A Dynamical Systems Approach to Traveling Wave Solutions for Liquid/Vapor Phase Transition

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Dedicated to Professor George Sell's 70th birthday

Abstract We study the existence of liquefaction and evaporation waves by the methods derived from dynamical systems theory. A traveling wave solution is a heteroclinic orbit with the wave speed as a parameter. We give sufficient and necessary conditions for the existence of such heteroclinic orbit. After analyzing the local unstable and stable manifolds of two equilibrium points, we show that there exists at least one orbit connecting the local unstable manifold of one equilibrium point to the local stable manifold of another equilibrium point. The method is known as the shooting method in the literature.

Mathematics Subject Classification (2010): Primary; Secondary

1 Introduction

Dynamic flows involving liquid/vapor phase transition is an important phenomenon occurring in many engineering processes. For retrograde fluids, i.e. fluids with high specific heat capacities, such flows can be approximated by assuming that the temperature is constant.

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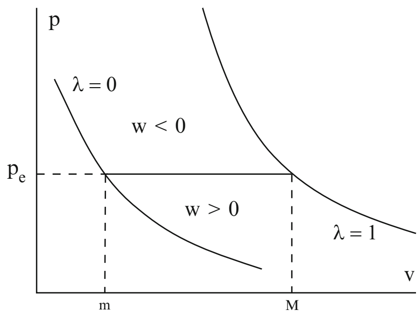

Fig. 1 The pressure function $p = p(\lambda, v)$ for some fixed λ

The one-dimensional case of the system describing such flows in Lagrangian coordinates is

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p(\lambda, v)_x = \varepsilon u_{xx}, \\ &\lambda_t = \frac{1}{\gamma} w(\lambda, v) + \beta \lambda_{xx}, \end{aligned} \tag{1}$$

where v is the specific volume, u the velocity of the fluid, λ the weight portion of vapor in the liquid/vapor mixture, ε the viscosity, β the diffusion coefficient and $\gamma > 0$ the typical reaction time. The pressure $p(\lambda, v)$ in (1) is a smooth function that satisfies

$$p_v < 0 < p_\lambda, \quad p_{vv} > 0. \tag{2}$$

Although $p_{vv} > 0$ was assumed in many previous papers, we do not use it in the proof of the existence of liquefaction and evaporation waves (Theorems 2.4 and 2.6).

Figure 1 shows the graph of a typical pressure function, where p_e is the equilibrium pressure at which liquid and vapor can coexist and m, M are the Maxwell points.

The function $w(\lambda, v)$ represents the rate of vapor initiation and growth:

$$w(v, \lambda) = (p(\lambda, v) - p_e)\lambda(\lambda - 1). \tag{3}$$

A traveling wave of (1.1) is a solution of the form $(u, v, \lambda)(s)$, where $s = \frac{x-ct}{\varepsilon}$ and c is the speed of the wave. With $u' = du/ds$, we have

$$\begin{aligned} &-cv' - u' = 0, \\ &-cu' + p' = u'', \\ &-c\lambda' = aw(\lambda, v) + b\lambda'', \\ &(u, v, \lambda)(\pm\infty) = (u_\pm, v_\pm, \lambda_\pm), \end{aligned} \tag{4}$$

where $a = \varepsilon/\gamma$, $b = \beta/\varepsilon$. Because λ is the weight portion of the vapor in the liquid/vapor mixture, we only admit solutions with $0 \le \lambda \le 1$.

From the third equation of (4), equilibrium points $(\lambda_\pm, v_\pm)$ must satisfy $w(\lambda, v) = 0$, which has three branches of solutions: $\lambda = 0$ or $\lambda = 1$ or $p(\lambda, v) = p_e$.

From the first two equations of (4), $c^2 v' + p' = u'' = -cv''$. Integrating from $-\infty$ to s we have:

$$-cdv/ds = c^2(v - v_-) + (p - p_-).$$

Let $s = b\xi$ and $u' = du/d\xi$. We have

$$\begin{aligned} \lambda'' &= -c\lambda' - abw(\lambda, v), \\ -\frac{c}{b}v' &= c^2(v - v_-) + (p(\lambda, v) - p_-). \end{aligned} \tag{5}$$

If the traveling wave connects $E_\pm$ with $(\lambda, v) = (\lambda_\pm, v_\pm)$, then

$$c^2(v_+ - v_-) + p(v_+, \lambda_+) - p(\lambda_-, v_-) = 0. \tag{6}$$

Definition 1.1. A liquefaction wave is a solution of (1.4) with

$$\lambda_- = 0,\ 0 < \lambda_+ \le 1,$$
$$p(\lambda_\pm, v_\pm) \ge p_e,\ c^2 + p_v(\lambda_\pm, v_\pm) < 0,$$

while an evaporation wave is that with

$$\lambda_- = 1,\ 0 \le \lambda_+ < 1,$$
$$p(\lambda_\pm, v_\pm) \le p_e,\ c^2 + p_v(\lambda_\pm, v_\pm) < 0.$$

A collapsing wave is a solution of (1.4) with

$$0 \le \lambda_- < 1,\ \ \lambda_+ = 1,$$
$$p(\lambda_\pm, v_\pm) \ge p_e,\ \ c^2 + p_v(\lambda_\pm, v_\pm) > 0,$$

while an explosion wave is that with

$$0 < \lambda_- \le 1,\ \ \lambda_+ = 0,$$
$$p(\lambda_\pm, v_\pm) \le p_e,\ \ c^2 + p_v(\lambda_\pm, v_\pm) > 0.$$

Recall that $\sqrt{-p_v}$ is the speed of sound. The definitions can be summarized in the following table:

	$p \ge p_e$	$p \le p_e$
subsonic $c^2 + p_v < 0$	liquefaction	evaporation
supersonic $c^2 + p_v > 0$	collapsing	explosion

Shearer [12–14], Fan [4] and Slemrod [15] studied the Liquid/vapor phase transition through the p-system of conservation laws of hyperbolic-elliptic mixed type. Fan [5–7] proved the existence of liquefaction and evaporation waves. He also studied the stability of a simplified system consisting of a system of two conservation laws and a KPP equation. See also [1–3, 8] for further discussions of the model.

