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Heteroclinic Bifurcation and Singularly Perturbed Boundary Value Problems

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We study a singularly perturbed boundary value problem in $\mathbf{R}^{m+n}$: $\dot{x} = f(x, y, \varepsilon)$, $\varepsilon\dot{y} = g(x, y, \varepsilon)$, $B_1(x(\omega_0), y(\omega_0), \varepsilon) = 0$, $B_2(x(\omega_0+\omega), y(\omega_0+\omega), \varepsilon) = 0$. Given a candidate for the 0th order approximation which exhibits both boundary layers and interior layers, we present a complete procedure to compute higher order expansions and a procedure to compute the real solution near a truncated asymptotic expansion assuming the hyperbolicity of the regular layers and some generic assumptions. Similar results concerning the existence of periodic solutions (relaxation oscillations) are also presented. Several ideas from dynamical systems theory are employed, e.g., exponential dichotomies, Fredholm alternatives, and heteroclinic bifurcations. © 1990 Academic Press, Inc.

1. INTRODUCTION

We study the singularly perturbed boundary value problem

$$
\begin{aligned}
&\dot{x} = f(x, y, \varepsilon),\\
&\varepsilon\dot{y} = g(x, y, \varepsilon),\\
&B_1(x(\omega_0), y(\omega_0), \varepsilon) = 0,\\
&B_2(x(\omega_0+\omega), y(\omega_0+\omega), \varepsilon) = 0,
\end{aligned}
\qquad \omega_0 \leqslant t \leqslant \omega_0+\omega \tag{1.1}
$$

where f, g, B_1, and B_2 are vector-valued nonlinear functions. We assume that a candidate for the 0th order asymptotic approximation of (1.1) is given which admits boundary layers near $t=\omega_0$ and $\omega_0+\omega$, and several interior layers connecting the regular layers. Our main assumption is the absence of any turning point in the regular layers, i.e., the matrix $g_y(x, y, 0)$ is hyperbolic along the 0th order regular approximations. (This is not a generic assumption. However, functions $g(x, y, \varepsilon)$ that satisfy the assumption form an open set in a suitable Banach space.) We prove that the candidate for the 0th order approximation is a genuine one, i.e., there exists

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a real solution of problem (1.1) nearby, provided some additional generic assumptions about system (1.1) are valid. We also provide procedures to compute the higher order approximations and procedures to compute the exact solution for a fixed ε.

Our treatment uses methods of dynamical systems theory. The idea of applying dynamical systems methods to singular perturbation problems can be traced back several decades. Among many contributions we mention the work of Vasil'eva [29], Hoppensteadt [17], Fife [8, 9], Fenichel [7], and Hale and Sakamoto [14]. However, it is recent developments in dynamical systems theory that make possible a systematic treatment of this subject. Among these developments is the theory of homoclinic and heteroclinic bifurcation, which aims at understanding and predicting the complicated behavior near a transverse homoclinic orbit. This theory proves to be a powerful tool for studying singular perturbation problems, because of the observation that transition layers are in fact heteroclinic orbits connecting the regular layers. The usual approach to homoclinic bifurcation is Melnikov's method; see Holmes and Marsden [16]. We shall use instead a version due to Chow, Hale, and Mallet-Paret [3] and Palmer [23] that uses exponential dichotomies, Lyapunov–Schmidt reduction, and the Fredholm alternative. To solve successively the linear recursive equations that determine the higher order approximations, we shall again make use the exponential dichotomies and the Fredholm alternative. Another area of dynamical systems that we shall use is the theory of the center manifold and its stable and unstable fibers (Fenichel [7]), which furnishes the best geometric insights into the occurrence of interior and boundary layers.

Our approach begins with the following observation. Consider a truncation of the asymptotic expansion of the solution to a certain order

$$(x, y)=\left(\sum_{j=0}^{p} \varepsilon^j x_j(t), \sum_{j=0}^{p} \varepsilon^j y_j(t)\right).$$

We expect it to be piecewise continuous and allow jumps between outer and inner approximations. Moreover, i.e., due to the truncation, Eq. (1.1) and its boundary condition will not be satisfied exactly, and some residual error is expacted. A function $(x(t, \varepsilon), y(t, \varepsilon))$ is said to be a formal approximation or a pseudo-solution of the boundary value problem (1.1) if it is piecewise continuous, and if the jump error, residual error, and boundary error are small. According to the shadowing lemma in the dynamical system theory, if the linearization around $(x(t, \varepsilon), y(t, \varepsilon))$ has an exponential dichotomy, then $(x(t, \varepsilon), y(t, \varepsilon))$ is a genuine approximation, i.e., there is an exact solution nearby. The above program has been carried out in Lin [19], where a nonautonomous problem similar to (1.1), but with the variable x absent, was studied. It was shown that the heteroclinic solution became transverse, with the angle of stable and unstable spaces being $0(\varepsilon)$, after adding higher order expansions, while for $\varepsilon = 0$ the angle was 0. The idea was in fact Malnikov's idea in disguise. Also the conditions to ensure transversality of the heteroclinic solution were exactly those that enabled us to compute the higher order expansions.

The current work is a genuine generalization in the sense that by setting $i = 1$, the problem in Lin [19] may be written in the form (1.1). Moreover, we now have an m-dimensional center manifold corresponding to the x variable and the linearization ceases to have an exponential dichotomy. Therefore, the concept of exponential trichotomy is introduced. The linearization in this paper is made around the 0th order approximation so that in the iteration process an ε-independent linear operator is obtained. In the previous work (Lin [19]), the linearization was around the pth order truncation, and thus ε-dependent. The advantage of such a change in numerical implementation is clear.

Another outstanding problem for system (1.1) is how to "project" the boundary conditions to the regular layers, which satisfy an m-dimensional system of equations

$$\begin{aligned} \dot{x} &= f(x, y, 0), \\ 0 &= g(x, y, 0). \end{aligned} \tag{1.2}$$

Not all the boundary conditions can be satisfied by the regular layers. The "cancellation law," which determines the induced boundary conditions for system (1.2), has been studied by many authors; see Wasow [30], Harris [15], O'Malley [22], and Flaherty and O'Malley [10]. Ours is a geometric condition which requires that the center stable manifold of the first regular layer intersect transversely the initial manifold, determined by the zero set of $B_1(x, y, 0) = 0$, etc. Many authors have required that the size of the boundary layers be small, or that the function g be linear in the second variable y. It can be verified that our geometrical condition in these cases can be simplified greatly and leads directly to the previous results, e.g., Tupčiev [28] or Harris [15].

Systems like (1.1) arise in various fields: morphogenetic and population dynamics, ecology, physiology, and chemistry. Fife [9] studied the system of second order equations

$$\begin{aligned} \varepsilon^2 \ddot{u} &= f(u, v), \\ \ddot{v} &= g(u, v), \qquad 0 \leqslant t \leqslant 1 \\ u(i) &= \alpha_i, \qquad v(i) = \beta_i, \quad i = 0, 1. \end{aligned}$$

He showed the existence of boundary and interior layers under some general assumptions. His result was improved by Ito [20]. Later we shall give a simple generalization and show how his assumption implies ours. Mimura, Tabata, and Hosono [21] studied a similar problem but with Neumann boundary condition. In both of the two examples the number of transition layers for a given system can range from a positive integer to infinity. See Sakamoto [26] also.

Closely related to system (1.1) is the problem of the existence of periodic solutions of a singularly perturbed system. Supposing that the 0th order outer layers and inner layers form a closed cycle, we ask if the system of differential equations with a small nonzero ε possesses a periodic solution near the closed cycle. Such periodic solutions, which appear in many applied fields, are usually called relaxation oscillations (Grasman [12]).

Our treatment of the problem of existence of periodic solutions is analogous to that of the boundary value problem (1.1), and thus the general results will be stated without proof. As an application we consider traveling wave solutions of the FitzHugh–Nagamo equation, which satisfy a singularly perturbed system in $\mathbf{R}^3$:

$$\begin{aligned} u' &= v, \\ v' &= \theta v - f(u) + w, \\ w' &= \varepsilon\theta^{-1}(u - \gamma w). \end{aligned}$$

We give a short proof of the existence of periodic traveling wave solutions for a typical cubic-type nonlinear functions $f(u)$. Other types of traveling wave solutions of the FitzHugh–Nagumo equation are also solutions of suitable boundary value problems, and can be treated by the methods of this paper. The relaxation oscillation in van der Pal's equation, however, does not satisfy the hyperbolicity conditions posed in this paper because of the existence of turning points on the slow manifold. We shall discuss turning points in a separate paper.

Our main results and hypotheses are stated in Section 2, which also includes the example adapted from Fife [9]. The Analytic hypotheses are rather complicated. However, the geometric idea behind them is natural and simple and therefore is also presented in Section 2. Basic definitions and lemmas concerning the linear variational equation of the nonlinear problem are given in Section 3. The solution of the linear boundary value problem in Section 4 admits several specified jump discontinuities and reminds us of the shadowing lemma in the dynamical system theory. We shall use Theorem 4.9 in Section 6, however, we first prove Theorem 4.1 in which the boundary value problem is stated in a more symmetric way which allows a shorter proof. In Section 5 we give a complete procedure for the construction of inner and outer expansions. The major tool in solving the linear recursive equations is developed by many authors, e.g., Palmer [23] in the study of the bifurcation of homoclinic orbits. We show that no matching in the y direction is needed while matching in the x direction is required and has to be compatible with the reduced boundary conditions in the slow manifold. The proof of the validity of the formal expansions obtained in Section 5 is given in Section 6, which is in fact a straightforward application of Theorem 4.9. We point out that a lot of difficulty comes from the fact that system (1.1) is autonomous. Perturbation of the length of time intervals occurs in Sections 4, 5, and 6, which makes the presentation rather awkward. Franke and Selgrade [11] have proved a shadowing lemma for autonomous systems where very complicated rescaling of time also occurred. Singularly perturbed periodic solutions are discussed in Section 7.

Since problem (1.1) is autonomous, solutions or formal approximations of solutions are invariant under a shift of time, i.e., if $u(t)$, $\alpha \leqslant t \leqslant \beta$ is a solution, so is $v(t) = u(t + \Delta)$, $\alpha - \Delta \leqslant t \leqslant \beta - \Delta$. The idea of allowing different shifts of time in different layers is very useful and it leads to the defining of local time in each layer, which resemble local coordinate charts in the theory of differentiable manifolds. We use $Z_i(t) = (X_i(t), Y_i(t))$, $t \in [a_i, b_i]$, $1 \leqslant i \leqslant I$ (or $Z_i(\varepsilon\tau)$, $\tau \in [\alpha_i, \beta_i]$, $\alpha_i = a_i/\varepsilon$, and $\beta_i = b_i/\varepsilon$), to describe a regular layer. We use $z_i(\tau) = (x_i(\tau), y_i(\tau))$, $\tau \in \mathbf{R}$, $1 \leqslant i \leqslant I-1$, to describe an interior layer between $Z_i(t)$ and $Z_{i+1}(t)$. The change of local time follows the following rule: $\tau = 0 \in \mathbf{R}$ in $z_i(\tau)$ corresponds to $t = b_i$ in $Z_i(t)$, i.e., $t = b_i - \varepsilon\tau$, and $t = a_{i+1}$ in $Z_{i+1}(t)$, i.e., $t = a_{i+1} + \varepsilon\tau$. Boundary layers are described by local time $\tau \in \mathbf{R}^+$ in $z_0(\tau)$ and $\tau \in \mathbf{R}^-$ in $z_I(\tau)$, with $\tau = 0$ in $z_0(\tau)$ identified with $t = a_1$ in $Z_1(t)$ and $\tau = 0$ in $z_I(\tau)$ identified with $t = b_I$ in $Z_I(t)$. The advantage of introducing the local time becomes obvious when expanding a_i, b_i, α_i, and β_i in power series of ε. The use of local time allows us to compute each expansion $a_i + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(a)$ (or $b_i + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(b)$) separately without interacting with the others. Throughout this paper we use the index a (or b) to indicate a constant or a function associated with the left (or right) end point of an interval.

Let $\{u_i(t),\ t \in [a_i, b_i]\}_{i=1}^{\nu}$ be a sequence of piecewise smooth solutions of an autonomous ordinary differential equation. If $u_i(b_i) = u_{i+1}(a_{i+1})$ and the trajectories are oriented such that the one of $u_{i+1}(t)$'s follows from that of $u_i(t)$'s. We define a "global solution" $u(t) = \bigvee_{i=1}^{\nu} \{u_i(t), t \in [a_i, b_i]\}$ by pasting the local solutions together, where $\bigvee$ is called the pasting operator and $u(t)$, $t \in [\omega_0, \omega_0 + \omega]$, is defined as follows :

(i) $\omega_0 \in \mathbf{R}$ is an arbitrary constant, $\omega = \sum_{i=1}^{\nu} (b_i - a_i)$.

(ii) $u(t) = u_j(t - \omega_0 - \sum_{i=1}^{j-1} (b_i - a_i) + a_j)$ if $\sum_{i=1}^{j-1} (b_i - a_i) \leqslant t - \omega_0 \leqslant \sum_{i=1}^{j} (b_i - a_i)$.

Similarly if $\{u_i(t),\ t \in [a_i, b_i]\}_{i=1}^{\nu}$ is a sequence of formal approximations of solutions of an autonomous ODE, we can still define a global formal approximation $u(t) = \bigvee_{i=1}^{\nu}\{u_i(t),\ t \in [a_i, b_i]\}$ as shows in (i) and (ii). Here we do not require $u_i(b_i) = u_{i+1}(a_{i+1})$, thus $u(t)$ may have jumps, which presumably are small.

Two functions $u_i(t)$, $t \in J_i$, $i = 1, 2$, are said to be orbitally close if the graphs of those functions are close to each other. Define the orbital distance as

$$\text{dist}(u_1, u_2) = \sup\{\delta(u_1, u_2), \delta(u_2, u_1)\},$$

where

$$\delta(u_1, u_2) = \sup_{t_1 \in J_1} \{ \inf_{t_2 \in J_2} (|u_1(t_1) - u_2(t_2)| + |t_1 - t_2|)\}.$$

Define a subset $E_J(\gamma, l)$ of continuous functions on J as

$$E_J(\gamma, l) = \{x(\cdot) \mid \sup_{t \in J}(|x(t)|\, e^{\gamma|t|}(1 + |t|^l)^{-1}) < \infty\},$$

which is a Banach space with the norm

$$\|x\|_{E_J(\gamma, l)} = \sup_{t \in J}(|x(t)|\, e^{\gamma|t|}(1 + |t|^l)^{-1}),$$

where γ is a real constant and $l \geqslant 0$ an integer. Let

$$E_J^k(\gamma, l) = \{x(t) \mid x(t), x'(t), ..., x^{(k)}(t) \in E_J(\gamma, l)\},$$

which is a Banach space with

$$\|x\|_{E_J^k(\gamma, l)} = \sum_{j=0}^{k} \|x^{(j)}\|_{E_J(\gamma, l)}.$$

We use "$\cdot$" to denote d/dt and "$'$" to denote $d/d\tau$, where $\tau = t/\varepsilon$ is a fast variable. The range and kernel of linear operators are denoted by $\mathscr{R}$ and $\mathscr{K}$.

2. Assumptions, Main Results, and an Example

We study the singularly perturbed boundary value problem

$$\begin{aligned} &\dot{x} = f(x, y, \varepsilon), \\ &\varepsilon\dot{y} = g(x, y, \varepsilon), \\ &B_1(x(\omega_0), y(\omega_0), \varepsilon) = 0, \\ &B_2(x(\omega_0 + \omega), y(\omega_0 + \omega), \varepsilon) = 0, \end{aligned} \qquad \omega_0 \leqslant t \leqslant \omega_0 + \omega \tag{2.1}$$

$x \in \mathbf{R}^m$, $m \geqslant 1$ and $y \in \mathbf{R}^n$, $n \geqslant 1$. f, g, B_1, and B_2 are C^∞ with all the derivatives being bounded. B_1: $\mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{d_1}$ and B_2: $\mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{d_2}$ with $d_1 + d_2 = m + n + 1$. $\varepsilon > 0$ is small. $\omega > 0$ is a parameter to be determined by the problem. $\omega_0 \in \mathbf{R}$ is an arbitrary constant, irrelevant to the problem in fact.

Assume that the 0th order slow manifold (or regular, or outer, or center manifold) has several branches

$$\mathscr{S}_i = \{(x, y) \mid y = G^i(x), G^i \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)\}, \qquad 1 \leqslant i \leqslant I,$$

where $\mathscr{S}_i$ consists of the zeros of $g(x, y, 0) = 0$. Let $(X_0^i(t), Y_0^i(t))$, $t \in [a_i, b_i]$, be a solution of the 0th order outer equation

$$\begin{aligned} \dot{x} &= f(x, y, 0), \\ 0 &= g(x, y, 0), \end{aligned} \tag{2.2}$$

which lies on $\mathscr{S}_i$, $1 \leqslant i \leqslant I$. We do not assume that $a_{i+1} = b_i$, however, we assume that $X_0^i(b_i) = X_0^{i+1}(a_{i+1})$, $1 \leqslant i \leqslant I-1$. Let $(x_0^i, y_0^i(\tau))$, $0 \leqslant i \leqslant I$, be a solution of the 0th order inner equation

$$\begin{aligned} x'(\tau) &= 0, \\ y'(\tau) &= g(x(\tau), y(\tau), 0), \end{aligned} \tag{2.3}$$

where x_0^i is a constant with $x_0^i = X_0^i(b_i)$ for $1 \leqslant i \leqslant I$ and $x_0^0 = X_0^1(a_i)$. $y_0^i(\tau)$ is defined for $\tau \in \mathbf{R}$ if $1 \leqslant i \leqslant I-1$, $\tau \in \mathbf{R}^+$ if $i = 0$, and $\tau \in \mathbf{R}^-$ if $i = I$. $y_0^i(\tau) \to Y_0^i(b_i)$ as $\tau \to -\infty$, $1 \leqslant i \leqslant I$, and $y_0^i(\tau) \to Y_0^{i+1}(a_{i+1})$ as $\tau \to +\infty$, $0 \leqslant i \leqslant I-1$. Moreover, the 0th order boundary conditions are satisfied, i.e.,

$$\begin{aligned} B_1(x_0^0, y_0^0(0), 0) &= 0, \\ B_2(x_0^I, y_0^I(0), 0) &= 0. \end{aligned} \tag{2.4}$$

We assume the normal hyperbolicity on $\mathscr{S}_i$ near the orbit of $(X_0^i(t), Y_0^i(t))$:

$$\sigma\{g_y(X_0^i(t), Y_0^i(t), 0)\} \cap \{|\mathrm{Re}\, \lambda| \leqslant \alpha_0\} = \phi, \qquad \text{for all} \quad t \in [a_i, b_i]. \tag{H$_1$}$$

The dimension of the stable and unstable spaces of g_y are denoted by d^- and $d^+ = n - d^-$. Assume that d^-, d^+, and $\alpha_0 > 0$ do not depend on $1 \leqslant i \leqslant I$.

We need to consider the linear homogeneous equation

$$y'(\tau) - g_y(x_0^i, y_0^i(\tau), 0)\, y(\tau) = 0, \tag{2.5}$$

and the adjoint equation

$$y'(\tau) + g_y^*(x_0^i, y_0^i(\tau), 0)\, y(\tau) = 0. \tag{2.6}$$

Assumption (H_1) and the fact $(x_0^i, y_0^i(\tau)) \to \mathscr{S}_{i+1}$ as $\tau \to +\infty$ imply that (2.5) has an exponential dichotomy for $\tau \in \mathbf{R}^+$, $0 \leqslant i \leqslant I-1$, and similarly (2.5) has an exponential dichotomy for $\tau \in \mathbf{R}^-$, $1 \leqslant i \leqslant I$. (See Lemma 3.4 of Palmer [23].) Let the solution map of (2.5) be $\hat{U}^i(\tau, \sigma)$ and the projections to the stable and unstable spaces be $\hat{Q}_s^i(\tau)$ and $\hat{Q}_u^i(\tau) = 1 - \hat{Q}_s^i(\tau)$. It should be clear that $y_0^i(\tau)'$ is a nontrivial bounded solution of (2.5). Assume that $y_0^i(\tau)'$, $\tau \in \mathbf{R}$, $1 \leqslant i \leqslant I-1$, is unique among such solutions up to a scalar factor, then from the general theory of exponential dichotomies and the Fredholm alternative, see Palmer [23], there exists a bounded solution $\psi_i(\tau)$, $\tau \in \mathbf{R}$, $1 \leqslant i \leqslant I-1$, of (2.6), which is unique up to a scalar factor. Moreover $\psi_i(\tau) \to 0$ exponentially as $\tau \to \pm\infty$. We need the generic assumptions

$$\Delta_i \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi_i^*(\tau) \cdot g_x(x_0^i, y_0^i(\tau), 0)\, d\tau \neq 0, \qquad 1 \leqslant i \leqslant I-1, \tag{H2}$$

$$\begin{aligned} &\Delta_i \cdot f(X_0^i(b_i), Y_0^i(b_i), 0) \neq 0, \\ &\Delta_i \cdot f(X_0^{i+1}(a_{i+1}), Y_0^{i+1}(a_{i+1}), 0) \neq 0, \qquad 1 \leqslant i \leqslant I-1. \end{aligned} \tag{H$_3$}$$

Consider the equations

$$\begin{aligned} &\Big\{ B_{1x}(x_0^0, y_0^0(0), 0) \\ &\quad + B_{1y}(x_0^0, y_0^0(0), 0) \int_{\infty}^{0} \hat{U}^0(0, s)\, \hat{Q}_u^0(s)\, g_x(x_0^0, g_0^0(s), 0)\, ds \Big\}\, x \\ &\quad + B_{1y}(x_0^0, y_0^0(0), 0)\, \hat{Q}_s^0(0)\, y = 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\Big\{ B_{2x}(x_0^I, y_0^I(0), 0) \\ &\quad + B_{2y}(x_0^I, y_0^I(0), 0) \int_{-\infty}^{0} \hat{U}^I(0, s)\, \hat{Q}_s^I(s)\, g_x(x_0^I, y_0^I(s), 0)\, ds \Big\}\, x \\ &\quad + B_{2y}(x_0^I, y_0^I(0), 0)\, \hat{Q}_u^I(0)\, y = 0. \end{aligned} \tag{2.8}$$

Equation (2.7) is the equation for the common targent vector (x, y) of $\mathscr{K}B_1$ and the center stable space of $(X_0^1(a_1), X_0^1(a_1))$; see (3.3). Equation (2.8) has a similar meaning.

The left-hand side of (2.7) defines a linear operator $\mathscr{B}_1 : (x, \hat{Q}^0_s(0)\, y) \in \mathbf{R}^m \times \mathscr{R}\hat{Q}^0_s(0) \to \mathbf{R}^{d_1}$. Similarly, the left-hand side of (2.8) defines a linear operator $\mathscr{B}_2 : (x, \hat{Q}^I_u(0)\, y) \in \mathbf{R}^m \times \mathscr{R}\hat{Q}^0_u(0) \to \mathbf{R}^{d_2}$. We assume that

$$\mathscr{B}_1 \text{ and } \mathscr{B}_2 \text{ are surjective.} \tag{H$_4$}$$

$$\begin{aligned} &\{\operatorname{span}[f(X^1_0(a_1), Y^1_0(a_1), 0)] \otimes \mathscr{R}\hat{Q}^0_s(0)\} \cap \mathscr{K}\mathscr{B}_1 = \{0\}, \\ &\{\operatorname{span}[f(X^I_0(b_I), Y^I_0(b_I), 0)] \otimes \mathscr{R}\hat{Q}^I_u(0)\} \cap \mathscr{K}\mathscr{B}_2 = \{0\}. \end{aligned} \tag{H$_5$}$$

(H$_4$) and (H$_5$) imply that

$$m + d^- \geqslant d_1 \geqslant d^- + 1 \qquad \text{and} \qquad m + d^+ \geqslant d_2 \geqslant d^+ + 1. \tag{2.9}$$

Conversely, if either one of (2.9) is valid then (H$_4$) and (H$_5$) are generic assumptions.

From (H$_5$), $(x, y) \in \mathscr{K}\mathscr{B}_1$ if and only if $y = G^0(x)$, where G^0 is a linear map with the domain $L_c(0) \subset \mathbf{R}^m$ and range $\subset \mathscr{R}\hat{Q}^0_s(0)$. Similarly, $(x, y) \in \mathscr{K}\mathscr{B}_2$ if and only if $y = G^{I+1}(x)$, where G^{I+1} is a linear map with the domain $R_c(I) \subset \mathbf{R}^m$ and range $\subset \mathscr{R}\hat{Q}^I_u(0)$. Obviously

$$\begin{aligned} \dim L_c(0) &= m + d^- - d_1, \\ \dim R_c(0) &= m + d^+ - d_2, \\ \dim L_c(0) + \dim R_c(0) &= m - 1. \end{aligned} \tag{2.10}$$

(H$_5$) also implies that

$$\begin{aligned} &f(X^1_0(a_1), Y^1_0(a_1), 0) \notin L_c(0), \\ &f(X^I_0(b_I), Y^I_0(b_I), 0) \notin R_c(I). \end{aligned}$$

Let $S^i(t, s)$ be the solution map for the linear equation

$$\begin{aligned} \dot{x}(t) - \{ &f_x(X^i_0(t), Y^i_0(t), 0) \\ &- f_y(X^i_0(t), Y^i_0(t), 0)\, g_y^{-1}(X^i_0(t), Y^i_0(t), 0)\, g_x(X^i_0(t), Y^i_0(t), 0)\}\, x(t) = 0. \end{aligned} \tag{2.11}$$

It is readily verified that $\dot{X}^i_0(t)$ or $f(X^i_0(t), Y^i_0(t), 0)$ is a solution of (2.11). Let Σ_1 and Σ_2 be two codimension one subspaces of $\mathbf{R}^m$. Let $[t_1, t_2] \subset [a_i, b_i]$ (or $[t_2, t_1] \in [a_i, b_i]$), and $\Sigma_1 \oplus f(X^i_0(t_1), Y^i_0(t_1), 0) = \Sigma_2 \oplus f(X^i_0(t_2), Y^i_0(t_2), 0) = \mathbf{R}^m$. We then define $S^i(t_2, t_1; \Sigma_2, \Sigma_1) : \Sigma_1 \to \Sigma_2$ as follows: $x_2 = S^i(t_2, t_1; \Sigma_2, \Sigma_1)\, x_1$ if there exists $\zeta \in \mathbf{R}$ such that

$$x_2 = S^i(t_2, t_1)\, x_1 + \zeta f(X^i_0(t_2), Y^i_0(t_2), 0).$$

It is obvious that $S^i(t_2, t_1; \Sigma_2, \Sigma_1)$ is an isomorphism: $\Sigma_1 \to \Sigma_2$ with the inverse $S^i(t_1, t_2; \Sigma_1, \Sigma_2)$.

DEFINITIONS. Let $TM_i = \{x \in \mathbf{R}^m \mid \Delta_i \cdot x = 0\}$ for $1 \leqslant i \leqslant I-1$.

Let TM_0 be an $(m-1)$-dimensional subspace of $\mathbf{R}^m$ with $L_c(0) \subset TM_0$ and $TM_0 \oplus \text{span}[f(X_0^1(a_1), Y_0^1(a_1), 0)] = \mathbf{R}^m$.

Let TM_I be an $(m-1)$-dimensional subspace of $\mathbf{R}^m$ with $R_c(I) \subset TM_I$ and $TM_I \oplus \text{span}[f(X_0^I(b_I), Y_0^I(b_I), 0)] = \mathbf{R}^m$.

Let $L_c(i) = S^i(b_i, a_i; TM_i, TM_{i-1}) L_c(i-1)$, and $R_c(i-1) = S^i(a_i, b_i; TM_{i-1}, TM_i) R_c(i)$, $1 \leqslant i \leqslant I$.

Observe that we have isomorphisms $L_c(i) \simeq L_c(i-1)$, $R_c(i-1) \simeq R_c(i)$, $1 \leqslant i \leqslant I$.

We assume that

$$L_c(i) \oplus R_c(i) = TM_i, \qquad 0 \leqslant i \leqslant I. \tag{H$_6$}$$

It is clear that $\dim L_c(i) = \dim L_c(0)$ and $\dim R_c(i-1) = \dim R_c(I)$ for $1 \leqslant i \leqslant I$. (H_6) is a generic assumption due to (2.10).

We now state our main results in Theorems 2.1 and 2.2. To simplify the notations, we shall denote $Z(t) = (X(t), Y(t)) \in \mathbf{R}^{m+n}$, $z(\tau) = (x(\tau), y(\tau)) \in \mathbf{R}^{m+n}$.

