)$ need to be updated, the domain $\mathcal{D}^{(i)}$ is not fixed. Li and Yu use a change of coordinates so that a boundary value problem in a fixed fan shaped domain is considered. In the new domain, the hyperbolic system (6.1) is no longer a PDE, it is an equation involving integral terms, so called the boundary value problem in functional form. For details see [10].

The method of characteristics uses an integral equation to calculate the value $\bar{v}^{(i)}$ in $\mathcal{D}^{(i)}$, so it is contractive if the time $\delta > 0$ is sufficiently small. Since the boundary values need update, we need that the functions in the right hand side of (6.15) is also contractive. The condition to verify is the characterizing number of a so called characterizing matrix H defined in [10].

The characterizing matrix is an $(n-j+k)n$ by $(n-j+k)n$ matrix

$$H := (\theta_{\ell i}) = \left(\frac{\partial \tilde{G}_\ell}{\partial \tilde{v}_i}\right).$$

Here, $\tilde{G}$ is an $(n-j+k)n$ vector constructed by concatenating $(G_r^{(i)}, G_s^{(i)})$, and $\tilde{v}$ is an $(n-j+k)n$ vector constructed by concatenating $(v_r^{(i)}, v_s^{(i)})$, with $-k \le i \le n-j-1$. The characterizing number of H is

$$|H| := \max_{\ell} \sum_{i} |\theta_{\ell i}|.$$

Li and Yu showed in [10] that if the characterizing number $|H| < 1$ and if $\delta > 0$ is sufficiently small, then the boundary value problem has a unique solution in $\mathcal{D}$. The smallness of $|H|$ is used to guarantee the convergence of the iteration of the method of characteristics in $\mathcal{D}$.

Unfortunately, the condition $|H| < 1$ for our problem is clearly not satisfied. A more general theorem of Li and Yu states that if there exits an $(n-j+k)n$ by $(n-j+k)n$ diagonal matrix γ such that the characterizing number $|\gamma H \gamma^{-1}| < 1$, then the free boundary value problem also has a unique solution. This is equivalent to defining $V_i^{(\ell)} = \gamma_i^{(\ell)} v_i^{(\ell)}, -k \le \ell \le n-j-1; i = 1, \dots, n$ and solving a new boundary value problem for $V_i^{(\ell)}$. The statement of the boundary value problem for $V_i^{(\ell)}$ is left to the readers. With

$$\gamma = \operatorname{diag}(\gamma_i^{(\ell)}, -k \le \ell \le n-j-1; i = 1, \dots, n),$$

the characterizing matrix $\gamma H \gamma^{-1}$ is related to the new boundary value problem for $V_i^{(\ell)}$.

If we exam (6.15) closely, we find that due to the special structure of the internal boundary conditions, there exits a partial order among $v_i^{(\ell)}$ such that the later variables depend on the earlier variables, but not the opposite. The partial order is given by following the characteristics that leave the outer boundaries and enter $\mathcal{D}$, until they hit the shock Γ, then following the departing characteristics that leave Γ, until they are parallel to one of the characteristics in $\mathcal{D}^L$ or $\mathcal{D}^R$. See Figure 6.1. If we successively scaling down $v_\ell^{(i)}$ by a small factor $\gamma_i^{(\ell)}$, the characterizing number for the new variable can be less than one.

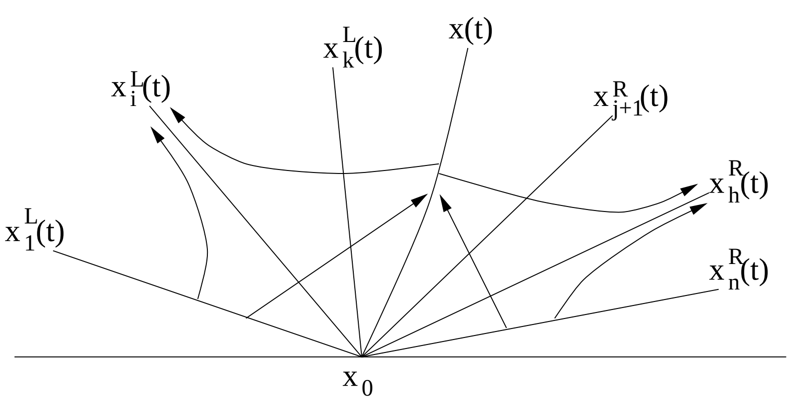

FIGURE 6.1. Define the scaling coefficients by tracing the characteristics.

Though such scaling is intuitively possible, for completeness, we will present it in detail. We describe our construction in five steps. Let c be any fixed constant that satisfies $0 < c < 1$.

1. Consider $-k \le i \le -1$. For such i, we have $\mathcal{D}^{(i)} \subset \mathcal{D}^L$. For $1 \le s \le n$, and $-k \le i \le \min\{-1, s-k-2\}$, define $\gamma_s^{(i)} = c^{i+k}$, and $V_s^{(i)} = \gamma_s^{(i)} v_s^{(i)}$. One can verify that on $\mathcal{D}^{(i)}, -k \le i \le -1$, for any $i+k+2 \le s \le n$, the sth characteristic curve is transverse to $y_i(t)$ where $V_s^{(i)} = cV_s^{(i-1)}$ is satisfied.

