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Twisted and Nontwisted Bifurcations Induced by Diffusion

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We discuss a diffusively perturbed predator–prey system. Freedman and Wolkowicz showed that the corresponding ODE can have a periodic solution that bifurcates from a homoclinic loop. When the diffusion coefficients are large, this solution represents a stable, spatially homogeneous time-periodic solution of the PDE. We show that when the diffusion coefficients become small, the spatially homogeneous periodic solution becomes unstable and bifurcates into spatially nonhomogeneous periodic solutions. The nature of the bifurcation is determined by the twistedness of an equilibrium/homoclinic bifurcation that occurs as the diffusion coefficients decrease. In the nontwisted case two spatially nonhomogeneous simple periodic solutions of equal period are generated, while in the twisted case a unique spatially nonhomogeneous double periodic solution is generated through period-doubling.

KEY WORDS: Reaction-diffusion equations; predator–prey systems; homoclinic bifurcations; periodic solutions.

1. INTRODUCTION

Suppose that the ODE system

$$U' = F(U, k), \qquad U \in \mathbb{R}^2, \qquad k \in \mathbb{R}, \qquad F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2) \tag{1.1}$$

has a homoclinic solution $U = q(t)$ when the parameter $k = k_\infty$. Assume also that for $k_\infty - \varepsilon < k < k_\infty$, there is a stable periodic solution $U = p(t, k)$ bifurcating from $q(t)$. We study the diffusively perturbed system

$$\begin{aligned} &U_t = DU_{\xi\xi} + F(U, k), \qquad 0 < \xi < 1 \\ &U_\xi(t, 0) = U_\xi(t, 1) = 0 \end{aligned} \tag{1.2}$$

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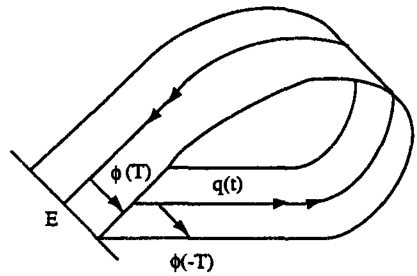
where $D = \text{diag}\{d_1, d_2\}$ is a positive diagonal matrix. The boundary conditions ensure that $U(t, \xi) = q(t)$ or $U(t, \xi) = p(t, k)$ is still a solution for system (1.2). Results from Refs. 8 and 13 indicate that when the diffusion coefficients are large, these spatially homogeneous (SH) solutions are stable. However, when the diffusion coefficients become small, SH solutions may lose stability and bifurcate into spatially nonhomogeneous (SN) solutions.

Such a bifurcation can create spatially nonhomogeneous patterns. Existing literature on pattern generation concentrates on small patterns generated through bifurcations of equilibria, or traveling waves constructed using transition layers [7, 25]. The mechanism of pattern generation studied in this paper is fundamentally different.

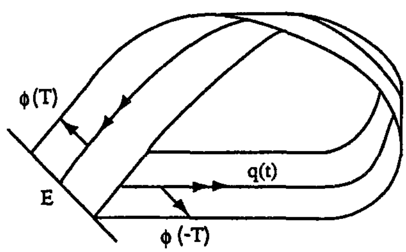
Since many ODE models are approximations to more realistic models where diffusion is present, examples that lead to systems (1.1) and (1.2) are plentiful. Freedman and Wolkowicz [12, 26] studied a two-species predator–prey system that models the group defense of prey against predation. They found a homoclinic solution $q(t)$ at a certain parameter value $k = k_\infty$. The homoclinic solution bifurcates into a long period solution $p(t, k)$ when $k_\infty - \varepsilon < k < k_\infty$. Suppose now diffusion is added to the system as in (1.2). The region of (d_1, d_2) where $p(t, k)$ is stable has been studied in Ref. 18, but the bifurcation when parameters cross the boundary Γ of the region has not been discussed. The purpose of the present paper is to discuss the bifurcation of $p(t, k)$ when (d_1, d_2) crosses Γ. The bifurcation of $q(t)$ into SN homoclinic solutions will be presented as the limit when the period is infinity.

The creation of SN periodic solutions is caused jointly by the homoclinic bifurcation in (1.1) and an equilibrium bifurcation in PDE modes in (1.2). Let $k = k_\infty$ such that (1.1) possesses a homoclinic solution $q(t)$ asymptotic to a hyperbolic equilibrium E. It is easy to find a curve Γ in the (d_1, d_2)-plane where the linearization of (1.2) at E has a simple zero eigenvalue and no other eigenvalue on the imaginary axis. When (d_1, d_2) crosses Γ transversely, the equilibrium E of (1.2) loses the hyperbolicity and two SN equilibria bifurcate from it. To describe bifurcations of $q(t)$ and $p(t, k)$ when (d_1, d_2) crosses Γ, we need the concept of the twistedness of the homoclinic solution $q(t)$. Let ϕ_c be a unit eigenvector corresponding to the zero eigenvalue, unique up to the multiplication by -1. It can be shown that the linearization of (1.2) around $q(t)$ has a solution $\phi(t)$ that approaches ϕ_c as $t \to -\infty$ and approaches $c^* \phi_c$ as $t \to +\infty$. Here c^* is a scalar function of (d_1, d_2). The limit of the solution $\phi(t)$ as $t \to -\infty$ is in fact a tangent vector to $W^{cu}_{loc}(E)$, transverse to the unstable eigenvector. See Fig. 1. We say that the homoclinic solution of (1.2) is twisted if $c^* < 0$, nontwisted if $c^* > 0$ and degenerate if $c^* = 0$.



A Nontwisted Homoclinic Orbit

T is a large constant.



A Twisted Homoclinic Orbit

Fig. 1. Twistedness of the homoclinic orbit is determined by comparing $\phi(-T)$ and $\phi(T)$.

All three cases have been found in Freedman and Wolkowicz's example by the numerical computation of c^*. See Fig. 10. An equivalent definition of twistedness will be given in Section 3. I have recently found that the change of twistedness is a generic phenomenon when diffusions are added to an ODE system that possesses a stable homoclinic orbit. But the proof will require a separate paper.

The bifurcation of $p(t, k)$ is determined by the twistedness of the homoclinic solution $q(t)$ at the point where (d_1, d_2) crosses Γ. Roughly speaking, two SN simple periodic solutions of equal periods are generated in the nontwisted case, while in the twisted case a unique SN symmetric double periodic solution is generated through period-doubling. A periodic solution $U(t, \xi)$ is said to be a simple periodic solution if its trajectory in a function space stays-near the orbit of $q(t)$ and hits a cross section Σ to the orbit of $q(t)$ precisely once. It is said to be a symmetric double periodic solution if it hits Σ precisely two times and satisfies a symmetry condition $U(t+T, \xi) = U(t, 1-\xi)$. Here $2T$ is the period of $U(t, \xi)$. Finally, when $q(t)$ is degenerate, it is possible to cross Γ in a way that no SN simple or symmetric double periodic solutions are generated. We do not discuss the existence of other types of SN solutions in this paper due to technical complications. The homoclinic twist bifurcation at a hyperbolic equilibrium was discovered in Ref. 27 and later studied in Refs. 3 and 15. But the homoclinic twist bifurcation discussed in this paper is new even in the ODE context.

In a separate paper [17], we show how our method can be used to prove the stability of the SN periodic solutions.

System (1.2) will be studied in the intermediate spaces $D_A(\theta)$ and $D_A(\theta+1)$, $0<\theta<1$. These function spaces allow solutions of (1.2) to have so-called maximal regularity and are normally used to study fully nonlinear parabolic equations [9]. Our system is not fully nonlinear, but to prove the smooth dependence of the solutions on d_1 and d_2, which are the coefficients of the highest derivatives in the equations, we need to use the maximal regularity of the solutions.

Solutions of (1.2) satisfy a reflection symmetry about the midpoint of the domain [0, 1], due to the special boundary conditions imposed there. For a function $U(\xi)$ defined on $\xi \in [0, 1]$, let $(RU)(\xi) = U(1-\xi)$, $0 \leqslant \xi \leqslant 1$. It can be verified if $U(t, \xi)$ is a solution to (1.2), so is $RU(t, \xi)$. Consequently, if U_1 is a SN periodic solution and is mutually disjoint with RU_1, then we have a pair of SN periodic solutions related by the symmetry. On the other hand, if U_1 has a nonempty intersection with RU_1, then U_1 is a $2T$ period SN solution satisfying $U(t+T, \xi) = RU(t, \xi)$. The R symmetry is very important in this paper since we can show that local center manifolds and flows on them respect the symmetry group R. We can also show that bifurcation functions derived by Lyapunov–Schmidt procedures are invariant with respect to R. A mapping f: $C[0, 1] \to C[0, 1]$ is invariant with respect to R if $f(RU) = Rf(U)$.

Suppose now the Neumann boundary conditions in (1.2) are replaced by periodic boundary conditions. In addition to the reflection symmetry, there is also a rotation symmetry, i.e., $U(t, \xi+\theta)$ is a solution if $U(t, \xi)$ is a solution. The bifurcation picture is quite different. Spatially nonhomogeneous tori may be generated instead of periodic solutions. See Ref. 17. Periodic boundary conditions will not be pursued further in this paper.

We introduce intermediate spaces $D_A(\theta)$ and $D_A(\theta+1)$ in Section 2. We then study invariant manifolds and their foliations in these spaces. These invariant manifolds and their foliations provide convenient coordinates to study dynamics of (1.2) near an equilibrium E. Some important lemmas regarding the symmetry R are also presented there.

The assumptions and the main results of this paper are given in Section 3.

In Section 4 we prove some lemmas needed in the sequel. In Section 5, we use a Lyapunov–Schmidt-type reduction to obtain a one-dimensional bifurcation equation whose solutions correspond to simple or symmetric double SN periodic solutions. Proofs of the main theorems are given in Section 6. In Section 7 we summarize our numerical results about the example from Ref. 12.

Recently Sandstede [23] has constructed center manifolds around some homoclinic solutions. It is hoped that a center manifold that is tangent to $q'(t)$ and $\phi(t)$ can be constructed some day. And it may bifurcate into a center manifold around the orbit of $p(t, k)$. Thus the twistedness of the homoclinic orbit should be passed to the twistedness of the center manifold of the periodic orbit. And the bifurcation of the periodic solutions should be determined by a one-dimensional return map on this center manifold. Thus, we naturally expect to see the occurrence of a simple or symmetric double periodic solution on this center manifold. However, I was unable to use the center manifold technique in this paper due to technical complications. On the contrary, the bifurcation function approach in this paper is easy to use. The trade-off is that only simple periodic solutions and symmetric double periodic solutions are discussed in this paper. Complete understanding of the dynamics near the homoclinic orbit is still open, especially around the degenerate point $c^* = 0$.

2. ABSTRACT PARABOLIC EQUATIONS, INVARIANT MANIFOLDS, AND FOLIATIONS

The PDE system (1.2) is studied in the intermediate spaces $D_A(\theta)$ and $D_A(\theta+1)$. Let A be a densely defined sectorial operator that generates a C_0 analytic semigroup e^{At} in a Banach space $\mathscr{X}$. For each $0<\theta<1$, define Banach spaces

$$D_A(\theta) = \{x \in \mathscr{X} \mid \lim_{t \to 0} t^{1-\theta} A e^{At} x = 0\}$$

$$D_A(\theta+1) = \{x \in D_A \mid Ax \in D_A(\theta)\}$$

The norms are

$$\|x\|_\theta = \sup_{0<t\leqslant 1} \|t^{1-\theta} A e^{At} x\|_{\mathscr{X}} + \|x\|_{\mathscr{X}}$$

$$\|x\|_{\theta+1} = \|Ax\|_\theta + \|x\|_{\mathscr{X}}$$

Intermediate spaces $D_A(\theta+m)$, $0<\theta<1$, $m \in \mathbb{N}^+$ can be defined similarly. Throughout this paper, let $\mathscr{X} = [L^2(0,1)]^2$, $A = \begin{pmatrix}\partial_{\xi\xi} & \\ & \partial_{\xi\xi}\end{pmatrix}$ and $D_A = \{u \in [H^2(0,1)]^2, u_\xi(0) = u_\xi(1) = 0\}$. See Ref. 9 for details about the intermediate spaces.

Denote $\mathscr{F}(U, d_1, d_2, k) = DU_{\xi\xi} + F(U, k)$. We can write (1.2) as an abstract nonlinear parabolic equation,

$$U' = \mathscr{F}(U, \mu) \tag{2.1}$$

where $\mu = (d_1, d_2, k)$ is a parameter. The solution for (2.1) with $U(0) = U_0$ will be denoted $U_*(t, U_0)$.

It is easy to see that $\mathscr{F}: D_A(\theta+m+1) \times \mathbb{R}^3 \to D_A(\theta+m)$, $m \geqslant 0$, is C^∞. Also, $F: D_A \to \mathscr{X}$ is C^∞ since F is C^∞. The following existence theorem is from Ref. 9.

Theorem 2.1. *For each $U_0 \in D_A(\theta+1)$, there exists $\tau > 0$ so that (2.1) admits a unique solution $U \in C^1([0,\tau]; D_A(\theta)) \cap C^0([0,\tau]; D_A(\theta+1))$. Moreover, U is a C^∞ function of U_0 and μ in the specified function spaces.*

Consider a time-dependent linear system

$$\begin{aligned} u' &= A(t)\, u + f(t) \\ u(s) &= x, \qquad a \leqslant s \leqslant t \leqslant b \end{aligned} \tag{2.2}$$

which comes from linearizing (2.1) around a particular solution. It is easy to verify that

(1) for all $t \in [a,b]$, $A(t): D_{A(t)} \to \mathscr{X}$ is sectorial, $D_{A(t)} = D_A$ with equivalent norms;

(2) for each $0<\theta<1$,

$$D_{A(t)}(\theta+1) = D_A(\theta+1) \qquad \text{for all} \quad t \in [a,b]$$

with equivalent norms;

(3) $A(\cdot) \in C([a,b]; L(D_A, \mathscr{X})) \cap C([a,b]; L(D_A(\theta+1), D_A(\theta)))$.

The following theorem is proved in Ref. 1.