THEOREM 2.1. *Suppose that* $\{(X_0^i(t), Y_0^i(t)\}_{i=1}^I$, $t \in [a_i, b_i]$ *is given which satisfies* (2.2) *and* $\{(x_0^i, y_0^i(\tau))\}_{i=1}^I$ *is given which satisfies* (2.3), *and* (H_1)–(H_6) *are satisfied. Then there exist formal power series*:

(i) $\sum_{j=0}^\infty \varepsilon^j X_j^i(t)$, $\sum_{j=0}^\infty \varepsilon^j Y_j^i(t)$, $1 \leqslant i \leqslant I$, $t \in [a_i - \delta, b_i + \delta]$, $\delta > 0$, *is a small constant.*

(ii) $\sum_{j=0}^\infty \varepsilon^j x_j^i(\tau)$, $\sum_{j=0}^\infty \varepsilon^j y_j^i(\tau)$, $0 \leqslant i \leqslant I$, *which are defined for* $\tau \in \mathbf{R}$ *if* $1 \leqslant i \leqslant I-1$; $\tau \in \mathbf{R}^+$ *if* $i = 0$ *and* $\tau \in \mathbf{R}^-$ *if* $i = I$.

(iii) $\sum_{j=1}^\infty \varepsilon^j \tau_j^i(a)$, $\sum_{j=1}^\infty \varepsilon^j \tau_j^i(b)$, $1 \leqslant i \leqslant I$.

The functions $X_j^i(t)$, $Y_j^i(t)$, $x_j^i(t)$, $y_j^i(t)$ *and the constants* $\tau_j^i(a)$, $\tau_j^i(b)$ *are calculated recursively by systems of linear equations and the auxiliary constants for the solutions of the linear equations are determined by an asymptotic matching principle. Moreover, for any integer* $p > 0$ *and* $0 < \beta < 1$, *the function*

$$\begin{aligned} z(t, p) = \bigvee_{i=1}^{I} \Bigg\{ &\left(\sum_{j=0}^{p} \varepsilon^j z_j^0(\tau), \tau \in [0, \varepsilon^{\beta-1}] \right) \\ &\vee \left(\sum_{j=0}^{p} \varepsilon^j Z_j^i(t), t \in \left[a_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(a) + \varepsilon^\beta, b_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(b) - \varepsilon^\beta \right] \right) \\ &\vee \left(\sum_{j=0}^{p} \varepsilon^j z_j^I(\tau), \tau \in [-\varepsilon^{\beta-1}, 0] \right) \Bigg\} \end{aligned}$$

is a formal approximation of (2.1) *with the jump errors as* $O(\varepsilon^{\beta(p+1)})$ *and boundary errors as* $O(\varepsilon^{(p+1)})$. *The residual errors in the slow variable t and fast variable* τ *are listed below*:

Residual errors in t	*equation of x*	*equation of y*
outer layers	$O(\varepsilon^{p+1})$	$O(\varepsilon^{p+1})$
inner layers	$O(\varepsilon^{\beta p})$	$O(\varepsilon^{\beta(p+1)})$

Residual errors in τ	*equations of x*	*equation of y*
outer layers	$O(\varepsilon^{p+2})$	$O(\varepsilon^{p+1})$
inner layers	$O(\varepsilon^{\beta p+1})$	$O(\varepsilon^{\beta(p+1)})$

General discussions of the asymptotic matching principle may be found in various places; see Eckhaus [5, 6]. We give a precise description for our purpose. Let the inner expansions at the two end points of the outer approximation be

$$\begin{aligned}\sum_{j=0}^{\infty} \varepsilon^j z_j(\tau, a, i) &\overset{\text{def}}{=} \sum_{j=0}^{\infty} \varepsilon^j Z_j^i\left(a_i + \sum_{k=1}^{\infty} \varepsilon^k \tau_k^i(a) + \varepsilon\tau\right),\\ \sum_{j=0}^{\infty} \varepsilon^j z_j(\tau, b, i) &\overset{\text{def}}{=} \sum_{j=0}^{\infty} \varepsilon^j Z_j^i\left(b_i + \sum_{k=1}^{\infty} \varepsilon^k \tau_k^i(b) + \varepsilon\tau\right),\end{aligned} \tag{2.12}$$

where $Z_j^i = (X_j^i, Y_j^i)$, $z_j(\tau, a, i) = (x_j(\tau, a, i), y_j(\tau, a, i))$, etc.

Asymptotic Matching Principle.

$$\begin{aligned} z_j(\tau, a, i) - z_j^{i-1}(\tau) &\in E_{\mathbf{R}^+}(\gamma, j), \qquad j \geqslant 0,\\ z_j(\tau, b, i) - z_j^{i}(\tau) &\in E_{\mathbf{R}^-}(\gamma, j), \qquad j \geqslant 0,\end{aligned} \tag{2.13}$$

where $0 < \gamma < \alpha_0$ is a constant.

Define the composite expansion $z_{\text{comp}}(t, p)$ in two steps. First for $t \in [a_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(a), b_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(b)]$, define

$$\begin{aligned} z_{\text{comp},i}(t, p) = {} & \sum_{j=0}^{p} \varepsilon^j Z_j^i(t) + \sum_{j=0}^{p} \varepsilon^j z_j^{i-1}\left(\frac{t-a_i}{\varepsilon} - \sum_{k=1}^{p} \varepsilon^{k-1}\tau_k^i(a)\right)\\ & + \sum_{j=0}^{p} \varepsilon^j z_j^i\left(\frac{t-b_i}{\varepsilon} - \sum_{k=1}^{p} \varepsilon^{k-1}\tau_k^i(b)\right)\\ & - \sum_{j=0}^{p} \varepsilon^j z_j\left(\frac{t-a_i}{\varepsilon} - \sum_{k=1}^{p} \varepsilon^{k-1}\tau_k^i(a), a, i\right)\\ & - \sum_{j=0}^{p} \varepsilon^j z_j\left(\frac{t-b_i}{\varepsilon} - \sum_{k=1}^{p} \varepsilon^{k-1}\tau_k^i(b), b, i\right). \end{aligned} \tag{2.14}$$

Next, define

$$z_{\text{comp}}(t, p) = \bigvee_{i=1}^{I} \left\{ z_{\text{comp},i}(t, p),\ t \in \left[a_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(a),\ b_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(b) \right] \right\}. \tag{2.15}$$

THEOREM 2.2. *Let* $z(t, \bar{p}) = (x(t, \bar{p}), y(t, \bar{p}))$, $t \in [\omega_0, \omega_0 + \bar{\omega}]$, *be the formal approximation as in Theorem* 2.1, *corresponding to some* $\bar{p} \geqslant 1$. *Then there is* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, *there exists a unique exact solution* $z_{\text{exact}}(t)$, $t \in [\omega_0, \omega_0 + \omega_{\text{exact}}]$, *of the boundary value problem* (2.1) *with* $\text{dist}(z_{\text{exact}}(t), z(t, \bar{p})) = O(\varepsilon)$, *and* $|\bar{\omega} - \omega_{\text{exact}}| = O(\varepsilon)$. *The composite expansion* $z_{\text{comp}}(t, p)$ *defined in* (2.14) *and* (2.15) *is uniformly valid in* $t \in [\omega_0, \omega_0 + \omega]$. *Moreover we have the following estimates for all* $p \geqslant 0$:

$$\text{dist}(z_{\text{exact}}(t), z(t, p)) = O(\varepsilon^{\beta(p+1)}), \tag{2.16}$$

$$\text{dist}(z_{\text{exact}}(t), z_{\text{comp}}(t, p)) = O(\varepsilon^{p+1}), \tag{2.17}$$

$$|\omega - \omega_{\text{exact}}| = O(\varepsilon^{p+1}). \tag{2.18}$$

It is useful to present a set of geometrical conditions which is parallel to the analytical hypotheses made in this section. Such geometrical conditions also help to explain how the 0th order approximations $(X_0^i(t), Y_0^i(t))$ and $(x_0^i(\tau), y_0^i(\tau))$ may be obtained.

Each $(x, y) \in \mathcal{S}_i$, $1 \leqslant i \leqslant I$, is an equilibrium point for Eq. (2.3). Hypothesis (H_1) implies that $\mathcal{S}_i$ is normally hyperbolic near the orbit of $(X_0^i(t), Y_0^i(t))$. There exist two families of invariant manifolds, namely stable fibers $W^s(x, G^i(x))$ and unstable fibers $W^u(x, G^i(x))$ passing through each $(x, G^i(x)) \in \mathcal{S}_i$. The orbit of $(x_0^i(\tau), y_0^i(\tau))$, $0 \leqslant i \leqslant I-1$, lies on $W^s(x, G^{i+1}(x))$ and the orbit of $(x_0^i(\tau), y_0^i(\tau))$, $1 \leqslant i \leqslant I$, lies on $W^u(x, G^i(x))$, for $x = X_0^{i+1}(a_{i+1})$ and $x = X_0^i(b_i)$, respectively. Consider $x_0^i = X_0^i(b_i) = X^{i+1}(a_{i+1})$, $1 \leqslant i \leqslant I-1$, as parameter in the equation

$$y' = g(x_0^i, y, 0). \tag{2.19}$$

One must find x_0^i such that (2.19) has a heteroclinic solution connecting $(x_0^i, G^i(x_0^i))$ and $(x_0^i, G^{i+1}(x_0^i))$. Here we have a standard heteroclinic perturbation problem. Our hypotheses imply that the set $M_i \stackrel{\text{def}}{=} \{x_0^i \mid \text{there is a heteroclinic orbit for (2.19)}\}$ is not empty. Moreover by (H_2), M_i is an $(m-1)$-dimensional submanifold in $\mathbf{R}^m$. and Δ_i is the normal of M_i, $1 \leqslant i \leqslant I-1$. See Hale and Lin [13] for a proof. (H_3) implies that each M_i is a local section for the induced slow flows on $\mathbf{R}^m$. See Fig. 1.

The flow on the slow manifold $\mathcal{S}_i$ is completely determined by its projection on $\mathbf{R}^m$, which satisfies the equation

$$\dot{X}(t) = f(X(t), G^i(X(t)), 0). \tag{2.20}$$

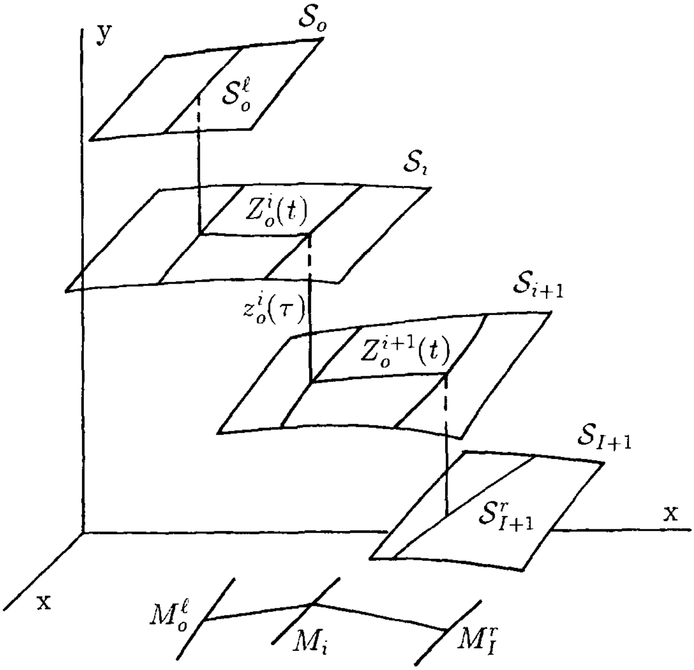

FIGURE 1

Therefore the reduced equation (2.20) is discontinuous when crossing a section M_i, $1 \leqslant i \leqslant I-1$. However, the trajectory

$$X(t) = \bigvee_{i=1}^{I} (X_0^i(t),\, t \in (a_i, b_i])$$

is continuous due to the fact $X_0^i(b_i) = X_0^{i+1}(a_{i+1})$.

Boundary conditions at the two end points $X_0^1(a_1)$ and $X_0^I(b_I)$ have to be specified in order to determine the 0th order approximation. Define

$$\mathscr{S}_0 = \{(x, y) \mid B_1(x, y, 0) = 0\},$$

$$\mathscr{S}_{I+1} = \{(x, y) \mid B_2(x, y, 0) = 0\},$$

$$\mathscr{S}^l_0 = \mathscr{S}_0 \cap \left\{ \bigcup_{x \in \mathbf{R}^m} W^s(x, G^1(x)))) \right\}, \tag{2.21}$$

$$\mathscr{S}^r_{I+1} = \mathscr{S}_{I+1} \cap \left\{ \bigcup_{x \in \mathbf{R}^m} W^u(x, G^I(x)) \right\}. \tag{2.22}$$

Both $\mathscr{S}^l_0$ and $\mathscr{S}^r_{I+1}$ are nonempty, for by our assumptions $(x_0^0, y_0^0(0)) \in \mathscr{S}^l_0$ and $(x_0^I, y_0^I(0)) \in \mathscr{S}^r_{I+1}$. ($H_4$) is equivalent to:

$(H_4)'$: The nonlinear mappings B_1 and B_2 are (locally) surjective. $\mathscr{S}_0$ and $\mathscr{S}_{I+1}$ are two (local) smooth submanifold in $\mathbf{R}^{m+n}$. The intersections in (2.21) and (2.22) are transverse (locally).

(H_5) is equivalent to $(H_5)'$ and $(H_5)''$.

$(H_5)'$:

$$TW^s(x_0^0, G^1(x_0^0)) \cap T\mathscr{S}_0 = \{0\} \qquad \text{at} \quad (x_0^0, y_0^0(0)),$$

$$TW^u(x_0^I, G^I(x_0^I)) \cap T\mathscr{S}_{I+1} = \{0\} \qquad \text{at} \quad (x_0^I, y_0^I(0)).$$

Define

$$M_0^l = \{x \in \mathbf{R}^m \mid \mathscr{S}_0 \cap W^s(x, G^1(x)) \neq \varnothing\},$$

$$M_I^r = \{x \in \mathbf{R}^m \mid \mathscr{S}_{I+1} \cap W^u(x, G^I(x)) \neq \varnothing\}.$$

It is not difficult to show that locally M_0^l and M_I^r are smooth submanifolds in $\mathbf{R}^m$. Let

$$(x, y) \in \mathscr{S}_0 \cap W^s(x, G^1(x)), \qquad x \in M_0^l.$$

(x, y) is locally unique and $(x, y) \to (x, G^1(x))$ is a diffeomorphism through the stable fiber $W^s(x, G^1(x))$. A similar situation also holds for $M_I^r \to \mathscr{S}^r_{I+1}$. Notice that $\dim M_0^l = d_2 - d^+ - 1$ and $\dim M_I^r = d_1 - d^- - 1$.

$(H_5)''$: The flow of (2.20) for $i = 1$ (or $i = I$) is not tangent to M_0^l (or M_I^r) near $(X_0^1(a_1), Y_0^1(a_1))$ (or $(X_0^I(b_I), Y_0^I(b_I))$).

The construction of the 0th order approximation $(X_0^i(t), Y_0^i(t))$, $1 \leqslant i \leqslant I$, can be stated as follows:

Find a continuous trajectory starting at M_0^l, ending at M_I^r, and passing through each M_i, $1 \leqslant i \leqslant I$, successively. The trajectory has to satisfy (2.20) on each $\mathscr{S}_i$ when moving from M_{i-1} to M_i. See Fig. 1.

It is clear from (H_6) that such a trajectory is locally unique. Details of how to compute such a trajectory shall not be discussed here though it is a problem of fundamental interest, since the method employed will be considerably different. All the hypotheses made above can be localized in an obvious way. We emphasize again that our analytical hypotheses are merely detailed descriptions of the sets of geometrical conditions. It is precisely the same conditions that ensure the solvability of higher order approximations and the validity of the formal power approximations.

The following example is a simple extension of Fife [9].

EXAMPLE 2.3. Consider

$$\begin{aligned} \ddot{u} &= f(u, v), \\ \varepsilon^2 \ddot{v} &= g(u, v), \qquad 0 \leqslant t \leqslant 1, \\ u(i) &= \alpha_i, \\ v(i) &= \beta_i, \qquad i = 0, 1, \end{aligned}$$

where $u \in \mathbf{R}^m$, $v \in \mathbf{R}^n$, $f: \mathbf{R}^{m+n} \to \mathbf{R}^n$, $g: \mathbf{R}^{m+n} \to \mathbf{R}^n$. Setting $\dot{u} = u_1$, $\varepsilon\dot{v} = v_1$, and adding $\dot{t} = 1$ to the equation we have a $(2m+2n+1)$-dimensional system, with $m+n+1$ initial conditions and $m+n+1$ terminal conditions:

$$\dot{t} = 1,$$
$$\dot{u} = u_1,$$
$$\dot{u}_1 = f(u, v),$$
$$\varepsilon\dot{v} = v_1,$$
$$\varepsilon\dot{v}_1 = g(u, v);$$
$$\text{B.C.}: \begin{cases} t = 0, \\ u = \alpha_0, \\ v = \beta_0; \end{cases} \qquad \begin{cases} t = 1, \\ u = \alpha_1, \\ v = \beta_1. \end{cases}$$

Let $x = (t, u, u_1)$ be the slow variable and $y = (v, v_1)$ be the fast variable. Assume that equation

$$0 = v_1,$$
$$0 = g(u, v),$$

has two branches of solutions

$$v = h_0(u) \qquad \text{and} \qquad v = h_1(u).$$

Assume that

$$\sigma\{g_v(u, h_i(u))\} \cap \overline{\mathbf{R}^-} = \varnothing, \qquad \overline{\mathbf{R}^-} = \mathbf{R}^- \cup \{0\}. \tag{2.23}$$

Let $A = \begin{pmatrix} 0 & 1 \\ g_v & 0 \end{pmatrix}$, and it is not hard to show the following:

(i) $\lambda \in \sigma(A)$ if and only if $\lambda = \pm\sqrt{\lambda_j}$, where $\{\lambda_j\}_{j=1}^n = \sigma\{g_v\}$. Therefore A is hyperbolic, with n-dimensional stable and unstable projections, denoted by Q_s and Q_u, respectively.

(ii) $\binom{v}{v_1} \in \mathcal{N}(\lambda - A)^k$ implies $\binom{v}{-v_1} \in \mathcal{N}(-\lambda - A)^k$, $k \geq 1$, is an integer. Therefore $\binom{v}{v_1} \in \mathcal{R}Q_s$ if and only if $\binom{v}{-v_1} \in \mathcal{R}Q_u$.

(iii)

$$\left\{v \,\middle|\, \begin{pmatrix} v \\ v_1 \end{pmatrix} \in \mathcal{R}Q_s\right\} = \left\{v_1 \,\middle|\, \begin{pmatrix} v \\ v_1 \end{pmatrix} \in \mathcal{R}Q_s\right\} = \mathbf{R}^n,$$

$$\left\{v \,\middle|\, \begin{pmatrix} v \\ v_1 \end{pmatrix} \in \mathcal{R}Q_u\right\} = \left\{v_1 \,\middle|\, \begin{pmatrix} v \\ v_1 \end{pmatrix} \in \mathcal{R}Q_u\right\} = \mathbf{R}^n.$$

Consider

$$
\begin{aligned}
v' &= v_1, \\
v_1' &= g(u, v),
\end{aligned}
\tag{2.24}
$$

with $u = \text{constant}$ as a parameter. Assume that $\Gamma \subset \mathbf{R}^m$ is a smooth codimension-1 smooth submanifold such that for $u \in \Gamma$, (2.24) has a heteroclinic solution $(v(\tau), v_1(\tau)) \to (h_i(u), 0)$, $i = 0, 1$, as $\tau \to \mp\infty$, respectively. Assume that $(v'(\tau), v_1'(\tau))$ is the only bounded solution for the linearization of (2.24) around $(v(\tau), v_1(\tau))$, then the formal adjoint equation shall have a unique bounded solution $(\psi(\tau), \psi_1(\tau))$ up to a scalar multiple. Assume that

$$\int_{-\infty}^{\infty} \psi_1(\tau)^* \, g_u(u, v(\tau))\, d\tau \neq 0. \tag{2.25}$$

Note that (2.23) implies (H_1) and (2.25) implies (H_2). In the case $m = n = 1$, (2.25) is equivalent to a condition in Fife [8, 9]. See also Lin [19] for a discussion of the equivalence.

The initial manifold $\mathscr{S}_0 = \{(t, u, u_1, v, v_1) \mid t = 0,\ u = \alpha_0,\ v = \beta_0,\ u_1 \in \mathbf{R}^m,\ v_1 \in \mathbf{R}^n\}$ and the terminal manifold $\mathscr{S}_{l+1} = \{(t, u, u_1, v, v_1) \mid t = 1,\ u = \alpha_1,\ v = \beta_1,\ u_1 \in \mathbf{R}^m,\ v_1 \in \mathbf{R}^n\}$ are explicitly given. However, it seems to be very difficult to describe the stable fibers and unstable fibers attaching to points $(u, h_i(u))$, $i = 0, 1$. A special case with $m = 1$, $n = 1$ has been studied by Fife [9]. We expect that conditions like (H_4) and (H_5) can only be checked numerically in general cases. Many authors assumed that

(a) $g(u, v)$ is linear in v, or

(b) $\beta_0 - h_0(\alpha_0)$ and $\beta_1 - h_1(\alpha_1)$ are small.

In both cases (a) and (b), the stable and unstable fibers can be computed (or approximated) by the generalized eigenspaces corresponding to the stable and unstable eigenvalues, respectively. Based on (iii), it is clear that if (a) or (b) holds, we have that

$$\mathscr{S}_0 \pitchfork \bigcup \{(t, u, u_1, v, v_1) \mid t \in \mathbf{R}, u \in \mathbf{R}^m, u_1 \in \mathbf{R}^m, (v, v_1) \in W^{\mathrm{s}}(h_0(u), 0)\},$$

$$\mathscr{S}_{l+1} \pitchfork \bigcup \{(t, u, u_1, v, v_1) \mid t \in \mathbf{R}, u \in \mathbf{R}^m, u_1 \in \mathbf{R}^m, (v, v_1) \in W^{\mathrm{u}}(h_1(u), 0)\},$$

and

$$T\mathscr{S}_0 \cap \{0\} \times \{0\} \times \{0\} \times TW^{\mathrm{s}}(h_0(u), 0) = \{0\},$$

$$T\mathscr{S}_{l+1} \cap \{0\} \times \{0\} \times \{0\} \times TW^{\mathrm{u}}(h_1(u), 0) = \{0\},$$

at the points of intersections, where W^{s} and W^{u} denote the stable and unstable manifolds of the equilibria $(h_i(u), 0)$ of Eq. (2.24). We can also obtain easily that $M_0^l = \{(t, u, u_1) \mid t = 0,\ u = \alpha_0,\ u_1 \in \mathbf{R}^m\}$ and $M_l^r = \{(t, u, u_1) \mid t = 1,\ u = \beta_0,\ u_1 \in \mathbf{R}^m\}$.

We have to solve the following two initial value problems in order to compute a 0th order approximation in the slow manifold:

$$\begin{aligned} \dot{t} &= 1, \\ \dot{u} &= u_1, \\ \dot{u}_1 &= f(u, h_0(u)), \qquad t \geqslant 0, \end{aligned} \tag{2.26}$$

with $t(0) = 0$, $u(0) = \alpha_0$ being given, and $u_1(0) \in \mathbf{R}^m$ as a parameter, and

$$\begin{aligned} \dot{t} &= 1, \\ \dot{u} &= u_1, \\ \dot{u}_1 &= f(u, h_1(u)), \qquad t \leqslant 1, \end{aligned} \tag{2.27}$$

with $t(1) = 1$, $u(1) = \alpha_1$ being given, and $u_1(1) \in \mathbf{R}^m$ as a parameter. Let the solution of (2.26) be $\phi_0 : (t, u_1(0)) \to (t, u, u_1)$ and the solution of (2.27) be $\phi_1 : (t, u_1(1)) \to (t, u, u_1)$. Let the trajectories of ϕ_0 and ϕ_1 intersect $\mathbf{R} \times \Gamma \times \mathbf{R}^m \subset \mathbf{R}^{2m+1}$ at two m-dimensional curves Γ_0 and Γ_1. Assume that

$$\Gamma_0 \pitchfork \Gamma_1 \qquad \text{in} \quad \mathbf{R} \times \Gamma \times \mathbf{R}^m.$$

Let $(t^*, u^*, u_1^*) \in \Gamma_0 \cap \Gamma_1$, $0 \leqslant t^* \leqslant 1$. Based on (t^*, u^*, u_1^*) we can compute $u_1(0)$ and $u_1(1)$. We assume that $u_1^* \neq 0$ and

$$\int_{-\infty}^{\infty} \psi_1(\tau)^*\, g_u(u^*, v(\tau))\, d\tau \cdot u_1^* \neq 0. \tag{2.28}$$

Clearly (2.28) implies (H_3), and $\dot{t} = 1$ implies $(\mathrm{H}_5)''$.

We have given a set of sufficient conditions such that Theorems 2.1 and 2.2 apply to this example. It is easy to verify that our conditions are natural generalization of Fife [9] for a case with $m = n = 1$.

3. Preliminaries

Most of our analysis depends on the properties of the linear variational equation around the approximate solutions. Here the concept of the exponential dichotomy has to be extended to the exponential trichotomy due to the presence of the slow motions on the slow manifolds. We refer to Coppel [4] and Palmer [23] for the basic properties of the exponential dichotomies. See also Sacker and Sell [25] and Sacker [24]. Many properties of the exponential trichotomy can be derived from the corresponding ones of the exponential dichotomy.

Consider a linear ODE in $\mathbf{R}^n$

$$\dot{x}(t) - A(t)\, x(t) = h(t), \qquad t \in J, \tag{3.1}$$

where $A(t)$ is a continuous and uniformly bounded matrix-valued function. Let $T(t, s)$ be the solution map for the linear homogeneous equation associated with (3.1).

DEFINITION 3.1. We say that (3.1), or $T(t, s)$, has an exponential trichotomy in J if there exist projections $P_c(t)$, $P_s(t)$, and $P_u(t) = I - P_c(t) - P_s(t)$, $t \in J$, and there are constants $K \geqslant 1$ and $\alpha > \sigma > 0$ such that

$$\begin{aligned}
&T(t, s)\, P_\nu(s) = P_\nu(t)\, T(t, s), && t \geqslant s \text{ in } J,\ \nu = \text{c, u, s},\\
&|T(t, s)\, P_c(s)| \leqslant K e^{\sigma|t-s|}, && t, s \text{ in } J,\\
&|T(t, s)\, P_s(s)| \leqslant K e^{-\alpha(t-s)}, && t \geqslant s \text{ in } J,\\
&|T(s, t)\, P_u(t)| \leqslant K e^{-\alpha(t-s)}, && t \geqslant s \text{ in } J.
\end{aligned}$$

We say that (3.1) has an exponential dichotomy in J if it has an exponential trichotomy with $P_c(t) = 0$ and $P_s(t) + P_u(t) = I$.

LEMMA 3.2. *Assume that* $J = \mathbf{R}^+$, $\lim_{t \to +\infty} A(t) = A(+\infty)$, *and* $\dot{x}(t) - A(\infty)\, x(t) = 0$ *has an exponential dichotomy with the exponent* $\alpha > 0$ *and projections* P_s *and* P_u. *Then* (3.1) *has an exponential dichotomy in* $\mathbf{R}^+$, *with the exponent* $\tilde{\alpha}$ *and projections* $\tilde{P}_s(t)$ and $\tilde{P}_u(t)$. *Moreover* $0 < \tilde{\alpha} < \alpha$ *can be chosen arbitrarily close to* α *and* $\tilde{P}_s(t) - P_s \to 0$ *as* $t \to +\infty$.

LEMMA 3.3. *Assume that* $|A(t)| \leqslant M$ $\forall J$, *and* $A(t)$ *has* d^-*-eigenvalues with real part* $\leqslant -\alpha < 0$ *and* $d^+ = n - d^-$ *eigenvalues with real part* $\geqslant \alpha > 0$ *for all* $t \in J$. *Assume that for any* $0 < \varepsilon < \alpha$, *there exists* $0 < \delta = \delta(M, \alpha, \varepsilon)$ *such that if* $|A(t_2) - A(t_1)| \leqslant \delta$ *for* $|t_2 - t_1| \leqslant h$, *where* $h > 0$ *is a fixed number not greater than the length of* J, *then* (3.1) *has an exponential dichotomy in* J *with the constant* $K = K(M, \alpha, \varepsilon)$ *and exponent* $\alpha - \varepsilon$. *Moreover,* $P_s(t)$ *approaches the spectral projection to the stable eigenspace of* $A(t)$ *for each fixed* t, *as* $\delta \to 0$.

The proof of Lemma 3.2 can be found in Palmer [23] and the proof of 3.3 in Coppel [4].

DEFINITION 3.4. Let $\mathscr{F}$: $E_J^1(\gamma, l) \to E_J(\gamma, l)$, $x \to h$, be defined as $h(t) = \dot{x}(t) - A(t)\, x(t)$. *Let* $\mathscr{F}^*$: $E_J^1(\gamma, l) \to E_J(\gamma, l)$, $y \to g$ be defined as $g(t) = \dot{y}(t) + A(t)^* y(t)$.