The following methods were used in [5, 7] in proving the existence of traveling waves:

(1) The Leray–Schauder degree theory.
(2) The theory of monotone systems of parabolic PDEs.

A common feature to methods (1) and (2) is the adding of a small diffusion term ηv_{xx} to the system,

$$v_t = \eta v_{xx} + c^2(v - v_-) + p - p_-,$$
$$\lambda_t = b\lambda_{xx} + a(p(\lambda, v) - p_e)\lambda(\lambda - 1).$$

First one finds traveling waves for the system with small $\eta > 0$. Then one shows that there exists a sequence $\eta_n \to 0$ such that the corresponding solutions $(\lambda^{\eta_n}, v^{\eta_n}) \to (\lambda, v)$. The limit is a traveling wave corresponding to $\eta = 0$.

We briefly describe the use of the Leray–Schauder degree theorem to our system. Consider the modified system:

$$\eta v'' + cv' = -\theta(c^2(v - v_-) + p - p_-),$$
$$b\lambda'' + c\lambda' = -\theta\, a w(v, \lambda),\ -L < \xi < M,$$
$$(v, \lambda)(-L) = (v_-, \lambda_-), \quad (v, \lambda)(M) = (\bar{v}_+, \bar{\lambda}_+).$$

By choosing $(\bar{v}_+, \bar{\lambda}_+)$ properly, once can show that there is a strictly monotone solution for all $\theta \in [0, 1]$.

Write the system as an integral equation $T(x, \theta) = x$. Then $T : \bar{\Omega} \times [0, 1] \to X$ is a compact operator in a real normed space. Moreover the solution exists if $\theta = 0$. From the Leray–Schauder degree, if we assume:

(i) $T(x, \theta) \neq x$ for $x \in \partial\Omega,\ \theta \in [0, 1]$.
(ii) The Leray–Schauder degree $D_I(T(\cdot, 0) - I, \Omega) \neq 0$,

Then for any $0 \leq \theta \leq 1$, $T(x, \theta) = x$ has at least one solution in Ω.

We then find convergent subsequences of monotone solutions such that

(i) $L(n) \to \infty, M(n) \to \infty$
(ii) $\bar{v}_n \to v_+, \bar{\lambda}_n \to \lambda_+$
(iii) $\eta_n \to 0$.

The limit of solutions is the traveling wave solution to the system with $\eta = 0$. Next, we briefly describe the use of the "Method for Monotone Systems of PDES" to our system. Consider

$$v_t = \eta v_{xx} + c^2(v - v_-) + p - p_-,$$
$$\lambda_t = b\lambda_{xx} + a(p - p_e)\lambda(\lambda - 1).$$

We can rewrite the system as

$$U_t = AU_{xx} + F(U, c),$$

where

$$F(U,c) = \begin{pmatrix} c^2(v - v_-) + p - p_- \\ a(p - p_e)\lambda(\lambda - 1) \end{pmatrix},$$
$$\nabla F = \begin{pmatrix} c^2 + p_v & p_\lambda \\ aP_v\lambda(\lambda - 1) & a(p - p_e)(2\lambda - 1) + aP_\lambda\lambda(\lambda - 1) \end{pmatrix}.$$

Under the sufficient conditions, we can verify that

(1) The system is monotone: off-diagonal terms of ∇F are positive.
(2) The eigenvalues of ∇F at U_- are negative.
(3) The wave speed c is sufficiently large.
(4) Other conditions for a monotone system are satisfied.

Then there exists a monotone solution U for small η. By letting $\eta \to 0$ we find the limit of solutions which corresponds to solutions for the system with $\eta = 0$.

Using a geometric/dynamical system's method (shooting method), Fan and Lin simplified the proof of the existence of evaporation and liquefaction waves obtained in [5,7]. We also rigorously proved the existence of collapsing and explosion waves that were only verified numerically before. Define $h(\lambda, v) := c^2(v - v_-) + p(\lambda, v) - p_-$. Then

$$\mathcal{C} := \{(\lambda, v) : h(\lambda, v) = 0\}$$

is the isocline for v due to (5). Based on (6),

$$h(\lambda, v) := c^2(v - v_\pm) + p(\lambda, v) - p_\pm.$$

In this paper we summarize our results obtained by the shooting method from 2005 to 2008 as follows:

Theorem 1.1. *(1) The sufficient and necessary conditions for the existence of collapsing waves are:*

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_\pm, v_\pm) \geq 0.$$

(2) The sufficient conditions for the existence of explosion waves are:

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_-, v_-) \geq 0.$$

The necessary conditions for the existence of explosion waves are:

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_+, v_+) \geq 0.$$

(3) *If* $\lambda_+ = 0, 1$, *then the sufficient conditions for the existence of liquefaction or evaporation waves are:*

$$c^2 + p_v(\lambda, v) < 0, \text{ if } v_- \leq v \leq v_+, \ \lambda = 0, 1,$$

and

$$c^2 \geq 4ab|p(\lambda_-, v_-) - p_e|.$$

(4) *If* $p_+ = p_e, 0 < \lambda_+ < 1$, *then the sufficient conditions for the existence of liquefaction or evaporation waves are:*

$$c^2 \geq 4ab|p(\lambda_-, v_-) - p_e|,$$
$$c^2 + p_v(\lambda, v) < 0, \text{ if } \lambda = 0 \leq \lambda \leq \lambda_+, \ v_- \leq v \leq v_+,$$

and along the isocline $\mathcal{C}$ *for* v,

$$\sup_{\lambda} \left\{ \frac{ab p_\lambda(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} \right\} < 1.$$