In fact, if $k+1 \le s \le n$, the right-going sth characteristics leaving $y_{-k}(t)$ covers the entire domain $\mathcal{D}^L$, so we allow $-k \le i \le -1$. If $1 \le s \le k$, the right-going sth characteristics leaving $y_{-k}(t)$ cannot pass $y_{s-k-1}(t)$. It covers only $\mathcal{D}^{(-k)}, \ldots, \mathcal{D}^{(s-k-2)}$. Thus, we allow only $-k \le i \le s-k-2$.

2. Consider $0 \le i \le n-j-1$. For such i, we have $\mathcal{D}^{(i)} \subset \mathcal{D}^R$. For $1 \le r \le n$, and $\max\{0, r-j\} \le i \le n-j-1$, define $\gamma_r^{(i)} = c^{n-j-i-1}$, and $V_r^{(i)} = \gamma_r^{(i)} v_r^{(i)}$. One can verify that on $\mathcal{D}^{(i)}, 0 \le i \le n-j-2$, for any $1 \le r \le i+j$, the rth characteristic curve is transverse to $y_{i+1}(t)$ where $V_r^{(i)} = cV_r^{(i+1)}$ is satisfied.

In fact, if $1 \le r \le j$, the left-going rth characteristic leaving $y_{n-j}(t)$ covers the entire $\mathcal{D}^R$, so we allow $0 \le i \le n-j-1$. If $j+1 \le r \le n$, the left-going r characteristic leaving $y_{n-j}(t)$ cannot pass $y_{r-j}(t)$. It covers only $\mathcal{D}^{(r-j)}, \ldots, \mathcal{D}^{(n-j-1)}$. Thus, we allow only $r-j \le i \le n-j-2$.

3. Let $0 < d$ be a small constant. Define

$$\gamma_{\hat{r}}^{(-1)} = d, \quad \hat{r} = 1, \ldots, k,$$
$$\gamma_{\hat{s}}^{(0)} = d, \quad \hat{s} = j+1, \ldots, n.$$

Note that $\hat{r}, \hat{s}$ correspond to indices of departing characteristics from $y_0(t)$. With $V_{\hat{r}}^{(-1)} = \gamma_{\hat{r}}^{(-1)} v_{\hat{r}}^{(-1)}$ and $V_{\hat{s}}^{(0)} = \gamma_{\hat{s}}^{(0)} v_{\hat{s}}^{(0)}$, we deduce from (6.13), (6.14) that

$$V_{\hat{s}}^{(0)} = K_{\hat{s}}^{(0)}(x, t, V_s^{(-1)}, V_r^{(0)}), \hat{s} = j+1, \ldots, n; s = k+1, \ldots, n; r = 1, \ldots, j,$$
$$V_{\hat{r}}^{(-1)} = K_{\hat{r}}^{(-1)}(x, t, V_s^{(-1)}, V_r^{(0)}), \hat{r} = 1, \ldots, k; s = k+1, \ldots, n; r = 1, \ldots, j.$$

Here $K_{\hat{s}}^{(0)}$ and $K_{\hat{r}}^{(-1)}$ are C^1 functions coming from the rescaling of $G_{\hat{s}}^{(0)}$ and $G_{\hat{r}}^{(-1)}$. It is clear that we can choose $d > 0$ sufficiently small so that

$$\max_{\text{rows of } K_i^{(\ell)}} \{\sum_s |\frac{\partial K_i^{(\ell)}}{\partial V_s^{(-1)}}| + \sum_r |\frac{\partial K_i^{(\ell)}}{\partial V_r^{(0)}}|\} < 1.$$

Here $\ell = -1, i = \hat{r}$ or $\ell = 0, i = \hat{s}$.

4. We are now following the characteristics that leave $y_0(t)$ and enter $\mathcal{D}^L$. For $1 \le r \le k-1$ and $r-k-1 \le i \le -2$, define $\gamma_r^{(i)} = dc^{-i-1}$ and $V_r^{(i)} = \gamma_r^{(i)} v_r^{(i)}$. One can verify that on $\mathcal{D}^{(i)}, -k \le i \le -2$, for any $1 \le r \le i+k+1$, the rth characteristic is transverse to $y_{i+1}(t)$ where $V_r^{(i)} \le cV_r^{(i+1)}$ is satisfied.

5. We are now following the characteristics that leave $y_0(t)$ and enter $\mathcal{D}^R$. For $j+2 \le s \le n$ and $1 \le i \le s-j-1$, define $\gamma_s^{(i)} = dc^i$ and $V_s^{(i)} = \gamma_s^{(i)} v_s^{(i)}$. One can verify that on $\mathcal{D}^{(i)}, 1 \le i \le n-j-1$, for any $i+j+1 \le s \le n$, the sth characteristic is transverse to $y_i(t)$ where $V_s^{(i)} = cV_s^{(i-1)}$ is satisfied.