Clearly $\mathscr{F}$ and $\mathscr{F}^*$ are linear bounded. Assume that (3.1) has an exponential dichotomy in J with constant K and exponent α. Let $\gamma > 0$ be a constant with $|\gamma| < \alpha$.

LEMMA 3.5. (i) *If $J = \mathbf{R}^-$, then for any $h \in E_{\mathbf{R}^-}(\gamma, l)$ and $u \in \mathscr{R}P_u(0)$, there exists a unique solution $x \in E^1_{\mathbf{R}^-}(\gamma, l)$ of (3.1) with $P_u(0)\,x(0) = u$. The solution can be written as*

$$x(t) = T(t, 0)u + \int_0^t T(t, s)\,P_u(s)\,h(s)\,ds + \int_{-\infty}^t T(t, s)\,P_s(s)\,h(s)\,ds.$$

Moreover $\|x\|^1_{E_{\mathbf{R}^-}(\gamma, l)} \leqslant C\{\|h\|_{E_{\mathbf{R}^-}(\gamma, l)} + \|u\|\}$.

(ii) *If $J = \mathbf{R}^+$, then for any $h \in E_{\mathbf{R}^+}(\gamma, l)$ and $v \in \mathscr{R}P_s(0)$, there exists a unique solution $x \in E_{\mathbf{R}^+}(\gamma, l)$ of (3.1) with $P_s(0)\,x(0) = v$. The solution can be written as*

$$x(t) = T(t, 0)v + \int_0^t T(t, s)\,P_s(s)\,h(s)\,ds + \int_{\infty}^t T(t, s)\,P_u(s)\,h(s)\,ds.$$

Moreover $\|x\|_{E^1_{\mathbf{R}^+}(\gamma, l)} \leqslant C\{\|h\|_{E_{\mathbf{R}^+}(\gamma, l)} + \|v\|\}$.

(iii) *If $J = \mathbf{R}$, then for any $h \in E_{\mathbf{R}(\gamma, l)}$, there exists a unique solution $x \in E^1_{\mathbf{R}}(\gamma, l)$ of (3.1) with $\|x\|_{E^1_{\mathbf{R}}(\gamma, l)} \leqslant C\{\|h\|_{E_{\mathbf{R}^+}(\gamma, l)}\}$. The solution can be written as*

$$x(t) = \int_{-\infty}^t T(t, s)\,P_s(s)\,h(s)\,ds + \int_{\infty}^t T(t, s)\,P_u(s)\,h(s)\,ds.$$

LEMMA 3.6. *If (3.1) has exponential dichotomies in $\mathbf{R}^-$ and $\mathbf{R}^+$ with the same exponent α in $\mathbf{R}^-$ and $\mathbf{R}^+$, $|\gamma| < \alpha$. Then $\mathscr{F} : E^1_{\mathbf{R}}(\gamma, l) \to E_{\mathbf{R}}(\gamma, l)$ is Fredholm with Index $\mathscr{F} = \dim \mathscr{R}P_u^-(0) - \dim \mathscr{R}P_u^+(0)$. $h \in \mathscr{R}\mathscr{F}$ if and only if*

$$\int_{-\infty}^{+\infty} \psi^*(t)\,h(t) = 0$$

for all $\psi \in \mathscr{K}\mathscr{F}^*$. Indeed, $\mathscr{K}\mathscr{F}^* \subset E_{\mathbf{R}}(\alpha, 0)$.

Consider the following system in $\mathbf{R}^{m+n}$ which comes from the linearization of the inner layers:

$$\begin{aligned} &\dot{x} = 0 \\ &\dot{y} - (A(t)x + B(t)\,y) = 0, \qquad t \in \mathbf{R}\ (\text{or } \mathbf{R}^{\pm}). \end{aligned} \tag{3.2}$$

LEMMA 3.7. *If $A(t)$ and $B(t)$ are continuous for $t \in \mathbf{R}^+$, and if $\lim_{t\to\infty} A(t) = A(+\infty)$ and $\lim_{t\to\infty} B(t) = B(+\infty)$ with*

$$|A(t) - A(+\infty)| \leqslant C_1 e^{-\gamma t},$$
$$|B(t) - B(+\infty)| \leqslant C_2 e^{-\gamma t},$$

suppose $\sigma B(+\infty) \cap \{|\mathrm{Re}\,\lambda| \leqslant \alpha\} = \varnothing$. Then (3.2) has an exponential trichotomy in $\mathbf{R}^+$. Moreover, if $(x, y(t)) \in \mathscr{R}\mathscr{P}_{cs}(t)$, $t \in \mathbf{R}^+$ is a solution of (3.2), and $0 < \gamma_1 = \min\{\alpha, \gamma\}$, then

$$|y(t) + B(+\infty)^{-1} A(+\infty) x| \leqslant C e^{-\gamma_1 t}.$$

Similar results also hold for (3.2) defined in $\mathbf{R}^-$.

Proof. Exponential trichotomies in $\mathbf{R}^+$ are not unique, and we can define one by setting

$$\mathscr{R}P_u(t) = \{(x, y) \mid x = 0,\ y(t) \in Q_u(t)\}$$
$$\mathscr{R}P_s(t) = \{(x, y) \mid x = 0,\ y(t) \in Q_s(t)\}$$
$$\mathscr{R}P_c(t) = \left\{(x, y) \mid x \in \mathbf{R}^m,\ y(t) = \int_0^t U(t, s)\, Q_s(s)\, A(s) x\, ds \right.$$
$$\left. + \int_\infty^t U(t, s)\, Q_u(s)\, A(s) x\, ds \right\},$$

where $U(t, s)$ is the solution map for $\dot{y}(t) - B(t)\, y = 0$, which, according to Lemma 3.2, has an exponential dichotomy in $\mathbf{R}^+$ with projections $Q_s(t)$ and $Q_u(t)$. Now let $(x, y(t)) \in \mathscr{R}P_{cs}(t)$, $t \in \mathbf{R}^+$, be a solution of (3.2). Let $y(t) = -B(+\infty)^{-1} A(+\infty) x + z(t)$, and we have

$$\dot{z} - B(t) z = [A(t) - A(+\infty)] x - [B(t) - B(+\infty)]\, B(+\infty)^{-1} A(+\infty) x.$$

The right-hand side is bounded by $Ce^{-\gamma t}$ in norm. From Lemma 3.5(ii), we have $|z(t)| \leqslant Ce^{-\gamma_1 t}$. Q.E.D.

Suppose that $\dot{y}(t) - B(t)\, y(t) = 0$, with solution map $U(t, s)$, has exponential dichotomies in $\mathbf{R}^-$ and $\mathbf{R}^+$, respectively. Let the projections to the stable and unstable spaces be $Q_s(t)$ and $Q_u(t)$, $t \in \mathbf{R}^-$ or $\mathbf{R}^+$. Assume that

$$\dim \mathscr{R}Q_u(0^-) = \dim \mathscr{R}Q_u(0^+) = d^+$$
$$\dim \mathscr{R}Q_u(0^-) \cap \mathscr{R}Q_s(0^+) = 1.$$

From Lemma 3.6, $\mathrm{Ind}\, \mathscr{F} = 0$ and $\dim \mathscr{K}\mathscr{F} = 1$. Therefore, the adjoint equation $\dot{y} + B(t)^* y(t) = 0$ has a unique bounded solution $y = \psi(t)$ up to a scalar multiple.

LEMMA 3.8. *Assume that $A(t)$ is continuous and bounded, and*

$$\Delta \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} \psi(t)^* A(t)\, dt \neq 0,$$

then Eq. (3.2) *has nonunique exponential trichotomies in* $\mathbf{R}^-$ *and* $\mathbf{R}^+$, *respectively. Moreover we can always choose the trichotomies in* $\mathbf{R}^-$ *and* $\mathbf{R}^+$, *with the projections being* $P_s(t)$, $P_c(t)$, *and* $P_u(t)$, $t \in \mathbf{R}^-$ *or* $\mathbf{R}^+$, *such that*

$$\mathscr{R}P_u(0^+) = U(0) \oplus W_1(0),$$

$$\mathscr{R}P_s(0^-) = V(0) \oplus W_2(0),$$
$$\mathscr{R}P_c(0^-) = N(0) \oplus W_1(0),$$

$$\mathscr{R}P_c(0^+) = N(0) \oplus W_2(0),$$

$$\mathscr{R}P_u(0^-) = U(0) \oplus \Phi(0),$$

$$\mathscr{R}P_s(0^+) = V(0) \oplus \Phi(0),$$

where $\Phi(0)$, $N(0)$, $U(0)$, $V(0)$, $W_1(0)$, *and* $W_2(0)$ *are linearly independent, with the following properties*:

(i) $\Phi(0) \stackrel{\text{def}}{=} \{(x, y): x = 0,\ y \in \mathscr{R}Q_u(0^-) \cap \mathscr{R}Q_s(0^+)\}$ *is one dimensional*;

(ii) $N(0) \stackrel{\text{def}}{=} \{(x, y): \Delta \cdot x = 0,\ y \perp \Phi(0),\ (x, y) \in \mathscr{R}P_{cu}(0^-) \cap \mathscr{R}P_{cs}(0^+)\}$, $y = Lx$, *L is a linear map from* $\Delta^\perp \to \Phi(0)^\perp$, *$N(0)$ is* $(m-1)$ *dimensional*;

(iii) $U(0) = \mathscr{R}P_u(0^-) \ominus \Phi(0)$, $V(0) = \mathscr{R}P_s(0^+) \ominus \Phi(0)$, *$U(0)$ is* $(d^+ - 1)$ *dimensional and $V(0)$ is* $(d^- - 1)$ *dimensional*;

(iv) $W_1(0) \subset \mathscr{R}P_{cu}(0^-) \ominus \mathscr{R}P_u(0^-)$, $W_2(0) \subset \mathscr{R}P_{cs}(0^+) \ominus \mathscr{R}P_s(0^+)$, *$W_1(0)$, and $W_2(0)$ are both one dimensional.* $(x, y) \in W_1(0)$ *or* $W_2(0)$ *implies that* $\Delta \cdot x \neq 0$ *unless* $x = 0$.

We define $U(t)$, $V(t)$, $N(t)$, $W_1(t)$, and $W_2(t)$ by $U(t) = T(t, 0)\, U(0)$, etc. The results of Lemma 3.8 are depicted in Fig. 2.

Proof. The unstable space $\mathscr{R}P_u(t)$ for $t \in \mathbf{R}^-$ and the stable space $\mathscr{R}P_s(t)$ for $t \in \mathbf{R}^+$ are uniquely defined, i.e.,

$$\mathscr{R}P_u(t) = \{(x, y): x = 0,\ y \in \mathscr{R}Q_u(t)\} \qquad \text{for} \quad t \in \mathbf{R}^-,$$

$$\mathscr{R}P_s(t) = \{(x, y): x = 0,\ y \in \mathscr{R}Q_s(t)\} \qquad \text{for} \quad t \in \mathbf{R}^+.$$

Part (i) follows from our assumption on $\mathscr{R}Q_u(0^-) \cap \mathscr{R}Q_s(0^+)$.

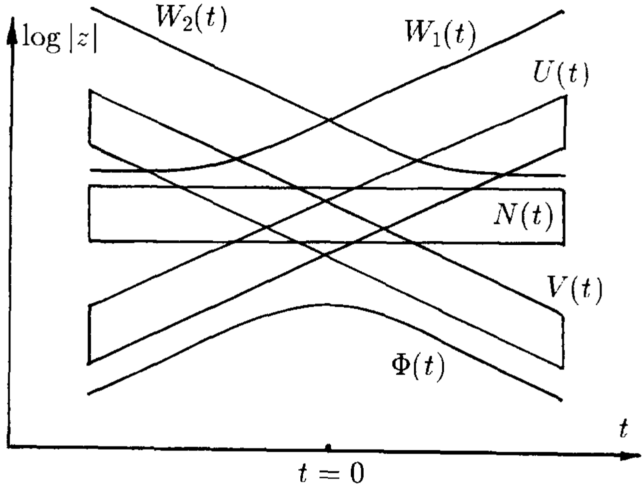

FIGURE 2

The center stable space $\mathscr{R}P_{cs}(t)$ for $t \in \mathbf{R}^+$ is uniquely defined, and

$$\mathscr{R}P_{cs}(t) = \left\{ (x, y) : y(t) = U(t, 0)\, Q_s(0^+)\, y(0) + \int_0^t U(t, s)\, Q_s(s)\, A(s)\, ds \cdot x + \int_\infty^t U(t, s)\, Q_u(s)\, A(s)\, ds \cdot x \right\}. \tag{3.3}$$

The center unstable space $\mathscr{R}P_{cu}(t)$ for $t \in \mathbf{R}^-$ is uniquely defined, and

$$\mathscr{R}P_{cu}(t) = \left\{ (x, y) : y(t) = U(t, 0)\, Q_u(0^-)\, y(0) + \int_0^t U(t, s)\, Q_u(s)\, A(s)\, ds \cdot x + \int_{-\infty}^t U(t, s)\, Q_s(s)\, A(s)\, ds \cdot x \right\}. \tag{3.4}$$

From those formulae, we conclude that $(x, y) \in \mathscr{R}Q_{cs}(0^+) \cap \mathscr{R}Q_{cu}(0^-)$ if and only if

$$\Delta \cdot x = 0.$$

Moreover $y = Lx$ is uniquely determined by x if the additional requirement

$$\Phi(0) \perp y(0)$$

is imposed. The proof of these facts follows closely from Palmer [23].

Define $N(0)$ as in (ii), and clearly $N(0) = \mathscr{R}P_{cu}(0^-) \cap \mathscr{R}P_{cs}(0^+) \ominus \Phi(0)$. Define $U(0)$ and $V(0)$ as in (iii), and clearly $U(0) \cap \mathscr{R}P_{cs}(0^+) = \{0\}$ and $V(0) \cap \mathscr{R}P_{cu}(0^-) = \{0\}$.

Let $\bar{x} \in \mathbf{R}^m$ be such that $\Delta \cdot \bar{x} \neq 0$. Define

$$\begin{aligned} w_1(t) &= \left\{(\bar{x}, y(t)) : y(t) = \left\{\int_0^t U(t,s)\, Q_u(s)\, A(s)\, ds \right.\right. \\ &\qquad \left.\left. + \int_{-\infty}^t U(t,s)\, Q_s(s)\, A(s)\, ds \right\} \bar{x} \right\} \qquad \text{for} \quad t \in \mathbf{R}^-, \\ w_2(t) &= \left\{(\bar{x}, y(t)) : y(t) = \left\{\int_0^t U(t,s)\, Q_s(s)\, A(s)\, ds \right.\right. \\ &\qquad \left.\left. + \int_{\infty}^t U(t,s)\, Q_u(s)\, A(s)\, ds \right\} \bar{x} \right\} \qquad \text{for} \quad t \in \mathbf{R}^+. \end{aligned} \tag{3.5}$$

It follows that $w_1(t) \in \mathscr{R}P_{cu}(t)$, $t \in \mathbf{R}^-$, and $w_2(t) \in \mathscr{R}P_{cs}(t)$, $t \in \mathbf{R}^+$. Moreover $w_1(0) \notin \mathscr{R}P_{cs}(0^+)$ and $w_2(0) \notin P_{cu}(0^-)$. Let $W_1(0) = \operatorname{span}\{w_1(0)\}$ and $W_2(0) = \operatorname{span}\{w_2(0)\}$. Property (iv) can easily be verified.

Finally, define $\mathscr{R}P_u(0^\pm)$, $\mathscr{R}P_s(0^\pm)$, and $\mathscr{R}P_c(0^\pm)$ as shown in the lemma, and set $\mathscr{R}P_v(t)$, $v = $ u, s, or c, $t \in \mathbf{R}^+$ or $\mathbf{R}^-$, by applying $T(t, 0)$ to $\mathscr{R}P_v(0^\pm)$, $t \in \mathbf{R}^+$ or $\mathbf{R}^-$, respectively. It remains to prove the desired exponential estimates to confirm that $P_v(t)$, $v = $ u, s, c which are completely determined by $\mathscr{R}P_v(t)$ are the desired projections which define exponential trichotomies. The proof is straightforward and shall be omitted. Q.E.D.

The following linear system shall be used in the study of regular layers. Define an evolution system in (x, y), with the help of an intermediate variable v,

$$\begin{aligned} &x'(\tau) - \varepsilon A(\varepsilon\tau)\, x(\tau) - \varepsilon B(\varepsilon\tau)\, v(\tau) = 0, \\ &v'(\tau) - D(\varepsilon\tau)\, v(\tau) = 0, \\ &v(\tau) = C(\varepsilon\tau)\, x(\tau) + y(\tau), \qquad \tau \in [a/\varepsilon, b/\varepsilon]. \end{aligned} \tag{3.6}$$

Here $a < b$. $\tau = t/\varepsilon$, $t \in [a, b]$. $' = d/d\tau$. $A(t)$, $B(t)$, $C(t)$, $D(t)$, and $(d/dt)\, D(t)$ are continuous in $t \in [a, b]$. $\sigma\{D(t)\} \cap \{|\mathrm{Re}\, \lambda| \leqslant \alpha\} = \varnothing$ for all $t \in [a, b]$. From Lemma 3.3, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $v' - D(\varepsilon\tau) v = 0$ has an exponential dichotomy in $[a/\varepsilon, b/\varepsilon]$, with the solution map being $U(\tau, \sigma)$ and the projections being $Q_u(\tau)$ and $Q_s(\tau)$. Let $S(t, s)$ be the solution map for $\dot{x}(t) - A(t)\, x(t) = 0$, and $T(\tau, \sigma)$ be the solution map for (3.6).

LEMMA 3.9. *Equation* (3.6) *has an exponential trichotomy in* $[a/\varepsilon, b/\varepsilon]$, $0<\varepsilon<\varepsilon_0$. *The constant and the exponent do not depend on* ε. *The center space is defined as*

$$\mathscr{R}P_c(\tau)=\{(x, y): C(\varepsilon\tau)x+y=0\}.$$

There exist constants C_2, $C_2>0$ *such that*

$$\begin{aligned} |T(\tau,\sigma)\,P_c(\sigma)| &\leqslant C_1, && a/\varepsilon\leqslant\sigma\leqslant\tau\leqslant b/\varepsilon,\\ \int_{\sigma_1}^{\tau} |T(\tau,\sigma)\,P_c(\sigma)|\,d\sigma &\leqslant C_2/\varepsilon, && a/\varepsilon\leqslant\sigma_1\leqslant\tau\leqslant\beta/\varepsilon. \end{aligned} \tag{3.7}$$

The stable space $\mathscr{R}P_s(\tau)$ *is the image of a linear isomorphism* $v_1(\tau)\to(x(\tau), y(\tau))$ *for all* $v_1(\tau)\in\mathscr{R}Q_s(\tau)$, *while the unstable space* $\mathscr{R}P_u(\tau)$ *is the image of a linear isomorphism* $v_2(\tau)\to(x(\tau), y(\tau))$ *for all* $v_2(\tau)\in\mathscr{R}Q_u(\tau)$. *Moreover*

$$|x(\tau)|+|y(\tau)-v_1(\tau)|\leqslant C\varepsilon\,|v_1(\tau)| \tag{3.8}$$

for $(x(\tau), y(\tau))\in\mathscr{R}P_s(\tau)$, *and*

$$|x(\tau)|+|y(\tau)-v_2(\tau)|\leqslant C\varepsilon\,|v_2(\tau)| \tag{3.9}$$

for $(x(\tau), y(\tau))\in\mathscr{R}P_u(\tau)$.

Proof. Consider a τ-dependent change of coordinates $(x, y)\to(x, v=C(\varepsilon\tau)x+y)$. Clearly $\mathscr{R}P_c(\tau)=\{v=0\}$ is invariant under $T(\tau,\sigma)$. And for $(x(\sigma), y(\sigma))\in\mathscr{R}P_c(\sigma)$,

$$T(\tau,\sigma)(x(\sigma), y(\sigma))=(S(\varepsilon\tau,\varepsilon\sigma)\,x(\sigma),\ -C(\varepsilon\tau)\,S(\varepsilon\tau,\varepsilon\sigma)\,x(\sigma)).$$

Therefore (3.7) is valid.

For $v_1(\tau)\in\mathscr{R}Q_s(\tau)$, define $(x(\tau), y(\tau))\in\mathscr{R}P_s(\tau)$ as

$$\begin{aligned} x(\tau) &= \int_{b,\varepsilon}^{\tau} S(\varepsilon\tau,\varepsilon\sigma)\,\varepsilon B(\varepsilon\sigma)\,U(\sigma,\tau)\,v_1(\tau)\,d\sigma\\ y(\tau) &= v_1(\tau)-C(\varepsilon\tau)\,x(\tau), \qquad a/\varepsilon\leqslant\tau\leqslant b/\varepsilon. \end{aligned} \tag{3.10}$$

From this the estimate (3.8) follows easily. We claim that

$$|T(\tau_1,\tau)(x(\tau), y(\tau))|\leqslant Ke^{-\gamma(\tau_1-\tau)}|v_1(\tau)|, \qquad \tau\leqslant\tau_1\leqslant b/\varepsilon \tag{3.11}$$

for some $K\geqslant 1$, $\gamma>0$. In fact $v_1(\sigma)=U(\sigma,\tau)\,v_1(\tau)$, $\sigma\geqslant\tau$ is a solution for the 2nd equation of (3.6) and an exponential estimate for $|v_1(\sigma)|$ holds due to the fact $v_1(\tau)\in \mathscr{R}Q_s(\tau)$, and (3.11) then follows from (3.8) with τ replaced by τ_1. There also exist C_3, $C_4>0$ such that

$$C_3|v_1(\tau)| \leqslant |x(\tau)| + |y(\tau)| \leqslant C_4|v_1(\tau)|.$$

Thus $|T(\tau_1, \tau)(x(\tau), y(\tau))| \leqslant K_1 e^{-\gamma(\tau_1-\tau)}(|x(\tau)|+|y(\tau)|)$, $\tau_1 \geqslant \tau$. Similarly, for $v_2(\tau)\in \mathscr{R}Q_u(\tau)$, define $(x(\tau), y(\tau))\in \mathscr{R}P_u(\tau)$ as

$$\begin{aligned} x(\tau) &= \int_{a/\varepsilon}^{\tau} S(\varepsilon\tau, \varepsilon\sigma)\, \varepsilon B(\varepsilon\sigma)\, U(\sigma, \tau)\, v_2(\tau)\, d\sigma, \\ y(\tau) &= v_2(\tau) - C(\varepsilon\tau)\, x(\tau), \qquad a/\varepsilon \leqslant \tau \leqslant b/\varepsilon. \end{aligned} \tag{3.12}$$

We can show (3.9) and

$$|T(\tau_1, \tau)(x(\tau), y(\tau))| \leqslant K_1 e^{-\gamma(\tau-\tau_1)}(|x(\tau)|+|y(\tau)|), \qquad \tau \geqslant \tau_1.$$

We can show that $\mathscr{R}P_s(\tau)$ and $\mathscr{R}P_u(\tau)$ are invariant under $T(\tau, \sigma)$. These together with $\mathscr{R}P_c(\tau)$ determine the projections $P_c(\tau)$, $P_s(\tau)$, and $P_u(\tau)$.

It can be shown that if $\varepsilon_0>0$ is small and $0<\varepsilon<\varepsilon_0$,

$$|P_c(\tau)| + |P_s(\tau)| + |P_u(\tau)| \leqslant K$$

for some $K>0$, based on (3.8) and (3.9). Q.E.D.

LEMMA 3.10. *If B is nilpotent, $B^k=0$ for some $k>1$ and $|B|\leqslant K$ for some $K\geqslant 1$. A is a matrix of the same order, $|A|\leqslant \delta<1$. If*

$$2(\delta K^{k-1})^{1/k} < 1,$$

then

$$\sum_{n=1}^{\infty} |(A+B)^n| < \infty.$$

Proof. $(A+B)^n = \sum C_{\sigma_1}\cdots C_{\sigma_n}$, where $\sigma_i=1$ or 2, $C_1=A$ and $C_2=B$. The total number of the terms in the sum is 2^n. Each nonzero product in the sum is of the form

$$A^{i1}B^{j1}A^{i2}B^{j2}\cdots A^{im}B^{jm}$$

with $\sum_{\alpha=1}^{m}(i_\alpha+j_\alpha)=n$ and $j_\alpha\leqslant k-1$. Let $n=lk$, and for each nontrivial term we have

$$|A^{i1}B^{j1}\cdots A^{im}B^{jm}| \leqslant \underbrace{(|A|\, K^{k-1})\cdots(|A|\, K^{k-1})}_{\text{more than } l\text{-tuple}} \leqslant (\delta K^{k-1})^l$$

since the total number of the A's is $i_1+\cdots+i_m\geqslant l$. We have the estimate for $r=\lim_{n\to\infty}\{|(A+B)^n|^{1/n}\}$:

$$r \leqslant \{2^n(\delta K^{k-1})^l\}^{1/n} = 2(\delta K^{k-1})^{1/k} < 1. \qquad \text{Q.E.D.}$$

4. A Linear Boundary Value Problem

The solvability of a linearized boundary value problem associated with (2.1) is the key to justify the correctness of our formal approximation. The unusual character of the linear boundary value problem is that the solution admits jumps which are part of the input data. It is reminiscent of our early work based on a modified shadowing lemma (Lin [19]). The major difference is that the linearization in this paper is made around the 0th order approximations while in the previous paper the linearization was made around the higher order truncations. The advantage of the new method is that the linear operator is now essentially independent of ε. The linearized equation in the outer layers and inner layers are quite different, mostly because the time spent on the regular region is $O(1/\varepsilon)$ (in the fast variable τ), therefore $O(\varepsilon)$ terms have to be retained, while the time spent in inner layers is shorter, thus the $O(\varepsilon)$ term may be dropped.

To simplify the notation, define $f^i(\varepsilon\tau) = f(X_0^l(\varepsilon\tau), Y_0^l(\varepsilon\tau), 0)$ for $i = 2l$, $1 \leqslant l \leqslant I$. Similarly define $f_x^i(\varepsilon\tau)$, $f_y^i(\varepsilon\tau)$, $g^i(\varepsilon\tau)$, $g_x^i(\varepsilon\tau)$, and $g_y^i(\varepsilon\tau)$ in the obvious way for $i = 2l$, $1 \leqslant l \leqslant I$. Next, define $f^i(\tau) = f(x_0^l(\tau), y_0^l(\tau), 0)$ for $i = 2l+1$, $0 \leqslant l \leqslant I$. Similarly, define $f_x^i(\tau)$, $f_y^i(\tau)$, $g^i(\tau)$, $g_x^i(\tau)$, and $g_y^i(\tau)$ in the obvious way for $i = 2l+1$, $0 \leqslant l \leqslant I$.

Consider the linear boundary value problem

$$z_i(\tau)' - A_i(\tau, \varepsilon)\, z_i(\tau) = F_i(\tau), \qquad \tau \in [\alpha_i, \beta_i], \qquad 1 \leqslant i \leqslant \nu,\ \nu = 2I+1, \tag{4.1}$$

$$z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + \zeta_i e_i = h_i, \qquad 1 \leqslant i \leqslant \nu - 1, \tag{4.2}$$

$$\bar{B}_1(-z_1(\alpha_1) + \zeta_0 e_0) = -\tilde{b}_1, \tag{4.3}$$

$$\bar{B}_2(z_\nu(\beta_\nu) + \zeta_\nu e_\nu) = \tilde{b}_2, \tag{4.4}$$

$$d^i \cdot z_i(\tau_i) = 0, \qquad 1 \leqslant i \leqslant \nu, \tag{4.5}$$

where $z_i = (x_i, y_i) \in \mathbf{R}^m \times \mathbf{R}^n$, $F_i(\tau) = (f_i(\tau), g_i(\tau)) \in \mathbf{R}^m \times \mathbf{R}^n$. $\bar{B}_1 = D_z B_1(x_0^0(0), y_0^0(0), 0)$: $\mathbf{R}^{m+n} \to \mathbf{R}^{d_1}$ and $\bar{B}_2 = D_z B_2(x_0^I(0), y_0^I(0), 0)$: $\mathbf{R}^{m+n} \to \mathbf{R}^{d_2}$ are matrices of rank d_1 and d_2, respectively. $\zeta_i \in \mathbf{R}$, $0 \leqslant i \leqslant \nu$, is an unknown parameter. $[\alpha_i, \beta_i]$ is given as follows:

(i) $[\alpha_i, \beta_i] = [a_l/\varepsilon + \varepsilon^{\beta-1} + \bar{a}_l,\ b_l/\varepsilon - \varepsilon^{\beta-1} + \bar{b}_l]$, where $0 < \beta < 1$, $\bar{a}_l$ and $\bar{b}_l$ are real polynomials in ε, if $i = 2l$, $1 \leqslant l \leqslant I$;

(ii) $[\alpha_i, \beta_i] = [-\varepsilon^{\beta-1}, \varepsilon^{\beta-1}]$, if $i = 2l+1$, $1 \leqslant l \leqslant I-1$;

(iii) $[\alpha_1, \beta_1] = [0, \varepsilon^{\beta-1}]$, and $[\alpha_\nu, \beta_\nu] = [-\varepsilon^{\beta-1}, 0]$.