Theorem 2.2. *Under the above conditions, there is a unique solution*

$$U \in C([s,b]; D_A(\theta+1)) \cap C^1([s,b]; D_A(\theta))$$

to (2.2) for each $x \in D_A(\theta+1)$ and $f \in C([s,b]; D_A(\theta))$. Denote the solution by $U(t) = T(t,s)\,x$ when $f=0$. Then $T(t,s)$ extends to $D_A(\theta)$ by continuity.

Finally, the variation of constants formula holds for solutions of (2.2) with $f \neq 0$:

$$U(t) = T(t, s)\, x + \int_s^t T(t, \xi)\, f(\xi)\, d\xi$$

Using the variation of constants formula, many familiar results of ODE systems can be extended to (2.1), with almost-identical proofs. The most useful ones in this paper are the smoothness of invariant manifolds and their foliations.

After a shift of coordinates, assume that $\{0\} \in D_A(\theta + 1)$ is an equilibrium of (2.1). Let $\tilde{A} = D_U \mathscr{F}(0, \mu_0)$ where $D_{\tilde{A}} = D_A$ and $\mu_0 = (d_{10}, d_{20}, k_\infty)$. Here $(d_{10}, d_{20}) \in \Gamma$ so that zero is an eigenvalue for $\tilde{A}$. Let $\mathscr{N} = \mathscr{F} - \tilde{A}U$. System (2.1) can be written as

$$U' = \tilde{A}U + \mathscr{N}(U, \mu)$$

Let

$$\sigma(\tilde{A}) = \sigma_- \cup \sigma_0 \cup \sigma_+$$

$$\operatorname{Re} \sigma_- \leqslant -\lambda_M$$

$$\operatorname{Re} \sigma_+ \geqslant \lambda_M$$

$$\operatorname{Re} \sigma_0 = 0$$

for some $\lambda_M > 0$. Let X, Y, and Z be the invariant subspaces corresponding to the spectral set σ_+, σ_-, and σ_0, respectively. We will identify $D_A(\theta + 1)$ with $X \times Y \times Z$ by writing $U = (x, y, z)$ if $U = x + y + z$. Since $X, Y, Z \subset D_A(\theta + 1)$, Rx, Ry, and Rz are defined by restricting R to subsets of $D_A(\theta + 1)$. It is also easy to verify that $RX = X$, $RY = Y$, and $RZ = Z$.

Since both $\mathscr{N}: D_A(\theta + 1) \times \mathbb{R}^3 \to D_A(\theta)$ and $\mathscr{N}: D_A \times \mathbb{R}^3 \to \mathscr{X}$ are C^∞, there exists a local center manifold that is C^ν for any $\nu > 0$ [6, 21]. Using the method of Ref. 5, which treats semilinear parabolic equations, we can prove that the center unstable and center stable manifolds are C^ν, and there exists C^ν invariant foliation of center unstable (center stable) manifolds by unstable (stable) fibers, if Lip $\mathscr{N}$ is small. The smallness of Lip $\mathscr{N}$ can be removed by modifying the equation outside a neighborhood of $\{0\}$, if we are interested only in local invariant manifolds and their foliations. In this following we show how to choose the modifier so that the reflection symmetry resulting from the Neumann boundary conditions will be preserved for the induced flow on the local center manifold.

First, we give coordinate-free definitions of global center unstable, center stable manifolds and their foliations. See Ref. 5 for similar definitions given to semilinear systems.

Definition 2.3. Let $0<\lambda_1<\lambda_2<\lambda_M$. The global center stable manifold is defined by

$$W^{cs}=\{U_0\in D_A(\theta+1)\mid U_*(t,U_0) \text{ exists for all } t\geqslant 0 \text{ and } \|U_*(t,U_0)\|_{\theta+1}\leqslant Ce^{t\lambda_1},\ t\geqslant 0\}$$

The global center unstable manifold is defined by

$$W^{cu}=\{U_0\in D_A(\theta+1)\mid U_*(t,U_0) \text{ exists for all } t\leqslant 0 \text{ and } \|U_*(t,U_0)\|_{\theta+1}\leqslant Ce^{-t\lambda_1},\ t\leqslant 0\}$$

Define the global center manifold by

$$W^c=W^{cu}\cap W^{cs}$$

For each $U_0\in W^{cs}$ (or W^{cu}), the stable fiber $W^s(U_0)$ [or unstable fiber $W^u(U_0)$] passing through U_0 is defined by

$$W^s(U_0)=\{V_0\in W^{cs}(0)\mid \|U_*(t,U_0)-U_*(t,V_0)\|_{\theta+1}\leqslant Ce^{-t\lambda_2},\ t\geqslant 0\}$$
$$W^u(U_0)=\{V_0\in W^{cu}(0)\mid \|U_*(t,U_0)-U_*(t,V_0)\|_{\theta+1}\leqslant Ce^{t\lambda_2},\ t\leqslant 0\}$$

Obviously, W^{cs} is forward invariant and W^{cu} is backward invariant. W^c is invariant. Also, each point on W^{cs} (or W^{cu}) belongs to one and only one stable (or unstable) fiber. The global foliations are forward or backward invariant in the sense that

$$U_*(t,W^s(U_0))\subset W^s(U_*(t,U_0)),\qquad t\geqslant 0$$
$$U_*(t,W^u(U_0))\subset W^u(U_*(t,U_0)),\qquad t\leqslant 0$$

Let $\mathcal{O}\subset D_A(\theta+1)$ be an open set containing the equilibrium $\{0\}$. Let $\tilde{\mathscr{F}}: D_A(\theta+1)\times\mathbb{R}^3\to D_A(\theta)$ and $\tilde{\mathscr{F}}: D_A\times\mathbb{R}^3\to\mathscr{X}$ be C^ν, $\nu>0$. Assume that $\tilde{\mathscr{F}}=\mathscr{F}$ in $\mathcal{O}\times\mathbb{R}^3$. Consider the system

$$U'=\tilde{\mathscr{F}}(U,\mu) \tag{2.3}$$

Definition 2.4. Assume that (2.3) has global invariant center stable and center unstable manifolds and invariant foliations as defined in Definition 2.3. Local invariant manifolds W^{cu}_{loc}, W^{cs}_{loc}, and W^c_{loc} for system (2.1) are the restrictions to $\mathcal{O}$ of the global invariant manifolds for system (2.3). Local invariant foliations of W^{cs}_{loc} and W^{cu}_{loc} for system (2.1) are the restrictions to $\mathcal{O}$ of the global invariant foliations of W^{cu} and W^{cs} for system (2.3).

Local invariant manifolds and local invariant foliations depend on the extension of $\mathscr{F}$ to $\tilde{\mathscr{F}}$ outside $\mathcal{O}$ and are thus not unique. Observe that Lip $\mathcal{N}$ is small inside $\mathcal{O}$ if the neighborhood $\mathcal{O}$ is small, due to the fact that $\mathcal{N}(0)=0$ and $\mathcal{N}'(0)=0$. The purpose of extending $\mathscr{F}$ to $\tilde{\mathscr{F}}$ is to have a small Lipschitz number for $\tilde{\mathcal{N}}=\tilde{\mathscr{F}}-\tilde{A}U$ outside $\mathcal{O}$.

Observe that $D_A(\theta+1)\subset[H^2(0,1)]^2$ is a continuous injection and $\|u\|_{H^2(0,1)}: H^2(0,1)\backslash\{0\}\to\mathbb{R}^+$ is C^ν for any $\nu>0$. Let $\psi: \mathbb{R}\to\mathbb{R}$ be C^∞ such that

$$\psi(s)=1 \quad \text{for } |s|\leqslant 1 \qquad \text{and} \qquad \psi(s)=0 \quad \text{for } |s|\geqslant 2, \qquad 0\leqslant\psi(s)\leqslant 1$$

Let $\tilde{\mathcal{N}}(U,\mu)=\mathcal{N}(\psi(|U|_{[H^2(0,1)]^2}/\rho)U,\mu)$, where $\rho>0$. It can be verified that

$$\tilde{\mathcal{N}}: [H^2(0,1)]^2\times\mathbb{R}^3\to[H^2(0,1)]^2$$

is C^ν, Lip $\tilde{\mathcal{N}}\to 0$ as $\rho\to 0$, and $\tilde{\mathcal{N}}=\mathcal{N}$ for $\|U\|_{[H^2(0,1)]^2}\leqslant\rho$. Recall that $D_A=\{U\in[H^2(0,1)]^2: \partial_x U=0 \text{ at } x=0,1\}$. After checking the boundary conditions, we find that both $\mathcal{N}, \tilde{\mathcal{N}}: D_A\to D_A$ are C^ν for any $\nu>0$. Since $D_A(\theta+1)\subset D_A\subset D_A(\theta)$, $\tilde{\mathcal{N}}: D_A(\theta+1)\to D_A(\theta)$ is C^ν with Lip $\tilde{\mathcal{N}}\to 0$ as $\rho\to 0$ in such space. We can prove the following theorem by using the method employed in Ref. 5.

Theorem 2.5. *For any $\nu>0$, there exists a small constant $\rho>0$ such that the global invariant manifolds for system (2.3) are C^ν embedded submanifolds in $D_A(\theta+1)$ if $|\mu-\mu_0|<\rho$. Moreover,*

$$W^{cs}=\{x=h_1(y,z,\mu)\}$$
$$W^{cu}=\{y=h_2(x,z,\mu)\}$$

where $(x,y,z)\in X\times Y\times Z$. The function h_i, $i=1,2$, is C^ν in all the variables, with $h_i(0,0,\mu_0)=0$, $Dh_i(0,0,\mu_0)=0$, and $Dh_i=O(\rho)$.

By a C^ν change of variable $(x,y,z)\to(x^1,y^1,z^1)$,

$$x^1=x-h_1(y,z,\mu)$$
$$y^1=y-h_2(x,z,\mu)$$
$$z^1=z$$

we can flatten these manifolds,

$$W^{cs} = \{x^1 = 0\}$$

$$W^{cu} = \{y^1 = 0\}$$

$$W^{c} = \{x^1 = 0, y^1 = 0\}$$

The change of variables preserves the symmetry, i.e., $U = (x, y, z) \to (x^1, y^1, z^1)$, *implies* $RU = (Rx, Ry, Rz) \to (Rx^1, Ry^1, Rz^1)$.

Proof. The existence and smoothness of such h_i, $i = 1, 2$, can be proved similar to Ref. 5. It can be verified that $\tilde{\mathcal{N}}(Rx, Ry, Rz, \mu) = R\tilde{\mathcal{N}}(x, y, z, \mu)$. Thus $RW = W$, where W stands for W^{cu}, W^{cs}, or W^{c}.

Let $U = (x, y, z) \in W^{cu}$. Then $RU \in W^{cu}$. Thus $h_2(Rx, Rz, \mu) = Ry = Rh_2(x, z, \mu)$. Similarly, $h_1(Ry, Rz, \mu) = Rh_1(y, z, \mu)$. It follows that $RU \to (Rx^1, Ry^1, Rz^1)$. □

We will use the new coordinates (x^1, y^1, z^1) to discuss invariant foliations for system (2.3). Let $U_0 \in W^c$. Then $U_0 = (0, 0, z_0)$ in the new coordinates. Denote $W^s(U_0)$ and $W^u(U_0)$ by $W^s(z_0)$ and $W^u(z_0)$. From Definition 2.3, we can show that different points on W^c do not belong to a same fiber $W^u(z_0)$ or $W^s(z_0)$. Furthermore, we also know that all fibers on W^{cu} (or W^{cs}) have to intersect W^c.

Theorem 2.6. *If* $\rho > 0$ *is small enough, then the stable fibers* $W^s(z_0)$, $z_0 \in W^c$ *form an invariant foliation of* W^{cs} *and the unstable fibers* $W^u(z_0)$, $z_0 \in W^c$ *form an invariant foliation of* W^{cu}, *for* $|\mu - \mu_0| < \rho$. *Moreover,*

$$W^s(z_0) = \{x^1 = 0, z^1 = z_0 + h_3(y^1, z_0, \mu) \text{ with } h_3(0, z_0, \mu) = 0\}$$

$$W^u(z_0) = \{y^1 = 0, z^1 = z_0 + h_4(x^1, z_0, \mu) \text{ with } h_4(0, z_0, \mu) = 0\}$$

The function h_i, $i = 3, 4$, *is* C^v *in all its variables,* $D_y h_3(0, 0, \mu_0) = 0$, $D_x h_4(0, 0, \mu_0) = 0$, *and* $Dh_i = O(\rho)$, $i = 3, 4$. *By a* C^v *change of variables* $(x^1, y^1, z^1) \to (x^2, y^2, z^2)$, *which is defined implicitly by*

$$x^1 = x^2$$

$$y^1 = y^2$$

$$z^1 = z^2 + h_3(y^2, z^2, \mu) + h_4(x^2, z^2, \mu)$$

we can flatten the fibers, so that

$$W^s(z_0) = \{x^2 = 0, z^2 = z_0\}$$

$$W^u(z_0) = \{y^2 = 0, z^2 = z_0\}$$

The change of variables preserves the symmetry i.e., if $(x^1, y^1, z^1) \to (x^2, y^2, z^2)$, *then* $(Rx^1, Ry^1, Rz^1) \to (Rx^2, Ry^2, Rz^2)$.

The change of variable here does not affect the flow on W^c.

Proof. The existence and smoothness of h_i, $i = 3, 4$, are proved similarly to that in Ref. 5.

(i) Since $W^s(Rz_0) = RW^s(z_0)$, and $W^u(Rz_0) = RW^u(z_0)$, we have $h_3(Ry^1, Rz_0, \mu) = Rh_3(y^1, z_0, \mu)$ and $h_4(Rx^1, Rz_0, \mu) = Rh_4(x^1, z_0, \mu)$. It follows that $(x^1, y^1, z^1) \to (x^2, y^2, z^2)$ implies that $(Rx^1, Ry^1, Rz^1) \to (Rx^2, Ry^2, Rz^2)$.

(ii) If $x^2 = 0$ and $y^2 = 0$, then $h_3(0, z^2, \mu) = 0$ and $h_4(0, z^2, \mu) = 0$. Therefore $z^1 = z^2$ on W^c. The equation for the flow on W^c is not changed. □

Define a function space $\mathscr{X}_n = \{(u_n \cos(n\pi \cdot), v_n \cos(n\pi \cdot)), (u_n, v_n) \in \mathbb{R}^2\}$. Obviously $\mathscr{X}_n$ is isomorphic to $\mathbb{R}^2$. Observe that $\sum \{\mathscr{X}_n, n \geq 0\}$ is dense in $[L^2(0, 1)]^2$.