As proofs of the existence of collapsing and explosion waves were presented in a separate paper [9], in the rest of this paper we will study the existence of liquefaction and evaporation waves only. The existence of liquefaction waves for $\lambda_- = 0$, $\lambda_+ = 1$ will be proved in Theorem 2.4 while the existence of liquefaction waves for $\lambda_- = 0$, $0 < \lambda_+ < 1$, $p_+ = p_e$ will be proved in Theorem 2.6. Similar proofs apply to the evaporation waves and will not be presented in this paper.

For liquefaction and evaporation waves, it will be shown in Lemma 2.1 that the wave speed c is positive. Therefore, from (6),

$$c = \sqrt{-\frac{p_(\lambda_+, v_+) - p(\lambda_-, v_-)}{v_+ - v_-}}. \tag{7}$$

In [5, 7], Fan proved that liquefaction and evaporation waves exist if the wave speed $c > 0$ satisfies $c \geq 2\sqrt{ab|p(\lambda_-, v_-) - p_e|}$. On the other hand, if the speeds satisfy $c \leq 2\sqrt{ab|p(\lambda_+, v_+) - p_e|}$, then there is no liquefaction or evaporation waves.

The locations of $(v_\pm, \lambda_\pm)$ for both waves are depicted in Fig. 2.

Recall that the pressure $p(\lambda, v)$ satisfies

$$p_v < 0 < p_\lambda,$$

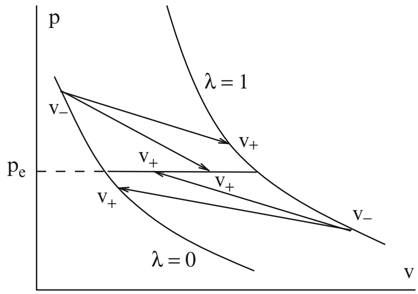

Fig. 2 The points $(v_\pm, \lambda_\pm)$ of liquefaction waves ($p_\pm \geq p_e$) and evaporation waves ($p_\pm \leq p_e$). The *arrows* point to the fronts of the waves

and the growth rate w is

$$w(\lambda, v) = (p - p_e)\lambda(\lambda - 1).$$

Since $p_\lambda > 0$, the function $p = p(\lambda, v)$ can be solved for $\lambda = \lambda^*(v, p)$. For each $v = v_0$, with $v_m < v_0 < v_M$, $w = w(\lambda, v_0)$ has three zeros: $\lambda = 0$, $\lambda_e(v_0) = \lambda^*(v_0, p_e)$, $\lambda = 1$. However, for $v_0 < v_m$ or $v_0 > v_M$, $w = w(\lambda, v_0)$ has two zeros: $\lambda = 0$, $\lambda = 1$.

Let $\mathcal{P} \subset \mathbb{R}^3$ be an open set bounded by finitely many smooth surfaces.

Definition 1.2. For each $P \in \mathcal{P}$ such that the $\Phi(\xi_0, P) \in \partial\mathcal{P}$ for some $\xi_0 > 0$, there exists the first touch time ξ_1 such that

$$\Phi(\xi_1, P) \in \partial\mathcal{P}, \text{ while } \Phi(\xi, P) \in \mathcal{P} \text{ for } 0 \leq \xi < \xi_1.$$

If the first touch time for P exists, then define the first touch point as $B(P) := \Phi(\xi_1, P)$.

The following lemma is related to the Wazewski's principle [10, 11]. It is not as general but works well on our system.

Lemma 1.2. *Assume that there exit two mutually disjoint open subsets of $\partial\mathcal{P}$: S_1 and S_2 such that,*

(1) For any $P \in S_j$, $j = 1, 2$, there exists a small $\varepsilon > 0$ such that $\Phi(\xi, P) \in \mathcal{P}$ for $-\varepsilon < \xi < 0$. Moreover, the flow $\Phi(\xi, \cdot)$ is transverse to S_1 or S_2.
(2) For any $P \in \mathcal{P}$ such that $\Phi(\xi, P) \in \partial\mathcal{P}$ for some $\xi > 0$, we have $B(P) \in S_1 \cup S_2$.
(3) There exists a smooth curve segment $\overline{P_1P_2}$ in $\mathcal{P}$ such that $B(P_1) \in S_1$, $B(P_2) \in S_2$.

Under these conditions, there exits a $P_0 \in \overline{P_1P_2}$ such that $\Phi(\xi, P_0)$ remains in $\mathcal{P}$ for all $\xi > 0$.

The shooting method alone does not provide information on the uniqueness of the traveling waves for each fixed wave speed c. In a work-in-progress paper by Fan and Lin, numerical computation combined with the shooting method has been performed on a similar system. For a given wave speed, these results suggest that each type of traveling wave for liquid/vapor phase transition may be unique.

2 Existence of Liquefaction and Evaporation Waves

In this section we present the proof of the existence of liquefaction waves for the case $\{\lambda_- = 0\} \to \{\lambda_+ = 1\}$, and the case $\{\lambda_- = 0\} \to \{p_+ = p_e\}$. The same proof applies to the evaporation waves with some minor changes.