Equation (4.1) is given as follows:

(i) For $i=2l$, $1\leqslant l\leqslant I$, (4.1) has the form

$$\begin{aligned}&x_i(\tau)'-\varepsilon f_x^i(\varepsilon\tau)\,x_i-\varepsilon f_y^i(\varepsilon\tau)\,y_i=f_i(\tau),\\&y_i(\tau)'-g_x^i(\varepsilon\tau)\,x_i-g_y^i(\varepsilon\tau)\,y_i-(g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau))'\,x_i\\&\quad+(g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau))\cdot\{\varepsilon f_x^i(\varepsilon\tau))\cdot\{\varepsilon f_x^i(\varepsilon\tau)\,x_i+\varepsilon f_y^i(\varepsilon\tau)\,y_i\}=g_i(\tau),\end{aligned}\tag{4.1$'$}$$

which can be rewritten as

$$\begin{aligned}&x_i(\tau)'-\varepsilon\{f_x^i(\varepsilon\tau)-f_y^i(\varepsilon\tau)\cdot g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau)\}\,x_i(\tau)\\&\quad-\varepsilon f_y^i(\varepsilon\tau)\,v_i(\tau)=f_i(\tau),\\&v_i(\tau)'-g_y^i(\varepsilon\tau)\,v_i(\tau)=g_i(\tau)+g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau)\,f_i(\tau),\\&\quad\text{where}\quad v_i(\tau)=y_i(\tau)+g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau)\,x_i(\tau).\end{aligned}\tag{4.1$''$}$$

(ii) For $i=2l+1$, $0\leqslant l\leqslant I$, (4.1) has the form

$$\begin{aligned}&x_i(\tau)'=f_i(\tau),\\&y_i(\tau)'-g_x^i(\tau)\,x_i(\tau)-g_y^i(\tau)\,y_i(\tau)=g_i(\tau).\end{aligned}\tag{4.1$'''$}$$

Let $U^i(\tau,\sigma)$ denote the solution map for the equation

(i) $y(\tau)'-g_y^i(\tau)\,y(\tau)=0,\qquad$ if $\;i=2l+1,\quad 0\leqslant l\leqslant I.$

(ii) $y(\tau)'-g_y^i(\varepsilon\tau)\,y(\tau)=0,\qquad$ if $\;i=2l,\qquad 1\leqslant l\leqslant I.$

From the hypothesis (H_1) and Lemmas 3.2 and 3.3, $U^i(\tau,\sigma)$ has an exponential dichotomy in $[\alpha_i,\beta_i]$ if $i=2l$, $1\leqslant l\leqslant I$, and $U^i(\tau,\sigma)$ has exponential dichotomies in $\mathbf{R}^+$ if $i=2l+1$, $0\leqslant l\leqslant I-1$, and in $\mathbf{R}^-$ if $i=2l+1$, $1\leqslant l\leqslant I$. Let the associated projections be $Q_{\mathrm{s}}^i(\tau)$ and $Q_{\mathrm{u}}^i(\tau)$ (onto the stable and unstable spaces, respectively). The constant τ_i in (4.5) is given by $\tau_i=t_i/\varepsilon\in[\alpha_i,\beta_i]$, where t_i does not depend on ε if $i=2l$, $1\leqslant l\leqslant I$. $\tau_i=0$ if $i=2l+1$, $0\leqslant l\leqslant I$.

Let $T^i(\tau,\sigma)$ be the solution map for the homogeneous equati on of (4.1). From Lemmas 3.8 and 3.9, $T^i(\tau,\sigma)$ has exponential trichotomy in $[\alpha_i,\beta_i]$ if $i=2l$, $1\leqslant l\leqslant I$, and $T^i(\tau,\sigma)$ has exponential trichotomies in $[\alpha_i,0]$ and $[0,\beta_i]$, respectively, if $i=2l+1$, $0\leqslant l\leqslant I$. We assume that the projections which define these trichotomies have been chosen such that Lemma 3.8 applies to the case $i=2l+1$, $1\leqslant l\leqslant I-1$, with $A(\tau)=g_x^i(\tau)$ and $B(\tau)=g_y^i(\tau)$, and such that Lemma 3.9 applies to the case $i=2l$, $1\leqslant l\leqslant I$, with $A(\varepsilon\tau)=f_x^i(\varepsilon\tau)-f_y^i(\varepsilon\tau)\,g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau)$, $B(\varepsilon\tau)=f_y^i(\varepsilon\tau)$, $C(\varepsilon\tau)=g_y^i(\varepsilon\tau)^{-1}g_x^i(\varepsilon\tau)$, and $D(\varepsilon\tau)=g_y^i(\varepsilon\tau)$. The linear subspaces associated with Lemmas 3.8 and

3.9 shall be denoted by $\mathscr{R}P^i_s$, $\mathscr{R}P^i_c$, $U^i(0)$, $V^i(0)$, $N^i(0)$, etc., for $1 \leqslant i \leqslant \nu$. We now define $d^i = (d^i_x, d^i_y) \neq 0$ as follows: $d^i \perp H(i)$, where

(i) $H(i) = \mathscr{R}P^i_s(0^-) \oplus \mathscr{R}P^i_u(0^+) \oplus N^i(0)$, $\quad i = 2l+1,\ 1 \leqslant l \leqslant I-1$;

(ii) $H(i) = \mathscr{R}P^i_s(\tau_i) \oplus \mathscr{R}P^i_u(\tau_i) \oplus \{(x, y): x \cdot f^i(\varepsilon\tau_i) = 0,$
$y = -g^i_y(\varepsilon\tau_i)^{-1} g^i_x(\varepsilon\tau_i) x\}$ $\quad$ if $i = 2l,\ 1 \leqslant l \leqslant I$;

(iii) $H(i) = \{(x, y): x \in TM_l\}$, $\quad$ if $i = 2l+1,\ l = 0$ or $l = I$.

$e_i \in \mathbf{R}^m \times \mathbf{R}^n$ is given as

$$e_i = (f^{i+1}(\varepsilon\alpha_{i+1}), -g^{i+1}_y(\varepsilon\alpha_{i+1})^{-1} g^{i+1}_x(\varepsilon\alpha_{i+1}) f^{i+1}(\varepsilon\alpha_{i+1})),$$
$$\text{if} \quad i = 2l-1,\ 1 \leqslant l \leqslant I,$$

$$e_i = (f^i(\varepsilon\beta_i), -g^i_y(\varepsilon\beta_i)^{-1} g^i_x(\varepsilon\beta_i) f^i(\varepsilon\beta_i)), \qquad \text{if} \quad i = 2l,\ 1 \leqslant l \leqslant I,$$

$$e_0 = \left(f^2(a_1), \int_{\infty}^{0} U^1(\tau, \sigma)\, Q^1_u(\sigma)\, g^1_x(\sigma)\, d\sigma \cdot f^2(a_1)\right),$$

$$e_\nu = \left(f^{\nu-1}(b_I), \int_{-\infty}^{0} U^\nu_s(\sigma)\, Q^\nu_s(\sigma)\, g^\nu_x(\sigma)\, d\sigma \cdot f^{\nu-1}(b_I)\right).$$

THEOREM 4.1. *There is* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0$ *the boundary value problem* (4.1)–(4.5) *admits a unique solution* $(\{z_i(\tau)\}_{i=1}^{\nu}, \{\zeta_i(\tau)\}_{i=0}^{\nu})$:

$$\sup_{1 \leqslant i \leqslant \nu} |z_i|_c + \sup_{0 \leqslant i \leqslant \nu} |\zeta_i| \leqslant C \Bigg\{ \sup_{1 \leqslant i \leqslant \nu-1} |h_i| + |\tilde{b}_1| + |\tilde{b}_2| + \sup_{1 \leqslant i \leqslant \nu} |g_i|_c$$
$$+ \frac{1}{\varepsilon} \{ |f_i|_c : i = 2l,\ 1 \leqslant l \leqslant I \} + \varepsilon^{\beta-1} \sup\{ |f_i|_c ; i = 2l+1,\ 0 \leqslant l \leqslant I \} \Bigg\}.$$

The proof of Theorem 4.1 is divided into several lemmas.

LEMMA 4.2. *Equations* (4.1) *and* (4.5) *admit a* (*nonunique*) *solution* $\tilde{z}_i(\tau) = (\tilde{x}_i(\tau), (\tilde{y}_i(\tau))$, *with*

$$|\tilde{z}_i|_c \leqslant C \left\{ \frac{1}{\varepsilon} |f_i|_c + |g_i|_c \right\}, \qquad i = 2l,\ 1 \leqslant l \leqslant I, \tag{4.6}$$

$$|\tilde{z}_i|_c \leqslant C \{ \varepsilon^{\beta-1} |f_i|_c + |g_i|_c \}, \qquad i = 2l+1,\ 0 \leqslant l \leqslant I. \tag{4.7}$$

Proof. Since $H(i) \subset \mathbf{R}^{m+n}$ is of codimension one, there exists a solution $\tilde{\phi}_i(\tau)$ of the homogeneous equation associated with (4.1) with $\tilde{\phi}_i(\tau_i) \notin H(i)$, $1 \leqslant i \leqslant \nu$. Moreover, $\tilde{\phi}_i(\tau)$ can be chosen such that

$$|\tilde{\phi}_i(\cdot)|_c \leqslant C |\tilde{\phi}_i(\tau_i)|$$

due to our definition of $H(i)$. Thus, we only need to solve (4.1), and (4.5) can be satisfied by adding a multiple of $\tilde{\phi}_i(\tau)$ to the solution.

For $i=2l$, $1 \leqslant l \leqslant I$, set $\bar{\bar{g}}_i(\tau) = g_i(\tau) + g_y^i(\varepsilon\tau)^{-1} g_x^i(\varepsilon\tau) f_i(\tau)$ and

$$v_i(\tau) = \int_{\alpha_i}^{\tau} U^i(\tau, \sigma)\, Q_s^i(\sigma)\, \bar{\bar{g}}_i(\sigma)\, d\sigma + \int_{\beta_i}^{\tau} U^i(\tau, \sigma)\, Q_u^i(\sigma)\, \bar{\bar{g}}_i(\sigma)\, d\sigma,$$

$$\tilde{x}_i(\tau) = \int_{\alpha_i}^{\tau} S^l(\varepsilon\tau, \varepsilon\sigma) \cdot \varepsilon \cdot \left\{ f_y^l(\varepsilon\sigma)\, v_i(\sigma) + \frac{1}{\varepsilon} f_i(\sigma) \right\} d\sigma,$$

$$\tilde{y}_i(\tau) = v_i(\tau) - g_y^i(\varepsilon\tau)^{-1} g_y^i(\varepsilon\tau)\, \tilde{x}_i(\tau).$$

$\tilde{z}_i(\tau) = (\tilde{x}_i(\tau), \tilde{y}_i(\tau))$ is obviously a solution of (4.1), satisfying (4.6).

For $i = 2l+1$, $1 \leqslant l \leqslant I-1$, extend $f_i(\sigma)$ and $g_i(\sigma)$ by 0 if $\sigma \notin [\alpha_i, \beta_i]$, so that $f_i(\sigma)$ and $g_i(\sigma)$ are defined for $\sigma \in \mathbf{R}$. We shall solve (4.1) and (4.5) in the extended domain $\sigma \in \mathbf{R}$. Set

$$\tilde{x}_i(\tau) = C_i \bar{x}_i + \int_0^{\tau} f_i(\sigma)\, d\sigma,$$

where $\bar{x}_i \in \mathbf{R}^m$ is such that $\Delta_l \cdot \bar{x}_i \neq 0$. Clearly we have

$$|\tilde{x}_i|_c \leqslant |C_i \bar{x}_i| + \varepsilon^{\beta-1} |f_i|_c$$

since $[\alpha_i, \beta_i] = [-\varepsilon^{\beta-1}, \varepsilon^{\beta-1}]$. We then look for a bounded solution $y_i(\tau)$ of the equation

$$\tilde{y}_i(\tau)' - g_y^i(\tau)\, \tilde{y}_i(\tau) = C_i g_x^i(\tau)\, \bar{x}_i + g_x^i(\tau) \int_0^{\tau} f_i(\sigma)\, d\sigma + g_i(\tau). \tag{4.8}$$

The right-hand side is a bounded function defined in $\mathbf{R}$. From Lemma 3.6, (4.8) has a unique bounded solution which satisfies

$$y_0^l(0)' \cdot \tilde{y}_i(0) = 0 \tag{4.9}$$

if and only if

$$\int_{-\infty}^{\infty} \psi_l(\tau)^* \cdot \left\{ C_i g_x^i(\tau)\, \bar{x}_i + g_x^i(\tau) \int_0^{\tau} f_i(\sigma)\, d\sigma + g_i(\tau) \right\} d\tau = 0, \tag{4.10}$$

where $\psi_l(\tau)$ is the unique bounded solution (up to a scalar multiple) of the adjoint equation of (4.8). From hypothesis (H_2) we can solve C_i so that (4.10) is valid. It is trivial to verify that

$$|C_i| \leqslant C(\varepsilon^{\beta-1} |f_i|_c + |g_i|_c).$$

Thus, we have (4.7).

Finally for $i=1$ and $i=\nu$, set

$$\tilde{x}_i(\tau)=\int_0^\tau f_i(\sigma)\,d\sigma, \qquad i=1 \text{ and } i=\nu,$$

$$\tilde{y}_i(\tau)=\int_0^\tau U^i(\tau,\sigma)\,Q^i_{\rm s}(\sigma)(g^i_x(\sigma)\,\tilde{x}_i(\sigma)+g_i(\sigma))\,d\sigma$$

$$+\int_\infty^\tau U^i(\tau,\sigma)\,Q^i_{\rm u}(\sigma)(g^i_x(\sigma)\,\tilde{x}_i(\sigma)+g_i(\sigma))\,d\sigma, \qquad i=1,$$

$$\tilde{y}_i(\tau)=\int_0^\tau U^i(\tau,\sigma)\,Q^i_{\rm u}(\sigma)(g^i_x(\sigma)\,\tilde{x}_i(\sigma)+g_i(\sigma))\,d\sigma$$

$$+\int_{-\infty}^\tau U^i(\tau,\sigma)\,Q^i_{\rm s}(\sigma)(g^i_x(\sigma)\,\tilde{x}_i(\sigma)+g_i(\sigma))\,dx, \qquad i=\nu.$$

Again $\tilde{z}_i=(\tilde{x}(\tau),\tilde{y}_i(\tau))$, $i=1,\nu$, is a solution of (4.1) with (4.7) being valid.

Our next step is to solve system (4.1)–(4.5) with $F_i(\tau)\equiv 0$.

LEMMA 4.3. *There exists a constant* $\varepsilon_0>0$ *such that if* $0<\varepsilon\leqslant\varepsilon_0$, *the boundary value problem* (4.1)–(4.5), *with* $F_i(\tau)\equiv 0$, *admits a unique solution* $(\{z_i(\tau)\}_{i=1}^{\nu},\{\zeta_i\}_{i=0}^{\nu})$. *Moreover*

$$\sup_{1\leqslant i\leqslant\nu}|z_i|_{\rm c}+\sup_{0\leqslant i\leqslant\nu}|\zeta_i|\leqslant C(\sup_{1\leqslant i\leqslant\nu-1}|h_i|+|\tilde{b}_1|+|\tilde{b}_2|).$$

Proof. The proof is based on an iteration scheme. If we choose $\{z_i\}_{i=1}^{\nu}\equiv 0$, $\{\zeta_i\}_{i=0}^{\nu}\equiv 0$, then $\{h_i\}_{i=1}^{\nu-1}$, $\tilde{b}_1$, and $\tilde{b}_2$ become the "error" terms. The purpose of the scheme is to project the errors onto the stable, unstable, and center spaces and pass them to the boundaries and eventually to be absorbed by the boundaries. (Recall the relations of $L_{\rm c}(0)$ and $R_{\rm c}(I)$ with ker $\bar{B}_1$ and ker $\bar{B}_2$.)

We start to define a codimension one subspace for each $1\leqslant i\leqslant\nu$ which admits the splitting

$$L^i_{\rm c}(\tau)\oplus R^i_{\rm c}(\tau)\subset\mathscr{R}P^i_{\rm c}(\tau), \qquad 1\leqslant i\leqslant\nu.$$

For $i=2l$, $1\leqslant l\leqslant I$, let

$$\Sigma_i=\{x\in\mathbf{R}^m\,|\,f^i(\varepsilon\tau_i)\cdot x=0\}.$$

Define

$$L^i_{\rm c}(\tau_i)=\{(x,y)\,|\,x\in S^l(\varepsilon\tau_i,a_l;\Sigma_i,TM_{l-1})\,L_{\rm c}(l-1),$$
$$y=-g^i_y(\varepsilon\tau_i)^{-1}g^i_y(\varepsilon\tau_i)x\}, \qquad \text{for } i=2l,\ 1\leqslant l\leqslant I,$$

$$R^i_{\rm c}(\tau_i)=\{(x,y)\,|\,x\in S^l(\varepsilon\tau_i,a_l;\Sigma_i,TM_{l-1})\,R_{\rm c}(l-1),$$
$$y=-g^i_y(\varepsilon\tau_i)^{-1}g^i_y(\varepsilon\tau_i)x\}, \qquad \text{for } i=2l,\ 1\leqslant l\leqslant I.$$

For $i=2l+1$, $1\leqslant l\leqslant I-1$, $(4.1)'''$ naturally extends to $\tau\in\mathbf{R}$, and the trichotomies extend to $\tau\in\mathbf{R}^-$ and $\tau\in\mathbf{R}^+$. According to Lemma 3.8, for each x, with $\Delta_l\cdot x=0$, there is a unique y such that $(x,y)\in\mathscr{R}P^l_{\mathrm{cu}}(0^-)\cap\mathscr{R}P^l_{\mathrm{cs}}(0^+)$, and such that $y^l_0(0)'\cdot y=0$. Denote the relation by $y=L^lx$. Define

$$L^i_{\mathrm{c}}(0)=\{(x,y)\mid x\in L_{\mathrm{c}}(l),\ y=L^lx\},$$

$$R^i_{\mathrm{c}}(0)=\{(x,y)\mid x\in R_{\mathrm{c}}(l),\ y=L^lx\}.$$

For $i=1$, $(4.1)'''$ extends to $\tau\in\mathbf{R}^+$. Define

$$L^1_{\mathrm{c}}(0)=\{(x,y)\mid (x,y)\in\mathscr{R}P^1_{\mathrm{cs}}(0)\cap\ker\bar{B}_1=\mathscr{K}\mathscr{B}_1\}.$$

From our discussion in Section 2, $L^1_{\mathrm{c}}(0)=\{(x,y)\mid x\in L_{\mathrm{c}}(0),\ y=G^0(x)\}$. Let

$$R^1_{\mathrm{c}}(0)=\left\{(x,y)\mid x\in R_{\mathrm{c}}(0),\ y=\int_{\infty}^{0}U^1(0,\sigma)\,Q^1_{\mathrm{u}}(\sigma)\,g^1_{\mathrm{v}}(\sigma)\,d\sigma\cdot x\right\}.$$

Similarly, for $i=\nu$, $(4.1)'''$ extends to $\tau\in\mathbf{R}^-$. Define

$$R^\nu_{\mathrm{c}}(0)=\{(x,y)\mid (x,y)\in\mathscr{R}\mathscr{P}^\nu_{\mathrm{cu}}(0)\cap\ker\bar{B}_2=\mathscr{K}\mathscr{B}_2\}.$$

From our discussion in Section 2, $R^\nu_{\mathrm{c}}(0)=\{(x,y)\mid x\in R_{\mathrm{c}}(I),\ y=G^{I+1}(x)\}$. Let

$$L^\nu_{\mathrm{c}}(0)=\left\{(x,y)\mid x\in L_{\mathrm{c}}(I),\ y=\int_{-\infty}^{0}U^\nu(0,\sigma)\,Q^\nu_{\mathrm{s}}(\sigma)\,g^\nu_x(\sigma)\,g^\nu_x(\sigma)\,d\sigma\cdot x\right\}.$$

Recall that $\tau_i=0$ for all $i=2l-1$, $1\leqslant l\leqslant I+1$. Finally, in all the cases let $L^i_{\mathrm{c}}(\tau)=T^i(\tau,\tau_i)\,L^i_{\mathrm{c}}(\tau_i)$ and $R^i_{\mathrm{c}}(\tau)=T^i(\tau,\tau_i)\,R^i_{\mathrm{c}}(\tau_i)$, $\tau\in[\alpha_i,\beta_i]$.

For convenience define $L^0_{\mathrm{c}}(\beta_0)=L^1_{\mathrm{c}}(\alpha_1)$ and $R^{\nu+1}_{\mathrm{c}}(\alpha_{\nu+1})=R^\nu_{\mathrm{c}}(\beta_\nu)$, and define $\mathscr{R}P^0_{\mathrm{u}}(\beta_0)$ to be a subspace of $\ker\bar{B}_1$ such that

$$\mathscr{R}P^0_{\mathrm{u}}(\beta_0)\oplus L^0_{\mathrm{c}}(\beta_0)=\ker\bar{B}_1. \tag{4.11}$$

Similarly, let $\mathscr{R}P^{\nu+1}_{\mathrm{s}}(\alpha_{\nu+1})$ be a subspace of $\ker\bar{B}_2$ such that

$$\mathscr{R}P^{\nu+1}_{\mathrm{s}}(\alpha_{\nu+1})\oplus R^{\nu+1}_{\mathrm{c}}(\alpha_{\nu+1})=\ker\bar{B}_2. \tag{4.12}$$

We remark that β_0 and $\alpha_{\nu+1}$ have no true meaning, they are introduced for the sake of notational symmetry. To complete the proof of Lemma 4.3, we need Lemmas 4.4 and 4.5 and Corollary 4.6.

LEMMA 4.4. $\mathbf{R}^{m+n}=\mathscr{R}P^i_{\mathrm{u}}(\beta_i)\oplus L^i_{\mathrm{c}}(\beta_i)\oplus\mathscr{R}P^{i+1}_{\mathrm{s}}(\alpha_{i+1})\oplus\mathbf{R}^{i+1}_{\mathrm{c}}(\alpha_{i+1})\oplus$ span$[e_i]$ *for all* $0\leqslant i\leqslant\nu$. *Let the projections corresponding to the above splitting be*

$$I=P(\mathscr{R}P^i_{\mathrm{u}}(\beta_i))+P(L^i_{\mathrm{c}}(\beta_i))+P(\mathscr{R}P^{i+1}_{\mathrm{s}}(\alpha_{i+1}))$$
$$+P(R^{i+1}_{\mathrm{c}}(\alpha_{i+1}))+P(\mathrm{span}[e_i]),$$

then the norms of all the projections are bounded by a constant $M \geqslant 1$ which does not depend on $\varepsilon \leqslant \varepsilon_0$.

Proof. We only prove the case $i=2l$, $1 \leqslant l \leqslant I$. Consider the limit of each subspace as $\varepsilon \to 0$. From (3.9) of Lemma 3.9, $\mathscr{R}P_u^i(\beta_i) \to \{(x, y): x=0, y \in \mathscr{R}Q_u^i(\beta_i)\}$ as $\varepsilon \to 0$. From Lemma 3.3, $\mathscr{R}Q_u^i(\beta_i) \to$ the unstable eigenspace of $g_y^i(\varepsilon\beta_i)\}$. From the definition of β_i, $i=2l$, it is clear that $\varepsilon\beta_i \to b_l$ and $\mathscr{R}P_u^i(\beta_i) \to E_1 \overset{\text{def}}{=} \{(x, y): x=0, y \in$ unstable eigenspace of $g_y^i(b_l)\}$, as $\varepsilon \to 0$. We can also show that $L_c^i(\beta_i) \to E_2 \overset{\text{def}}{=} \{(x, y): x \in L_c(l), y = -g_y^i(b_l)^{-1} g_x^i(b_l)x\}$ and $\operatorname{span}[e_i] \to E_5 = \operatorname{span}[(f^i(b_l), -g_y^i(b_l)^{-1} g_x^i(b_l) f^i(b_l)]$ based on $\varepsilon\beta_i \to b_l$ as $\varepsilon \to 0$. We then observe that $\alpha_{i+1} = -\varepsilon^{\beta-1} \to -\infty$ as $\varepsilon \to 0$. If $(x, y(\tau)) \in R_c^{i+1}(\tau)$, then $x \in R_c(l)$ and $(x, y(\tau)) \in \mathscr{R}P_{cu}^{i+1}(\tau)$. From Lemma 3.7, $y(\tau) \to -g_y^{i+1}(-\infty)^{-1} g_x^{i+1}(-\infty)x$. It follows that $R_c^{i+1}(\alpha_{i+1}) \to E_4 \overset{\text{def}}{=} \{(x, y): x \in R_c(l), y = -g_y^i(b_l)\, g_x^i(b_l)x\}$ since $g_y^{i+1}(-\infty) = g_y^i(b_l)$ and $g_x^{i+1}(-\infty) = g_x^i(b_l)$. From Lemma 3.2, $\mathscr{R}P_s^{i+1}(\alpha_{i+1}) = \{(x, y): x=0, y \in Q_s^{i+1}(\alpha_{i+1})\} \to E_3 \overset{\text{def}}{=} \{(x, y): x=0, y \in$ stable eigenspace of the matrix $g_x^i(b_l)\}$ as $\varepsilon \to 0$.

Finally, observe that $\mathbf{R}^{m+n} = \bigoplus_{i=1}^5 E_i$. If $\varepsilon_0 > 0$ is small and $0 < \varepsilon < \varepsilon_0$, each subspace under consideration is close to one of the spaces E_i, $1 \leqslant i \leqslant 5$, the desired result follows from the standard theory concerning the perturbation of E_i, $1 \leqslant i \leqslant 5$, and the projections determined by the mutually complementary subspaces. See Kato [18]. Q.E.D.

LEMMA 4.5. *For each $\bar{z} \in L_c^{i+1}(\alpha_{i+1})$, we can find $\bar{\bar{z}} \in L_c^i(\beta_i)$ and ζe_i such that*

$$|z - \bar{\bar{z}} - \zeta e_i| \leqslant C\varepsilon^\beta |\bar{z}|. \tag{4.13}$$

For each $\bar{z} \in R_c^i(\beta_i)$, we can find $\bar{\bar{z}} \in R_c^{i+1}(\alpha_{i+1})$ and ζe_i such that (4.13) *is still valid.*

Proof. We only give the proof for the case $i=2l$, $1 \leqslant l \leqslant I$, and $\bar{z} \in R_c^i(\beta_i)$ since the proof of the other cases is completely similar. We have by definition that $\beta_i = b_l/\varepsilon + \bar{b}_l - \varepsilon^{\beta-1}$ and $\alpha_{i+1} = -\varepsilon^{\beta-1}$. Let $\bar{z} = (\bar{x}, \bar{y}) \in R_c^i(\beta_i)$, then $\bar{y} = -g_y^i(\varepsilon\beta_i)^{-1} g_x^i(\varepsilon\beta_i)\bar{x}$. Moreover there exists $\bar{\zeta} \in \mathbf{R}$ such that

$$\tilde{x} \overset{\text{def}}{=} S^l(b_l, \varepsilon\beta_i)\bar{x} + \bar{\zeta} f^i(b_l) \in R_c(l) \subset TM_l.$$

According to our definition of $R_c^{i+1}(0)$, we find $\tilde{y} = \tilde{y}(\tilde{x}) \in \mathbf{R}^n$ such that $(\tilde{x}, \tilde{y}) \in R_c^{i+1}(0)$ with $|\tilde{y}| \leqslant C|\tilde{x}|$. Finally, define $\bar{\bar{z}} = (\bar{\bar{x}}, \bar{\bar{y}}) = T^{i+1}(-\varepsilon^{\beta-1}, 0)\,(\tilde{x}, \tilde{y}) \in R_c^{i+1}(\alpha_{i+1})$. And clearly we have $\bar{\bar{x}} = \tilde{x}$.