Recall that (2.1) comes from (1.2). The hypotheses on F will be specified in Section 3. In particular, they imply that

(1) Z is one dimensional, spanned by an eigenvector in $\mathscr{X}_1$, corresponding to the eigenvalue $\lambda = 0$; and

(2) X is one dimensional, spanned by an eigenvector in $\mathscr{X}_0$, corresponding to the eigenvalue $\lambda = \lambda_+$.

We may identify Z and X with $\mathbb{R}$. More precisely, let w be a unit vector in Z. For any $z \in Z$, there is a unique $\bar{z} \in \mathbb{R}$ such that $z = \bar{z}w$. We will identify z with $\bar{z}$ and drop the over-bar. The same comment also applies to X. It can be verified that if $x \in X$ and $z \in Z$, then $Rx = x$ and $Rz = -z$. We use $U \sim (x^2, y^2, z^2)$ to indicate that U corresponds to (x^2, y^2, z^2) in the new coordinates.

Theorem 2.7. *(a) If* $U \in \mathscr{X}_0$ *and if* $U \sim (x^2, y^2, z^2)$ *in the new coordinates, then* $z^2 = 0$ *and* $y^2 \in \mathscr{X}_0$. *The converse is also true.*

(b) For system (2.3), the flow on W^c has the form

$$x^2 = 0, \qquad y^2 = 0$$

$$\frac{d}{dt} z^2 = g(z^2, \mu)$$

where $g(0, \mu) = 0$, $D_{z^2} g(0, \mu_0) = 0$ and $g(-z^2, \mu) = -g(z^2, \mu)$.

Proof. (a) In the original coordinates, $U = x + y + z$ with $x \in X$, $y \in Y$, and $z \in Z$. If $U \in \mathscr{X}_0$, then it is obvious that $z = 0, x \in \mathscr{X}_0$, and $y \in \mathscr{X}_0$.

We first examine the changes of variables $(x, y, z) \to (x^1, y^1, z^1)$ as in Theorem 2.5. Consider the change of variable $y^1 = y - h_2(x, z, \mu)$. When $z = 0$, the graph $\{z = 0, y = h_2(x, 0, \mu)\} = W^{cu} \cap \{z = 0\}$ is one-dimensional. Now let us restrict the system to $\mathscr{X}_0$, where $E = \{0\}$ is hyperbolic. By the standard existence theorem of the unstable manifold for the ODE system, there exists a smooth function $\tilde{h}$ such that $W^u = \{y = \tilde{h}(x, \mu)\}$ for the restricted system. Clearly the graph $\{y = \tilde{h}(x, \mu)\} \subseteq \{y = h_2(x, 0, \mu)\}$. Since they are both one dimensional, we have that $\tilde{h}(x, \mu) = h_2(x, 0, \mu)$. This proves that $h_2(x, 0, \mu) \in \mathscr{X}_0$. Recall that $z = z^1$. Thus if $U \in \mathscr{X}_0$, $z^1 = 0$, and $y^1 \in \mathscr{X}_0$, and vice versa.

We now consider the second change of variable $(x^1, y^1, z^1) \to (x^2, y^2, z^2)$ as in Theorem 2.6. Since $y^2 = y^1$, $y^2 \in \mathscr{X}_0 \Leftrightarrow y^1 \in \mathscr{X}_0$. If $(0, y^1, z^1)$ is a point on $W^s(z_0)$, then $(0, Ry^1, Rz_1)$ is a point on $W^s(Rz_0)$. From Theorem 2.6, compare the z^1 coordinates, and observe that $Rz_0 = -z_0$, we have $R(z_0 + h_3(y^1, z_0, \mu)) = -z_0 + h_3(Ry^1, -z_0, \mu)$. However, $Rh_3 = -h_3$. Thus $h_3(Ry^1, -z_0, \mu) = -h_3(y^1, z_0, \mu)$. Similarly, we can show that $h_4(Rx^1, -z_0, \mu) = -h_4(x^1, z_0, \mu)$. Therefore if $y^2 = y^1 \in \mathscr{X}_0$, we have $Ry^1 = y^1$, $h_3(y^2, 0, \mu) = 0$, and $h_4(x^2, 0, \mu) = 0$. In this case, we have $z^1 = 0 \Leftrightarrow z^2 = 0$.

By combining the two changes of variables, we have verified the assertions of (a).

(b) The assertions $x^2 = 0$ and $y^2 = 0$ are obvious. If $U(t) \sim (0, 0, z^2(t))$ is a solution on W^c, so is $RU(t) \sim (0, 0, -z^2(t))$. Therefore $g(-z^2, \mu) = -g(z^2, \mu)$. □

3. ASSUMPTIONS AND MAIN RESULTS

We assume that the ODE system (1.1) satisfies the following hypotheses.

(H_1) $F\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is C^∞.

(H_2) At $k=k_\infty$, (1.1) possesses a homoclinic solution $U=q(t)$ asymptotic to an equilibrium $E=E(k_\infty)$.

(H_3) At $E(k_\infty)$, the Jacobian matrix

$$J=\begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix}$$

satisfies $a+d>0$ and $ad-bc<0$.

Hypothesis H_3 implies that $E(k_\infty)$ is hyperbolic with eigenvalues denoted $-\lambda_- < 0 < \lambda_+$, satisfying $\lambda_+ - \lambda_- < 0$. The equilibrium $E=E(k)$ continues to exist for all $k \approx k_\infty$. We will suppress k if no confusion should arise. The homoclinic orbit is stable from inside since $\lambda_+ - \lambda_- < 0$. Assume that the homoclinic orbit breaks in a certain direction when k moves away from k_∞, so that periodic solutions bifurcate from $q(t)$ for $k_\infty - \varepsilon < k < k_\infty$. More precisely, consider the linear variational equation of (1.1) around $U=q(t)$,

$$U' = \partial_U F(q(t), k_\infty)\, U \tag{3.1}$$

and its adjoint equation

$$\Psi' = -[\partial_U F(q(t), k_\infty)]^*\, \Psi \tag{3.2}$$

System (3.2) has a unique nontrivial bounded solution $\Psi(t)$ up to multiplying by nonzero constants. It is known that $\Psi(t) \sim \Psi_0 e^{-\lambda_+ t}$ and $q(-t) - E(k_\infty) \sim \phi_0 e^{-\lambda_+ t}$ as $t \to +\infty$ where Ψ_0 (or ϕ_0) is the left (or right) eigenvector of the matrix J corresponding to the eigenvalue λ_+. See [14]. For definiteness, assume that

$$\lim_{t\to+\infty} \Psi(t)(q(-t)-E)\, e^{2t\lambda_+} = -1 \tag{3.3}$$

We now assume that the breaking of the homoclinic solution $q(t)$ is in the direction determined by

(H_4) $\int_{-\infty}^{\infty} \Psi(t)\cdot \partial_k F(q(t), k_\infty)\, dt > 0$

From Silnikov [24], (3.3) and H_4 imply that there exists $\varepsilon > 0$ so that for $k_\infty - \varepsilon < k < k_\infty$, system (1.1) has a simple periodic solution $p(t, k)$ which is orbitally near $q(t)$ and is asymptotically stable. A more transparent relation indicating that the periodic solutions can only be found for $k < k_\infty$ with $k - k_\infty = O(e^{-T\lambda_+})$ is given in Ref. 16, where T is the period of

$p(t, k)$. There is a one-to-one correspondence between k and T. Moreover, there exists $C > 1$, independent of k, such that

$$C^{-1} e^{-T\lambda_+} \leqslant \frac{dk}{dT} \leqslant C e^{-T\lambda_+}$$

The proof of that can be obtained by the same method used in Ref. 16.

Consider eigenvalues for the linear variational equation around the equilibrium $E(k_\infty)$. It can be verified that each eigenfunction must be in one $\mathscr{X}_n$, $n \geqslant 0$, with an eigenvalue λ_n satisfying

$$\det \begin{pmatrix} \lambda + a + n^2\pi^2 d_1 & b \\ c & \lambda + d + n^2\pi^2 d_2 \end{pmatrix} = 0$$

The spaces $\mathscr{X}_n$ are defined in Section 2. For each $\mathscr{X}_n$, denote the eigenvalues corresponding to the nth Fourier mode $(\lambda_{n1}, \lambda_{n2})$ with $\operatorname{Re} \lambda_{n1} \geqslant \operatorname{Re} \lambda_{n2}$. Based on $a + d > 0$, we have $\operatorname{Re} \lambda_{n2} < 0$. An nth mode is unstable if and only if $\operatorname{Re} \lambda_{n1} > 0$. The critical case $\lambda_{n1} = 0$ occurs if

$$(a + n^2\pi^2 d_1)(d + n^2\pi^2 d_2) = bc$$

We can show that when decreasing (d_1, d_2), the first mode loses stability before the other Fourier modes. (Theorem 3.1). Thus, we are interested in parameter values where $\lambda_{11} = 0$. Define

$$\Gamma = \{(d_1, d_2) : (a + \pi^2 d_1)(d + \pi^2 d_2) = bc\}$$

Theorem 3.1. *The first quadrant, $\mathbb{R}^2_+$, is divided by Γ into two regions: $\mathscr{G}_+$ and $\mathscr{G}_-$, where $(a + \pi^2 d_1)(d + \pi^2 d_2) - bc < 0$ and > 0, respectively.*

- *(i)* $\lambda_{11} > 0$ *in* $\mathscr{G}_+$. *If d_1 and d_2 are sufficiently small, then* $(d_1, d_2) \in \mathscr{G}_+$.
- *(ii)* $\operatorname{Re} \lambda_{11} < 0$ *in* $\mathscr{G}_-$. *The region $\mathscr{G}_-$ is unbounded.*
- *(iii)* $\lambda_{11} = 0$ *on* Γ.
- *(iv)* $\lambda_{01} = \lambda_+ > 0$ *in* $\mathbb{R}^2_+$. *If* $(d_1, d_2) \in \mathscr{G}_- \cup \Gamma$, *then* $\operatorname{Re} \lambda_{nj} < 0$ *for* $(n, j) \neq (0, 1)$ *or* $(1, 1)$.
- *(v)* $\nabla \lambda_{11} = (\partial_{d_1} \lambda_{11}, \partial_{d_2} \lambda_{11}) \neq 0$ *for* $(d_1, d_2) \in \Gamma$. *In particular,* $\partial_{d_1} \lambda_{11} < 0$ *if* $d + \pi^2 d_2 > 0$ *and* $\partial_{d_2} \lambda_{11} < 0$ *if* $a + \pi^2 d_1 > 0$.

Theorem 3.1 will be proved in Section 6. Figure 2 depicts Γ, $\mathscr{G}_+$, and $\mathscr{G}_-$ for all possible cases except for a possible permutation of d_1 and d_2.

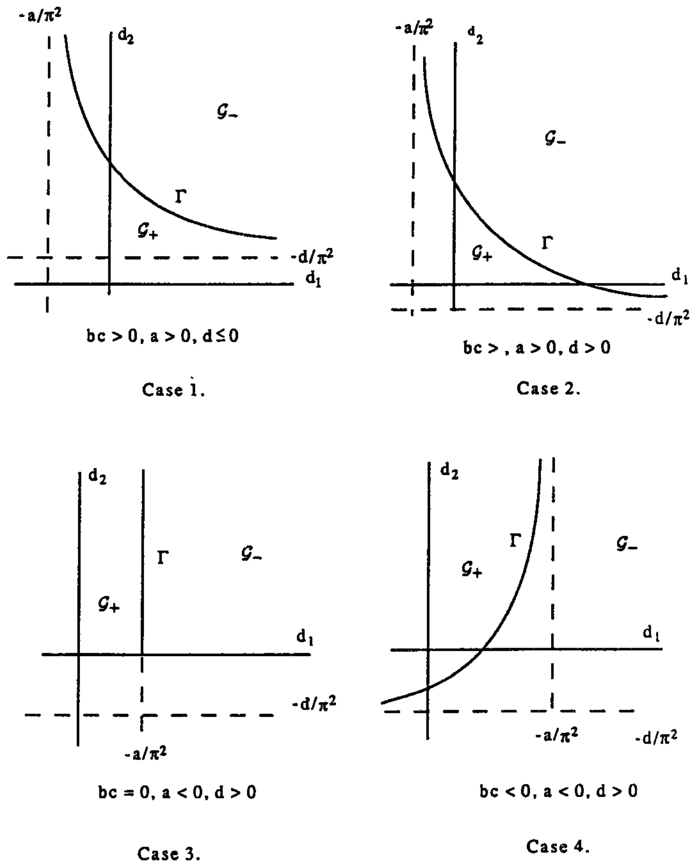


Fig. 2. The curve Γ divides the first quadrant of the (d_1, d_2) plane into two parts. All the possibilities are listed, except for permutations of d_1 and d_2.

When $bc > 0$, we may have $a > 0$, $d > 0$, which is Case 2 in Fig. 2. We may also have $a > 0$, $d \leqslant 0$, which is Case 1 in Fig. 2. It is impossible to have $a < 0$, $d < 0$, since $a + d > 0$. The other possible case is $a \leqslant 0$, $d > 0$, which is obtained from Case 1 by symmetry. When $bc = 0$, since $ad - bc < 0$, we have $ad < 0$. Thus we have either $a < 0$, $d > 0$, which is Case 3 in Fig. 2, or $a > 0$, $d < 0$, by symmetry. When $bc < 0$, again $ad - bc < 0$ implies $ad < 0$. Case $a < 0$, $d > 0$, is in Case 4, the other case $a > 0$, $d < 0$, is obtained by symmetry. Observe in Case 4, when increasing d_2, we can move from $\mathscr{G}_-$ to $\mathscr{G}_+$. It is interesting to note that the equilibrium may become more unstable by increasing one of the diffusion coefficient.

The following theorem was stated in Ref. 18.

Theorem 3.2. *For each positive $(d_1, d_2) \in \mathscr{G}_-$, there exists a smooth function $\varepsilon^*(d_1, d_2) > 0$ such that the SH periodic solution $p(t, k)$ is asymptotically stable in $D_A(\theta + 1)$ if $k_\infty - \varepsilon^* < k < k_\infty$.*

Theorem 3.2 can be proved by using notions of exponential dichotomies and roughness of exponential dichotomies in $D_A(\theta + 1)$. The proof is similar to the proof of Theorem 4.5 in Ref. 17. Since those methods are quite different from those used in this paper, we will not give details here.