System (5) can be written as a first order system of three variables (λ, μ, v):

$$\begin{aligned} \lambda' &= \mu, \\ \mu' &= -c\mu - abw(\lambda, v), \\ -\frac{c}{b}v' &= c^2(v - v_-) + (p(\lambda, v) - p_-). \end{aligned} \tag{8}$$

We look for a heteroclinic solution of (8) connecting the equilibrium points $E_\pm := \{(\lambda_\pm, \mu_\pm, v_\pm)\}$.

Equilibrium states are the zeros of the right hand side of (8).

$$\begin{aligned} \mu &= 0, \\ w(\lambda, v) &= 0, \\ c^2(v - v_-) + p(\lambda, v) - p_- &= 0. \end{aligned} \tag{9}$$

The solutions of $w = 0$ form three branches: $\lambda = 0,\ 1$ and $p(\lambda, v) = p_e$. The graph of (9) with a given c is a straight line in the (v, p) plane, see Fig. 2. However, the graph of (9) in the (λ, v) plane is the isocline $\mathcal{C}$ for v.

For each v with $v_m < v < v_M$, by solving $p(\lambda, v) = p_e$ for λ, we have

$$\lambda = \lambda_e(v).$$

The equilibrium E_- is on $\lambda = 0$ with $p_- > p_e$ or on $\lambda = 1$ with $p_- < p_e$. The equilibrium E_+ is on the line $\lambda = 1, p_+ > p_e$ or on $p = p_e, 0 < \lambda < 1$ (liquefaction wave). The equilibrium E_+ is on $\lambda = 0, p < p_e$ or on $p = p_e, 0 < \lambda < 1$ (evaporation wave).

Let $p_+ = p(\lambda_+, v_+)$. The wave speed c and $v_\pm$ are now related by (6): $c^2(v_+ - v_-) + (p_+ - p_-) = 0$.

From (6), for the liquefaction wave $p_+ < p_-$, we must have $v_+ > v_-$; while for the evaporation wave $p_+ > p_-$, we must have $v_+ < v_-$.

2.1 Eigenvalues and Eigenvectors at Equilibrium Points

In this section, we first show that if $c > 0$, then the equilibrium E_- corresponding to $\lambda = 0, 1$ is a saddle with exactly two positive eigenvalues and one negative eigenvalue, while the equilibrium E_+ has one positive eigenvalues and two negative eigenvalues. The traveling wave solution we look for is a heteroclinic solution connecting saddle to saddle. Moreover, as $\xi \to \pm\infty$, the orbit of the traveling wave starts at the two dimensional local unstable manifold $W^u_{loc}(E_-)$ and ends at the two dimensional local stable manifold $W^s_{loc}(E_+)$.

The linear variational system at equilibrium point is

$$\begin{pmatrix} \Lambda \\ M \\ V \end{pmatrix}' = A \begin{pmatrix} \Lambda \\ M \\ V \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 0 & 1 & 0 \\ -abw_\lambda & -c & -abw_v \\ -\frac{b}{c}p_\lambda & 0 & -bc - \frac{b}{c}p_v \end{pmatrix}.$$

Eigenvalues r are determined by

$$\det(rI - A) = \begin{vmatrix} r & -1 & 0 \\ abw_\lambda & r+c & abw_v \\ \frac{b}{c}p_\lambda & 0 & r + \frac{b}{c}(p_v + c^2) \end{vmatrix} = 0.$$

We first study eigenvalues at the equilibrium $E_\pm$ with $\lambda = 0$ or 1,

$$w_\lambda(\lambda, v) = (2\lambda - 1)(p - p_e), \quad w_v(\lambda, v) = 0,$$

$$\det(rI - A) = (r^2 + cr + w_\lambda)(r + \frac{b}{c}(p_v + c^2)).$$

Eigenvalues at $\lambda = 0, 1$ are

$$r_1 = -c/2 - \sqrt{(c/2)^2 - ab(2\lambda - 1)(p(\lambda, v) - p_e)},$$

$$r_2 = -c/2 + \sqrt{(c/2)^2 - ab(2\lambda - 1)(p(\lambda, v) - p_e)},$$

$$r_3 = -\frac{b}{c}(p_v(\lambda, v) + c^2).$$

Lemma 2.1. *(1) Assume that $c > 0$. At E_-, assume that $c^2 + p_v(\lambda, v) < 0$ and $(2\lambda - 1)(p - p_e) < 0$. Then the equilibrium E_- has two positive eigenvalues and one negative eigenvalue. At E_+, assume that $c^2 \geq 4ab|p(\lambda, v) - p_e|$ and $c^2 + p_v(\lambda, v) < 0$. Then if E_+ is on the line $\lambda = 0, 1$, it has two negative eigenvalues and one positive eigenvalue.*

(2) Assume that $c < 0$. At E_-, assume that $c^2 + p_v(\lambda, v) < 0$ and $(2\lambda - 1)(p - p_e) < 0$. Then the equilibrium E_- has two negative eigenvalues and one positive eigenvalue. At E_+, assume that $c^2 \geq 4ab|p(\lambda, v) - p_e|$ and $c^2 + p_v(\lambda, v) < 0$. Then if E_+ is on the line $\lambda = 0, 1$, it has two positive eigenvalues and one negative eigenvalue.

Proof. Proof of (1): We always have $r_3 > 0$ for $\lambda = 0$ or 1. If $\lambda = 0,\ p > p_e$ or if $\lambda = 1,\ p < p_e$, we have

$$ab(2\lambda - 1)(p(\lambda, v) - p_e) < 0,$$

and hence $r_1 < 0$ and $r_2 > 0$. Thus E_- has two unstable eigenvalues and one stable eigenvalue. If $\lambda = 1,\ p > p_e$ or if $\lambda = 0,\ p < p_e$, we use $c^2 \geq 4ab|p(\lambda, v) - p_e|$ to show r_1, r_2 are real and $r_1, r_2 < 0$. Thus E_+ has two stable eigenvalues and one unstable eigenvalue.