Since $T^{i+1}(\tau, 0)(\tilde{x}, \tilde{y}) \to (\tilde{x}, -g_y^i(b_l)^{-1} g_x^i(b_l)\tilde{x})$ exponentially fast as

$\tau \to -\infty$, see Lemma 3.7. To complete the proof of Lemma 4.5, it suffices to prove that

$$(\bar{x}, -g_y^i(\varepsilon\beta_i)^{-1}g_x^i(\varepsilon\beta_i)\bar{x}) + (\zeta f^i(\varepsilon\beta_i), -g_y^i(\varepsilon\beta_i)^{-1}g_x^i(\varepsilon\beta_i)\,\zeta f^i(\varepsilon\beta_i))$$
$$-(\tilde{x}, -g_y^i(b_l)^{-1}g_x^i(b_l)\tilde{x}) = O(\varepsilon^\beta \cdot |\tilde{x}|).$$

This is true since we have

$$\varepsilon\beta_i - b_l = O(\varepsilon^\beta)$$
$$|S^l(b_l, \varepsilon\beta_i)\bar{x} - \bar{x}| \leqslant C\varepsilon^\beta |\bar{x}|.$$

Q.E.D.

COROLLARY 4.6. (i) $|P(\mathscr{R}P_u^i(\beta_i))| + |P(\mathscr{R}P_s^{i+1}(\alpha_{i+1}))| + |P(R_c^{i+1}(\alpha_{i+1}))| + |P(\mathrm{span}[e_i])| = O(\varepsilon^\beta)$ *if the domain of all the operators is restricted to* $L_c^{i+1}(\alpha_{i+1})$.

(ii) $|P(\mathscr{R}P_u^i(\beta_i))| + |P(\mathscr{R}P_s^{i+1}(\alpha_{i+1}))| + |P(L_c^i(\beta_i))| + |P(\mathrm{span}[e_i])| = O(\varepsilon^\beta)$ *if the domain of all the operators is restricted to* $R_c^i(\beta_i)$.

Proof of Lemma 4.3 (*continued*). From (4.11) and (4.12), there exist unique elements $\bar{v}$ and $\bar{\bar{v}}$ such that

$$\bar{v} \in \mathscr{R}P_s^1(\alpha_1) \oplus R_c^1(\alpha_1) \oplus \mathrm{span}[e_0],$$
$$\bar{\bar{v}} \in \mathscr{R}P_u^\nu(\beta_\nu) \oplus L_c^\nu(\beta_\nu) \oplus \mathrm{span}[e_\nu],$$
$$\bar{B}_1 \bar{v} = -\tilde{b}_1, \tag{4.14}$$
$$\bar{B}_2 \bar{\bar{v}} = \tilde{b}_2, \tag{4.15}$$

with

$$|\bar{v}| \leqslant c|\tilde{b}_1| \qquad \text{and} \qquad |\bar{\bar{v}}| \leqslant c|\tilde{b}_2|.$$

We can rexrite (4.3) and (4.4) as

$$\begin{aligned} -\bar{v} - z_1(\alpha_1) + \zeta_0 e_0 &\in \ker \bar{B}_1, \\ z_\nu(\beta_\nu) - \bar{\bar{v}} + \zeta_\nu e_\nu &\in \ker \bar{B}_2. \end{aligned} \tag{4.16}$$

We are ready to define the iteration scheme. Let

$$\begin{aligned} h_i^1 &= h_i, \qquad 1 \leqslant i \leqslant \nu - 1, \\ h_0^1 &= \bar{v} \qquad \text{and} \qquad h_\nu^1 = \bar{\bar{v}}. \end{aligned} \tag{4.17}$$

Define for $k \geqslant 1$:

$$z_i^k = -T^i(\tau, \alpha_i)\{P(\mathscr{R}P_s^i(\alpha_i)) + P(R_c^i(\alpha_i))\}\, h_{i-1}^k$$
$$+ T^i(\tau, \beta_i)\{P(\mathscr{R}P_u^i(\beta_i)) + P(L_c^i(\beta_i))\}\, h_i^k, \qquad 1 \leqslant i \leqslant v, \quad (4.18)$$

$$z_0^k(\beta_0) = \{P(\mathscr{R}P_u^0(\beta_0)) + P(L_c^0(\beta_0))\}\, h_0^k \qquad (4.18)'$$

$$z_{v+1}^k(\alpha_{v+1}) = \{P(\mathscr{R}P_s^{v+1}(\alpha_{v+1})) + P(R_c^{v+1}(\alpha_{v+1}))\}\, h_v^k \qquad (4.18)''$$

$$\zeta_i^k = \frac{1}{|e_i|^2} e_i \cdot P(\text{span}[e_i])\, h_i^k, \qquad 0 \leqslant i \leqslant v, \quad (4.19)$$

$$h_i^{k+1} = h_i^k - [z_i^k(\beta_i) - z_{i+1}^k(\alpha_{i+1}) + \zeta_i^k e_i], \qquad 0 \leqslant i \leqslant v. \quad (4.17)'$$

We claim that

$$z_i(\tau) = \sum_{k=1}^{\infty} z_i^k(\tau), \qquad 1 \leqslant i \leqslant v,$$
$$\zeta_i = \sum_{k=1}^{\infty} \zeta_i^k, \qquad 0 \leqslant i \leqslant v. \qquad (4.20)$$

is the desired solution for system (4.1)–(4.5), with $F_i(\tau) \equiv 0$. The proof is given in the following two lemmas.

LEMMA 4.7. *If* $\sum_{k=1}^{\infty} |h_i^k| < \infty$ *for all* $0 \leqslant i \leqslant v$, *then* (4.20) *is a solution for system* (4.1)–(4.5) (*with* $F_i(\tau) \equiv 0$).

Proof. $\sum_{k=1}^{\infty} |h_i^k| < \infty$ implies that $\sum_{k=1}^{\infty} |z_i^k(\cdot)|_c < \infty$ and $\sum_{k=1}^{\infty} |\zeta_i^k| < \infty$ from (4.18), (4.19), and the estimates for $T^i(\tau, \alpha_i)$, $T^i(\tau, \beta_i)$ on each indicated subspace. Therefore $\{z_i(\tau)\}$ and $\{\zeta_i\}$ are well defined by (4.20) with

$$\sup_{1 \leqslant i \leqslant v} |z_i|_c + \sup_{0 \leqslant i \leqslant v} |\zeta_i| \leqslant C \sup_{0 \leqslant i \leqslant v} \sum_{k=1}^{\infty} |h_i^k|. \qquad (4.21)$$

$z_i(\tau)$, $1 \leqslant i \leqslant v$, is a solution of (4.1) since each $z_i^k(\tau)$ is such a solution. Add equation (4.17)′ through $k=1$ to $k=\infty$, and we have

$$h_i^1 = \sum_{k=1}^{\infty} z_i^k(\beta_i) - \sum_{k=1}^{\infty} z_{i+1}^k(\alpha_{i+1}) + \sum_{k=1}^{\infty} \zeta_i^k e_i, \qquad 1 \leqslant i \leqslant v-1.$$

From (4.17), we obtain (4.2). For $i=0$, we have

$$h_0^1 = \bar{v} = \{P(\mathscr{R}P_u^0(\beta_0)) + P(L_c^0(\beta_0))\} \sum_{k=1}^{\infty} h_0^k - z_1(\alpha_1) + \zeta_0 e_0.$$

From (4.16) and (4.14), we have (4.3). Similarly, (4.4) can be verified. From the definitions of d^i and $z_i^k(\tau)$, it is also clear that $z(\tau_i) \in H(i) = \mathscr{R}P_u^i(\tau_i^+) \oplus \mathscr{R}P_s^i(\tau_i^-) \oplus L_c^i(\tau_i) \oplus R_c^i(\tau_i)$ and $d^i \perp H(i)$, thus (4.5) is valid.

It remains to prove the following lemma.

LEMMA 4.8. *There is a constant* $\varepsilon_0 > 0$ *such that* $0 < \varepsilon \leqslant \varepsilon_0$, *then*

$$\sup_{0 \leqslant i \leqslant v} \sum_{k=1}^{\infty} |h_i^k| < C\Big(\sup_{1 \leqslant i \leqslant v-1} |h_i| + |\tilde{b}_1| + |\tilde{b}_2|\Big),$$

where the constant C does not depend on ε.

Proof. It is straightforward to verify that

$$z_i^k(\beta_i) - z_{i+1}^k(\alpha_{i+1}) + \zeta_i^k e_i = h_i^k + \cdots,$$

where $\cdots$ consists of functions of h_{i-1}^k and h_{i+1}^k only. Therefore by (4.17′)

$$\begin{aligned} h_i^{k+1} = {} & T^i(\beta_i, \alpha_i)\{P(\mathscr{R}P_s^i(\alpha_i)) + P(R_c^i(\alpha_i))\}\, h_{i-1}^k \\ & + T^i(\alpha_{i+1}, \beta_{i+1})\{P(\mathscr{R}P_u^{i+1}(\beta_{i+1})) + P(L_c^{i+1}(\beta_{i+1}))\}\, h_{i+1}^k, \end{aligned} \tag{4.22}$$

for $0 \leqslant i \leqslant v$, provided that we define $h_{-1}^k = h_{v+1}^k \equiv 0$. Define an equivalent norm for $\{h_i^k\}_{i=0}^v$ as

$$\|\{h_i^k\}_{i=0}^v\| = \sup_{0 \leqslant i \leqslant v} \{|h_i^k(L_c)| + |h_i^k(R_c)| + |h_i^k(P_u)| + |h_i^k(P_s)| + |h_i^k(e_i)|\},$$

where

$$h_i^k(L_c) = P(L_c^i(\beta_i))\, h_i^k, \;\; h_i^k(R_c) = P(R_c^{i+1}(\alpha_{i+1}))\, h_i^k, \;\; h_i^k(P_u) = P(\mathscr{R}P_u^i(\beta_i))\, h_i^k,$$

$$h_i^k(P_s) = P(\mathscr{R}P_s^{i+1}(\alpha_{i+1}))\, h_i^k \qquad \text{and} \qquad h_i^k(e_i) = P(\mathrm{span}[e_i])\, h_i^k.$$

It suffices to show $\sum_{k=1}^{\infty} \|\{h_i^k\}_{i=0}^v\| < \infty$. Observe that

$$\begin{aligned} |h_i^k(e_i)| \leqslant C|h_i^k| \leqslant C \sup_{0 \leqslant i \leqslant v} \{ & |h_i^{k-1}(L_c)| + |h_i^{k-1}(R_c)| \\ & + |h_i^{k-1}(P_u)| + |h_i^{k-1}(P_s)|\} \end{aligned}$$

from (4.22). Thus it suffices to obtain the estimates for

$$\sup_{0 \leqslant i \leqslant v} \sum_{k=1}^{\infty} \{|h_i^k(L_c)| + |h_i^k(R_c)| + |h_i^k(P_u)| + |h_i^k(P_s)|\}.$$

We shall use matrices to write (4.22). Let $\mathbf{H}(k)$ be a $(v+1) \times (m+n-1)$-dimensional column vector,

$$\mathbf{H}^t(k) = (\ldots, h_i^k(L_c)^t, h_i^k(P_u)^t, h_i^k(R_c)^t, h_i^k(P_s)^t, \ldots),$$

where $0 \leqslant i \leqslant v$, and "t" denotes the transpose. Equation (4.22) is equivalent to the equation

$$\mathbf{H}(k+1) = \mathscr{M}\mathbf{H}(k),$$

where $\mathcal{M}$ is a $[(v+1)\times(m+n-1)]^2$ matrix, which is of block tri-diagonal form, $\mathcal{M}_{ij}\equiv 0$ for all $0\leqslant j\leqslant v$:

$$\mathcal{M}=\begin{pmatrix} \mathcal{M}_{00} & \mathcal{M}_{01} & 0 & & & 0 \\ \mathcal{M}_{10} & \mathcal{M}_{11} & \mathcal{M}_{12} & 0 & & \\ 0 & \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \mathcal{M}_{v-1,v} \\ 0 & & & & \mathcal{M}_{v,v-1} & \mathcal{M}_{vv} \end{pmatrix}$$

$$\mathcal{M}_{i,i-1}=\begin{pmatrix} 0 & 0 & P(L_c^i(\beta_i))\,T^i(\beta_i,\alpha_i)\,P(R_c^i(\alpha_i)) & P(L_c^i(\beta_i))\,T^i(\beta_i,\alpha_i)\,P(\mathcal{R}P_s^i(\alpha_i)) \\ 0 & 0 & P(\mathcal{R}P_u^i(\beta_i))\,T^i(\beta_i,\alpha_i)\,P(R_c^i(\alpha_i)) & P(\mathcal{R}P_u^i(\beta_i))\,T^i(\beta_i,\alpha_i)\,P(\mathcal{R}P_s^i(\alpha_i)) \\ 0 & 0 & P(R_c^{i+1}(\alpha_{i+1}))\,T^i(\beta_i,\alpha_i)\,P(R_c^i(\alpha_i)) & P(R_c^{i+1}(\alpha_{i+1}))\,T^i(\beta_i,\alpha_i)\,P(\mathcal{R}P_s^i(\alpha_i)) \\ 0 & 0 & P(\mathcal{R}P_s^{i+1}(\alpha_{i+1}))\,T^i(\beta_i,\alpha_i)\,P(R_c^i(\alpha_i)) & P(\mathcal{R}P_s^{i+1}(\alpha_{i+1}))\,T^i(\beta_i,\alpha_i)\,P(\mathcal{R}P_s^i(\alpha_i)) \end{pmatrix}.$$

Except for the (3, 3)th entry in $\mathcal{M}_{i,i-1}$, which is bounded by KM^2, all the other entries in the 3rd column are bounded by $C\varepsilon^{\beta}$ (Corollary 4.6), and all the entries in the 4th column are bounded by $KM^2e^{-\alpha(\beta_i-\alpha_i)/2}\leqslant C\varepsilon^{\beta}$, for we have $|\beta_i-\alpha_i|\geqslant\varepsilon^{\beta-1}$. We remark that

$$|T^i(\beta_i,\alpha_i)\,P(\mathcal{R}P_s^i(\alpha_i))|\leqslant KMe^{-\alpha(\beta_i-\alpha_i)/2}$$

even for $i=2l+1$, $1\leqslant l\leqslant v-1$, while exponential trichotomy does not exist in the whole interval $[\alpha_i,\beta_i]$. (See Lemma 3.8.) Similarly,

$$\mathcal{M}_{i,i+1}=\begin{pmatrix} P(L_c^i(\beta_i))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(L_c^{i+1}(\beta_{i+1})) & P(L_c^i(\beta_i))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(\mathcal{R}P_u^{i+1}(\beta_{i+1})) & 0 & 0 \\ P(\mathcal{R}P_u^i(\beta_i))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(L_c^{i+1}(\beta_{i+1})) & P(\mathcal{R}P_u^i(\beta_i))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(\mathcal{R}P_u^{i+1}(\beta_{i+1})) & 0 & 0 \\ P(R_c^{i+1}(\alpha_{i+1}))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(L_c^{i+1}(\beta_{i+1})) & P(R_c^{i+1}(\alpha_{i+1}))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(\mathcal{R}P_u^{i+1}(\beta_{i+1})) & 0 & 0 \\ P(\mathcal{R}P_s^{i+1}(\alpha_{i+1}))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(L_c^{i+1}(\beta_{i+1})) & P(\mathcal{R}P_s^{i+1}(\alpha_{i+1}))\,T^i(\alpha_{i+1},\beta_{i+1})\,P(\mathcal{R}P_u^{i+1}(\beta_{i+1})) & 0 & 0 \end{pmatrix}.$$

Except for the (1, 1)th entry which is bounded by KM^2, all the other entries in the 1st and the 2nd column are bounded by $C\varepsilon^{\beta}$.

We can write $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, where $\mathcal{M}_1$ is a block upper triangular matrix which consists of only the (1, 1)th entry of each $\mathcal{M}_{i,i+1}$. $\mathcal{M}_2$ is a block lower triangular matrix which consists of only the (3, 3)th entry of each $\mathcal{M}_{i,i-1}$:

$$|\mathcal{M}_i| \leqslant CM^2, \qquad i = 1, 2$$

$$|\mathcal{M}_3| \leqslant C\varepsilon^{\beta}.$$

It is easily verified that $\mathcal{M}_1\mathcal{M}_2 = \mathcal{M}_2\mathcal{M}_1 = 0$. Therefore

$$(\mathcal{M}_1 + \mathcal{M}_2)^i = \mathcal{M}_1^i + \mathcal{M}_2^i, \qquad \forall i \geqslant 0.$$

$$(\mathcal{M}_1 + \mathcal{M}_2)^{\nu+1} = 0.$$

We can show that $\sum_{k=1}^{\infty} |(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)^k| < \infty$, which implies the desired result of this lemma, by virtue of Lemma 3.10, provided that ε_0 is sufficiently small and $0 < \varepsilon \leqslant \varepsilon_0$. This completes the proof of Lemma 4.8.

We now prove that the solutions of system (4.1)–(4.5) are unique. Consider the modified system

$$\begin{aligned} &z_i(\tau)' - A_i(\tau, \varepsilon)\, z_i(\tau) = 0, && \tau \in [\alpha_i, \beta_i], \quad 1 \leqslant i \leqslant \nu, \\ &z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + \zeta_i e_i = h_i, && 1 \leqslant i \leqslant \nu - 1, \\ &B_1(-z_1(\alpha_1) + \zeta_0 e_0) = -\tilde{b}_1, && \\ &B_2(z_\nu(b_\nu) + \zeta_\nu e_\nu) = \tilde{b}_2, && \\ &d^i \cdot z_i(\tau_i) = D_i. && 1 \leqslant i \leqslant \nu. \end{aligned}$$

It is easy to modify our proof of the existence theorem for system (4.1)–(4.5) to show that the system presented above admits at least one solution for any given $\{h_i\}$, $\tilde{b}_1$, $\tilde{b}_2$, and $\{D_i\}$. Using the relation $z_i(\beta_i) = T^i(\beta_i, \alpha_i)\, z_i(\alpha_i)$, this system is in fact a linear algebraic equation which has an $[(m+n)(\nu-1) + d_1 + d_2 + \nu = (m+n)\nu + \nu + 1]$-dimensional inhomogeneous term and an unknown vector of the same dimension. It is basic fact from the linear algebra that the existence of a solution for any inhomogeneous term implies the uniqueness for such a system.

The proof of Lemma 4.3 has been completed. Q.E.D.

Proof of Theorem 4.1 (*continued*). By Lemma 4.2, we construct $\{\tilde{z}_i(\tau)\}$ which satisfies (4.1) and (4.5) with estimates (4.6) and (4.7). Thus

$$\begin{aligned} \sup_{1 \leqslant i \leqslant \nu} \{|\tilde{z}_i|_c\} \leqslant C \Bigg\{ &\frac{1}{\varepsilon} \sup\{|f_i|_c : i = 2l,\ 1 \leqslant l \leqslant I\} \\ &+ \varepsilon^{\beta-1} \sup\{|f_i|_c : i = 2l+1,\ 0 \leqslant l \leqslant I\} \\ &+ \sup\{|g_i|_c : 1 \leqslant i \leqslant \nu\} \Bigg\}. \end{aligned}$$

Next we use Lemma 4.3 to solve the boundary value problem

$$z_i(\tau)' - A_i(\tau, \varepsilon)\, z_i(\tau) = 0, \qquad 1 \leqslant i \leqslant \nu,$$

$$z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + \zeta_i e_i = h_i - (\tilde{z}_i(\beta_i) - \tilde{z}_{i+1}(\alpha_{i+1})), \qquad 1 \leqslant i \leqslant \nu - 1,$$

$$\bar{B}_1(-z_1(\alpha_1) + \zeta_0 e_0) = -\tilde{b}_1 + \bar{B}_1 \tilde{z}_1(\alpha_1),$$

$$\bar{B}_2(z_\nu(\beta_\nu) + \zeta_\nu e_\nu) = \tilde{b}_2 - \bar{B}_2 \tilde{z}_\nu(\beta_\nu),$$

$$d^i \cdot z_i(\tau_i) = 0.$$

Based on the estimate for $\sup\{|\tilde{z}_i|_c\}$, we have the desired estimate for the solution $z_i(\tau)$. Finally, $\{\tilde{z}_i(\tau) + z_i(\tau)\}_{i=1}^{\nu}$ and $\{\zeta_i\}_{i=0}^{\nu}$ are the desired solution of (4.1)–(4.5), and the estimate for this solution follows easily from those for $\{\tilde{z}_i(\tau)\}$ and $\{z_i(\tau)\}$. This completes the proof of Theorem 4.1.

THEOREM 4.9. *There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, the linear boundary value problem*

$$z_i(\tau)' - A_i(\tau, \varepsilon)\, z_i(\tau) = F_i(\tau), \qquad \tau \in [\alpha_i, \beta_i], \quad 1 \leqslant i \leqslant \nu, \tag{4.1}$$

$$z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + \zeta_1 e_i = h_i, \qquad 1 \leqslant i \leqslant \nu - 1, \tag{4.2}$$

$$\bar{B}_1(z_1(\alpha_1)) = \tilde{b}_1, \tag{4.3$'$}$$

$$\bar{B}_2(z_\nu(\beta_\nu)) = \tilde{b}_2, \tag{4.4$'$}$$

$$d^i \cdot z_i(\tau_i) = 0, \qquad 2 \leqslant i \leqslant \nu - 1, \tag{4.5$'$}$$

admits a unique solution $\{z_i(\tau)\}_{i=1}^{\nu}$, $\{\zeta_i\}_{i=1}^{\nu-1}$. Moreover

$$\begin{aligned}
\sup_{1 \leqslant i \leqslant \nu} |z_i|_c + \sup_{1 \leqslant i \leqslant \nu - 1} |\zeta_i| &\\
\leqslant C\Big(\sup_{1 \leqslant i \leqslant \nu - 1} |h_i| + |\tilde{b}_1| + |\tilde{b}_2| + \sup_{1 \leqslant i \leqslant \nu} |g_i|_c &\\
+ \frac{1}{\varepsilon} \sup\{|f_i|_c : i = 2l, 1 \leqslant l \leqslant I\} &\\
+ \varepsilon^{\beta - 1} \sup\{|f_i|_c : i = 2l + 1, 0 \leqslant l \leqslant I\}\Big). &
\end{aligned}$$

Proof. With the same inhomogeneous terms $\{F_i\}_{i=1}^{\nu}$, $\{h_i\}_{i=1}^{\nu-1}$, $\tilde{b}_1$, and $\tilde{b}_2$, apply Theorem 4.1 to get a solution $\{\tilde{z}_i(\tau)\}_{i=1}^{\nu}$, $\{\tilde{\zeta}_i\}_{i=0}^{\nu}$ for the system (4.1)–(4.5). Let

$$w_1(\tau) = -T^1(\tau, \alpha_1)\, \zeta_0 e_0,$$

$$w_\nu(\tau) = T^\nu(\tau, \beta_\nu)\, \zeta_\nu e_\nu,$$

and let

$$
\begin{aligned}
z_1^{(1)}(\tau) &= \bar{z}_1(\tau) + w_1(\tau), & z_\nu^{(1)}(\tau) &= \bar{z}_\nu(\tau) + w_\nu(\tau), \\
z_i^{(1)}(\tau) &= \bar{z}_i(\tau), & 2 &\leqslant i \leqslant \nu - 1, \\
\zeta_1^{(1)} &= \bar{\zeta}_1 + \zeta_0, & \zeta_{\nu-1} &= \bar{\zeta}_{\nu-1} + \zeta_\nu, \\
\zeta_i^{(1)} &= \bar{\zeta}_i, & 2 &\leqslant i \leqslant \nu - 1.
\end{aligned}
$$

It is obvious that $(\{z_i^{(1)}\}_{i=1}^{\nu}, \{\zeta_i^{(1)}\}_{i=1}^{\nu-1})$ thus obtained satisfies (4.1), (4.3)′, (4.4)′ and (4.5)′. However, (4.2) is not satisfied, with error terms $h_1^{(1)} = \zeta_0 \cdot (e_1 - T^1(\beta_1, \alpha_1)\, e_0)$, $h_{\nu-1}^{(1)} = \zeta_\nu(e_{\nu-1} - T^\nu(\alpha_\nu, \beta_\nu)\, e_\nu)$, and $h_i^{(1)} = 0$, $2 \leqslant i \leqslant \nu - 2$. From the definition of e_0 and e_ν,

$$
\begin{aligned}
|T^1(\tau, 0)\, e_0 - (f^2(a_1), -g_y^2(a_1)^{-1} g_x^2(a_1)\, f^2(a_1))| &\leqslant c e^{-\gamma\tau}, & \tau &\geqslant 0, \\
|T^\nu(\tau, 0)\, e_\nu - (f^{\nu-1}(b_l), -g_y^{\nu-1}(b_l)^{-1} g_x^{\nu-1}(b_l)\, f^{\nu-1}(b_l))| &\leqslant c e^{\gamma\tau}\ , & \tau &\leqslant 0,
\end{aligned}
$$

by virtue of the fact that $e_0 \in \mathscr{R}P_{\text{cs}}^1(0)$ and $e_\nu \in \mathscr{R}P_{\text{cu}}^\nu(0)$; see Lemma 3.5 and Lemma 3.7. From the definition of e_1 and $e_{\nu-1}$, we have

$$
\begin{aligned}
|(f^2(a_1), -g_y^2(a_1)^{-1} g_x^2(a_1)\, f^2(a_1)) - e_1| &= O(\varepsilon^\beta), \\
|(f^{\nu-1}(b_l), -g_y^{\nu-1}(b_l)^{-1} g_x^{\nu-1}(b_l)\, f^{\nu-1}(b_l)) - e_{\nu-1}| &= O(\varepsilon^\beta).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|h_1^{(1)}| &= O(\varepsilon^\beta)\, |\zeta_0|, \\
|h_{\nu-1}^{(1)}| &= O(\varepsilon^\beta)\, |\zeta_\nu|.
\end{aligned}
$$

Suppose that $\varepsilon_0 > 0$ is small and $0 < \varepsilon \leqslant \varepsilon_0$, and we can apply Theorem 4.1 again with the inhomogeneous terms $\{F_i\} \equiv 0$, $\tilde{b}_1 = 0$, $\tilde{b}_2 = 0$, and $\{h_i\} = \{-h_i^{(1)}\}$. Again apply the above procedure to adjust the solution and obtain an approximation of the solution of system (4.1), (4.2), (4.3)′, (4.4)′, and (4.5)′, but with

$$
\begin{aligned}
|h_1^{(2)}| &= O(\varepsilon^\beta)\, |h_1^{(1)}| \\
|h_{\nu-1}^{(2)}|| &= O(\varepsilon^\beta)\, |h_{\nu-1}^{(1)}| \\
h_i^{(2)} &= 0, \qquad 2 \leqslant i \leqslant \nu - 2,
\end{aligned}
$$

Apply the indicated procedure repeatedly and we have

$$
|h_i^{(j)}| \to 0, \qquad \text{as } j \to \infty, \quad 1 \leqslant i \leqslant \nu - 1
$$

$$
\sum_j |h_i^{(j)}| < \infty.
$$

Finally, $z_i = \bar{z}_i + \sum_{j=1}^{\infty} z_i^{(j)}$, $1 \leqslant i \leqslant \nu$, and $\zeta_i = \bar{\zeta}_i + \sum_{j=1}^{\infty} \zeta_i^{(j)}$, $1 \leqslant i \leqslant \nu - 1$, is the desired solution of Theorem 4.9. The estimate for the solution also follows easily. The uniqueness of the solution of Theorem 4.9 can be proved exactly like the uniqueness of Theorem 4.1. Q.E.D.

5. Formal Power Series Solutions and Matching Principles

Let $f(t, \varepsilon)$ be continuous and defined on $t \in J$ and $|\varepsilon| \leqslant \varepsilon_0$, J is an interval in $\mathbf{R}$, bounded or unbounded, open or closed, and $\varepsilon_0 > 0$ is a constant. We say $f(t, \varepsilon) = O(\varepsilon^m)$ if for any compact subinterval $J_1 \subset J$, there exists a constant $C(J_1)$ such that $|f(t, \varepsilon)| \leqslant C(J_1)|\varepsilon^m|$, $t \in J_1$. We say that $f(t, \varepsilon) = o(\varepsilon^m)$ if $f(t, \varepsilon)/\varepsilon^m \to 0$ uniformly in any compact J_1, as $\varepsilon \to 0$. These notations are slightly different from the standard ones which require the uniformity in the whole interval J. We write the asymptotic expansion of $f(t, \varepsilon)$ as

$$f(t, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \phi_j(t) \tag{5.1}$$

if $f(t, \varepsilon) = \sum_{j=0}^{m} \varepsilon^j \phi_j(t) + O(\varepsilon^{m+1})$ for all $m \geqslant 0$. It is immediately obvious that if each $(\partial^i/\partial\varepsilon^i) f(t, \varepsilon)$ exists and is continuous in (t, ε), then the asymptotic expansion of $f(t, \varepsilon)$ exists and is completely determined by Taylor's formula. Conversely, given any formal power series $\sum_{j=0}^{\infty} \varepsilon^j \phi_j(t)$, there exists a (nonunique) asymptotic sum $f(t, \varepsilon)$ such that (5.1) holds (Borel–Ritt). By exploiting (5.1), we can define the sum, the product, and the composition with usual functions of any numbers of formal power series. We can also define the differentiation, integration, or the change of variables of the formal series using (5.1).