It is obvious from the construction that the characterizing number of the boundary value problem for the new variable $V^{(i)}, -k \leq i \leq n-j-1$ is $|\gamma H \gamma^{-1}| < 1$. Thus from Chapter 4 of [10], the new boundary value problem for $V^{(i)}$ has a unique solution. Therefore, the original boundary value problem has a unique solution.

7. Some examples

The results of this paper are based on some generic conditions which are not easy to verify by hand. In this section, we will give some very simple examples which are somewhat artificial. In the first example, we show that an initial shock profile of an over compressive conservation law may fail to propagate as one shock, giving rise to two non-over compressive shocks. This is why over compressive shocks are not considered in this paper. In the second example we present a shock solution of a non conservation system modified from the first example.

Consider a system of two conservation laws:

$$u_{it} + u_i u_{ix} = 0, \quad i = 1, 2,$$

with initial conditions:

$$u_1(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0. \end{cases},$$

$$u_2(x,0) = \begin{cases} \delta + 1, & x < 0,\ 0 < \delta < 0.5, \\ -\delta(1+x), & x \geq 0. \end{cases}$$

If considered separately, each u_i has a shock $x_i(t)$ starting at $x_i(0) = 0$. But the shock speed for u_1 is $s = 0.5$ by the usual (RH) condition while the speed for u_2 is non-constant, though starting with the same speed $s = \frac{d}{dt} x_2(0) = 0.5$. The inconsistency of the shock speed means that the single shock for the system can not be continued to $t > 0$.

It is easy to verify that the above system satisfies (H1) and (H2'). However, since $k = 0$ and $j = n = 2$, condition (H3): $k + 1 \geq j$, is not satisfied. the single shock we try to continue is over compressive.

Let us look at the shock profile:

$$\begin{aligned} &q_i'' = (q_i - s) q_i', \quad \text{or} \\ &q_i' = q_i^2/2 - s q_i - w, \quad w = (u_i^L)^2/2 - s u_i^L. \end{aligned}$$

The solution $q_i, i = 1, 2$ can be written by using hyperbolic tangent functions. The manifold $W^u(u^L) \cap W^s(u^R)$ is two dimensional, because the shock profile system is invariant under the two-parameter shifting $(q_1(t + t_1), q_2(t + t_2))$. As predicted by Lemma 5.5, the adjoint system has $m = 2$ linearly independent bounded solutions. It is impossible to continue the shock since two (GRH) conditions must be satisfied while s is the only parameter at our disposal.

We will denote the shock profile of u_1 connecting $u^L = 1$ to $u^R = 0$ by $q^*(\zeta)$.

Let $0 < \alpha, \beta < 0.5$. In the second example, we consider a non conservation system.

$$\begin{aligned} u_{1t} + (u_1 + \alpha u_2) u_{1x} &= 0, \\ u_{2t} + (\beta u_1 + u_2) u_{2x} &= 0. \end{aligned} \tag{7.1}$$

If at $(x_0, t_0) = (0, 0)$ the above has a shock profile, it must satisfy

$$\begin{aligned} u_1'' &= (u_1 + \alpha u_2 - s) u_1', \\ u_2'' &= (\beta u_1 + u_2 - s) u_2'. \end{aligned} \tag{7.2}$$

A solution for (7.2) can be constructed based on the function q^*. Let

$$p_1 = \frac{q^*(1-\alpha)}{1-\alpha\beta}, \quad p_2 = \frac{q^*(1-\beta)}{1-\alpha\beta}.$$

With $s = 0.5$, $(u_1, u_2) = (p_1, p_2)$ is a heteroclinic solution of (7.2) connecting $(u_1^L, u_2^L) = (\frac{(1-\alpha)}{1-\alpha\beta}, \frac{(1-\beta)}{1-\alpha\beta})$ to $(u_1^R, u_2^R) = (0, 0)$.

It is easy to verify that $k = 0, j = 2$. Thus the profile we construct represents an over compressive shock for the non conservation system (7.1). The shock profile is invariant under the one-parameter shifting $(p_1(t + t_0), p_2(t + t_0))$. Therefore, we expect that the connection between the left and right states is one-dimensional. It is my belief that generically, for almost every small (α, β), Hypotheses (H1), (H2), (H4) of this paper are satisfied. A proof would use many tools in the bifurcation theory of homoclinic orbits with non hyperbolic equilibria and is better left to a separate paper. Then, from Lemma 5.2, the adjoint equation of the linearized system has one linearly independent solution and the (GRH) consists of only one equation. To satisfy the (GRH) condition, we only need to find the correct wave speed s. Therefore, for any piecewise smooth initial condition $(u_1(x, 0), u_2(x, 0))$ with

$$(u_1(0-, 0), u_2(0-, 0)) = (\frac{(1-\alpha)}{1-\alpha\beta}, \frac{(1-\beta)}{1-\alpha\beta}), \quad (u_1(0+, 0), u_2(0+, 0)) = (0, 0),$$

the shock solution can be continued to $t > 0$ for a short time.

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Department of Mathematics, North Carolina State University,
Raleigh, NC 27695-8205, xblin@math.ncsu.edu