The proof of (2) is completely similar and shall be omitted. □

Since in the case (2), a heteroclinic connection from E_- to E_+ usually does not happen, we shall assume $c > 0$.

2.2 Existence of Liquefaction Waves for $\lambda_- = 0,\ \lambda_+ = 1$

In this section, we consider the liquefaction wave connecting $\lambda_- = 0$ to $\lambda_+ = 1$. The liquefaction wave that connects $\lambda_- = 0$ to $p_+ = p_e$ shall be constructed later. Since liquefaction and evaporation waves are subsonic waves, cf. Definition 1.1, we assume that the waves satisfy the following assumption in this section:

(H1) $c^2 + p_v(\lambda, v) < 0$, if $v_- \leq v \leq v_+$ and $\lambda = 0, 1$.

The traveling waves satisfy the following system of equations:

$$\lambda' = \mu, \quad \mu' = -c\mu - abw(\lambda, v), \tag{10}$$

$$\frac{-c}{b} v' = c^2(v - v_-) + (p(\lambda, v) - p_-). \tag{11}$$

As from Lemma 2.1, we assume that $c > 0$.

The isocline for v means $\mathcal{C} := \{(\lambda, v) : v' = 0\}$. Clearly $(\lambda, v) \in \mathcal{C}$ if $h(\lambda, v) = 0$. It is easy to see that on the two equilibrium points, $(\lambda_\pm, v_\pm) \in \mathcal{C}$.

Due to (H1), on the line $\lambda_- = 0$ we have $c^2 + p_v < 0$. If $v > v_-$, then $h(0, v) < 0$. Therefore $v' > 0$ if $v_- < v < v_+$ and $\lambda = 0$. Similarly, due to (H1) again, if $\lambda = 1$, we can show that $v' < 0$ if $v_- < v < v_+$. Now for each $v \in (v_-, v_+)$, there exists a unique $\lambda \in (0, \lambda_+)$ such that $h(\lambda, v) = 0$, denoted by

$$\lambda = \lambda_c(v).$$

Due to the fact $p_\lambda > 0$, $\lambda_c(v)$ is a smooth function of $v \in (v_-, v_+)$.

In general $\lambda_c(v)$ may not be a monotone function as depicted in Fig. 3.

The isocline for v divides the rectangle $(\lambda_-, \lambda_+) \times (v_-, v_+)$ into two parts. Let

$$\mathcal{N} := \{(\lambda, v) : v_- < v < v_+,\ 0 < \lambda < \lambda_c(v)\}.$$

If $(\lambda, v) \in \mathcal{N}$, then $v'(\xi) > 0$. Let EF be the curve on which $p = p_e$ and $v_- < v < v_+, \lambda_- < \lambda < \lambda_+$, see Fig. 3. Then $EF \subset \mathcal{N}$ where $v' > 0$. This can be shown as follows. Since on EF, $p = p_e < p_+$ and $v < v_+$, we have

$$h(\lambda, v) = c^2(v - v_+) + (p - p_+) < 0.$$

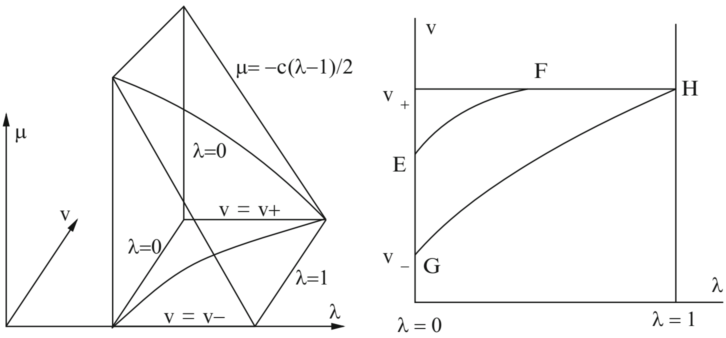

Fig. 3 The pentahedron and the top view of its base $\mathcal{F}_b$. On the curve EF, $p(\lambda,v) = p_e$. On the isocline GH, $v' = 0$

Consider a pentahedron shaped solid $\mathcal{P}$ in (λ,μ,v) space bounded by the five surfaces:

$$\text{Left side } \mathcal{F}_\ell := \{\lambda = 0\};$$
$$\text{Back side } \mathcal{F}_k := \{v = v_+,\ 0 < \lambda < 1,\ 0 < \mu < -c(\lambda-1)/2\};$$
$$\text{Front side } \mathcal{F}_f := \{v_- < v < v_+,\ \lambda = \lambda_c(v),\ 0 < \mu < -c(\lambda-1)/2\};$$
$$\text{Slant side } \mathcal{F}_s := \{c(\lambda-1)/2 + \mu = 0,\ 0 < \lambda < \lambda_c(v), v_- < v < v_+\};$$
$$\text{Bottom side } \mathcal{F}_b := \{\mu = 0,\ 0 < \lambda < 1\}.$$

The bottom side is further divided into $\mathcal{F}_b = \mathcal{F}_{b1} \cup \mathcal{F}_{b2}$, with

$$\mathcal{F}_{b1} := \mathcal{F}_b \cap \{p(\lambda,v) \geq p_e\},$$
$$\mathcal{F}_{b2} := \mathcal{F}_b \cap \{p(\lambda,v) < p_e\}.$$

For each interior point P of $\mathcal{P}$, let $B(P)$ be the first touch point on $\partial\mathcal{P}$ as in Definition 1.2.