We look for a power series solutions $(\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t), \sum_{j=0}^{\infty} \varepsilon^j Y_j^i(t))$ in the regular region or $(\sum_{j=0}^{\infty} \varepsilon^j x_j^i(\tau), \sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau))$ in the interior or boundary layer region, which formally satisfies Eq. (1.1) in the indicated region, and satisfies boundary conditions at the initial and the terminal points. Moreover the jumps between the outer and the inner layers are $o(\varepsilon^p)$ for all $p > 0$ (matching of the inner and the outer expansions).

We shall denote functions with the argument $(X_0^i(t), Y_0^i(t), 0)$, e.g., $f(X_0^i(t), Y_0^i(t), 0)$, by $\bar{f}^i(t)$; and denote functions with the argument $(x_0^i(\tau), y_0^i(\tau), 0)$, e.g., $f(x_0^i(\tau), y_0^i(\tau), 0)$, by $\hat{f}^i(\tau)$. Note that these notations are different from those in Section 4.

5.1. *Formal Power Series Solutions in the Regular Regions*

Let $\{X_j\}_{j=0}^{\infty}$ and $\{Y_j\}_{j=0}^{\infty}$ be any sequences of real vectors. Consider the formal asymptotic expansion

$$f\left(\sum_{j=0}^{\infty} \varepsilon^j X_j, \sum_{j=0}^{\infty} \varepsilon^j Y_j, \varepsilon\right)$$
$$= f(X_0, Y_0, 0) + \sum_{j=1}^{\infty} \varepsilon^j \{f_x(X_0, Y_0, 0)\, X_j + f_y(X_0, Y_0, 0)\, Y_j$$
$$+ F_j(X_1, Y_1, ..., X_{j-1}, Y_{j-1}, ..., D^\alpha f(X_0, Y_0, 0), ...)\}, \tag{5.2}$$

where $F_j(\cdots)$ is a sum of multilinear forms on $X_1, Y_1, ..., X_{j-1}, Y_{j-1}$ and each term has the form

$$D_x^{i_1} D_y^{i_2} D_\varepsilon^{i_3} f(X_0, Y_0, 0)\, X_1^{k_1} \cdots X_{j-1}^{k_{j-1}} Y_1^{h_1} \cdots Y_{j-1}^{h_{j-1}}, \tag{5.3}$$

where $i_3 \geq 0$ is an integer, and $k_1, ..., k_{j-1}, h_1, ..., h_{j-1}, i_1$, and i_2 are multi-indices, satisfying $k_1 + \cdots + k_{j-1} = i_1$, $h_1 + \cdots + h_{j-1} = i_2$, and $|k_1| + 2|k_2| + \cdots + (j-1)|k_{j-1}| + |h_1| + 2|h_2| + \cdots + (j-1)|h_{j-1}| + i_3 = j$. Consider the recursive equations

$$\begin{aligned} \dot{X}_0^i(t) &= f(X_0^i(t), Y_0^i(t), 0) \\ 0 &= g(X_0^i(t), Y_0^i(t), 0) \end{aligned} \tag{5.4$_0$}$$

$$\begin{aligned} \dot{X}_j^i(t) &= f_x(X_0^i(t), Y_0^i(t), 0)\, X_j^i(t), + f_y(X_0^i(t), Y_0^i(t), 0)\, Y_j^i(t) \\ &\quad + F_j(X_1^i, Y_1^i, ..., X_{j-1}^i, Y_{j-1}^i, ..., D^\alpha f(X_0^i, Y_0^i, 0), ...), \\ \dot{Y}_{j-1}^i(t) &= g_x(X_0^i(t), Y_0^i(t), 0)\, X_j^i(t), + g_y(X_0^i(t), Y_0^i(t), 0)\, Y_j^i(t) \\ &\quad + G_j(X_1^i, Y_1^i, ..., X_{j-1}^i, Y_{j-1}^i, ..., D^\alpha g(X_0^i, Y_0^i, 0), ...), \end{aligned} \tag{5.4$_j$}$$

where G_j comes from the Taylor expansion of $g(\sum_{j=0}^{\infty} \varepsilon^j X_j^i, \sum_{j=0}^{\infty} \varepsilon^j Y_j^i, \varepsilon)$ and each term has the same structure as (5.2) and (5.3).

Assume that the time that the trajectory stays near the ith slow manifold is $t \in [a_i + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(a),\ b_i + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(b)]$, $1 \leq i \leq I$. We also need to determine $\{\tau_j^i(a)\}_{j=1}^{\infty}$ and $\{\tau_j^i(b)\}_{j=1}^{\infty}$, $1 \leq i \leq I$, recursively. Taking into account the perturbation of a_i and b_i we have to compute each $X_j^i(t)$, $0 \leq j$, in a neighborhood of $[a_i, b_i]$, say $[a_i - \delta, b_i + \delta]$, although initially $X_0^i(t)$ and $Y_0^i(t)$ are defined in $[a_i, b_i]$ only.

The 0th order term $(X_0^i(t), Y_0^i(t))$ is known to satisfy $(5.4)_0$. If $X_0^i(t), ..., X_{j-1}^i(t), Y_0^i(t), ..., Y_{j-1}^i$ have been computed, our assumption (H_1) implies that

$$Y_j^i(t) = -\bar{g}_y^i(t)^{-1} \bar{g}_x^i(t)\, X_j^i(t) + (\text{a function of } t) \tag{5.5$''$}$$

from the 2nd equation of $(5.4)_j$. From the first equation of $(5.4)_j$, we have

$$\dot{X}_j^i(t) = (\bar{f}_x^i(t) - \bar{f}_y^i(t)\, \bar{g}_y^i(t)^{-1} \bar{g}_x^i(t))\, X_j^i(t) + (\text{a function of } t).$$

Using the solution map $S^i(t, s)$ defined previously in Section 2, we have

$$X_j^i(t) = S^i(t, a_i)\, X_j^i(a_i) + (\text{a function of } t). \qquad (5.5)'$$

The problem would be completely solved if we can find $X_j^i(a_i)$ or $X_j^i(b_i)$, which are related by

$$X_j^i(b_i) = S^i(b_i, a_i)\, X_j^i(a_i) + \bar{C}_j^i, \qquad (5.5)$$

where the constant is computable and does not depend on X_j^i and Y_j^i. In order to eliminate the trivial perturbation of a shift in the time t, which has already been taken care of by the series $\sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(a)$ and $\sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(b)$, we assume that

$$\theta_i \cdot X_j^i(t_i) = 0, \qquad j \geqslant 1 \qquad (5.6)$$

where θ_i is such that $\theta_i \cdot \dot{X}_0^i(t_i) \neq 0$, $t_i \in [a_i, b_i]$. Equation (5.6) implies

$$\begin{aligned} \theta(a, i) \cdot X_j^i(a_i) &= \bar{C}_j(a, i), \\ \theta(b, i) \cdot X_j^i(b_i) &= \bar{C}_j(b, i), \end{aligned} \qquad (5.7)$$

where $\theta(a, i) = S^*(a_i, t_i)\, \theta_i$ and $\theta(b, i) = S^*(b_i, t_i)\, \theta_i$, and $*$ denotes the adjoint of an operator. $\bar{C}_j(a, i)$ and $\bar{C}_j(b, i)$ are computable without knowing X_j^i and Y_j^i.

Equations (5.5) and (5.7) shall be employed in the matching procedure to determine $X_j^i(a_i)(X_j^i(b_i))$, $\tau_{j-1}^i(a)$, and $\tau_{j-1}^i(b)$ recursively.

5.2. *Formal Series Solutions for the Interior Layers*

The equation for the interior layer is

$$\begin{aligned} x'(\tau) &= \varepsilon f(x(\tau), y(\tau), \varepsilon), \\ y'(\tau) &= g(x(\tau), y(\tau), \varepsilon), \end{aligned} \qquad (5.8)$$

where $\tau = 0$ corresponds to $t = b_i + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(b)$ in the outer layer $Z_i(t)$ and $t = a_{i+1} + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^{i+1}(a)$ in the outer layer $Z_{i+1}(t)$. We look for the formal series expansion $(\sum_{j=0}^{\infty} \varepsilon^j x_j^i(\tau), \sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau))$, $1 \leqslant i \leqslant I-1$, which formally satisfies (5.8) and matches with $Z_i(t)$ (and $Z_{i+1}(t)$) as $\tau \to -\infty$ ($\tau \to \infty$). We have the recursive equations

$$\begin{aligned} x_0^i(\tau)' &= 0, \\ y_0^i(\tau)' &= g(x_0^i(\tau), y_0^i(\tau), 0). \end{aligned} \qquad (5.9)_0$$

We have assumed that $x_0^i(\tau)$ ($\equiv$ constant x_0^i) and $y_0^i(\tau)$ are given, $y_0^i(\tau) \to G^i(x_0^i)(G^{i+1}(x_0^i))$ as $\tau \to -\infty$ ($\tau \to \infty$):

$$x_1^i(\tau)' = f(x_0^i, y_0^i(\tau), 0), \tag{5.9$_1$}$$

$$y_1^i(\tau)' = g_x(x_0^i, y_0^i(\tau), 0)\, x_1^i(\tau) + g_y(x_0^i, y_0^i(\tau), 0)\, y_1^i(\tau) + g_\varepsilon(x_0^i, y_0^i(\tau), 0),$$

$$\begin{aligned} x_j^i(\tau)' &= \hat{f}_x^i(\tau)\, x_{j-1}^i(\tau) + \hat{f}_y^i(\tau)\, y_{j-1}^i(\tau) \\ &\quad + F_{j-1}(x_1^i, y_1^i, ..., x_{j-2}^i, y_{j-2}^i(\tau), ..., D^\alpha f(x_0^i, y_0^i(\tau), 0), ...), \\ y_j^i(\tau)' &= \hat{g}_x^i(\tau)\, x_j^i(\tau) + \hat{g}_y^i(\tau)\, y_j^i(\tau) \\ &\quad + G_j(x_1^i, y_1^i, ..., x_{j-1}^i, y_{j-1}^i(\tau), ..., D^\alpha g(x_0^i, y_0^i(\tau), 0), ...), \qquad j \geqslant 2. \end{aligned} \tag{5.9$_j$}$$

Let the growth condition at $\tau = \pm\infty$ be

$$y_j^i(\tau) \in E_{\mathbf{R}}(0, j). \tag{5.10}$$

We also require that

$$y_j^i(0) \perp y_0^i(0)'. \tag{5.11}$$

Assume $x_0^i, y_0^i, ..., x_{j-1}^i, y_{j-1}^i$ have been obtained, then from the first equation of $(5.9)_j$,

$$x_j^i(\tau) = x_j^i(0) + (\text{a function of } \tau \text{ which is in } E(0, j)), \tag{5.12}$$

$$y_j^i(\tau) = \hat{g}_y^i(\tau)\, y_j^i(\tau) + \hat{g}_x^i(\tau)\, x_j^i(0) + (\text{a function of } \tau \text{ in } E(0, j)). \tag{5.13}$$

In order that (5.13) has a solution $y_j^i(\tau) \in E(0, j)$, we need to choose $x_j^i(0)$ such that

$$\int_{-\infty}^{\infty} \psi_i(\tau)^* \{\hat{g}_x^i(\tau)\, x_j^i(0) + (\text{a function of } \tau)\}\, d\tau = 0,$$

or

$$\Delta_i \cdot x_j^i(0) = d_j^i. \tag{5.14}$$

If (5.14) is valid, there exists a unique solution $y_j^i(\tau)$ of (5.13) satisfying both (5.10) and (5.11) for all $1 \leqslant i \leqslant I-1$, $1 \leqslant j$; see Lemma 3.6.

5.3. *Formal Series Solutions for the Boundary Layers.*

The formal series solution $(\sum_{j=0}^{\infty} x_j^i(\tau), \sum_{j=0}^{\infty} y_j^i(\tau))$, $i = 0$ or I, satisfies the same equations $(5.9)_0$, $(5.9)_1$, and $(5.9)_j$ as the interior layers do. The growth conditions are

$$y_0^i(\tau) \in E_{\mathbf{R}^+}(0, j), \tag{5.15}$$

$$y_j^I(\tau) \in E_{\mathbf{R}^-}(0, j). \tag{5.16}$$

Let $i=0$ first. Assume that $x_1^0, y_1^0, ..., x_{j-1}^i, y_{j-1}^i$ have been computed, and we have from $(5.9)_j$ that

$$x_j^0(\tau)=x_j^0(0)+(\text{a function of } \tau \text{ which is in } E_{\mathbf{R}^+}(0,j)), \tag{5.17}$$

$$y_j^0(\tau)'=\hat{g}_y^0(\tau)\, y_j^0(\tau)+\hat{g}_x^0(\tau)\, x_j^0(0)+(\text{a function of } \tau \text{ in } E_{\mathbf{R}^+}(0,j)). \tag{5.18}$$

In order to satisfy (5.15), we have (Lemma 3.5)

$$\begin{aligned} y_j^0(\tau)=&\,\hat{U}^0(\tau,0)\,\hat{Q}_s^0(0)\,y_j^0(0)+\int_0^\tau \hat{U}^0(\tau,\sigma)\,\hat{Q}_s^0(\sigma)\,\hat{g}_x^0(\sigma)\,d\sigma\cdot x_j^0(0)\\ &+\int_\infty^\tau \hat{U}^0(\tau,\sigma)\,\hat{Q}_u^0(\sigma)\,\hat{g}_x^0(\sigma)\,d\sigma\cdot x_j^0(0)\\ &+(\text{functions of } \tau \text{ in } E\mathbf{R}_+(0,j)), \end{aligned} \tag{5.19}$$

$$\begin{aligned} y_j^0(0)=&\,\hat{Q}_s^0(0)\,y_j^0(0)+\int_\infty^0 \hat{U}^0(0,\sigma)\,\hat{Q}_u^0(\sigma)\,\hat{g}_x^0(\sigma)\,d\sigma\cdot x_j^0(0)\\ &+(\text{a vector in } \mathscr{R}\hat{Q}_u^0(0)). \end{aligned}$$

From the Taylor expansion of $B_1(\sum \varepsilon^j x_j^0(0), \sum \varepsilon^j y_j^0(0), \varepsilon)=0$, we have the recursive equations

$$B_1(x_0^0(0), y_0^0(0), 0)=0,$$

$$\begin{aligned} &B_{1x}(x_0^0, y_0^0(0), 0)\, x_j^0(0)+B_{1y}(x_0^0, y_0^0(0), 0)\, y_j^0(0)\\ &\quad+B_{1j}(x_1^0, y_1^0(0), ..., x_{j-1}^0, y_{j-1}^0(0), ..., D^\alpha B_1(x_0^0, y_0^0(0), 0), ...)=0, \end{aligned} \tag{5.20}$$

where B_{1j} has the same structure as F_j. By our assumption $x_1^0, ..., y_{j-1}^0(0)$ are known and B_{1j} is a given vector in $\mathbf{R}^{d_1}$. Substitute (5.19) into (5.20), and we have

$$\begin{aligned} &\left\{B_{1x}+B_{1y}\cdot\int_\infty^0 \hat{U}^0(0,\sigma)\,\hat{Q}_u^0(\sigma)\,\hat{g}_x^0(\sigma)\,d\sigma\right\} x_j^0(0)\\ &\quad+B_{1y}\cdot\hat{Q}_s^0(0)\,y_j^0(0)+C_j^0=0, \end{aligned} \tag{5.21}$$

where $C_j^0\in\mathbf{R}^{d_1}$ can be explicitly computed.

Let M_1, M_2 be two linearly independent subspaces. Define the projection $P(M_1, \mathrm{M}_2)$ in the space $M_1\oplus M_2$ with $\mathscr{K}P(M_1, M_2)=M_2$ and $\mathscr{R}P(M_1,M_2)=M_1$. As was pointed out immediately after (H_5) in Section 2,

$$\{(x,y): x\in L_c(0),\ y=G^0(x)\}\subset\mathscr{K}B_1.$$

Therefore (5.21) reduces to

$$\begin{aligned}&\mathscr{B}_1(P(R_c(0)\oplus \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0), L_c(0))\, x_j^0(0),\\ &\quad \hat{Q}_s^0(0)\, y_j^0(0) - G^0 P(L_c(0), R_c(0)\\ &\qquad \oplus \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0))\, x_j^0(0)) + C_j^0 = 0. \qquad (5.22)\end{aligned}$$

Since $R_c(0)\oplus \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), Y_0^1(a_1), 0)\otimes \mathscr{R}\hat{Q}_s^0(0)$ is complementary to $\mathscr{K}\mathscr{B}_1$ and $\mathscr{B}_1$ is surjective, we can solve (5.22) to obtain

$$P(R_c(0)\oplus \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0), L_c(0))\, x_j^0(0) = \hat{C}_j^0, \qquad (5.22)'$$

$$\begin{aligned}&\hat{Q}_s^0(0)\, y_j^0(0) - G^0 P(L_c(0), R_c(0)\\ &\qquad \oplus \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0))\, x_j^0(0) = \hat{d}_j^0. \qquad (5.22)''\end{aligned}$$

Similarly, the same argument applies to $(x_j^I(\tau), y_j^I(\tau))$ and we conclude that based on $B_2(\sum \varepsilon^j x_j^I(0), \sum \varepsilon^j y_j^I(0), \varepsilon) = 0$, (5.16), and ($H_5$), we are able to obtain

$$P(L_c(I)\oplus \operatorname{span} f(X_0^I(b_I), Y_0^I(b_I), 0), R_c(I))\, x_j^I(0)) = \hat{C}_j^{I+1}, \quad (5.22)'''$$

$$\begin{aligned}&\hat{Q}_u^I(0)\, y_j^I(0) - G^{I+1} P(R_c(I), L_c(I)\\ &\qquad \oplus \operatorname{span} f(X_0^I(b_I), Y_0^I(b_I), 0))\, x_j^I(0) = \hat{d}_j^I. \qquad (5.22)^{iv}\end{aligned}$$

5.4. *Matching of the Interior and Boundary Layers with the outer Layers*

The matching principle employed here is due to Van Dyke (see Eckhaus [5]). Since $\tau = 0$ in the interior or boundary layers is identified with $t = b_i + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^i(b)$ in $Z_i(t)$ and $t = a_{i+1} + \sum_{j=1}^{\infty} \varepsilon^j \tau_j^{i+1}(a)$ in $Z_{i+1}(t)$ of the outer layers, after a change of variable we have the so-called inner expansion of the outer layers,

$$\sum_{j=0}^{\infty} \varepsilon^j Z_j^i\left(b_i + \sum_{k=1}^{\infty} \varepsilon^k \tau_k^i(b) + \varepsilon\tau\right) = \sum_{j=0}^{\infty} \varepsilon^j z_j(\tau, b, i), \qquad (5.23)$$

$$\sum_{j=0}^{\infty} \varepsilon^j Z_j^i\left(a_i + \sum_{k=1}^{\infty} \varepsilon^k \tau_k^i(a) + \varepsilon\tau\right) = \sum_{j=0}^{\infty} \varepsilon^j z_j(\tau, a, i), \qquad (5.24)$$

where $Z = (X, Y)$, $z = (x, y) \in \mathbf{R}^m \times \mathbf{R}^n$. Observe that in computing $x_j(\tau, b, i)$, $X_0^i, ..., X_j^i(t)$, $\tau_1^i(b), ..., \tau_j^i(b)$ are needed. Assume that $X_0^i, ..., X_{j-1}^i(t)$, $\tau_1^i(b), ..., \tau_{j-1}^i(b)$ are known, and we have

$$\begin{aligned}x_j(0, b, i) &= X_j^i(b_i) + \dot{X}_0^i(b_i)\, \tau_j^i(b) + \cdots\\ &= X_j^i(b_i) + f(X_0^i(b_i), Y_0^i(b_i), 0)\, \tau_j^i(b) + \cdots, \qquad (5.25)\end{aligned}$$

where $\cdots$ stands for a known vector.

Since (5.23) formally satisfies the first two equations of (2.1), we have

$$\begin{aligned} x_j(\tau, b, i)' &= f_x(X_0^i(b_i), Y_0^i(b_i), 0)\, x_{j-1}(\tau, b, i) \\ &\quad + f_y(X_0^i(b_i), Y_0^i(b_i), 0)\, y_{j-1}(\tau, b, i) \\ &\quad + F_{j-1}(x_1, y_1, ..., x_{j-2}, y_{j-2}, ..., D^\alpha f(X_0^i(b_i), Y_0^i(b_i), 0), ...), \\ y_j(\tau, b, i)' &= g_x(X_0^i(b_i), Y_0^i(b_i), 0)\, x_j(\tau, b, i) \qquad (5.26)_j \\ &\quad + g_y(X_0^i(b_i), Y_0^i(b_i), 0)\, y_j(\tau, b, i) \\ &\quad + G_j(x_1, y_1, ..., x_{j-1}, y_{j-1}, ..., D^\alpha g(X_0^i(b_i), Y_0^i(b_i), 0), ...). \end{aligned}$$

It is clear that each $z_i(\tau, b, i) \in E_{\mathbf{R}^-}(0, j)$ and $z_i(\tau, a, i) \in E_{\mathbf{R}^+}(0, j)$. Recall that $x_j^i(\tau)$ and $y_j^i(\tau) \in E_{\mathbf{R}^-}(0, j)$ if $1 \leqslant i \leqslant I$, and $x_j^i(\tau)$ and $y_j^i(\tau) \in E_{\mathbf{R}^+}(0, j)$ if $0 \leqslant i \leqslant I-1$. The matching principles from (2.13) are

$$z_j^i(\tau) - z_i(\tau, b, i) \in E_{\mathbf{R}^-}(\gamma, j), \qquad \text{if} \quad 1 \leqslant i \leqslant I; \qquad (5.27)$$

$$z_j^i(\tau) - z_i(\tau, a, i+1) \in E_{\mathbf{R}^+}(\gamma, j), \qquad \text{if} \quad 0 \leqslant i \leqslant I-1. \qquad (5.28)$$

If $j=0$, then $z_0(\tau, b, i) = Z_0^i(b_i)$ and $z_0(\tau, a, i+1) = Z_0^{i+1}(a_{i+1})$. Obviously in this case (5.27) and (5.28) are valid, due to the hypotheses on $z_0^i(\tau) = (x_0^i, y_0^i(\tau))$. Assume that (5.27) and (5.28) are valid for $0 \leqslant j \leqslant j_0 - 1$. We show that by choosing the proper subsidiary conditions we may have (5.27) and (5.28) for $j = j_0$. Comparing the 1st equations of $(5.26)_j$ and $(5.9)_j$, we have

$$\begin{aligned} x_j^i(\tau) - x_j(\tau, b, i) &= x_j^i(0) - x_j(0, b, i) \\ &\quad + \int_0^\tau \{\text{a function of } \sigma \text{ which is in } E_{\mathbf{R}^-}(\gamma, j-1)\}\, d\sigma, \qquad \tau \geqslant 0,\ 1 \leqslant j \leqslant I. \end{aligned}$$

In order to have $x_j^i(\tau) - x_j(\tau, b, i) \in E_{\mathbf{R}^-}(\gamma, j)$, we must have

$$x_j^i(0) - x_j(0, b, i) + \int_0^{-\infty} \{\text{a function of } \sigma \text{ which is in } E_{\mathbf{R}^-}(\gamma, j-1)\}\, d\sigma = 0.$$

Rewrite it as

$$x_j^i(0) - X_j^i(b_i) - f(X_0^i(b_i), Y_0^i(b_i), 0)\, \tau_j^i(b) = C_j(b, i), \qquad 1 \leqslant i \leqslant I, \qquad (5.29)$$

where we have employed (5.25). Similarly, we obtain

$$\begin{aligned} x_j^i(0) - X_j^{i+1}(a_{i+1}) - f(X_0^{i+1}(a_{i+1}), Y_0^{i+1}(a_{i+1}), 0)\, \tau_j^{i+1}(a) \\ = C_j(a, i+1), \qquad 0 \leqslant i \leqslant I-1, \qquad (5.30) \end{aligned}$$

in order to have $x_j^i(\tau) - x_j(\tau, a, i+1) \in E_{\mathbf{R}^+}(\gamma, j)$. Both $C_j(b, i)$ and $C_j(a, i+1)$ are explicitly computable.

Next , assuming that (5.27) and (5.28) are valid for the x-component, we claim that (5.27) and (5.28) are valid for the y-component without any additional subsidiary conditions. The proof is based on a comparison of the second equations of $(5.9)_j$ and $(5.26)_j$ and an application of Lemma 3.5. Details may be found in Lin [19], thus shall not be rendered here.

5.5. *Determine the* jth *Order Expansions in the Interior, Boundary, and Outer Layers*

It remains to compute $X_j^i(a_i)$, $X_j^i(b_i)$, $\tau_j^i(a)$, and $\tau_j^i(b)$ for $1 \leqslant i \leqslant I$, and $x_j^i(0)$, $0 \leqslant i \leqslant I$. The equations to be satisfied are

$$X_j^i(b_i) = S^i(b_i, a_i)\, X_j^i(a_i) + \bar{C}_j^i, \qquad 1 \leqslant i \leqslant I, \tag{5.5}$$

$$\begin{aligned} \theta(a, i) \cdot X_j^i(a_i) &= \bar{C}_j(a, i), \qquad 1 \leqslant i \leqslant I, \\ \theta(b, i) \cdot X_j^i(b_i) &= \bar{C}_j(b, i), \qquad 1 \leqslant i \leqslant I, \end{aligned} \tag{5.7}$$

where $\theta(a, i) \cdot f(X_j^i(a_i), Y_j^i(a_i), 0) \neq 0$ and $\theta(b, i) \cdot f(X_j^i(b_i), Y_j^i(b_i), 0) \neq 0$;

$$\Delta_i \cdot x_j^i(0) = d_j^i, \qquad 1 \leqslant i \leqslant I-1. \tag{5.14}$$

$$P(R_c(0) \oplus \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0), 0), L_c(0))\, x_j^0(0) = \hat{C}_j^0, \tag{5.22$'$}$$

$$P(L_c(I) \oplus \operatorname{span} f(X_0^I(b_I), Y_0^I(b_I), 0), 0), R_c(I))\, x_j^I(0) = \hat{C}_j^{I+1}, \tag{5.22$'''$}$$

$$x_j^i(0) - X_j^i(b_i) - f(X_j^i(b_i), Y_j^i(b_i), 0)\, \tau_j^i(b) = C_j(b, i), \qquad 1 \leqslant i \leqslant I, \tag{5.29}$$

$$\begin{aligned} x_j^i(0) - X_j^{i+1}(a_{i+1}) - f(X_0^i(a_{i+1}), Y_0^i(a_{i+1}), 0)\, \tau_j^{i+1}(a) \\ = C_j(a, i+1), \qquad 0 \leqslant i \leqslant I-1. \end{aligned} \tag{5.30}$$

Let $P(a, i)$ ($P(b, i)$) be the projection in $\mathbf{R}^m$, with the range being TM_{i-1} (TM_i) for $1 \leqslant i \leqslant I$, and the kernel being $\operatorname{span} f(X_0^i(a_i), Y_0^i(a_i), 0)$ ($\operatorname{span} f(X_0^i(b_i), Y_0^i(b_i), 0)$). From (5.29) and (5.30) we have

$$\begin{aligned} P(a, i+1)(x_j^i(0) - X_j^{i+1}(a_{i+1})) \\ = P(a, i+1)\, C_j(a, i+1), \qquad 0 \leqslant i \leqslant I-1, \end{aligned} \tag{5.31}$$

$$P(b, i)(x_j^i(0) - X_j^i(b_i)) = P(b, i)\, C_j(b, i), \qquad 1 \leqslant i \leqslant I. \tag{5.32}$$

From (5.14),

$$\Delta_i (I - P(b, i))\, x_j^i(0) = d_j^i.$$

We can solve $(I - P(b, i))\, x_j^i(0)$ from this. Similarly, we can solve $(I - P(a, i+1))\, x_j^i(0)$. Let

$$(I - P(b, i))\, x_j^i(0) = d_j(b, i), \qquad 1 \leqslant i \leqslant I-1, \tag{5.33}$$

$$(I - P(a, i+1))\, x_j^i(0) = d_j(a, i+1), \qquad 1 \leqslant i \leqslant I-1. \tag{5.34}$$

Subtract (5.32) from (5.31), and we have

$$P(b, i)\, X_j^i(b_i) - P(a, i+1)\, X_j^{i+1}(a_{i+1}) = C_{ij}, \qquad 1 \leqslant i \leqslant I-1. \quad (5.35)$$