The result in Theorem 3.2 is not very precise since $\varepsilon^*(d_1, d_2) \to 0$ as $(d_1, d_2) \to \Gamma$. For a given k (or period T), the loss of stability for $p(t, k)$ does not happen exactly at Γ.

To describe what happens near Γ, two new notions are introduced: (1) the stability of the equilibrium for the flow on W^c_{loc} and (2) the twistedness of the homoclinic orbit when following $q(t)$ from $t = -\infty$ to $t = +\infty$.

When $(d_1, d_2) \in \Gamma, \lambda_{11} = 0$, the equilibrium $E(k_\infty)$ has a one-dimensional center manifold W^c_{loc} that is tangent to the one-dimensional eigenspace corresponding to $\lambda_{11} = 0$. The flow on W^c_{loc} is described in Theorem 2.7. When $\lambda_{11} = 0$, it has the form

$$z' = -\hat{c} z^3 + \text{h.o.t.}$$

$$x = 0, \qquad y = 0$$

(H_5) When $\lambda_{11} = 0$, the equilibrium E is stable on $W^c_{\text{loc}}(E)$ in the sense that $\hat{c} > 0$.

Numerical computation in Section 7 shows that in Freedman and Wolkowicz's example, the condition $\hat{c} > 0$ is valid for all $(d_1, d_2) \in \Gamma$ in the range specified by $0 < \pi^2 d_1 < 3$.

Twistedness of the homoclinic solution $q(t)$ has been described in Section 1. Because of its importance, we will give a simple and equivalent definition. Let $k = k_\infty$ and $(d_1, d_2) \in \Gamma$. Linearizing (1.2) around $q(t)$, we have

$$U'(t) = D U_{\xi\xi}(t) + \partial_U F(q(t), k_\infty)\, U(t) \tag{3.4}$$

The subspace of the first Fourier mode $\mathscr{X}_1$ is invariant under (3.4). Since $\mathscr{X}_1$ is two dimensional, (3.4) on $\mathscr{X}_1$ reduces to an ODE on (u_1, v_1).

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \mathscr{A}(t) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \tag{3.5}$$

where $\mathscr{A}(t) \to \mathscr{A}(\infty) = \begin{pmatrix} -\pi^2 d_1 - a & -b \\ -c & -\pi^2 d_2 - d \end{pmatrix}$ as $t \to \pm\infty$. According to a theorem in Ref. 14, each solution of (3.5) approaches a solution of the linear autonomous equation,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \mathscr{A}(\infty)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

with an exponentially small error. Therefore, there is a unique solution $(u_1(t), v_1(t))$ to (3.5), up to multiplying by scalar constants, that approaches an eigenvector of the zero eigenvalue of $\mathscr{A}(\infty)$ as $t \to -\infty$. Let $\tilde{U}(t) = (u_1(t) \cos \pi x, v_1(t) \cos \pi x)$ be the unique solution of (3.4) that is in $\mathscr{X}_1$ and approaches an eigenvector ϕ_c, corresponding to $\lambda_{11} = 0$ as $t \to -\infty$. By the same argument, when $t \to +\infty$, $\tilde{U}(t)$ approaches another eigenvector associated to $\lambda_{11} = 0$, denoted $c^*\phi_c$, where c^* is a function of $(d_1, d_2) \in \Gamma$.

Definition 3.3. Let $\lim_{t \to +\infty} \tilde{U}(t) = c^*\phi_c$. The homoclinic solution $q(t)$ is said to be nontwisted if $c^* > 0$, twisted if $c^* < 0$, or degenerate if $c^* = 0$.

Remark. In $D_A(\theta + 1)$, solutions of (3.4) that approach ϕ_c as $t \to -\infty$ are not unique. They have the form $U(t) = \tilde{U}(t) + C\dot{q}(t)$, where C is an arbitrary constant. Since $\dot{q}(t) \to 0$ exponentially as $t \to +\infty$, we have $\lim_{t \to \infty} U(t) = c^*\phi_c$ for any $C \in \mathbb{R}$. Therefore the twistedness defined in Definition 3.3 is precisely the one given in Section 1.

Let $k = k_\infty$ and $(d_1, d_2) \in \Gamma$. From Theorem 3.1, $\nabla\lambda_{11} \neq 0$. It is also obvious that $\nabla\lambda_{11}$ intersects Γ transversely at (d_1, d_2). We can make a smooth change of variable $\mathscr{B}$: $(d_1, d_2) \to (l, m)$ in a neighborhood of Γ so that $m = \lambda_{11}$ and l is the arc length on Γ when $m = 0$, after assigning $l = 0$ to an arbitrary point on Γ. The new coordinates flatten Γ, i.e., $\Gamma = \{m = 0, l \in \tilde{I}\}$, where $\tilde{I} \subset \mathbb{R}$ is an open interval.

For $m \approx 0$ and $l \in \tilde{I}$, we look for simple period T or symmetric double period $2T$ SN solutions, where $T > \bar{t}$, $\bar{t}$ being a large constant. In the parameter space (T, l, m) we want to find regions where such SN solutions exist.

For $(d_1, d_2) \in \Gamma$, $\mathscr{B}(d_1, d_2) = (l_0, 0)$, and the twistedness $c^* = c^*(l_0)$ is a function of l_0.

Throughout this paper, assume that the hypotheses H_1–H_5 are satisfied.

Theorem 3.4. *For each $l_0 \in \tilde{I}$, $c^*(l_0) \neq 0$, there exist a large constant $\bar{t} > 0$ and an open set $\mathcal{O} \subset \mathbb{R}^2$ containing $(l_0, 0)$, the size of which depends* *on l_0, such that two C^1 functions $L(T, l, m)$ and $r(T, l, m)$ can be defined for $T > \bar{t}$ and $(l, m) \in \mathcal{O}$ with the following properties:*

(1) $r(\infty, l_0, 0) = c^*(l_0)$, *where* $r(\infty, l_0, 0) = \lim_{t \to +\infty} r(t, l_0, 0)$;

(2) $L(T, l, m) = e^{mT} + O(e^{-\alpha T})$, $0 < m < \alpha$;

(3) $\partial/\partial m\{L(T, l, m)\, r(T, l, m)\} > 0$ *(or* < 0*) when* $c^*(l_0) > 0$ *(or* < 0*).*

Moreover, the existence and uniqueness of simple or symmetric double SN periodic solutions to (2.1) are determined by the following conditions:

(i) If $0 < c^*(l_0) < 1$ *or* $1 < c^*(l_0)$, *then there is no simple period T SN solution when* $0 \leqslant L(T, l, m)\, r(T, l, m) \leqslant 1$; *there are precisely two simple period T SN solutions* $U_1(t, \xi)$ *and* $U_2(t, \xi)$ *when* $L(T, l, m)\, r(T, l, m) > 1$. *The two solutions are related by* $U_2(t, \xi) = U_1(t, 1 - \xi)$.

(ii) If $c^*(l_0) = 1$, *then there exist two simple period T SN solutions when* $L(T, l, m)\, r(T, l, m) > 1$. *There exists* $\delta > 0$ *such that the number of solutions is precisely two when* $L(T, l, m)\, r(T, l, m) \geqslant 1 + \delta$ *for some* $\delta > 0$; *and there is no simple period T SN solution when* $0 \leqslant L(T, l, m)\, r(T, l, m) \leqslant 1 - \delta$.

(iii) If $-1 < c^*(l_0) < 0$ *or* $c^*(l_0) < -1$, *then there is precisely one SN symmetric double period 2T solution* $U(t, \xi)$ *when* $L(T, l, m)\, r(T, l, m) < -1$. *There is no such SN period 2T solution when* $-1 \leqslant L(T, l, m)\, r(T, l, m) \leqslant 0$.

(iv) If $c^*(l_0) = -1$, *then there is at least one symmetric double period 2T SN solution when* $L(T, l, m)\, r(T, l, m) < -1$. *Such a solution is unique when* $L(T, l, m)\, r(T, l, m) < -1 - \delta$ *for some* $\delta > 0$. *There is no such SN period 2T solution when* $-1 + \delta \leqslant L(T, l, m)\, r(T, l, m) \leqslant 0$.

Corollary. *When* $m > 0$, *there is a pair of SN equilibria* E_1, E_2 *bifurcating from E. The results above also show the bifurcation of SN homoclinic or heteroclinic solutions asymptotic to* E_1 *and/or* E_2 *as a special case when* $T = \infty$. *If* $c^* \neq 0$, *the limit of the curve* $Lr = 1$ *is identical to* Γ. *When crossing* Γ *at a point where* $c^*(l_0) > 0$, *the bifurcation of a pair of homoclinic solutions, each asmptotic to* E_1 *or* E_2 *occurs. When crossing* Γ *at a point where* $c^*(l_0) < 0$, *the bifurcation of a pair of heteroclinic solutions connecting* E_1 *and* E_2 *occurs.*

Theorem 3.5. *For each* $l_0 \in \bar{I}$ *with* $c^*(l_0) = 0$ *and* $(d/dl)\, c^*(l_0) \neq 0$, *there exist constants* $\varepsilon > 0$ *and* $\bar{t} > 0$ *such that functions* $l^*(m)$, $|m| < \varepsilon$ *and* $\delta(T) = ce^{-mT}$, $T > \bar{t}$, *for some* $c > 0$ *can be defined. If* $|l - l^*(m)| < \delta(T)$, $|m| < \varepsilon$, *and* $T > \bar{t}$, *then there is no simple period T or symmetric double* *period 2T SN solution to (2.1), inside a* $(\delta(T))^{1/2}$ *neighborhood of the orbit of* $q(t)$.

Theorem 3.4 provides fairly accurate information about the bifurcation to simple or symmetric double periodic SN solutions when crossing the curve $L(T, l, m)\, r(T, l, m) = 1$ not near the points $c^*(l_0) = \pm 1$ or $c^*(l_0) = 0$. First, Theorem 3.4 (3) assures that $L(T, l, m)\, r(T, l, m)$ is monotonic in term of m. From the asymptotic forms (1) and (2), it is also clear that the sign of $Lr - 1$ changes when m is increased form negative to positive, provided that T is large. When crossing the curve $Lr = 1$ near $c^*(l_0) = \pm 1$, bifurcation to a simple or symmetric double periodic SN solution will occur but the precise moment is unknown. Our method does not predict the existence or uniqueness of such solutions in a narrow strip around $Lr = 1$. Theorem 3.5, on the other hand, assures that when $c^*(l_0) = 0$, we can pass $m = 0$ through a small tubular neighborhood of $l = l^*(m)$ without creating any simple period T or symmetric double period $2T$ *SN* solution. The size of the tubular neighborhood shrinks to zero as $T \to +\infty$.

The regions in the (d_1, d_2) plane mentioned in Theorem 3.4 and Theorem 3.5 are depicted in Fig. 3, where we assume that $L(T, l, m) = e^{mT}$, $r(T, l, m) = c^*(l)$, and $l^*(m) = 0$. In the shaded area, the existence and uniqueness of a simple period T (or symmetric double period $2T$) SN solution are guaranteed except near $c^*(l) = \pm 1$. The tubular neighborhood near $l = 0$ where crossing $m = 0$ without causing bifurcation to a simple or symmetric double period SN solution is also shown. A sketch of all kinds of homoclinic, heteroclinic, and periodic solutions is in Fig. 4.

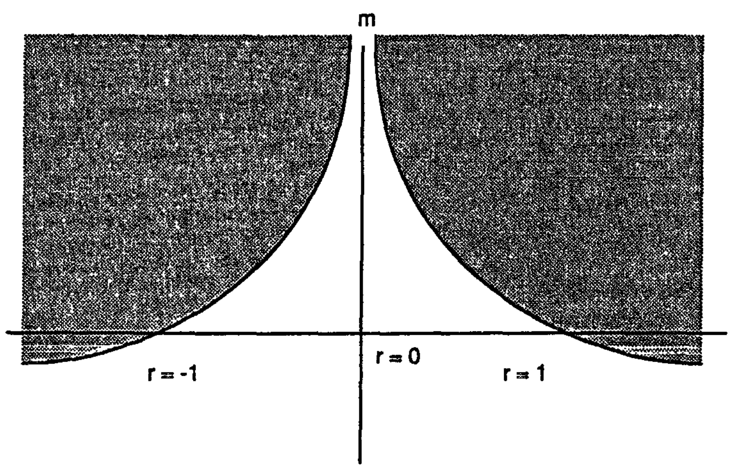


Fig. 3. A sketch of the bifurcation diagram in the (l, m) plane. SN simple or symmetric double periodic solutions occur in the shaded areas.

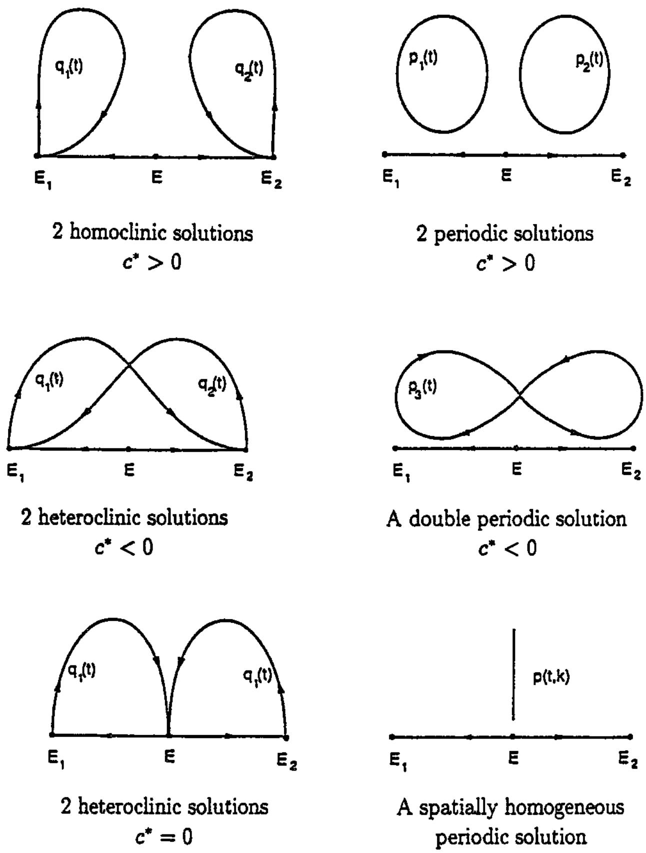


Fig. 4. A sketch of all kinds of homoclinic, heteroclinic, and periodic solutions.