(1) Since $d\lambda/d\xi = \mu > 0$ inside $\mathcal{P}$, $B(P) \notin \mathcal{F}_\ell$.
(2) On $\mathcal{F}_k$, we have $v' = -\frac{b}{c}(p(v_+,\lambda) - p_+)$. Since $p_\lambda > 0$ and $\lambda < 1$, we have $p(v_+,\lambda) < p(v_+,1) = p_+$. Thus, $v' > 0$. It is possible $B(P) \in \mathcal{F}_k$.
(3) On $\mathcal{F}_f$, we have $\lambda' > 0$ and $v' = 0$. Let the outward normal of $\mathcal{F}_f$ be $\mathbf{n} = \{(\lambda,\mu,v) = (1,0,-d\lambda_c(v)/dv)$, and let the vector field be $\mathbf{f}$. Then $\mathbf{n}\cdot\mathbf{f} > 0$. The flow starts on $\mathcal{F}_f$ must leave $\mathcal{P}$ transversely. It is possible that $B(P) \in \mathcal{F}_f$ for some $P \in \mathcal{P}$.
(4) On the interior of $\mathcal{F}_{b2}$ we have $d\mu/d\xi < 0$ due to $w > 0$ for $0 < \lambda < 1$ and $p < p_e$. It is possible that $B(P) \in \mathcal{F}_{b2}$.
(5) On the interior of $\mathcal{F}_{b1}$, we have $d\mu/d\xi = -abw(\lambda,v) > 0$ due to $w < 0$ for $0 < \lambda < 1$ and $p > p_e$. Thus $B(P)$ is not in the interior of $\mathcal{F}_{b1}$.

If $B(P) \in \{p = p_e\} \cap \mathcal{F}_{b1}$ at the first touch time $\xi_1 > 0$, then from (10) it is easy to verify that $\mu(\xi_1) = \mu'(\xi_1) = 0$ and $\mu''(\xi_1) < 0$. Therefore, there exists $\delta > 0$ such that $\mu(\xi) < 0$ if $\xi_1 - \delta \leq \xi < \xi_1$, contradicting to ξ_1 being the first touch time. So $B(P)$ cannot be on the line $\{p = p_e\} \cap \mathcal{F}_{b1}$.

The following lemma shows that $B(P) \notin \mathcal{F}_s$.

Lemma 2.2. *The first touch point with the boundary $B(P)$ is not on the slant side $\mathcal{F}_s$.*

Proof. The inward normal of the slant side $\mathcal{F}_s := \{c(\lambda - 1)/2 + \mu = 0\}$ is

$$\mathbf{n} = (\mathbf{n}_\lambda, \mathbf{n}_\mu, \mathbf{n}_v) = (-c/2, -1, 0).$$

The vector field is

$$\mathbf{f} = (\mathbf{f}_\lambda, \mathbf{f}_\mu, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on $\mathcal{F}_s$,

$$\mathbf{n} \cdot \mathbf{f} = -c\mu/2 + c\mu + abw(\lambda, v) > 0.$$

Using $\mu = -c(\lambda - 1)/2$, we have

$$\mathbf{n} \cdot \mathbf{f} = (1 - \lambda)((c/2)^2 - ab\lambda(p - p_e)). \tag{12}$$

Since (λ, v) satisfies

$$c^2(v - v_-) + (p - p_-) \leq 0, \quad \text{and } v \geq v_-,$$

we have $p \leq p_-$. Therefore,

$$(c/2)^2 > ab(p_- - p_e) \geq ab\lambda(p(\lambda, v) - p_e).$$

It follows that $\mathbf{n} \cdot \mathbf{f} > 0$, see (12). Therefore $B(P) \notin \mathcal{F}_s$. □

Let us check the edges of $\mathcal{P}$ (not including E_+). The four edges that lie on $\lambda = 0$ cannot contain $B(P)$ as shown by (1).

Among the other four edges, two of them bound $\mathcal{F}_s$, so they cannot contain $B(P)$ due to $\mathbf{n} \cdot \mathbf{f} > 0$ as in Lemma 2.2. What left are the two more edges that bound $\mathcal{F}_{b1}$ (not including $\{p = p_e\}$). They cannot contain $B(P)$ due to $\mu' > 0$.

We have shown that if $B(P)$ is the point where $\Phi(\xi, P)$ first hits the boundary of $\mathcal{P}$, either it lies on $S_1 := \mathcal{F}_f$ or it lies on $S_2 := \mathcal{F}_{b2} \cup \mathcal{F}_k \cup \{\mu = 0, v = v_+, 0 < \lambda < \lambda_e(v_+)\}$.

Moreover, $B(P)$ cannot belong to the four boundaries of $\mathcal{F}_\ell$, the three boundaries of $\mathcal{F}_s$ and the three boundaries of $\mathcal{F}_f$. The point $B(P)$ can belong to the common boundary of $\mathcal{F}_k$ and $\mathcal{F}_{b2}$ but not the common boundaries of $\mathcal{F}_k$ and $\mathcal{F}_{b1}$.

Lemma 2.3. *There exist* $P_1, P_2 \in W^u_{loc}(E_-) \cap \mathcal{P}$ *such that* $B(P_1) \in S_1$ *and* $B(P_2) \in S_2$.

Proof. To start the shooting method, we list facts about $W^u_{loc}(E_-)$: Let $r_1 < 0 < r_2$ be the two eigenvalues for (10). Let $r_3 > 0$ be the eigenvalue with the eigenvector $(0,0,1)$.