We compute

$$P(R_c(0), L_c(0) + \operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0))\, x_j^0(0) \quad (5.36)$$

and

$$P(\operatorname{span} f(X_0^1(a_1), Y_0^1(a_1), 0).\ R_c(0) + L_c(0))\, x_j^0(0) \quad (5.37)$$

from (5.22)'. We then obtain

$$P(R_c(0), L_c(0))\, P(a, 1)\, X_j^1(a_1) \quad (5.38)$$

from (5.31). Similarly, we obtain

$$P(L_c(I), R_c(I) + \operatorname{span} f(X_0^I(b_I), Y_0^I(b_I), 0))\, x_j^I(0), \quad (5.39)$$

$$P(\operatorname{span} f(X_0^I(b_I), Y_0^I(b_I), 0), R_c(I) + L_c(I))\, x_j^I(0), \quad (5.40)$$

$$P(L_c(I), R_c(I))\, P(b, I)\, X_j^I(b_I). \quad (5.41)$$

Observe that we have the obvious formulas

$$P(b, i)\, S^i(b_i, a_i) = P(b, i)\, S^i(b_i, a_i)\, P(a, i), \qquad 1 \leqslant i \leqslant I,$$

$$P(b, i)\, S^i(b_i, a_i)|_{\mathscr{R}P(a,i)} = S^i(b_i, a_i; TM_i, TM_{i-1}), \qquad 1 \leqslant i \leqslant I,$$

$$P(R_c(i), L_c(i))\, S^i(b_i, a_i; TM_i, TM_{i-1})$$
$$= S^i(b_i, a_i; TM_i, TM_{i-1})\, P(R_c(i-1), L_c(i-1)), \qquad 1 \leqslant i \leqslant I.$$

Therefore from (5.5) we have

$$P(R_c(i), L_c(i))\, P(b, i)\, X_j^i(b_i)$$
$$= S(b_i, a_i; TM_i, TM_{i-1})\, P(R_c(i-1), L_c(i-1))\, P(a, i)\, X_j^i(a_i)$$
$$+ P(R_c(i), L_c(i))\, P(b, i)\, \bar{C}_j^i.$$

Similarly, we have

$$P(L_c(i-1), R_c(i-1))\, P(a, i)\, X_j^i(a_i)$$
$$= S(a_i, b_i; TM_{i-1}, TM_i)\, P(L_c(i), R_c(i))\, P(b, i)\, X_j^i(b_i)$$
$$- P(L_c(i-1), R_c(i-1))\, P(a, i)\, S(a_i, b_i)\, \bar{C}_j^i,$$

from which we compute

$$P(R_c(i), L_c(i))\, P(b, i)\, X_j^i(b_i)$$

and

$$P(R_c(i), L_c(i))\, P(a, i+1)\, X_j^{i+1}(a_{i+1}), \qquad 1 \leqslant i \leqslant I-1, \tag{5.42}$$

based on (5.38), where (5.35) has been employed. We can also compute

$$P(L_c(i), R_c(i))\, P(b, i)\, X_j^i(b_i)$$

and

$$P(L_c(i), R_c(i))\, P(a, i+1)\, X_j^{i+1}(a_{i+1}), \qquad 1 \leqslant i \leqslant I-1, \tag{5.43}$$

based on (5.41), where (5.35) has also been employed. Add the results of (5.42) and (5.43), and we obtain

$$P(a, i+1)\, X_j^{i+1}(a_{i+1}), \qquad 0 \leqslant i \leqslant I-1$$

and

$$P(b, i)\, X_j^i(b_i), \qquad 1 \leqslant i \leqslant I. \tag{5.44}$$

From (5.31) and (5.32), we obtain

$$P(a, i+1)\, x_j^i(0), \;\; 0 \leqslant i \leqslant I-1$$

and

$$P(b, i)\, x_j^i(0), \qquad 1 \leqslant i \leqslant I. \tag{5.45}$$

From (5.33) and (5.34), we obtain

$$x_j^i(0), \qquad 1 \leqslant i \leqslant I-1,$$

$P(\text{span}\, f(X_0^1(a_1), Y_0^1(a_1), 0), TM_0)\, x_j^0(0)$ and $P(\text{span}\, f(X_0^I(b_I), Y_0^I(b_I), 0), TM_I)\, x_j^I(0)$ may be obtained from $(5.22)^I$ and $(5.22)^{III}$. Therefore we obtain $x_j^0(0)$ and $x_j^I(0)$. This completes the calculation of

$$x_j^i(0), \qquad 0 \leqslant i \leqslant I. \tag{5.46}$$

Let us write

$$X_j^i(a_i) = P(a, i)\, X_j^i(a_i) + \rho_j(a, i)\, f(X_0^i(a_i), Y_0^i(a_i), 0),$$

$$X_j^i(b_i) = P(b, i)\, X_j^i(b_i) + \rho_j(b, i)\, f(X_0^i(b_i), Y_0^i(b_i), 0).$$

It is clear that $\rho_j(a, i)$ and $\rho_j(b, i)$ may be obtained form (5.7), and this completes the calculation of

$$X_j^i(a_i),\ X_j^i(b_i), \qquad 1 \leqslant 1 \leqslant I. \tag{5.47}$$

Finally, we compute

$$\tau_j^i(a),\ \tau_j^i(b), \qquad 1 \leqslant 1 \leqslant I, \tag{5.48}$$

from (5.29) and (5.30).

Let us recall how (5.46), (5.47), and (5.48) imply $\{x_j^i(\tau)\}_{i=0}^I$, $\{y_j^i(\tau)\}_{i=0}^I$, $\{X_j^i(t)\}_{i=1}^I$, and $\{Y_j^i(t)\}_{i=1}^I$. $X_j^i(t)$ may be obtained from (5.5)′ and $Y_j^i(t)$ from (5.5)″. $x_j^i(\tau)$ may be obtained from (5.12) for $1 \leqslant i \leqslant I-1$ and $y_j^i(\tau)$, $1 \leqslant i \leqslant I-1$, is uniquely solvable from (5.13), (5.10), and (5.11) since (5.14) is valid. From (5.18)′, to compute $y_j^0(\tau)$ we need $\hat{Q}_s^0(0)\, y_j^0(0)$, which can be obtained in (5.22)″. Similarly $\hat{Q}_u^I(0)\, y_j^I(0)$ can be obtained in $(5.22)^{IV}$ and $y_j^I(\tau)$ is computable.

6. Proof of the Main Results

We have obtained in Section 5 the formal power series expansions for the outer layers $\sum_{j=0}^\infty \varepsilon^j X_j^i(t)$, $\sum_{j=0}^\infty \varepsilon^j Y_j^i(t)$, $1 \leqslant i \leqslant I$, $t \in [a_i, b_i]$, and the formal power expansions for the interior and boundary layers $\sum_{j=0}^\infty \varepsilon^j x_j^i(\tau)$, $\sum_{j=0}^\infty \varepsilon^j y_j^i(\tau)$, $\tau \in \mathbf{R}$, $1 \leqslant i \leqslant I-1$, $\tau \in \mathbf{R}^+$ for $i=0$, $\tau \in \mathbf{R}^-$ for $i=I$. The inner and outer layers are matched through the determination of the formal power series $\sum_{j=1}^\infty \varepsilon^j \tau_j^i(a)$ and $\sum_{j=1}^\infty \varepsilon^j \tau_j^i(b)$ which serve as the perturbation of a_i and b_i, $1 \leqslant i \leqslant I$. The matching is achieved through the asymptotic matching principle described in Section 2. First the inner expansions of the outer layers are calculated in (2.12), then the auxiliary parameters are determined so that (2.13) is satisfied for all $j \geqslant 0$, $1 \leqslant i \leqslant I$.

Proof of Theorem 2.1. Since $(5.4)_j$ is obtained from the Taylor expansion of (2.1), it is clear that the residual error for the truncations of the outer expansion $\{(\sum_{j=0}^p \varepsilon^j X_j^i(t), \sum_{j=0}^p \varepsilon^j Y_t^i(t)) \colon t \in [a_i + \sum_{j=1}^p \varepsilon^j \tau_j^i(a) + \varepsilon^\beta, b_i + \sum_{j=1}^p \varepsilon^j \tau_j^i(b) - \varepsilon^\beta]\}_{i=1}^I$ is $O(\varepsilon^{p+1})$ in the slow variable t. Therefore in the fast variable τ the residual error is $O(\varepsilon^{p+2})$ for the equation of x and $O(\varepsilon^{p+1})$ for the equation of y.

Similarly, $(5.9)_j$ is obtained from the Taylor expansion of (2.1) in the fast variable τ, thus for the truncations of the inner expansion $\{(\sum_{j=0}^p \varepsilon^j x_j^i(\tau), \sum_{j=0}^p \varepsilon^j y_j^i(\tau)) \colon \tau \in [-\varepsilon^{\beta-1}, \varepsilon^{\beta-1}]$ for $1 \leqslant i \leqslant I-1$, $\tau \in [0, \varepsilon^{\beta-1}]$ for $i=0$ and $\tau \in [-\varepsilon^{\beta-1}, 0]$ for $i=I\}$, the residual error is $O(\varepsilon^{p+1})$ in every compact subinterval which does not depend on ε. For a uniform estimate for $|\tau| \leqslant \varepsilon^{\beta-1}$, recall that $x_j^i(\tau) \in E(0, j)$ and $F_j \in E(0, j)$, therefore the residual error for the equation of x is $O(\varepsilon|\varepsilon\tau|^p) = O(\varepsilon^{\beta p+1})$ and the residual error for the equation of y is $O(|\varepsilon\tau|^{p+1}) = O(\varepsilon^{\beta(p+1)})$. If translated into the slow variable t, the residual error is $(O(\varepsilon^{\beta p}), O(\varepsilon^{\beta(p+1)}))$ in the equation of (x, y), respectively.

The boundary error is $B_i(x, y, \varepsilon) = O(\varepsilon^{p+1})$, $i = 1, 2$, following from the Taylor expansion of $B_i(x, y, \varepsilon)$, $i = 1, 2$, easily.

The jump error may be obtained with the aid of $\sum_{j=0}^{\infty} \varepsilon^j z_j(\tau, a, i)$ (or $\sum_{j=0}^{\infty} \varepsilon^j z_j(\tau, b, i)$). For example,

$$\left|\sum_{j=0}^{p} \varepsilon^j Z_j^i\left(b_i + \sum_{k=1}^{p} \varepsilon^k \tau_k^i(b) - \varepsilon^{\beta}\right) - \sum_{j=0}^{p} \varepsilon^j z_j^i(-\varepsilon^{\beta-1})\right|$$

$$\leqslant \left|\sum_{j=0}^{p} \varepsilon^j Z_j^i\left(b_i + \sum_{k=1}^{p} \varepsilon^k \tau_k^i(b) - \varepsilon\tau\right) - \sum_{j=0}^{p} \varepsilon^j z_j(-\tau, b, i)\right|_{\tau=\varepsilon^{\beta-1}}$$

$$+ \left|\sum_{j=0}^{p} \varepsilon^j z_j(-\tau, b, i) - \sum_{j=0}^{p} \varepsilon^j z_j^i(-\tau)\right|_{\tau=\varepsilon^{\beta-1}}$$

$$\leqslant C|\varepsilon \cdot \varepsilon^{\beta-1}|^{p+1} + \text{h.o.t.} \leqslant C|\varepsilon|^{\beta(p+1)},$$

from the matching principle (2.12) and (2.13).

Proof of Theorem 2.2. Let the formal approximation obtained from a truncation of the formal power series expansion be denoted by

$$z_i(\tau, p) = \begin{cases} \sum_{j=0}^{p} \varepsilon^j Z_j^l(\varepsilon\tau), & i = 2l, \quad 1 \leqslant l \leqslant I, \\ \sum_{j=0}^{p} \varepsilon^j z_j^l(\tau), & i = 2l+1, \quad 0 \leqslant l \leqslant I, \end{cases}$$

where $z_i(\tau, p) = (x_i(\tau, p), y_i(\tau, p))$, $1 \leqslant i \leqslant v$, etc. From Theorem 2.1,

$$x_i(\tau, p)' - \varepsilon f(x_i(\tau, p), y_i(\tau, p), \varepsilon) \overset{\text{def}}{=} f_i(\tau)$$

$$= \begin{cases} O(\varepsilon^{p+2}) & \text{if } i = 2l, \quad 1 \leqslant l \leqslant I, \\ O(\varepsilon^{\beta p+1}) & \text{if } i = 2l+1, \quad 0 \leqslant l \leqslant I; \end{cases}$$

$$y_i(\tau, p)' - g(x_i(\tau, p), y_i(\tau, p), \varepsilon) \overset{\text{def}}{=} g_i(\tau)$$

$$= \begin{cases} O(\varepsilon^{p+1}) & \text{if } i = 2l, \quad 1 \leqslant l \leqslant I, \\ O(\varepsilon^{\beta(p+1)}) & \text{if } i = 2l+1, \quad 0 \leqslant l \leqslant I; \end{cases}$$

$$z_i(\beta_i, p) - z_{i+1}(\alpha_{i+1}, p) \overset{\text{def}}{=} h_i = O(\varepsilon^{\beta(p+1)});$$

$$B_1(x_1(\alpha_1, p), y_1(\alpha_1, p), \varepsilon) \overset{\text{def}}{=} \tilde{b}_1 = O(\varepsilon^{p+1});$$

$$B_2(x_v(\beta_v, p), y_v(\beta_v, p), \varepsilon) \overset{\text{def}}{=} \tilde{b}_2 = O(\varepsilon^{p+1}),$$

where $z_i(\tau, p)$ is defined for $\tau \in [\alpha_i, \beta_i]$, $1 \leqslant i \leqslant v = 2I + 1$, with

(i) $\alpha_i = a_l/\varepsilon + \sum_{j=1}^{p} \varepsilon^{j-1} \tau_j^l(a) + \varepsilon^{\beta-1}$, $\beta_i = b_l/\varepsilon + \sum_{j=1}^{p} \varepsilon^{j-1} \tau_j^l(b) - \varepsilon^{\beta-1}$, if $i = 2l$, $1 \leqslant l \leqslant I$;

(ii) $\alpha_i = -\varepsilon^{\beta-1}$, $\beta_i = \varepsilon^{\beta-1}$ if $i = 2l+1$, $1 \leqslant l \leqslant I-1$;

(iii) $\alpha_1 = 0$, $\beta_1 = e^{\beta-1}$, $\alpha_v = -\varepsilon^{\beta-1}$, and $\beta_v = 0$.

Write the exact solution as $(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau))$, where $\tau \in [\alpha_i + \Delta\alpha_i, \beta_i + \Delta\beta_i]$. We assume $\Delta\alpha_i = \Delta\beta_i = 0$ if $i = 2l+1$, $0 \leqslant l \leqslant I$. The equation for $z_i(\tau) = (x_i(\tau), y_i(\tau))$ is

$$x_i(\tau)' = \varepsilon f(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon) - \varepsilon f(x_i(\tau, p), y_i(\tau, p), \varepsilon) - f_i(\tau), \tag{6.1$_f$}$$

$$y_i(\tau)' = g(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon) - g(x_i(\tau, p), y_i(\tau, p), \varepsilon) - g_i(\tau), \qquad 1 \leqslant i \leqslant v, \tag{6.1$_g$}$$

$$z_i(\beta_i + \Delta\beta_i) - z_{i+1}(\alpha_{i+1} + \Delta\alpha_{i+1}) = z_{i+1}(\alpha_{i+1} + \Delta\alpha_{i+1}, p) - z_i(\beta_i + \Delta\beta_i, p), \qquad 1 \leqslant i \leqslant v-1, \tag{6.2}$$

$$B_1(x_1(\alpha_1, p) + x_1(\alpha_1), y_1(\alpha_1, p) + y_1(\alpha_1), \varepsilon) = 0, \tag{6.3}$$

$$B_2(x_v(\beta_v, p) + x_v(\beta_v), y_v(\beta_v, p) + y_v(\beta_v), \varepsilon) = 0, \tag{6.4}$$

$$d^i \cdot z_i(\tau_i) = 0, \qquad 2 \leqslant i \leqslant v-1, \tag{6.5}$$

where d^i and τ_i, $1 \leqslant i \leqslant v$, are defined in Section 4. We shall use the results of Section 4, and to this end write the equation in the linear variational form:

For $i = 2l$, $1 \leqslant l \leqslant I$,

$$x_i(\tau)' - \varepsilon f_x^i(\varepsilon\tau) x_i - \varepsilon f_y^i(\varepsilon\tau) y_i = \mathscr{F}_f^i, \tag{6.1$'_f$}$$

$$[y_i(\tau) + g_y^i(\varepsilon\tau)^{-1} g_x^i(\varepsilon\tau) x_i(\tau)]' - g_y^i(\varepsilon\tau)[y_i(\tau) + g_y^i(\varepsilon\tau)^{-1} g_x^i(\varepsilon\tau) x_i(\tau)] = \mathscr{F}_g^i, \tag{6.1$'_g$}$$

where

$$\begin{aligned}
\mathscr{F}_f^i &= \mathscr{F}_f^i(f_i, x_i, y_i, \varepsilon) \\
&= \varepsilon f(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon) \\
&\quad - \varepsilon f(x_i(\tau, p), y_i(\tau, p), \varepsilon) - f_i(\tau) \\
&\quad - \varepsilon D_z f(x_i(\tau, p), y_i(\tau, p), \varepsilon) z_i(\tau) + \varepsilon(D_z f(x_i(\tau, p), y_i(\tau, p), \varepsilon) \\
&\quad - D_z f(x_i(\tau, 0), y_i(\tau, 0), 0)) z_i(\tau) \\
&= O(\varepsilon |z_i|^2 + \varepsilon^2 |z_i| + |f_i|);
\end{aligned}$$

$$
\begin{aligned}
\mathscr{F}^i_g &= \mathscr{F}^i_g(g_i, x_i, y_i, \varepsilon)\\
&= g(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon)\\
&\quad - g(x_i(\tau, p), y_i(\tau, p), \varepsilon)\\
&\quad - D_z g(x_i(\tau, p), y_i(\tau, p), \varepsilon)\, z_i(\tau) + (D_z g(x_i(\tau, p), y_i(\tau, p), \varepsilon)\\
&\quad - D_z g(x_i(\tau, 0), y_i(\tau, 0), 0))\, z_i(\tau)\\
&\quad - g_i(\tau) + [g^i_y(\varepsilon\tau)^{-1} g^i_x(\varepsilon\tau)]'\, x_i(\tau)\\
&\quad + g^i_y(\varepsilon\tau)^{-1} g^i_x(\varepsilon\tau)\{\varepsilon f(x_i(\tau; p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon)\\
&\quad - \varepsilon f(x_i(\tau, p), y_i(\tau, p), \varepsilon) - f_i(\tau)\}\\
&= O(|z_i|^2 + \varepsilon |z_i| + |g_i| + |f_i|).
\end{aligned}
$$

For $i = 2l + 1$, $0 \leqslant l \leqslant I$,

$$x_i(\tau)' = \mathscr{F}^i_f, \tag{6.1}''_f$$

$$y_i(\tau)' - g^i_x(\tau)\, x_i(\tau) - g^i_y(\tau)\, y_i(\tau) = \mathscr{F}^i_g, \tag{6.1}''_g$$

where

$$
\begin{aligned}
\mathscr{F}^i_f &= \mathscr{F}^i_f(f_i, x_i, y_i, \varepsilon)\\
&= \varepsilon f(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon)\\
&\quad - \varepsilon f(x_i(\tau, p), y_i(\tau, p), \varepsilon) - f_i(\tau)\\
&= O(\varepsilon |z_i| + |f_i|);
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{F}^i_g &= g(x_i(\tau, p) + x_i(\tau), y_i(\tau, p) + y_i(\tau), \varepsilon) - g(x_i(\tau, p), y_i(\tau, p), \varepsilon)\\
&\quad - D_z g(x_i(\tau, p), y_i(\tau, p), \varepsilon)\, z_i(\tau)\\
&\quad + [D_z g(x_i(\tau, p), y_i(\tau, p), \varepsilon) - D_z g(x_i(\tau, 0), y_i(\tau, 0), 0)]\, z_i(\tau) - g_i(\tau)\\
&= O(|z_i|^2 + \varepsilon^\beta |z_i| + |g_i|).
\end{aligned}
$$

To compute the jumps we shall need the following estimates for $i = 2l$, $1 \leqslant l \leqslant I$, which can be verified directly:

$$
\begin{aligned}
&|z_i'| \leqslant C(|z_i| + |f_i| + |g_i|),\\
&|z_i(\cdot, p)''| \leqslant C\varepsilon^2,\\
&|z_i(\cdot, 0)' - z_i(\cdot, p)'| \leqslant C\varepsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
&z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + z_i(\beta_i, 0)'\, \Delta\beta_i\\
&\quad - z_{i+1}(\alpha_{i+1}, 0)'\, \Delta\alpha_{i+1} = \mathscr{H}^i, \qquad 1 \leqslant i \leqslant \nu - 1
\end{aligned} \tag{6.2}'
$$

where

$$\begin{aligned}
\mathscr{H}^i &= \mathscr{H}^i(h_i, z_i, \Delta\beta_i, \Delta\alpha_{i+1}, \varepsilon) \\
&= z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) - (z_i(\beta_i + \Delta\beta_i) \\
&\quad - z_{i+1}(\alpha_{i+1} + \Delta\alpha_{i+1})) \\
&\quad + z_{i+1}(\alpha_{i+1} + \Delta\alpha_{i+1}, p) - z_i(\beta_i + \Delta\beta_i, p) \\
&\quad - z_{i+1}(\alpha_{i+1}, p) + z_i(\beta_i, p) - h_i \\
&\quad - z_{i+1}(\alpha_{i+1}, p)' \Delta\alpha_{i+1} + z_i(\beta_i, p)' \Delta\beta_i \\
&\quad + [z_i(\beta_i, 0)' - z_i(\beta_i, p)'] \Delta\beta_i \\
&\quad + [z_{i+1}(\alpha_{i+1}, p)' - z_{i+1}(\alpha_{i+1}, 0)'] \Delta\alpha_{i+1} \\
&= O(|z_i'| \, |\Delta\beta_i| + |z_{i+1}'| \, |\Delta\alpha_{i+1}| + |z_i''(\cdot, p)| \\
&\quad \times |\Delta\beta_i|^2 + |z_{i+1}''(\cdot, p)| \, |\Delta\alpha_{i+1}|^2 + |h_i| \\
&\quad + |z_i(\cdot, 0)' - z_i(\cdot, p)'| \, |\Delta\beta_i| + |z_{i+1}(\cdot, 0)' - z_{i+1}(\cdot, p)' |\Delta\alpha_{i+1}|) \\
&= O(|h_i| + \varepsilon^2(|\Delta\beta_i| + |\Delta\alpha_{i+1}|) + \varepsilon^2(|\Delta\beta_i|^2 + |\Delta\alpha_{i+1}|^2) \\
&\quad + (|z_i| + |f_i| + |g_i|)|\Delta\beta_i| + |(z_{i+1}| + |f_{i+1}| + |g_{i+1}|)|\Delta\alpha_{i+1}|).
\end{aligned}$$

Recall that $\Delta\alpha_i = \Delta\beta_i = 0$ for $i = 2l+1$, $0 \leqslant l \leqslant l$. Therefore either $z_i(\beta_i, 0)' = \varepsilon f^i(\varepsilon\beta_i)$, $i = 2l$, or $z_{i+1}(\alpha_{i+1}, 0)' = \varepsilon f^{i+1}(\varepsilon\alpha_{i+1})$, $i+1 = 2l$. Define $\zeta_i = \varepsilon\, \Delta\beta_i$ (or $-\varepsilon\, \Delta\alpha_{i+1}$), $e_i = f^i(\varepsilon\beta_i)$ (or $f^{i+1}(\varepsilon\alpha_{i+1})$) for $i = 2l$ (or $i+1 = 2l$), and we can rewrite (6.2)′ as

$$z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + \zeta_i e_i = \mathscr{H}^i \tag{6.2''}$$

with $\mathscr{H}^i = O(|h_i| + \varepsilon|\zeta_i| + |\zeta_i|^2 + (1/\varepsilon)(|z_{2l}| + |f_{2l}| + |g_{2l}|)\, |\zeta_i|)$.

The boundary equations can be written as

$$\bar{B}_1 z_1(\alpha_1) = \mathbf{b}_1, \tag{6.3'}$$

$$\bar{B}_2 z_\nu(\beta_\nu) = \mathbf{b}_2, \tag{6.4'}$$

where

$$\begin{aligned}
\mathbf{b}_1 = \mathbf{b}_1(\tilde{b}_1, z_1, \varepsilon) &= [D_z B_1(x_1(\alpha_1, 0), y_1(\alpha_1, 0), 0) \\
&\quad - D_z B_1(x_1(\alpha_1, p), y_1(\alpha_1, p), y_1, p), \varepsilon)] \, z_1(\alpha_1) \\
&\quad - \tilde{b}_1 - \{B_1(x_1(\alpha_1, p) + x_1(\alpha_1), y_1(\alpha_1, p) + y_1(\alpha_1), \varepsilon) \\
&\quad - B_1(x_1(\alpha_1, p), y_1(\alpha_1, p), \varepsilon) \\
&\quad - D_z B_1(x_1(\alpha_1, p), y_1(\alpha_1, p), \varepsilon)\, z_1(\alpha_1)\} \\
&= O(\varepsilon|z_1| + |z_1|^2 + |\tilde{b}_1|);
\end{aligned}$$

$$\begin{aligned}\mathbf{b}_2 = \mathbf{b}_2(\tilde{b}_2, z_\nu, \varepsilon) &= [D_z B_2(x_\nu(\beta_\nu, 0), y_\nu(\beta_\nu, 0), 0) \\ &\quad - D_z B_2(x_\nu(\beta_\nu, p), y_\nu(\beta_\nu, p), \varepsilon)]\, z_\nu(\beta_\nu) - \tilde{b}_2 \\ &\quad - \{B_2(x_\nu(\beta_\nu, p) + x_\nu(\beta_\nu), y_\nu(\beta_\nu, p) + y_\nu(\beta_\nu), \varepsilon) \\ &\quad - B_2(x_\nu(\beta_\nu, p), y_\nu(\beta_\nu, p), \varepsilon) \\ &\quad - D_z B_2(x_\nu(\beta_\nu, p), y_\nu(\beta_\nu, p), \varepsilon)\, z_\nu(\beta_\nu)\} \\ &= O(\varepsilon |z_\nu| + |z_\nu|^2 + |\tilde{b}_2|).\end{aligned}$$

We look for solutions of the system $(6.1)'_f$, $(6.1)'_g$, $(6.1)''_f$, $(6.1)''_g$, $(6.2)''$, $(6.3)'$, $(6.4)'$, and (6.5) in the Banach space

$$(\{z_i\}_{i=1}^{\nu}, \{\zeta_i\}_{i=1}^{\nu-1}) \in \prod_{i=1}^{\nu} C^1[\alpha_i - \delta, \beta_i + \delta] \times \prod_{i=1}^{\nu-1} \mathbf{R}^{m+n},$$

where $\delta > 0$, with the ε-dependent norm defined as

$$\|\cdot\|_\varepsilon = \frac{1}{\varepsilon} \sup_{1 \leqslant i \leqslant \nu} |z_i|_{C^1[\alpha_i - \delta, \beta_i + \delta]} + \frac{1}{\varepsilon} \sup_{1 \leqslant i \leqslant \nu - 1} |\zeta_i|.$$

Let an open subset $\mathscr{O}_\delta(\varepsilon)$ be defined as

$$\mathscr{O}_\delta(\varepsilon) = \{\{z_i\}_{i=1}^{\nu}, \{\zeta_i\}_{i=1}^{\nu-1} : \|\{z_i\}, \{\zeta_i\}\|_\varepsilon < \delta\}.$$

Applying Theorem 4.9 to our system we have an abstract equation

$$\begin{aligned}(\{z_i\}_{i=1}^{\nu}, \{\zeta_i\}_{i=1}^{\nu-1}) &= \mathscr{A}^{-1} \cdot (\{\mathscr{F}^i_f, \mathscr{F}^i_g\}_{i=1}^{\nu}, \{\mathscr{H}^i\}_{i=1}^{\nu-1}, \mathbf{b}_1, \mathbf{b}_2) \\ &= \Phi(\{z_i\}_{i=1}^{\nu}, \{\zeta_i\}_{i=1}^{\nu}, \{f_i\}_{i=1}^{\nu}, \{g_i\}_{i=1}^{\nu}, \{h_i\}_{i=1}^{\nu-1}, \tilde{b}_1, \tilde{b}_2).\end{aligned} \tag{6.6}$$

We observe that although Theorem 4.9 only gives the solution for the linear equation is $\tau \in [\alpha_i, \beta_i]$, however, $z_i(\tau)$ extends to $[\alpha_i - \delta, \beta_i + \delta]$, here $\mathscr{A}^{-1}$ is the extended solution map. Moreover, estimates for $\sup_{1 \leqslant i \leqslant \nu} |z_i'|_{C^1[\alpha_i - \delta, \beta_i + \delta]}$ can be easily obtained from the estimate in Theorem 4.9 and the equation itself. We look for a fixed point of the mapping

$$\Phi\colon \prod_{i=1}^{\nu} C^1[\alpha_i - \delta, \beta_i + \delta] \times \prod_{i=1}^{\nu-1} \mathbf{R}^{m+n} \circlearrowleft.$$

Now for any $0 < \beta < 1$, if $p \geqslant 1$ is such that $\beta(p+1) > 1$, we have

$$\begin{aligned}&\|\Phi(0, 0, \{f_i\}, \{\langle g_i\}, \{h_i\}, \tilde{b}_1, \tilde{b}_2)\|_\varepsilon \\ &\quad = O\left(\frac{|h_i|}{\varepsilon} + \frac{|\tilde{b}_1|}{\varepsilon} + \frac{|\tilde{b}_2|}{\varepsilon} + \frac{|g_i|}{\varepsilon} + \frac{|f_i|_{i=2l}}{\varepsilon^2} + \frac{|f_i|_{i=2l+1}}{\varepsilon^{2-\beta}}\right) \\ &\quad = O(\varepsilon^{\beta(p+1)-1}).\end{aligned}$$

It is also straightforward to verify that if $\delta > 0$ is sufficiently small, then there is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$,

$$\Phi: \mathcal{C}_\delta(\varepsilon) \to \mathcal{C}_\delta(\varepsilon)$$

and is a contraction in the indicated norm. Therefore, we have a unique fixed point $(\{z_i\}_{i=1}^{\nu}, \{\zeta_i\}_{i=1}^{\nu-1})$ in $\mathcal{C}_\delta(\varepsilon)$. Moreover

$$\sup_{1 \leqslant i \leqslant \nu} |z_i|_{C^1[\alpha_i - \delta, \beta_i + \delta]} + \sup_{1 \leqslant i \leqslant \nu - 1} |\zeta_i| = O(\varepsilon^{\beta(p+1)}).$$

Observe that in order to form a solution of (2.1), we need ε to be sufficiently small such that $\Delta\alpha_{i+1}$ (or $\Delta\beta_i) = (1/\varepsilon)\,\zeta_i = O(\varepsilon^{\beta(p+1)-1}) < \delta$. It is also clear that

$$|\omega - \omega_{\text{exact}}| \leqslant \varepsilon \cdot \sum_{i=1}^{\nu} (|\Delta\alpha_i| + |\Delta\beta_i|) = O(\varepsilon^{\beta(p+1)}). \tag{6.7}$$

We can now define

$$z_{\text{exact}}(t) = \bigvee_{i=1}^{\nu} \{z^i_{\text{exact}}(\tau),\ t = \varepsilon\tau,\ \tau \in [\alpha_i + \Delta\alpha_i, \beta_i + \Delta\beta_i]\},$$

where

$$z^i_{\text{exact}}(\tau) = z_i(\tau, p) + z_i(\tau), \qquad \tau \in [\alpha_i + \Delta\alpha_i, \beta_i + \Delta\beta_i].$$

To obtain the estimates for $\text{dist}(z_{\text{exact}}(t), z(t, p))$, we have to choose for each τ in the domain of $z^i_{\text{exact}}(\tau)$ (or $z_i(\tau, p)$) a τ_1 in the domain of $z_i(\tau, p)$ (or $z^i_{\text{exact}}(\tau)$) such that

$$|z_i(\tau_1, p) - z^i_{\text{exact}}(\tau)| \text{ (or } |z^i_{\text{exact}}(\tau_1) - z_i(\tau, p)|)$$

is small. For $\tau \in [\alpha_i + \Delta\alpha_i, \beta_i + \Delta\beta_i] \cap [\alpha_i, \beta_i]$, let $\tau_1 = \tau$ and we have $|z_i(\tau, p) - z^i_{\text{exact}}(\tau)| \leqslant |z_i|_c$. Otherwise choose $|\tau_1 - \tau| = |\Delta\alpha_i|$ (or $|\Delta\beta_i|) = O(\varepsilon^{\beta(p+1)-1})$, and thus $|z_i(\tau, p) - z^i_{\text{exact}}(\tau_1)| = O(\varepsilon^{\beta(p+1)})$ (or $|z^i_{\text{exact}}(\tau) - z_i(\tau_1, p)| = O(\varepsilon^{\beta(p+1)})$). Here recall $z_i(\ , p)' = O(\varepsilon)$ for $i = 2l$.