4. SOME LEMMAS

The result in Lemma 4.1 is our major tool to study a solution $U(t)$, $0 \leqslant t \leqslant t_0$, that stays in a small neighborhood of a nonhyperbolic equilibrium. Following an idea of Silnikov, we show that if t_0 can be arbitrarily large, $U(t) = (x(t), y(t), z(t))$, $0 \leqslant t \leqslant t_0$, is determined by, and depends continuously on, its boundary values: $y(0)$, $z(0)$, and $x(t_0)$. Using exponential dichotomies we can easily show that $x(t) = O(e^{-\alpha(t_0 - t)})$ and $y(t) = O(e^{-\alpha t})$ for some $\alpha > 0$. However, in the center direction, the flow is not exponentially decaying either moving forward or backward. Following the approach of Ref. 4, we will compare the Z coordinates of $U(t)$ with a (nonunique) solution $U_0(t)$ on $W^c_{\mathrm{loc}}(E)$. Let P_x, P_y, P_z be the spectral projections from $D_A(\theta+1)$ onto X, Y, Z. In the flat coordinates, we show that $\|P_z(U(t)-U_0(t))\|$ is small and approaches zero uniformly for $t\in[0,t_0]$ as $t_0\to+\infty$. If we are interested only in dynamics in the Z direction, $U(t)$ can be replaced by $U_0(t)$ on $W^c_{\text{loc}}(E)$ with a very small error.

It is also clear that the smallness of $P_z(U_0(t)-U(t))$ strongly depends on a good choice of coordinates. Since $x(t_0)$ and $y(0)$ are not small as $t_0\to\infty$, an undesired change of variables may destroy the smallness of $P_z(U_0(t)-U(t))$.

Let $\mathcal{O}\subset\mathcal{D}_{\mathscr{A}}(\theta+\infty)$ be a small neighborhood of an equilibrium $U=0$ where the flat coordinates introduced in Section 2 are used in $\mathcal{O}$. We now consider the abstract parabolic equation (2.1) written in the flat coordinates,

$$\begin{aligned} x' &= A_1x+g_1(x,y,z,\mu)\\ y' &= A_2y+g_2(x,y,z,\mu)\\ z' &= A_3z+g_3(x,y,z,\mu)\end{aligned}\tag{4.1}$$

Here $A_1=A|_X$, $A_2=A|_Y$, and $A_3=A|_Z$. $\operatorname{Re}\sigma(A_1)>\lambda_M>0$, $\operatorname{Re}\sigma(A_2)<-\lambda_M<0$, and $\operatorname{Re}\sigma(A_3)=0$. The functions g_i, $i=1,2,3$, are C^ν, $\nu\geqslant 2$, in all the variables. Since the coordinates are flat, it can be verified that $g_1(0,y,z,\mu)=0$, $g_2(x,0,z,\mu)=0$, and $g_3(0,y,z,\mu)=g_3(x,0,z,\mu)=g_3(0,0,z,\mu)$. Moreover, $D_Ug_i(0,0,0,\mu_0)=0$, $i=1,2,3$. The equation for the flow on the center manifold is

$$z'=A_3z+g_3(0,0,z,\mu)\tag{4.2}$$

Let $\Phi(t,z_0,\mu)$ be the solution map for (4.2), with $\Phi(0,z_0,\mu)=z_0$. We have the following.

Lemma 4.1. *For any $\alpha_0,\beta>0$ with $0<\beta<\alpha_0<\lambda_M$, there exist positive constants $\varepsilon_M,\delta_M,\mu_M$, and t_m with the following properties. The constant ε_M is small enough so that $\{U=(x,y,z)\mid \|x\|_X\leqslant\varepsilon_M,\ \|y\|_Y\leqslant\varepsilon_M,\ \|z\|_Z\leqslant\varepsilon_M\}\subset\mathcal{O}$. If $|\mu|<\mu_M$, $t_0\geqslant t_m$ and $z_0\in Z$, satisfying*

$$\|\Phi(t,z_0,\mu)\|_Z\leqslant\varepsilon_M \quad \text{for} \quad t\in[0,t_0]$$

and if $|x_0|+|y_0|<\delta_M$, then there exists a unique solution $U(t)\in\mathcal{O}$, $t\in[0,t_0]$, to Eq. (4.1), satisfying the boundary conditions

$$x(t_0)=x_0,\qquad y(0)=y_0,\qquad \text{and}\qquad z(0)=z_0$$

The solution can be written in the form

$$U(t) = (x^{\$}(t),\ y^{\$}(t),\ \Phi(t) + z^{\$}(t)), \qquad 0 \leqslant t \leqslant t_0$$

where $\Phi(t) = \Phi(t, z_0, \mu)$, $z^{\$}(0) = 0$. $w^{\$}(t) = w^{\$}(t; t_0, x_0, y_0, z_0, \mu)$, $w = x, y$, *or* z, *are* $C^{\nu-1}$ *functions in all the variables if* g_i, $i = 1, 2, 3$, *is* C^{ν}. *Moreover, let* r *be a multiindex with* $0 \leqslant |r| \leqslant \nu - 1$. *Suppose that* α_1 *satisfies* $0 < \beta < \alpha_1 < \alpha_0 - |r|\beta$. *Then*

$$|D^r x^{\$}(t)|_X \leqslant Ce^{\alpha_1(t - t_0)}$$

$$|D^r y^{\$}(t)|_Y \leqslant Ce^{-\alpha_1 t}$$

$$|D^r z^{\$}(t)\|_Z \leqslant Ce^{-\alpha_1 t_0 + \beta t}, \qquad 0 \leqslant t \leqslant t_0$$

The proof for Lemma 4.1 in the ODE case can be found in Refs. 4, 10, and 19. The proof for systems of abstract parabolic equations is similar and will not be rendered here.

Since the small eigenvalue $\lambda_{11} = m$ and since the flow on the center manifold is odd, we can rewrite (4.2) as the following:

$$z' = mz - \hat{c}z^3 + z^5 h_1(z, \mu) \tag{4.3}$$

Here $\hat{c} = \hat{c}(\mu) > 0$ due to H_5, and $|\mu| \leqslant \mu_M$. The function h_1 is C^{ν} for all $\nu > 0$ and $h_1(-z, \mu) = h_1(z, \mu)$. Equation (4.3) has three equilibria $z = 0$ and $z = \pm z_E$ where $z_E \approx \sqrt{m/\hat{c}}$ provided that $m > 0$ and m is small.

In Lemma 4.2 and Lemma 4.3 we present some estimates on the function $\Phi(t, z_0, \mu)/z_0$, which measures the degree of expansion or contraction on the center manifold. The importance of these estimates will be clear in the next two sections, where bifurcation functions and their approximations are introduced. The proofs are technical and can be skipped on the first reading. In fact, the results in Lemma 4.2 and Lemma 4.3 are easy to verify for the truncated equation

$$z' = mz - \hat{c}z^3$$

All we try to show in these lemmas is that the perturbation term $z^5 h_1(z, \mu)$ does not change the solution significantly.

Let $\varepsilon > 0$ be a small constant. By plotting the phase diagram of (4.3) on $(-\varepsilon, \varepsilon)$ (see Fig. 5), it can be verified that $|\Phi(t, z_0, \mu)| < \varepsilon$ provided $|z_0| < \varepsilon$, and m and μ_M are small. In Lemma 4.2, we show that $\Phi(t, z_0, \mu)/z_0$ is monotonic with respect to z_0 in $(0, \varepsilon)$ if $t > 0$ is fixed. We also give formulas that will provide some lower bounds on $|(\partial/\partial z_0)(\Phi(t, z_0, \mu)/z_0)|$ in the future.

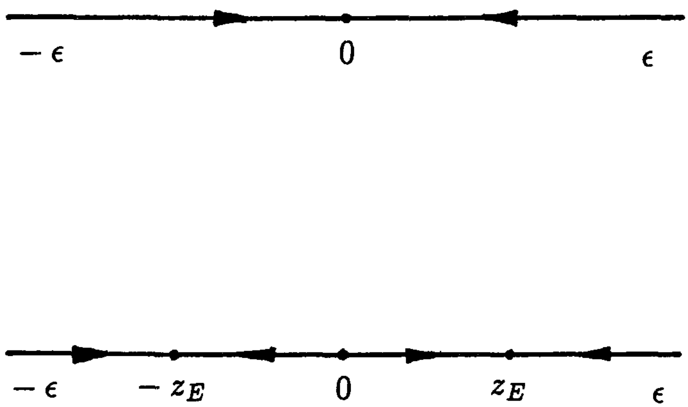


Fig. 5. Phase diagrams for the flow on the center manifold.

Lemma 4.2. *There exists* $\varepsilon > 0$ *such that*

$$\operatorname{sign}\left\{\frac{\partial}{\partial z_0}\left(\frac{\Phi(t, z_0, \mu)}{z_0}\right)\right\} = -\operatorname{sign} z_0$$

if $0 < |z_0| < \varepsilon$. *Moreover, we can show the following.*

(i) *If* $m \leqslant 0$, *then*

$$\frac{\partial}{\partial z_0}\left[\frac{\Phi}{z_0}\right] = \frac{C_1(z_0^2 - \Phi^2)}{m - \hat{c}z_0^2 + z_0^4 h_1(z_0, \mu)} \cdot \frac{\Phi}{z_0^2}$$

where C_1 *is a function of* z_0. $C_1 \approx \hat{c}$ *if* m *and* ε *are small.*

(ii) *If* $m \geqslant 0$ *and* $z_0^2 \neq z_E^2$, *then we have*

$$\frac{\partial}{\partial z_0}\left[\frac{\Phi}{z_0}\right] = \frac{C_2(z_0^2 - \Phi^2)}{z_E^2 - z_0^2} \cdot \frac{\Phi}{z_0^2}$$

where C_2 *is a function of* z_0. $C_2 \approx 1$ *if* m *and* ε *are small.*

(iii) *If* $m > 0$ *and* $z_0^2 = z_E^2$, *then*

$$\frac{\partial}{\partial z_0}\left[\frac{\Phi}{z_0}\right] = (e^{-m't} - 1)/z_0$$

where $-m' = (\partial/\partial z)[mz - \hat{c}z^3 + z^5 h_1(z, \mu)]|_{z=z_E} \approx -2m$ *if* m *and* ε *small.*

Proof. Since $\Phi(t, z_0, \mu)$ is an odd function of z_0, it suffices to consider $z_0 > 0$. Let $z = \Phi(t, z_0, \mu)$. It is well-known that $U(t) = \partial\Phi/\partial z_0$ satisfies the linear variational equation for (4.3) with $U(0) = 1$; so does $\partial\Phi(t, z_0, \mu)/\partial t/\partial\Phi(0, z_0, \mu)/\partial t$. Therefore, they must be identical. Using (4.3) to replace $\partial\Phi(t, z_0, \mu)/\partial t$, we have

$$\frac{\partial\Phi}{\partial z_0} = \frac{mz - \hat{c}z^3 + z^5 h_1(z, \mu)}{mz_0 - \hat{c}z_0^3 + z_0^5 h_1(z_0, \mu)} \tag{4.4}$$

$$\begin{aligned}\frac{\partial}{\partial z_0}\left(\frac{\Phi}{z_0}\right) &= \left\{\frac{mz - \hat{c}z^3 + z^5 h_1(z, \mu)}{mz_0 - \hat{c}z_0^3 + z_0^5 h_1(z_0, \mu)} \frac{z_0}{z} - 1\right\} \frac{z}{z_0^2} \\ &= \frac{\hat{c}(z_0^2 - z^2) + z^4 h_1(z, \mu) - z_0^4 h_1(z_0, \mu)}{m - \hat{c}z_0^2 + z_0^4 h_1(z_0, \mu)} \cdot \frac{z}{z_0^2}\end{aligned} \tag{4.5}$$

Since $h_1(z, \mu)$ is an even function of z, we have

$$z^4 h_1(z, \mu) - z_0^4 h_1(z_0, \mu) = C_3(z_0^2 - z^2)$$

where C_3 is a function of z_0 and is small if both z and z_0 are small. This proves (i).

For any fixed $t > 0$, $m > 0$, let $z_0 \to z_E$, then $z = \Phi \to z_E$. From (4.4), and the fact that $(\partial\Phi/\partial z_0) \to e^{-m't}$ as $z_0 \to z_E$, t fixed, we have

$$\lim_{z_0 \to z_E} \frac{mz - \hat{c}z^3 + z^5 h_1(z, \mu)}{mz_0 - \hat{c}z_0^3 + z_0^5 h_1(z_0, \mu)} = e^{-m't}$$

where $-m' = (\partial/\partial z)[mz - \hat{c}z^3 + z^5 h_1(z, \mu)]|_{z = z_E} \approx -2m$, since $z_E \approx \sqrt{m/\hat{c}}$. Therefore, (iii) follows from the first line of (4.5).

When $m > 0$, since z_E is a nonzero equilibrium, $m - \hat{c}z_E^2 + z^4 h_1(z_E, \mu) = 0$. Therefore

$$\begin{aligned}m - \hat{c}z_0^2 + z_0^4 h_1(z_0, \mu) &= m - \hat{c}z_0^2 + z_0^4 h_1(z_0, \mu) - [m - \hat{c}z_E^2 + z_E^4 h_1(z_E, \mu)] \\ &= C_4(z_E^2 - z_0^2)\end{aligned}$$

where C_4 is a function of z_0 and is close to $\hat{c}$. From this, (ii) follows from the second line of (4.5).