(1) The line $\{\lambda = 0\}$ is on $W^u_{loc}(E_-)$. On this line, we have

$$v' > 0 \text{ if } v > v_-, \quad v' < 0 \text{ if } v < v_-.$$

(2) $W^u_{loc}(E_-)$ is two dimensional with two linearly independent tangent vectors

$$(\Lambda, M, V) = (0,0,1) \text{ and } (\Lambda, M, V) = (1, r_2, 0).$$

Based on this, we can express the local unstable manifold as

$$W^u_{loc}(E_-) = \{(\lambda, v, \mu) : -\varepsilon_1 < \lambda < \varepsilon_1,\ v_- - \varepsilon_2 < v < v_- + \varepsilon_2,\ \mu = \mu^*(\lambda, v)\}.$$

Moreover, if $\lambda > 0$, then $\mu^* > 0$.

Recall that the isocline $\mathcal{C}$ for v can be expressed as $\lambda = \lambda_c(v)$ with $\lambda_c(v_-) = 0$, and $d\lambda_c(v_-)/dv > 0$. Choose $\bar{v}$ with $v_- < \bar{v} < v_- + \varepsilon_2$ so that $0 < \lambda_c(\bar{v}) < \varepsilon_1$.

For each $0 < \lambda_1 < \lambda_c(\bar{v})$, define a line segment $\overline{P_1P_2}$ on $W^u_{loc}(E_-)$:

$$\overline{P_1P_2} := \{(\lambda, v, \mu) : \lambda_1 \le \lambda \le \lambda_c(\bar{v}), v = \bar{v}, \mu = \mu^*(\lambda, v)\}.$$

It is parameterized by λ with $\lambda = \lambda_c(\bar{v})$ corresponding to P_1 and $\lambda = \lambda_1$ corresponding to P_2. It is also clear that $\overline{P_1P_2}$ is in $\mathcal{P}$ except for the point P_1.

Since the flow on the v-axis is transversal to the plane $\{v = v^+\}$, assuming that λ_1 is sufficiently small so that P_2 is close to the v-axis on $W^u(E_-)$, then the orbit $\Phi(\xi, P_2)$ stays close to the v-axis until it hits $v = v_+$ transversely. It is easy to show that $B(P_2)$ will hit the boundary of $\mathcal{P}$ in S_2. On the other hand, P_1 is on S_1 and the flow $\Phi(\xi, P_1)$ leaves $\mathcal{P}$ transversely at $P_1 \in S_1$. □

Theorem 2.4. *Consider the liquefaction waves with* $\lambda_- = 0$, $\lambda_+ = 1$. *Assume* $c^2 + p_v(\lambda, v) < 0$ *if* $\lambda = 0, 1$ *and* $v \in [v_-, v_+]$ *and* $c^2 > 4ab|p(\lambda_-, v_-) - p_e|$. *Then there exists a liquefaction wave connecting* E_- *to* E_+. *The* (λ, v) *components of the wave are monotone.*

Proof. There exits an relatively open subset $\mathcal{O}_1$ of $\overline{P_1P_2}$ containing every P such that $B(P) \in S_1$. There exits also an relatively open subset $\mathcal{O}_2$ of $\overline{P_1P_2}$ containing every P such that $B(P) \in S_2$. It is also clear that $\mathcal{O}_1$ and $\mathcal{O}_2$ are mutually disjoint. Since $P_1 \in \mathcal{O}_1$, $P_2 \in \mathcal{O}_2$ and $\overline{P_1P_2}$ is a connected set,

$$\overline{P_1P_2} - (\mathcal{O}_1 \cup \mathcal{O}_2) \tag{13}$$

is nonempty. Let P be a point from (13). Then $\Phi(\xi,P)$ cannot hit the boundary of $\mathcal{P}$. It must stay inside $\mathcal{P}$ and approach the equilibrium E_+. Also $\Phi(\xi,P) \to E_-$ as $\xi \to -\infty$ since $P \in W^u_{loc}(E_-)$. □

2.3 Existence of Liquefaction Waves for $\lambda_- = 0,\ 0 < \lambda_+ < 1,\ p_+ = p_e$

Assume $c > 0$ as before so that the equilibrium E_- corresponding to $\lambda = 0, 1$ is a saddle with exactly two positive eigenvalues and one negative eigenvalue. We do not know exactly what are the eigenvalues for E_+ when $p_+ = p_e, \lambda \neq 0, 1$. However, it s not used in the proof of Theorem 2.6.

Assume that

$$c^2 + p_v(\lambda, v) < 0$$

is satisfied throughout the region $\lambda \in [0, \lambda_e], v \in [v_-, v_+]$. The isocline $\mathcal{C} := \{v' = 0\}$ can be solved from the equation

$$c^2(v - v_+) + p(\lambda, v) - p_+ = 0,$$

and the solution can be expressed as

$$v = v_c(\lambda),\ 0 \le \lambda \le \lambda_+,$$
$$\frac{dv_c(\lambda)}{d\lambda} = \frac{-p_\lambda}{c^2 + p_v} > 0.$$

We look for the liquefaction wave connecting $\lambda_- = 0$ to $p_+ = p_e$. The traveling wave satisfies (10) and (11). As from Lemma 2.1, we assume that $c > 0$.

Consider a pentahedron shaped solid $\mathcal{P}$ in (λ, μ, v) space bounded by the five surfaces (Fig. 4):

Left side $\mathcal{F}_\ell := \{\lambda = 0\}$;

Back side $\mathcal{F}_k := \{v = v_+,\ 0 < \lambda < \lambda_e,\ 0 < \mu < -c(\lambda - \lambda_e)/2\}$;

Front side $\mathcal{F}_f := \{0 < \lambda < \lambda_e,\ v = v_c(\lambda),\ 0 < \mu < -c(\lambda - \lambda_e)/2\}$;

Slant side $\mathcal{F}_s := \{c(\lambda - \lambda_e)/2 + \mu = 0,\ 0 < \lambda < \lambda_e,\ v_c(\lambda) < v < v_+\}$;

Bottom side $\mathcal{F}_b := \{\mu = 0,\ 0 < \lambda < \lambda_e\}$.