We have proved (2.16) for large p such that $\beta(p+1) > 1$. For an arbitrary integer $p \geqslant 0$, we can choose $\bar{p}$ so large that $\beta(\bar{p}+1) > p+1$. Accordingly $\text{dist}(z_{\text{exact}}(t), z(t, \bar{p})) = O(\varepsilon^{(p+1)})$. It is straightforward to verify that

$$\text{dist}(z(\tau, p), z(t, \bar{p})) = O(\varepsilon^{\beta(p+1)}).$$

Therefore the desired result in (2.16) follows. To prove (2.17), we need the estimates

$$\text{dist}(z_{\text{comp}}(t, \bar{p}) - z(t, \bar{p})) = O(\varepsilon^{\beta(\bar{p}+1)}), \tag{6.8}$$

$$\text{dist}(z_{\text{comp}}(t, \bar{p}) - z_{\text{comp}}(t, p)) = O(\varepsilon^{(p+1)}). \tag{6.9}$$

The estimate in (2.17) then follows easily. To show (6.8), we only need to compate $z_{\text{comp},i}(t, \bar{p})$ (see (2.14)) with $z(t, \bar{p})$ on $[a_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(a), b_i + \sum_{j=1}^{p} \varepsilon^j \tau_j^i(b)]$, which is further divided into regular and boundary regions with the length of the boundary layers being ε^β. We then use the property of $\sum_{j=0}^{p} \varepsilon^j z_j((t-a_i)/\varepsilon - \sum_{j=1}^{p} \varepsilon^{j-1} \tau_j^i(a), a, i)$, i.e., it is close to $\sum_{j=0}^{p} Z_j^i(t)$ in the boundary layer near $t = a_i$, and is close to $\sum_{j=0}^{p} \varepsilon^j z_j^{i-1}((t-a_i)/\varepsilon - \sum_{j=1}^{p} \varepsilon^{j-1} \tau_j^i(a))$ in the regular layer and the boundary layer near $t = b_i$. Similar consideration is also given to another term $\sum_{j=0}^{p} \varepsilon^j z_j((t-b_i)/\varepsilon - \sum_{j=1}^{p} \varepsilon^{j-1} \tau_j^i(b), b, i)$ in $z_{\text{comp},i}(t, p)$. Estimate (6.9) can be proved by the same argument. Let ω corresponding to $\bar{p}$ be denoted by $\omega(\bar{p})$. From (6.7) we have $|\omega(\bar{p}) - \omega_{\text{exact}}| = O(\varepsilon^{\beta(\bar{p}+1)})$. It is easy to show that $|\omega(\bar{p}) - \omega(p)| = O(\varepsilon^{(p+1)})$. Thus (2.18) follows easily. Q.E.D.

7. Singularly Perturbed Periodic Solutions

Problems concerning the existence of periodic solutions to the singularly perturbed system

$$\begin{aligned} \dot{x} &= f(x, y, \varepsilon), \\ \varepsilon \dot{y} &= g(x, y, \varepsilon), \end{aligned} \tag{7.1}$$

$x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$, are closely related to the boundary value problem (1.1). We shall use all the notations in Section 2. As in Section 2, assume that the 0th order slow manifold $\mathcal{S}_i$, $1 \leqslant i \leqslant I$, is hyperbolic near $(X_0^i(t), Y_0^i(t))$, $t \in [a_i, b_i]$, which is a solution of (2.2), i.e., we assume condition (H_1) as in Section 2. Again the dimensions of the stable and unstable spaces of g_y are denoted by d^- and d^+, which are independent of $1 \leqslant i \leqslant I$. Assume that $x_0^i = X_0^i(b_i) = X_0^{i+1}(a_{i+1})$, $1 \leqslant i \leqslant I$, where $X_0^{I+1}(a_{I+1})$ is $X_0^1(a_1)$; $(x_0^i, y_0^i(\tau))$, $\tau \in \mathbf{R}$, $1 \leqslant i \leqslant I$, is a heteroclinic solution of (2.3), connecting $\mathcal{S}_i$ to $\mathcal{S}_{i+1}$, where $\mathcal{S}_{I+1}$ is $\mathcal{S}_1$. The outer layers $\{(X_0^i(t), Y_0^i(t)), t \in [a_i, b_i]\}$, $1 \leqslant i \leqslant I$, and the inner layers $\{(x_0^i, y_0^i(\tau)), \tau \in \mathbf{R}\}$, $1 \leqslant i \leqslant I$, thus form a closed cycle. We expect that under some additional conditions there exists a unique periodic solution of (7.1) near the 0th order closed cycle when $\varepsilon > 0$ is small.

Consider the linear homogeneous equation (2.5) and the adjoint equation (2.6). Assume that $y_0^i(\tau)'$ and $\psi_i(\tau)$, $\tau \in \mathbf{R}$, $1 \leqslant i \leqslant I$, are the only bounded solutions of (2.5) and (2.6), respectively, up to a scalar multiple. Assume (H_2) and (H_3) as in Section 2, but with $1 \leqslant i \leqslant I$. We need to define the hyperbolicity of the closed cycle of the reduced flow on $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_I$. Let $S^i(t, s)$ be the solution map for the linear equation (2.11). Define $S^i(t_2, t_1; \Sigma_2, \Sigma_1)$ for $[t_1, t_2] \subset [a_i, b_i]$ and $\Sigma_1 \oplus f(X_0^i(t_1), Y_0^i(t_1), 0) = \Sigma_2 \oplus f(X_0^i(t_2), Y_0^i(t_2), 0) = \mathbf{R}^m$ just like those in Section 2.

DEFINITIONS. Let $TM_i = \{x \in \mathbf{R}^m \mid \Delta_i \cdot x = 0\}$ for $1 \leqslant i \leqslant I$, $TM_0 = TM_I$. The periodic solution $\bigvee_{i=1}^{I} (X_0^i(t), t \in [a_i, b_i])$ of (2.20) is said to be hyperbolic if the composite map $\prod_{i=1}^{I} S^i(b_i, a_i; TM_i, TM_{i-1}): TM_0 \to TM_0$ is hyperbolic.

The main assumption of this section is:

($H_{4,pd}$) The periodic solution $\bigvee_{i=1}^{I} (X_0^i(t), t \in [a_i, b_i])$ of (2.20) is hyperbolic.

Observe that ($H_{4,pd}$) implies the existence of a stable space $L_c(0)$ and an unstable space $R_c(0)$ of the map $\prod_{i=1}^{I} S^i(b_i, a_i; TM_i, TM_{i-1})$ with $L_c(0) \oplus R_c(0) = TM_0$. We can define $L_c(i)$ and $R_c(i)$, $1 \leqslant i \leqslant I$, just as in Section 2. We also remark that the linearized flow around the periodic solution $\bigvee_{i=1}^{I} (X_0^i(t), t \in [a_i, b_i])$ of (2.20) has an exponential trichotomy with a one-dimensional center space spanned by $\{\dot{X}_0^i(t)\}_{i=1}^{I}$. Here we need a generalization of the concept of the exponential trichotomy to piecewise continuous linear systems. Exponential trichotomy on the slow manifold was used to study singularly perturbed periodic solutions by Bettelli and Lazzari [31].

THEOREM 7.1. *Suppose that* $\{X_0^i(t), Y_0^i(t)\}_{i=1}^{I}$, $t \in [a_i, b_i]$, *is given which satisfies* (2.2) *and* $\{x_0^i, y_0^i(\tau)\}_{i=1}^{I}$ *is given which satisfies* (2.3) *and* (H_1)–($H_{4,pd}$) *as made in this section are satisfied. Then there exist formal power series*:

(i) $\sum_{j=0}^{\infty} \varepsilon^j X_j^i(t)$, $\sum_{j=0}^{\infty} \varepsilon^j Y_j^i(t)$, $1 \leqslant i \leqslant I$, $t \in [a_i, b_i]$,

(ii) $\sum_{j=0}^{\infty} \varepsilon^j x_j^i(\tau)$, $\sum_{j=0}^{\infty} \varepsilon^j y_j^i(\tau)$, $1 \leqslant i \leqslant I$, $\tau \in \mathbf{R}$, *and*

(iii) $\sum_{j=0}^{\infty} \varepsilon^j \tau_j^i(a)$, $\sum_{j=0}^{\infty} \varepsilon^j \tau_j^i(b)$, $1 \leqslant i \leqslant I$,

with the functions $X_j^i(t)$, $Y_j^i(t)$, $x_j^i(\tau)$, $y_j^i(\tau)$ *and the constants* $\tau_j^i(a)$, $\tau_j^i(b)$ *computable by systems of linear equations and the auxiliary constants for the solutions of the linear equations determined by the asymptotic matching principle.*

Moreover, for any integer $p > 0$ *and* $0 < \beta < 1$, *if* $z(t, p)$ *and* $z_{\mathrm{comp}}(t, p)$ *are defined as in Section* 2, *with* $z_j^0(\tau) \stackrel{\mathrm{def}}{=} z_j^I(\tau)$, *for* $t \in [0, \omega]$, *and are periodic with period* $\omega = \sum_{i=1}^{I} \{(b_i - a_i) + \sum_{j=1}^{p} \varepsilon^j (\tau_j^i(b) - \tau_j^i(a))\}$, *then* $z(t, p)$ *and* $z_{\mathrm{comp}}(t, p)$ *are formal approximations of* (7.1). *The jump errors for* $z(t, p)$ *are* $O(\varepsilon^{\beta(p+1)})$ *while* $z_{\mathrm{comp}}(t, p)$ *is continuous. The residual errors for* $z(t, p)$ *and* $z_{\mathrm{comp}}(t, p)$ *are as listed in Theorem* 2.1.

Finally, there is $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, *there exists a unique exact periodic solution* $z_{\mathrm{exact}}(t)$ *of system* (7.1) *with period* ω_{exact}, $|\omega - \omega_{\mathrm{exact}}| = O(\varepsilon)$, *and* $\mathrm{dist}(z_{\mathrm{exact}}(t), z(t, p)) = O(\varepsilon)$. *We have the estimates*

$$\text{dist}(z_{\text{exact}}(t), z(t, p)) = O(\varepsilon^{\beta(p+1)}),$$
$$\text{dist}(z_{\text{exact}}(t), z_{\text{comp}}(t, p)) = O(\varepsilon^{p+1}),$$
$$|\omega - \omega_{\text{exact}}| = O(\varepsilon^{p+1}).$$

The proof of Theorem 7.1 is similar to the proof of Theorems 2.1 and 2.2, or even simpler at certain points. We still need a splitting on each TM_i, $L_c(i) \oplus R_c(i) = TM_i$, which comes from the hyperbolicity of the solution $\bigvee_{i=1}^{I} (X_0^i(t),\ t \in [a_i, b_i])$, (see $H_{4,\text{pd}}$)). When computing the higher order expansions, there is no need to consider matching of y variables, however, there are matching conditions which determine $X_j^i(a_i)$, $X_j^i(b_i)$, $\tau_j^i(a)$, $\tau_j^i(b)$, and $x_j^i(0)$ for $1 \leqslant i \leqslant I$, $j \geqslant 1$. The solvability of a system of algebraic equations which determines $X_j^i(a_i)$, etc., relies on the hyperbolicity of the linear composite map $\prod_{i=1}^{I} S^i(b_i, a_i; TM_i, TM_{i-1})$: $TM_0 \to TM_0$. In order to show the validity of the formal series expansions we consider a linear periodic system

$$\begin{aligned} &z_i(\tau)' - A_i(\tau, \varepsilon)\, z_i(\tau) = F_i(\tau), \qquad \tau \in [\alpha_i, \beta_i], \\ &z_i(\beta_i) - z_{i+1}(\alpha_{i+1}) + \zeta_i e_i = h_i, \\ &d^i \cdot z_i(\tau_i) = 0, \end{aligned} \tag{7.2}$$

for $-\infty < i < \infty$. System (7.2) is a linearization to (7.1) for $1 \leqslant i \leqslant 2I$ just as system (4.1), (4.2), and (4.5) is to system (2.1). For $1 > i$ and $i > 2I$, the coefficients of (7.2) come from a $2I$ periodic extension in the index i. System (7.2) can be solved by the method of iteration, just as system (4.1)–(4.5) in Section 4. Again the essential role is played by the hyperbolicity in the variables, x and y, similar to the situation in Section 4. Details shall not be rendered here.

We remark that hypothesis ($H_{4,\text{pd}}$) can be weakened. Theorem 7.1 is valid if $\prod_{i=1}^{I} S^i(b_i, a_i; TM_i, TM_{i-1})$: $TM_0 \to TM_0$ is nondegenerate, i.e., one is not an eigenvalue for that map. However, we would not have a splitting $TM_i = L_c(i) \oplus R_c(i)$.

EXAMPLE. Consider the traveling wave solution of the FitzHugh–Nagumo equation which satisfies a system of ordinary differential equations in $\mathbf{R}^3$,

$$u' = v, \tag{7.3}$$

$$v' = \theta v - [f(u) - w], \tag{7.4}$$

$$w' = \varepsilon \theta^{-1}(u - \gamma \omega). \tag{7.5}$$

For $\varepsilon = 0$, $w' = 0$, w is a parameter for Eqs. (7.3) and (7.4). The function $f(u)$ has the qualitative form of a cubic polynomial, and for definiteness, we take $f(u) = -u(u-\beta)(u-1)$, where $0<\beta<\frac{1}{2}$. For $\bar{w}<w<\bar{\bar{w}}$, the equation $f(u)-w=0$ has three zeros, $u_1(w)<u_0(w)<u_2(w)$. See Fig. 3.

LEMMA 7.2. *There is $\theta_0>0$ such that for $0\leqslant\theta\leqslant\theta_0$, system (7.3), (7.4) possesses a unique heteroclinic solution connecting $(u_1(w),0)$ to $(u_2(w),0)$ if $w=w_1(\theta)$. Here $w_1(\theta)$ is a C^∞ function, $w_1(\theta)'<0$* for $0\leqslant\theta\leqslant\theta_0$, *$w_1(\theta_0)=0$, and $w_1(0)$ is such that $\int_{u_1(w_1(0))}^{u_2(w_1(0))}[f(u)-w_1(0)]\,du=0$. Also for $0\leqslant\theta\leqslant\theta_0$, system (7.3), (7.4) possesses a unique heteroclinic solution connecting $(u_2(w),0)$ to $(u_1(w),0)$ if $w=w_2(\theta)$. Here $w_2(\theta)$ is a C^∞ function, $w_2(\theta)'>0$ for $0\leqslant\theta\leqslant\theta_0$, and $w_2(0)=w_1(0)$.*

For the cubic polynomial $f(u) = -u(u-\beta)(u-1)$, $0<\beta<\frac{1}{2}$. The lemma can be proved by computing the heteroclinic solution and the parameter $w_1(\theta)$ explicitly; see Casten, Cohen, and Lagerstrom [2]. The same results also hold for the more general cubic type function $f(u)$. A proof can be obtained by phase-plane analysis; see Smoller [27].

Let $w\in(\bar{w},\bar{\bar{w}})$ and consider the equilibria $(u_1(w),0)$ and $(u_2(w),0)$ for (7.3) and (7.4). Since $df(u_1)<0$ and $df(u_2)<0$, it is clear the $(u_1,0)$ and $(u_2,0)$ are saddle points with eigenvalues

$$\frac{\theta}{2}\pm\frac{\sqrt{\theta^2-4df(u)}}{2}, \tag{7.6}$$

where $u=u_1(w)$ or $u_2(w)$. If $\tilde{\theta}\geqslant 0$ and $\tilde{w}\in(\bar{w},\bar{\bar{w}})$ are such that system (7.3) and (7.4) has a heteroclinic solution $(\tilde{u}(\tau),\tilde{v}(\tau))$ connecting $(u_1(\tilde{w}),0)$ to $(u_2(\tilde{w}),0)$ or $(u_2(\tilde{w}),0)$ to $(u_1(\tilde{w}),0)$, the conditions on (θ,ω) near $(\tilde{\theta},\tilde{w})$ for system (7.3) and (7.4) to have a hetroclinic solution near $(\tilde{u}(\tau),\tilde{v}(\tau))$

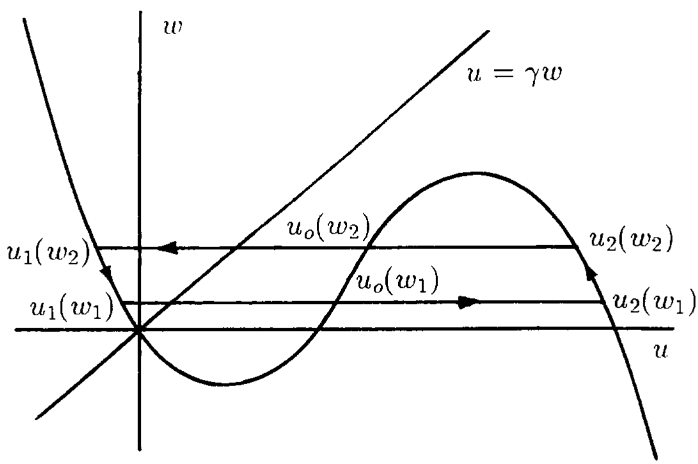

FIGURE 3

can be obtained by the method of Lyapunov–Schmidt with application of exponential dichotomies (Palmer [23], see also Hale and Lin [13]). Since the unstable manifold for $(u_1(\tilde{w}), 0)$ or $(u_2(\tilde{w}), 0)$ is of dimension one, if $(\tilde{u}(\tau), \tilde{v}(\tau))$ is a heteroclinic solution from $(u_1(w), 0)$ to $(u_2(w), 0)$ (or $(u_2(w), 0)$ to $(u_1(w), 0)$), then $(\tilde{u}(\tau)', \tilde{v}(\tau)')$ spans the one-dimensional space of bounded solutions of the linearized equations of system (7.3) and (7.4) around $(\tilde{u}(\tau), \tilde{v}(\tau))$,

$$\phi_1' = \phi_2,$$
$$\phi_2' = \theta\phi_2 - df(\tilde{u}(\tau))\,\phi_1.$$

Under these conditions, it can be shown that (θ, w) has to satisfy a bifurcation equation

$$G(\theta, w) = 0,$$

whose solution near $(\tilde{\theta}, \tilde{w})$ corresponds to a unique heteroclinic solutions near $(\tilde{u}(\tau), \tilde{v}(\tau))$. Moreover,

$$\frac{\partial G(\tilde{\theta}, \tilde{w})}{\partial \theta} = \int_{-\infty}^{\infty} \psi_2(\tau)\,\tilde{v}(\tau)\,d\tau, \tag{7.7}$$

$$\frac{\partial G(\tilde{\theta}, \tilde{w})}{\partial w} = \int_{-\infty}^{\infty} \psi_2(\tau)\,d\tau, \tag{7.8}$$

and here $(\psi_1(\tau), \psi_2(\tau))$ is the unique bounded solution, up to a scalar multiple of the adjoint system

$$\psi_1(\tau)' - df(\tilde{u}(\tau))\,\psi_2(\tau) = 0,$$
$$\psi_2(\tau)' + \psi_1(\tau) + \theta\psi_2(\tau) = 0.$$

Observe that if $X(\tau)$ is a fundamental matrix of a linear equation, $(X^{-1}(\tau))^*$ is a fundamental matrix of its formal adjoint equation. Let $\binom{\tilde{u}(\tau)'}{\tilde{v}(\tau)'}$ form one column of the fundamental matrix $X(\tau)$ of the linearization of system (7.3) and (7.4). We readily find that a column of $(X^{-1}(\tau))^*$ is $(\det X(\tau))^{-1}\binom{-\tilde{v}(\tau)'}{\tilde{u}(\tau)'}$. Since $\det X(\tau) = ce^{\theta\tau}$, without loss of generality, $c = 1$, we have $\psi_1(\tau) = -\tilde{v}(\tau)'\,e^{-\theta\tau}$ and $\psi_2(\tau) = \tilde{u}(\tau)'\,e^{-\theta\tau}$. Substituting into (7.7) and (7.8) we have

$$\frac{\partial G(\tilde{\theta}, \tilde{w})}{\partial \theta} = \int_{-\infty}^{\infty} [\tilde{v}(\tau)]^2,\, e^{-\theta\tau}\,d\tau > 0$$

$$\frac{\partial G(\tilde{\theta}, \tilde{w})}{\partial w} = \int_{-\infty}^{\infty} e^{-\theta\tau}\,\tilde{u}(\tau)'\,d\tau = \begin{cases} \int_{u_1}^{u_2} e^{-\theta\tau}\,du, & \text{case 1,} \\ \int_{u_2}^{u_1} e^{-\theta\tau}\,du, & \text{case 2.} \end{cases}$$

Case 1 (or case 2) occurs if $(\tilde{u}(\tau), \tilde{v}(\tau))$ connects $(u_1(w), 0)$ to $(u_2(w), 0)$ (or $(u_2(w), 0)$ to $(u_1(w), 0)$). Therefore $\partial G(\tilde{\theta}, \tilde{w})/\partial \omega > 0$ in case 1 and $\partial G(\tilde{\theta}, \tilde{w})/\partial w < 0$ in case 2.

Remark 7.3. According to the partial derivatives of $G(\tilde{\theta}, \tilde{w})$, $G(\theta, w) = 0$ has a C^∞ solution $w = w(\theta)$ locally, with $dw(\theta)/d\theta < 0$ in case 1, and $dw(\theta)/d\theta > 0$ in case 2. Based on this, Lemma 7.3 can be proved by homotopy continuation, starting from the easiest case $\theta = 0$. We shall not elaborate it here.

Let $\theta \in (0, \theta_0)$ be fixed, ε be a small parameter. The fast system (7.3)–(7.5), when $\varepsilon = 0$, has a heteroclinic solution connecting $(u_1(w), 0)$ to $(u_2(w), 0)$ if $w = w_1(\theta)$. It also has a heteroclinic solution connecting $(u_2(w), 0)$ to $(u_1(w), 0)$ if $w = w_2(\theta)$. Note that $w_2(\theta) > w_1(\theta)$.

Assume also that $\gamma > 0$ is small so that the point $(u_2(w_2), w_2)$ lies under the line $u = \gamma w$. We now consider the slow system of equations

$$\varepsilon \frac{du}{dt} = v,$$

$$\varepsilon \frac{dv}{dt} = \theta v - f(u) + w,$$

$$\frac{dw}{dt} = \theta^{-1}(u - \gamma w), \qquad t = \varepsilon\tau.$$

If $\varepsilon = 0$, the flow on the slow manifold $\mathscr{S} = \{v = 0, f(u) = w\}$ is governed by

$$v = 0,$$

$$w = f(u),$$

$$\frac{dw}{dt} = \theta^{-1}(u - \gamma w), \qquad t = \varepsilon\tau.$$

Clearly, the slow flow connects $(u_2(w_1), w_1)$ to $(u_2(w_2), w_2)$ and $(u_1(w_2), w_2)$ to $(u_1(w_1), w_1)$ along $\mathscr{S}$. Together with the fast flow which connects $(u_1(w_1), 0)$ to $(u_2(w_1), 0)$ (in the $u-v$ plane) and $(u_2(w_2), 0)$ to $(u_1(w_2), 0)$, we have a closed cycle. We shall verify that all the hypotheses of Theorem 7.1 are satisfied. The hyperbolicity of the two branches of slow manifold $\{(u, v, w) \mid \bar{w} < w < \bar{\bar{w}},\ v = 0,\ u = u_i(w),\ i = 1, 2\}$, i.e., (H_1), follows from the eigenvalues of the equilibria, as in (7.6).

Hypothesis (H_2) follows from $\partial G(\theta, w_i(\theta))/\partial w \neq 0$, $i = 1, 2$. The slow manifold $\mathscr{S}$ is one-dimensional and the flow on $\mathscr{S}$ passes $(u_i(w_j), 0, w_j)$, $i, j = 1, 2$, at nonzero speed, thus (H_3) is also satisfied. Finally, $(\mathrm{H}_{4,\mathrm{pd}})$ is trivially satisfied since $\mathscr{S}$ is one dimensional. Therefore Theorem 7.1 applies to this example. We have shown the following

THEOREM 7.4. *For each* $0<\theta<\theta_0$, *there is* $\varepsilon^*(\theta)>0$ *such that for* $0<\varepsilon<\varepsilon^*(\theta)$, *system* (7.3)–(7.5) *possesses a unique periodic solution near the closed cycle formed by two pieces of heteroclinic orbits and two pieces of orbits of the slow flow as indicated in Fig.* 3. *The computation of asymptotic expansions and the exact solution can be achieved by using Theorem* 7.1, *with the error estimates also given in Theorem* 7.1.

The results in Theorem 7.4 are known and have been proven by the topological method (Carpenter [1]) and asymptotic method (Casten *et al.* [2]). The purpose of presenting this example is to illustrate how our method can be effectively applied to practical problems. Other types of traveling waves in the FitzHugh–Nagumo equation as well as the Hodgkin–Huxley equation can be treated as various boundary value problems in the same spirit.

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