When $m \leqslant 0$, zero is an attractor on W^c_{loc}. If $z_0 > 0$, then $z_0^2 > \Phi^2$. Thus $(\partial/\partial z_0)[\Phi/z_0] < 0$, based on (i). When $m > 0$, $z_0 = z_E$, $(\partial/\partial z_0)[\Phi/z_0] < 0$, based on (iii). When $m > 0$, $z_0 \neq z_E$, $z_0 > 0$, z_E attracts z_0. From the phase diagram (see Fig. 5), it is clear that $z_0^2 - \Phi^2$ and $z_E^2 - z_0^2$ always have different signs. Thus $(\partial/\partial z_0)[\Phi/z_0] < 0$, based on (ii). □

In the next lemma, we derive some estimates on the rate of contraction or repelling for the equilibria zero and/or $\pm z_E$ on the center manifold. These estimates are to be used in conjunction with Lemma 4.2.

Lemma 4.3. *There exist $\bar{z} > 0$ and $\bar{m} > 0$ with the flowing properties.*

(a) Let $|z_0| < \bar{z}$, $0 < m < \bar{m}$, and $\Phi = \Phi(t_0, z_0, \mu)$. For any $\tau_0 > 0$, if $mt_o > \tau_0 > 0$, then there exists $\eta = \eta(\tau_0) > 0$ such that either

$$z_0^2 \leqslant (1-\eta)\,\Phi^2 \qquad \text{or} \qquad |\Phi^2 - z_E^2| \leqslant (1-\eta)\,|z_0^2 - z_E^2|$$

(b) If $-\bar{m} \leqslant m < 0$ and $-mt_o > \tau_0$, then

$$\Phi^2 \leqslant (1-\eta)\,z_0^2$$

Proof. Let $w = z^2$. Define $h(w, \mu) = 2mw - 2\hat{c}w^2 + 2w^3 h_1(\sqrt{w}, \mu)$. We have $w' = h(w, \mu)$. Let the solution map be $w(t, w_0)$. Since h_1 is a C^∞, even function of z, it can be shown that h is C^∞. Let $\hat{w} = m/3\hat{c}$ and $w_E = z_E^2 = (m/\hat{c}) + O(m^2)$. It is easy to see that $(\partial^2 h/\partial w^2) < 0$ if $\bar{z}$ is small and $w < \bar{z}^2$. Therefore, using Taylor's formula with remainder, we have

$$\frac{\partial}{\partial w}\left(\frac{h(w,\mu)}{w}\right) = \frac{(\partial/\partial w)\,h \cdot w - h}{w^2} < 0$$

Here we have used the fact that $h(0, \mu) = 0$. Similarly, since $h(w_E, \mu) = 0$,

$$\frac{\partial}{\partial w}\left(\frac{h(w,\mu)}{w - w_E}\right) < 0$$

Consider case (a), $m > 0$, first.

(i) If $w_0 > w_E$, then $w(t) = w(t, w_0) > w_E$ for all $t > 0$. Since $h(w_E, \mu) = 0$ and $(\partial/\partial w)(h/w) < 0$,

$$\frac{h(w,\mu)}{w - w_E} \leqslant \frac{\partial}{\partial w} h(w_E, \mu) = -2m + O(m^2) < -m$$

if $0 < m < \bar{m}$. Let $e^{-\tau_0} = 1 - \eta$. From $(w - w_E)' \leqslant -m(w - w_E)$, we have

$$\begin{aligned} w(t) - w_E &\leqslant e^{-mt_0}(w_0 - w_E) \\ &\leqslant e^{-\tau_0}(w_0 - w_E) \\ &\leqslant (1-\eta)(w_0 - w_E) \end{aligned}$$

(ii) Observe that $\hat{w} < w_E$. If $0 < w(t_0/2, w_0) \leqslant \hat{w}$, then $0 < w(t) \leqslant \hat{w}$, for $0 \leqslant t \leqslant t_0/2$. Since h/w is monotonic,

$$\begin{aligned}\frac{h(w,\mu)}{w} &\geqslant \frac{h(\hat{w},\mu)}{\hat{w}} \\ &= 2m - 2\hat{c}\hat{w} + O(\hat{w}^2) \\ &= 2m - \frac{2m}{3} + O(m^2) \\ &> m\end{aligned}$$

if m is small. Let $1-\eta = e^{-\tau_0/2}$. From $w' \geqslant mw$, we have

$$w(t_0) > w\left(\frac{t_0}{2}\right) \geqslant e^{mt_0/2} w_0$$

Therefore $w_0 \leqslant (1-\eta)\, w(t_0)$.

(iii) If $\hat{w} \leqslant w(t_0/2) < w_E$, then for $(t_0/2) \leqslant t \leqslant t_0$, $\hat{w} \leqslant w(t) < w_E$. Observe that $w_E/\hat{w} = 3 + O(m)$. Using the monotonicity of h/w,

$$\begin{aligned}\frac{h(w,\mu)}{w-w_E} &\leqslant \frac{h(\hat{w},\mu)}{\hat{w}-w_E} \\ &= \left(\frac{\hat{w}}{\hat{w}-w_E}\right)(2m - 2\hat{c}\hat{w} + O(\hat{w}^2)) \\ &= \frac{1}{-2+O(m)}\left(\frac{4m}{3} + O(m^2)\right) \\ &= -\frac{2m}{3} + O(m^2) \\ &< -\frac{m}{2}\end{aligned}$$

if m is small. Let $1-\eta = e^{-\tau_0/4}$. From $(w-w_E)' \geqslant -(m/2)(w-w_E)$, we have

$$\begin{aligned}|w(t_0) - w_E| &\leqslant \left| w\left(\frac{t_0}{2}\right) - w_E \right| e^{-(m/2)(t_0/2)} \\ &\leqslant |w_0 - w_E|\, e^{-\tau_0/4} \\ &\leqslant |w_0 - w_E|\,(1-\eta)\end{aligned}$$

Case (b), $m < 0$, can be proved similarly to case (a), (i). □

Lemma 4.4. *Assume that* $ad-bd<0, d_1>0, d_2>0,$ *and* $(a+\pi^2 d_1)(d+\pi^2 d_2) \geqslant bc$. *Then* $f(\xi)=(a+\xi\pi^2 d_1)(d+\xi\pi^2 d_2)-bc$ *satisfies* $f'(\xi)>0$ *for all* $\xi \geqslant 1$ *and* $f(\xi)>0$ *for all* $\xi>1$.

Proof. The assumption implies that $f(1) \geqslant 0$. It is easy to verify that

$$\begin{aligned}\frac{d}{d\xi} f(\xi) &= 2\xi\pi^4 d_1 d_2 + a\pi^2 d_2 + d\pi^2 d_1 \\ &> \frac{1}{\xi}\{(a+\xi\pi^2 d_1)(d+\xi\pi^2 d_2)-bc\} \\ &= \frac{1}{\xi} f(\xi)\end{aligned}$$

From this the desired result follows. □

The following lemma relates hypothesis H_4 with the breaking of the homoclinic orbit $q(t)$. It is a variation of a well-known result on the homoclinic bifurcation using Melnikov's integral. See Ref. 16.

Lemma 4.5. *Consider the ODE system (1.1). Let* Σ *be a cross section intersecting the orbit of* $q(t)$ *transversely. Let* $q(0) \in \Sigma$. *Assume that* $T_{q(t)} W^u(E) \cap T_{q(t)} W^s(E)$ *is one dimensional—spanned by* $\dot{q}(t)$. *Let* $t_1>0$ *and* $\vec{v} \perp \{T_{q(t_1)} W^u(E) + T_{q(t_1)} W^s(E)\}$. *Then for each* $k \approx k_\infty$, *there exist a unique* $g(k) \in \mathbb{R}$ *and a piecewise smooth solution* $U(t,k)$ *of (1.1) that is* C^1 *in* $(-\infty, t_1) \cup (t_1, \infty)$. *Moreover,* $U(0,k) \in \Sigma \cap W^u(E)$ *and* $U(t_1^+, k) \in W^s(E)$ *with* $U(t_1^+, k) - U(t_1^-, k) = g(k)\vec{v}$. *Here* $U(t_1^-, k)$ *and* $U(t_1^+, k)$ *denote the left and right limit at* t_1. *Finally, if* H_4 *is valid, then* $(d/dk)\, g(k) \neq 0$.

5. BIFURCATION EQUATIONS FOR SIMPLE AND SYMMETRIC DOUBLE PERIODIC SOLUTIONS

Let $\bar{x}>0$ be a small constant and $\Sigma=\{x=\bar{x}\}$ be a cross section that intersects the orbit of $q(t)$ transversely at $(\bar{x},0,0) \in \mathcal{O}$. Assume that $q(0) \in \Sigma$. Trajectories near the homoclinic orbit must hit $\Sigma \cap \mathcal{O}$ at least once. We can make Σ smaller so that trajectories starting from Σ must reenter $\mathcal{O}$ after a fixed time t_1. The cross section Σ is used to fix the phase we are not construcing a Poincaré mapping: $\Sigma \to \Sigma$.

First, consider a simple periodic solution of period $T=t_0+t_1$. Since t_1 is fixed, the period T is determined by t_0. The solution can be divided into an outer solution $U_*(t)=(x_*(t), y_*(t), z_*(t))$, $0 \leqslant t \leqslant t_1$, and an inner solution $U^*(t)=(x^*(t), y^*(t), z^*(t))$, $0 \leqslant t \leqslant t_0$. In the sequel, we use superscript (subscript) to denote inner (outer) solutions. Let the outer solution be specified by an initial value problem with the initial value $U_*(0)=(\bar{x}, y_1, z_1) \in \Sigma$ and let the solution be denoted by $U_*(t; \bar{x}, y_1, z_1, \mu)$. Let the inner solution be specified by the boundary value problem as in Lemma 4.1 with the boundary conditions $x^*(t_0)=\bar{x}$, $y^*(0)=y_0$, and $z^*(0)=z_0$, and stays in $\mathcal{O}$ for all $t \in [0, t_0]$. See Fig. 6. By Lemma 4.1, such an inner solution is unique and is denoted

$$x^*(t)=x^{\$}(t; t_0, \bar{x}, y_0, z_0, \mu)$$
$$y^*(t)=y^{\$}(t; t_0, \bar{x}, y_0, z_0, \mu)$$
$$z^*(t)=\Phi(t; z_0, \mu)+z^{\$}(t; t_0, \bar{x}, y_0, z_0, \mu)$$

Define

$$x^*(t_0, y_0, z_0, \mu)=x^*(0)$$
$$y^*(t_0, y_0, z_0, \mu)=y^*(t_0)$$
$$z^*(t_0, y_0, z_0, \mu)=z^*(t_0)$$
$$\hat{x}(y_1, z_1, \mu)=x_*(t_1; \bar{x}, y_1, z_1, \mu)$$
$$\hat{y}(y_1, z_1, \mu)=y_*(t_1; \bar{x}, y_1, z_1, \mu)$$
$$\hat{z}(y_1, z_1, \mu)=z_*(t_1; \bar{x}, y_1, z_1, \mu)$$

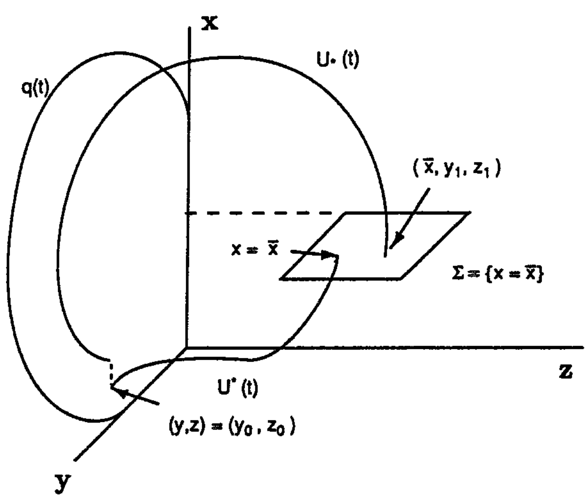


Fig. 6. A sketch of the inner solution U^* and outer solution U_*.

The end points of outer and inner solutions must match. We have the following equations:

$$G_1 \stackrel{\text{def}}{=} \hat{x}(y_1, z_1, \mu) - x^*(t_0, y_0, z_0, \mu) = 0 \tag{5.1}$$

$$y_1 = y^*(t_0, y_0, z_0, \mu) \tag{5.2}$$

$$z_1 = z^*(t_0, y_0, z_0, \mu) \tag{5.3}$$

$$y_0 = \hat{y}(y_1, z_1, \mu) \tag{5.4}$$

$$z_0 = \hat{z}(y_1, z_1, \mu) \tag{5.5}$$

See Fig. 6.

Substituting (5.4) and (5.5) into (5.2), we have

$$y_1 = y^*(t_0, \hat{y}(y_1, z_1, \mu), \hat{z}(y_1, z_1, \mu), \mu) \tag{5.6}$$

Using the smallness of $\partial y^*/\partial y_0$ and $\partial y^*/\partial z_0$ (see Lemma 4.1), we solve y_1 from (5.6) by a contraction principle to yield

$$y_1 = \tilde{y}(t_0, z_1, \mu) \tag{5.7}$$

Lemma 5.1.

(i) There exists a constant $\alpha_1 > 0$ *such that* $|\tilde{y}| + |\partial \tilde{y}/\partial z_1| = O(e^{-\alpha_1 t_0})$.

(ii) $\tilde{y}(t_0, -z_1, \mu) = R\tilde{y}(t_0, z_1, \mu)$.

(iii) When $z_1 = 0$, $(\bar{x}, \tilde{y}(t_0, 0, \mu), 0) \in \mathscr{X}_0$.

Proof. (i) follows from Lemma 4.1.

Since $(x_*(t), y_*(t), z_*(t))$ satisfies initial values $(x_*(0), y_*(0), z_*(0)) = (\bar{x}, y_1, z_1)$, then $(Rx_*(t), Ry_*(t), Rz_*(t))$ satisfies initial values $(R\bar{x}, Ry_1, Rz_1)$. Therefore

$$w_*(t_1; R\bar{x}, Ry_1, Rz_1, \mu) = Rw_*(t_1; \bar{x}, y_1, z_1, \mu)$$

where $w_* = x_*$, y_*, or z_*. Next, since $(Rx^*(t), Ry^*(t), Rz^*(t))$ satisfies boundary values $Rx^*(t_0) = R\bar{x} = \bar{x}$, $Ry^*(0) = Ry^0$, $Rz^*(0) = Rz^0$,

$$w^*(t_0, Ry^0, Rz^0, \mu) = Rw^*(t_0, y^0, z^0, \mu)$$

where $w^* = x^*$, y^*, or z^*. Based on these facts, using the uniqueness of the fixed point, (ii) can be verified from (5.6).