The bottom side is further divided into $\mathcal{F}_b = \mathcal{F}_{b1} \cup \mathcal{F}_{b2}$, with

$$\mathcal{F}_{b1} := \mathcal{F}_b \cap \{p(\lambda, v) \ge p_e\},$$
$$\mathcal{F}_{b2} := \mathcal{F}_b \cap \{p(\lambda, v) < p_e\}.$$

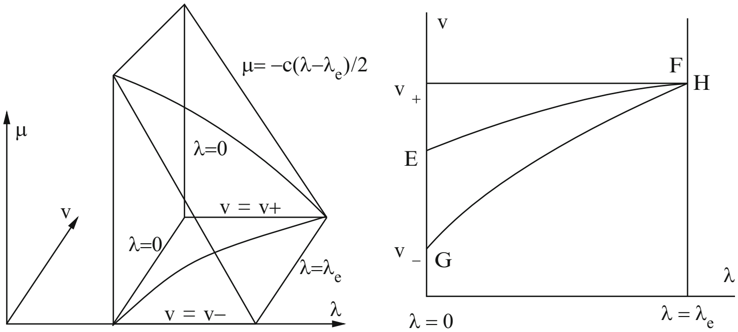

Fig. 4 The pentahedron and the top view of its base $\mathcal{F}_b$. On the curve EF, $p(\lambda,v) = p_e$. On the isocline GH, $v' = 0$

Let P be an interior point of $\mathcal{P}$ and $B(P)$ be the first touch point of the orbit at $\partial\mathcal{P}$. Just as in §2.2, we can show:

(1) $B(P) \notin \mathcal{F}_\ell$.
(2) It is possible that $B(P) \in \mathcal{F}_k$.
(3) It is possible that $B(P) \in \mathcal{F}_f$.
(4) It is possible that $B(P) \in \mathcal{F}_{b2}$.
(5) $B(P)$ is not in the interior of $\mathcal{F}_{b1}$. Also $B(P)$ cannot be on the line $\{p = p_e\} \cap \mathcal{F}_{b1}$.

The following lemma shows that $B(P) \notin \mathcal{F}_s$.

Lemma 2.5. *Assume that along the isocline $\mathcal{C}$, we have*

$$\sup_\lambda \left\{ \frac{abp_\lambda(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} \right\} < 1.$$

Then $B(P)$ is not on the slant side $\mathcal{F}_s$.

Proof. The inward normal of the slant side $\mathcal{F}_s := \{c(\lambda - \lambda_e)/2 + \mu = 0\}$ is

$$\mathbf{n} = (\mathbf{n}_\lambda, \mathbf{n}_\mu, \mathbf{n}_v) = (-c/2, -1, 0).$$

The vector fields are

$$\mathbf{f} = (\mathbf{f}_\lambda, \mathbf{f}_\mu, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on $\mathcal{F}_s$,

$$\mathbf{n} \cdot \mathbf{f} = -c\mu/2 + c\mu + abw(\lambda, v) > 0.$$

Using $\mu = -c(\lambda - \lambda_e)/2$, we have

$$\mathbf{n} \cdot \mathbf{f} = (\lambda_e - \lambda)\left((c/2)^2 - ab\frac{w(\lambda, v)}{\lambda - \lambda_e}\right). \tag{14}$$

Recall that $p_v < 0$, thus $\partial w/\partial v > 0$. Since on $\mathcal{F}_s$, we have $v_c(\lambda) < v < v_+$, therefore

$$\frac{w(\lambda, v)}{\lambda - \lambda_e} < \frac{w(\lambda, v_c(\lambda))}{\lambda - \lambda_e} \leq \frac{1}{4}\left|\frac{p(\lambda, v_c(\lambda)) - p(\lambda_e, v_c(\lambda_e))}{\lambda - \lambda_e}\right|, \tag{15}$$

by the fact $\lambda(1 - \lambda) < 1/4$ and $p(\lambda_e, v_c(\lambda_e)) = 0$.

The difference quotient can be estimated by

$$\begin{aligned}\sup_\lambda \left|\frac{dp(\lambda, v_c(\lambda))}{d\lambda}\right| &= \sup_\lambda |p_\lambda + p_v(dv_c(\lambda)/d\lambda)| \\ &= \sup_\lambda \left|p_\lambda + p_v\frac{-p_\lambda}{c^2 + p_v}\right| = \sup_\lambda \left|\frac{c^2 p_\lambda}{c^2 + p_v}\right|.\end{aligned}$$

If the assumption of the lemma is satisfied, then from from (14) and (15), we have $\mathbf{n} \cdot \mathbf{f} > 0$. □

Theorem 2.6. *Consider the liquefaction waves with* $\lambda_- = 0$, $0 < \lambda_+ < 1$, $p_+ = p_e$. *Assume that* $c^2 + p_v(\lambda, v) < 0$ *throughout the region and*

$$\sup_\lambda \left\{\frac{abp_\lambda(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|}\right\} < 1,$$

along the isocline $\mathcal{C}$ *for* v. *Then there exists a liquefaction wave connecting* E_- *to* E_+ *with* $p_+ = p_e$. *The* (λ, v) *components of the wave are monotone.*

Proof. The rest of the proof of the existence of the liquefaction waves follows exactly like the case where $\lambda_+ = 1$. □

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