When $z_1 = 0$, in (5.6), let $y_1 \in \mathscr{X}_0$. Since $\mathscr{X}_0$ is invariant under the flow, then $y_* \in \mathscr{X}_0$ and $z_* \in \mathscr{X}_0$, i.e., $z_* = 0$. Since we can solve the boundary value problem, as described in Lemma 4.1, in $\mathscr{X}_0$, therefore, the right-hand side of (5.6), i.e., y^* is in $\mathscr{X}_0$. We then can solve (5.6) by the contraction principle in $\mathscr{X}_0 \cap Y$. This implies that the unique solution $\tilde{y}(t_0, 0, \mu) \in \mathscr{X}_0$. (iii) then follows from Theorem 2.7. □

We now substitute (5.7) into (5.1). Recall that $\mu = (l, m, k)$. G_1 is now a function of (t_0, l, m, k, z_1).

$$G_1(t_0, l, m, k, z_1) = \hat{x}(\tilde{y}(t_0, z_1, \mu), z_1, \mu) - x^*(t_0, \hat{y}, \hat{z}, \mu) \qquad (5.8)$$

where the arguments of $\hat{y}$ and $\hat{z}$ are $(\tilde{y}(t_0, z_1, \mu), z_1, \mu)$.

Lemma 5.2.

(i) $G_1(t_0, l, m, k, -z_1) = G_1(t_0, l, m, k, z_1)$.

(ii) $(\partial/\partial k)\, G_1 \neq 0$.

Proof. (i) The functions $\tilde{y}, \hat{y}, \hat{z}, \hat{x}$, and x^* are all invariant under the reflection R. Therefore $G_1(t_0, l, m, k, Rz_1) = RG_1(t_0, l, m, k, z_1)$. Assertion (i) then follows from the facts $Rz_1 = -z_1$ and $RG_1 = G_1$.

(ii) Set $t_0 = +\infty, z_1 = 0$, and $\mu = \mu_0$, where $\mu_0 = (l_0, m_0, k_\infty)$ with $m_0 = 0$ [or $(l_0, m_0) \in \Gamma$]. We then have $x^* = 0$ from Lemma 4.1 and $\tilde{y} = 0$ from Lemma 5.1. We now show that the function $G_1(\infty, l_0, 0, k, 0)$ is the Melnikov function as in Lemma 4.5.

Since $\mathscr{X}_0$ is invariant under systems (2.1), and $(\bar{x}, 0, 0) \in \mathscr{X}_0$, we have $x_*(t_1, \bar{x}, 0, 0, \mu) \in \mathscr{X}_0$. In $\mathscr{X}_0$, the equilibrium E is hyperbolic with $W^s_{loc}(E) = \{x = 0\}$, $W^u_{loc}(E) = \{y = 0\}$. Thus $(\bar{x}, 0, 0) \in W^u_{loc}(E)$. Consequently, $U(t_1)$, with the initial condition $(\bar{x}, 0, 0)$ is in $W^u(E) \cap \mathscr{X}_0$. Observe that $(1, 0, 0)$ is a vector orthogonal to $\dot{q}(t_1)$, where t_1 is a large constant such that $q(t_1)$ has reentered $\mathcal{O}$. Thus, $G_1(\infty, l_0, 0, k, 0)$ is the function $g(k)$ in Lemma 4.5. From Lemma 4.5 and hypothesis H_4, we have $(\partial/\partial k)$ $G_1(\infty, l_0, 0, k_\infty, 0) \neq 0$. Observe that G_1 is a C^1 function in a neighborhood of $(\infty, l_0, k_\infty, 0)$. Thus, $(\partial/\partial k)\, G_1 \neq 0$ for (t_0, l, m, k, z_1) near $(\infty, l_0, 0, k_\infty, 0)$. □

Since $G_1(\infty, l_0, 0, k_\infty, 0) = 0$, reflecting the existence of the homoclinic solution $q(t)$ at $k = k_\infty$, we can use Lemma 5.2(ii) to solve $k = k^*(t_0, l, m, z_1)$ from (5.8), if $t_0 \approx \infty$, $l \approx l_0$, $m \approx 0$, and $z_1 \approx 0$. From Lemma 5.2 (i),

$$k^*(t_0, l, m, -z_1) = k^*(t_0, l, m, z_1) \qquad (5.9)$$

We now substitute $k = k^*t_0, l, m, z_1)$ into (5.3), to obtain a bifurcation function,

$$G_2(t_0, l, m, z_1) \stackrel{\text{def}}{=} z^*(t_0, \hat{y}(\tilde{y}(t_0, z_1, \mu), z_1, \mu), \hat{z}(\tilde{y}(t_0, z_1, \mu), z_1, \mu), \mu) \quad (5.10)$$

where $\mu = (l, m, k^*(t_0, l, m, z_1))$. A solution of the equation

$$z_1 = G_2(t_0, l, m, z_1) \quad (5.11)$$

corresponds to a simple period $T = t_0 + t_1$ solution to (2.1).

Lemma 5.3. $G_2(t_0, l, m, -z_1) = -G_2(t_0, l, m, z_1)$. *In particular,* $G_2(t_0, l, m, 0) = 0$. *The solution corresponds to* $z_1 = 0, k = k^*(t_0, l, m, 0)$ *is a period* $T = t_0 + t_1$ *SH solution.*

Proof. Since the functions $\tilde{y}$, $\hat{y}$, $\hat{z}$, and z^* in the definition of G_2 are all invariant under the symmetry R, so is G_2. since $G_2 \in Z$, we have $RG_2 = -G_2$. This proves that G_2 is an odd function of z_1.

As in the proof of Lemma 5.2, $\tilde{y}(t_0, 0, \mu) \in \mathscr{X}_0$. The outer solution with initial condition $(\bar{x}, \tilde{y}, 0) \in \mathscr{X}_0$ must be in $\mathscr{X}_0$. Therefore the periodic solution corresponding to $z_1 = 0$ is in $\mathscr{X}_0$. □

Next we consider a symmetric double periodic solution $U(t)$ of period $2T$. The bifurcation equation for the existence of such solution can be derived much the same way as for the simple period T solution. Therefore we discuss it only briefly. From our definition, $U(t + T) = RU(t), t \in \mathbb{R}$. Assuming that $U(0) \in \Sigma = \{x = \bar{x}\}$, we define

$$U_*(t) = U(t), \qquad 0 \leqslant t \leqslant t_1$$
$$U^*(t) = U(t + t_1), \qquad 0 \leqslant t \leqslant t_0$$
$$T = t_0 + t_1$$

The matching conditions on the outer and inner solutions are

$$U^*(0) = U_*(t_1)$$
$$U^*(t_0) = RU_*(0)$$

As before, let the outer solution $U_*(t)$ be determined by the initial value $U_*(0) = (\bar{x}, y_1, z_1)$ and the inner solution $U^*(t)$ be determined by the boundary condition $(x^*(t_0), y^*(0), z^*(0)) = (\bar{x}, y_0, z_0)$. Then we still have (5.1), (5.4), and (5.5), but (5.2) and (5.3) change to

$$y_1 = Ry^*(t_0, y_0, z_0, \mu) \quad (5.2)'$$

$$z_1 = Rz^*(t_0, y_0, z_0, \mu) \quad (5.3)'$$

Notice that the functions Ry^* and Rz^* have similar smallness properties like y^* and z^*. We will use the same notations as for simple periodic solutions if no confusion occurs. As before, we can solve y_1 to get (5.7), and Lemma 5.1 is still valid. Here and afterward we use the same notations for functions $\tilde{y}$, G_1, k^*, G_2 when deriving bifurcation equations for both simple and symmetric double periodic solutions. Define G_1 as in (5.8). Again, we have Lemma 5.2. Solving $k = k^*(t_0, l, m, z_1)$ as before, we still have (5.9). Substituting $k = k^*$ into (5.3)′, and defining the function $G_2(t_0, l, m, z_1)$ as in (5.10), we have found that the solutions of the equation

$$-z_1 = G_2(t_0, l, m, z_1) \tag{5.11$'$}$$

correspond to symmetric double period $2T$ solutions to (2.1).

Analogous to Lemma 5.3, we have

$$G_2(t_0, l, m, -z_1) = -G_2(t_0, l, m, z_1)$$

However, when $z_1 = 0$, we really have obtained a SH simple period T solution tracing its orbit twice, since $U(t+T) = RU(t) = U(t)$ in this case.

6. PROOF OF THE MAIN RESULTS

Proof of Theorem 3.1. Since $\lambda_{11}\lambda_{12} = (a + \pi^2 d_1)(d + \pi^2 d_2) - bc$, which is negative (positive or zero) in $\mathscr{G}_+$ ($\mathscr{G}_-$ or Γ), assertions (i), (ii), and (iii) follow from the fact that $\operatorname{Re} \lambda_{12} < 0$.

Let $(d_1, d_2) \in \Gamma \cup \mathscr{G}_-$. From Lemma 4.4, $\lambda_{n1}\lambda_{n2} = (a + n^2\pi^2 d_1)(d + n^2\pi^2 d_2) - bc$ is an increasing function for $n \geqslant 1$. It follows that $\lambda_{n1}\lambda_{n2} > 0$ for $n \geqslant 2$. From $\operatorname{Re} \lambda_{n2} < 0$ we have $\operatorname{Re} \lambda_{n1} < 0$, $n \geqslant 2$. This proves (iv) for $n \geqslant 2$. The proof for the cases $n = 0, 1$ are obvious and will be omitted.

Let $\alpha =^{\text{def}} a + \pi^2 d_1$ and $\beta =^{\text{def}} d + \pi^2 d_2$. $\alpha + \beta > 0$ since $a + d > 0$.

$$\lambda_{11}, \lambda_{12} = \frac{1}{2}\left\{ -(\alpha + \beta) \pm \sqrt{(\alpha+\beta)^2 + 4bc} \right\}$$

$$\frac{\partial}{\partial \alpha} \lambda_{11} = \frac{1}{2}\left\{ -1 + (\alpha - \beta)[(\alpha - \beta)^2 + 4bc]^{-1/2} \right\}$$

When $(d_1, d_2) \in \Gamma$, we have $\alpha\beta - bc = 0$. Therefore $(\partial/\partial\alpha)\,\lambda_{11} = \frac{1}{2}\{-1 + (\alpha - \beta)/(\alpha + \beta)\} = -\beta/(\alpha + \beta)$. Thus, $(\partial\lambda_{11}/\partial d_1) < 0$ if $\beta > 0$. Similarly $(\partial\lambda_{11}/\partial d_2) < 0$ if $\alpha > 0$. It is impossible to have both $\alpha \leqslant 0$ and $\beta \leqslant 0$ since $\alpha + \beta > 0$. This proves (v). □

As mentioned earlier, we use only one set of notations $\tilde{y}, k^*, G_1$, and G_2 for functions employed when deriving bifurcation functions for both simple and symmetric double periodic solutions. This allows us to treat both problems simultaneously.

Since the proof of the main results is technical, it may be useful to preview the main idea used here. Consider finding a simple SN periodic solution. Recall that the bifurcation equation

$$z_1 = z^*(t_0, \hat{y}, \hat{z}, \mu)$$

where $\hat{y}$ and $\hat{z}$ are as in (5.10), has a trivial solution $z_1 = 0$, which corresponds to a SH solution. Since we are not interested in such solution, it is reasonable to look for solutions of the equation $z^*/z_1 = 1$.

Let $\mu = (l, m, k^*(t_0, l, m, z_1))$ and let

$$z_0 = \hat{z}(\tilde{y}(t_0, z_1, \mu), z_1, \mu) \tag{6.1}$$

Then $z_0 = 0$ if $z_1 = 0$. We look for solutions of

$$\frac{z^*}{z_0} \cdot \frac{z_0}{z_1} = 1$$

We can show, in the limiting case, $z^*/z_0 \approx \Phi/z_0$; the latter is the rate of expansion on the local center manifold. Also, $z_0/z_1 \approx c^*(l)$; the latter is the rate of expansion along the outer solution, and its sign represents the twistedness of the homoclinic orbit. In particular, based on Lemma 4.2, we can show that z^*/z_0 is monotonic for $z_0 > 0$ or $z_0 < 0$ (Lemma 6.2). The proof of our main results would be easier if the outer solution were the multiplication by c^* and the inner solution were the expansion by the rate Φ/z_0. However, such approximations have some small errors. Care must be exercised to ensure that the error terms do not disturb the main terms.

Denote

$$r_1(t_0, l, m) = \{\lim_{z_1 \to 0} z_0/z_1\} \tag{6.2}$$

Then

$$r_1(t_0, l, m) = \frac{\partial \hat{z}}{\partial y_1} \frac{\partial \tilde{y}}{\partial z_1}(t_0, 0, \mu) + \frac{\partial \tilde{z}(\tilde{y}, 0, \mu)}{\partial z_1} + \frac{\partial \hat{z}}{\partial k} \cdot \frac{\partial k^*(t_0, l, m, 0)}{\partial z_1}$$

Assuming now that $t_0 = +\infty$, we have $\tilde{y} = 0$ and $(\partial \tilde{y}/\partial z_1) = 0$ (Lemma 5.1). Also, $k^*(\infty, l_0, 0, 0) = k_\infty$, reflecting the existence of the homoclinic solution $q(t)$ at $k = k_\infty$. From (5.9), we can show that $(\partial/\partial z_1)\, k^*(\infty, l, m, 0) = 0$. We have shown the first part of the following.

Lemma 6.1. *$r_1(\infty, l_0, 0) = (\partial\hat{z}/\partial z_1)(0, 0, (l_0, 0, k_\infty)) = c^*(l_0)$, where $c^*(l_0)$ is defined in Definition* 3.3.

Lemma 6.1 offers an easy way to compute $r_1(\infty, l_0, 0)$ since $c^*(l_0)$ can be obtained by computing the ODE system (3.5) in $\mathscr{X}_1$. The proof of the second equality of Lemma 6.1 is deferred to Section 7.

In the first part of this section we assume that $c^*(l_0) \neq 0$ for an $l_0 \in \bar{I}$. Then, based on Lemma 6.1, $(\partial z_0/\partial z_1) \neq 0$ when $z_1 = 0$. We can solve z_1 form (6.1) to obtain the inverse function

$$z_1 = \tilde{z}_1(t_0, l, m, z_0) \tag{6.3}$$

Here $\tilde{z}_1$ is a smooth function defined for $t_0 \approx +\infty, m \approx 0, l \approx l_0$. Assume that the domain of $\tilde{z}_1$ is so small that

$$\text{sign}\{\tilde{z}_1/z_0 \,|\, z_0 \neq 0\} = \text{sign}\, c^*(l_0)$$

Let

$$L_1(t_0, l, m) = \lim_{z_0 \to 0} \frac{z^*(t_0, \hat{y}(\tilde{y}(t_0, \tilde{z}_1, \mu), \tilde{z}_1, \mu), z_0, \mu)}{z_0} \tag{6.4}$$

where $\tilde{z}_1$ is given in (6.3). From Lemma 4.1, we have $z^*(\cdots) = \Phi(t_0, z_0, \mu) + z^s(\cdots)$, where $\cdots$ represents the variables from the r.h.s. of (6.4), and $|z^s| + |Dz^s| = O(e^{-\alpha_1 t_0})$. It follows that

$$L_1(t_0, l, m) = \frac{\partial\Phi(t_0, 0, \mu)}{\partial z_0} + O(e^{-\alpha_1 t_0}), \qquad \alpha_1 > 0$$

Since $\Phi(t, z_0, \mu)$ satisfies the equation $z' = mz - \hat{c}z^3 + \text{h.o.t.}$ [see (4.3)], we have $(\partial/\partial z_0)\, \Phi(t_0, 0, \mu) = e^{mt_0}$. Therefore,

$$L_1(t_0, l, m) = e^{mt_0} + O(e^{-\alpha_1 t_0}) \tag{6.5}$$

Recall the monotonicity of $\Phi(t_0, z_0, \mu)/z_0, z_0 \neq 0$, proved in Lemma 4.2. Using the smallness of $z^s = z^*(\cdots) - \Phi(t_0, z_0, \mu)$, we have the following results.

Lemma 6.2. *Let $\pm z_E$ be the nonzero equilibria of (4.3) if $m > 0$. Let $\mu_0 = (l_0, m_0, k_\infty)$ where $l_0 \in \bar{I}$ and $m_0 = 0$. For each $0 < \eta < 1$ there exist $\varepsilon = \varepsilon(\eta) > 0$ and $\bar{m} > 0$ such that if $0 < |z_0| < \varepsilon$, $|\mu - \mu_0| < \varepsilon$, $t_0 > (1/\varepsilon)$, and $m < \bar{m}$, then we have the following result.*

$$\frac{d}{dz_0}\left[z^*(t_0,\hat{y}(\tilde{y}(t_0,\tilde{z}_1,\mu),\tilde{z}_1,\mu),z_0,\mu)/z_0\right] \tag{6.6}$$

$$=\begin{cases} \dfrac{C_5(z_0^2-\Phi^2)}{z_E^2-z_0^2}\dfrac{\Phi}{z_0^2}, & \textit{if}\ \ m>0,\ z_E^2\neq z_0^2,\ z_0^2\leqslant(1-\eta)\,\Phi^2 \\ & \textit{or}\ |z_E^2-\Phi^2|\leqslant(1-\eta)|\,z_E^2-z_0^2| \qquad (6.6a) \\ C_6(e^{-m't_0}-1)/z_0, & \textit{if}\ \ m>0,\ z_0^2=z_E^2 \text{ and } e^{-m't_0}\leqslant 1-\eta, \\ & \textit{where}\ -m'=\lim\limits_{z\to z_E}\dfrac{\partial h}{\partial z}(\ ) \qquad (6.6b) \\ \dfrac{C_7(z_0^2-\Phi^2)}{m-\hat{c}z_0^2+z_0^4h_1(z^0,\mu)}\dfrac{\Phi}{z_0^2}, & \textit{if}\ \ m\leqslant 0,\ \textit{and}\ \Phi^2\leqslant(1-\eta)\,z_0^2 \qquad (6.6c) \end{cases}$$

Here $C_5, C_6>\frac{1}{2}$, *and* $C_7>(\hat{c}/2)$ *are functions of* (t_0, l, m, z_0). *In all three cases* $\operatorname{sign}\{(\partial/\partial z_0)(z^*/z_0)\} = -\operatorname{sign}\{z_0\}$.

Proof. Assume that $\bar{m}$ is small so that $|z_E|<\varepsilon/2$.

$$\left|\frac{d}{dz_0}\left(\frac{z^\$}{z_0}\right)\right|=\frac{|z^\$-(dz^\$/dz_0)\,z_0|}{|z_0^2|}$$

$$\leqslant\frac{1}{|z_0^2|}\left\{\left|z^\$(z_0)-\frac{d}{dz_0}z^\$(0)\cdot z_0\right|+\left|\frac{d}{dz_0}z^\$(0)\cdot z_0-\frac{d}{dz_0}z^\$(z_0)\cdot z_0\right|\right\}$$

$$\leqslant C\sup\left\|\frac{d^2z^\$}{dz_0^2}\right\|\cdot|z_0|$$

Here the fact that $|z_0|<\varepsilon$ is used. We now use Lemma 4.1. We may assume that β in that lemma is arbitrarily small at the cost of selecting a smaller ε. Let $0<\alpha<\alpha_0-3\beta$; we have

$$\left|\frac{d}{dz_0}\left(\frac{z^\$}{z_0}\right)\right|\leqslant C_8\,|z_0|\,e^{-\alpha t_0}$$

(i) Suppose that $m>0$, $z_E^2\neq z_0^2$, and $z_0^2\leqslant(1-\eta)\,\Phi^2$. Let C_2 be the constant as in Lemma 4.2, case (ii). Then

$$\left|\frac{C_2(z_0^2-\Phi^2)}{z_E^2-z_0^2}\frac{\Phi}{z_0^2}\right|\geqslant\left|\frac{C_2\eta\Phi}{z_E^2-z_0^2}\right|\geqslant 3C_8\,|z_0|\,e^{-\alpha t_0}$$

if $t_0>(1/\varepsilon)$ and $\varepsilon>0$ are sufficiently small. Here we have used the facts that $\Phi^2\geqslant z_0^2$ and $|z_E^2-z_0^2|$ is small. Therefore $|(d/dz_0)(z^\$/z_0)|<\frac{1}{3}\,|(d/dz_0)(\Phi/z_0)|$. Since $C_2\approx 1$, $C_5>\frac{1}{2}$. Therefore, (6.6a) follows from Lemma 4.2, case (ii).

Suppose now that $m>0, z_E^2 \neq z_0^2$, and $|z_E^2-\Phi^2| \leqslant (1-\eta)\, |z_E^2-z_0^2|$. Then

$$\frac{|z_0^2-z_E^2+z_E^2-\Phi^2|}{|z_E^2-z_0^2|} \geqslant 1-(1-\eta)=\eta$$

$$\frac{C_2\, |z_0^2-\Phi^2|}{|z_E^2-z_0^2|} \frac{|\Phi|}{z_0^2} \geqslant C_2\eta \frac{|\Phi|}{z_0^2} \geqslant 3C_8\, |z_0|\, e^{-\alpha t_0} \tag{6.7}$$

The last inequality is based on the fact $|\Phi| \geqslant |z_0|$ and $|z_0|$ is small. Based on a similar argument, (6.6a) follows from Lemma 4.2, case (ii).

(ii) If $m>0$, $z_0^2=z_E^2$, and $e^{-m't_0} \leqslant 1-\eta$, then

$$|(e^{-m't_0}-1)/z_0| \geqslant \eta/|z_0| \geqslant 3C_8\, |z_0|\, e^{-\alpha t_0}$$

if $|z_0|<\varepsilon$ and $t_0>(1/\varepsilon)$, $\varepsilon>0$ is sufficiently small. As before, (6.6b) then follows from Lemma 4.2 (iii).

(iii) If $m \leqslant 0$ and $\Phi^2 \leqslant (1-\eta)\, z_0^2$, then

$$\left|\frac{C_1(z_0^2-\Phi^2)}{m-\hat{c}z_0^2+z_0^4h_1(z_0,\mu)} \frac{\Phi}{z_0^2}\right| \geqslant \left|\frac{C_1\eta\Phi}{m-\hat{c}z_0^2+z_0^4h_1(z_0,\mu)}\right| \geqslant 3C_8\, |z_0|\, e^{-\alpha t_0}$$

When deriving the last inequality, we assume that ε is sufficiently small so that $\sup_z\{h(z)\}=\tilde{m}$ with $\tilde{m}<\alpha$. Then $|\Phi| \geqslant |z_0|\, e^{-\tilde{m}t_0}$ by the Gronwall inequality. Therefore, if t_0 is sufficiently large and $m-\hat{c}z_0^2+z_0^4h_1(z_0,\mu)$ is small, the last inequality holds. Equations (6.6c) then follows from Lemma 4.2, case (i).

The proof of Lemma 6.2 has been completed. □

Lemma 6.3. *Under the same conditions of Lemma 6.2, in any of the three cases, (6.6a), (6.6b) or (6.6c), if we choose smaller $\varepsilon>0$, we have [see (6.3) for $\tilde{z}_1$]*

$$\frac{\partial}{\partial z_0}\left(\frac{z^*}{\tilde{z}_1}\right) \begin{cases} >0, & \text{if } \tilde{z}_1<0 \\ <0, & \text{if } \tilde{z}_1>0 \end{cases}$$

Proof. Observe that

$$\frac{\partial}{\partial z_0}\left(\frac{z^*}{\tilde{z}_1}\right)=\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\frac{z_0}{\tilde{z}_1}\right)=\frac{z_0}{\tilde{z}_1}\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)+\frac{z^*}{z_0}\frac{\partial}{\partial z_0}\left(\frac{z_0}{\tilde{z}_1}\right)$$

Since $\tilde{z}_1$ is an odd function of z_0, we have

$$\left|\frac{\partial}{\partial z_0}\left(\frac{z_0}{\tilde{z}_1}\right)\right| = \left|\left(\tilde{z}_1 - \frac{\partial \tilde{z}_1}{\partial z_0} z_0\right)\Big/\tilde{z}_1^2\right| \leqslant C_9 \, |z_0^3|/\tilde{z}_1^2$$

We first show that $|(\partial/\partial z_0)(z^*/z_0)| > 2C_9 \, |z_0 z^*/\tilde{z}_1|$, since this will imply that the sign of $\partial/\partial z_0(z^*/\tilde{z}_1)$ is determined by the sign of $(z_0/\tilde{z}_1)(\partial/\partial z_0)(z^*/z_0)$. Based on $|\Phi| \geqslant |z_0| \, e^{-\hat{m}t_0}$ and $|z^s| \leqslant C \, |z_0| \, e^{-\alpha_1 t_0}$, we have for large t_0, $|z^*| < 2|\Phi|$. We then need to show

$$\left|\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)\right| > C_{10} \, |\Phi| \tag{6.8}$$

where $C_{10} = 4C_9 C_{11}$ with $C_{11} = \sup |z_0/\tilde{z}_1|$.

(i) If $m > 0$, $z_E^2 \neq z_0^2$, and $z_0^2 \leqslant (1-\eta)\,\Phi^2$, then from (6.6a),

$$\left|\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)\right| > \frac{1}{2}\frac{|\eta\Phi|}{|z_E^2 - z_0^2|} > C_{10} \, |\Phi|$$

provided that $|z_E^2 - z_0^2| < (\eta/(2C_{10}))$, which can be achieved by choosing smaller ε.

(ii) If $m > 0$, $z_E^2 \neq z_0^2$, and $|z_E^2 - \Phi^2| \leqslant (1-\eta)\,|z_E^2 - \Phi^2|$, then from (6.6a) and (6.7),

$$\left|\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)\right| > \frac{|\eta\Phi|}{2z_0^2} > C_{10} \, |\Phi|$$

provided that $z_0^2 < (\eta/(2C_{10}))$, which is valid if ε is sufficiently small.

(iii) If $m > 0$, $z_0^2 = z_E^2$, and $e^{-m't_0} \leqslant 1-\eta$, then from (6.6b),

$$\left|\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)\right| \geqslant \frac{\eta}{2\,|z_0|} \geqslant C_{10} \, |\Phi|$$

Here we need $|z_0\Phi| < (\eta/(2C_{10}))$, which is valid if ε is small.

(iv) If $m \leqslant 0$ and $\Phi^2 \leqslant (1-\eta)\,z_0^2$, then from (6.6c),

$$\left|\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)\right| \geqslant \frac{|\eta\Phi|}{2\,|m - \hat{c}z_0^2 + z_0^4 h_1(z_0, \mu)|} \geqslant C_{10} \, |\Phi|$$

provided that $|m - \hat{c}z_0^2 + \cdots| \leqslant (\eta/(2C_{10}))$, that is valid if ε is small.

In all cases, the sign of $(\partial/\partial z_0)(z^*/\tilde{z}_1)$ agrees with that of $(z_0/\tilde{z}_1)$ $(\partial/\partial z_0)(z^*/z_0)$. From Lemma 6.2

$$\operatorname{sign}\left\{\frac{\partial}{\partial z_0}\left(\frac{z^*}{z_0}\right)\right\} = -\operatorname{sign}\{z_0\}$$

Therefore $\operatorname{sign}\{(\partial/\partial z_0)(z^*/\tilde{z}_1)\} = -\operatorname{sign}\{\tilde{z}_1\}$. □

Corollary 6.4. *Under the conditions of Lemma 6.2, we have that* $|z^*/z_1|$ *is a decreasing function of* $|z_1|$.

Recall that t_1 is fixed and $T = t_0 + t_1$ depends solely on t_0. Let

$$\begin{aligned} L(T, l, m) &= L_1(t_0, l, m)\, e^{mt_1} \\ r(T, l, m) &= r_1(t_0, l, m)\, e^{-mt_1} \end{aligned} \tag{6.9}$$

We shall prove that with such functions L and r, Theorem 3.4 is valid.

If no arguments are given, for notational simplicity, Φ means $\Phi(t_0, z_0, \mu)$, z_0 is defined in (6.1), $z^* = G_2(t_0, l, m, z_1)$ is defined in (5.10), $z_1 = \tilde{z}_1$ is defined in (6.3), and z^s means $z^s(t_0, \hat{y}(\tilde{y}(t_0, \tilde{z}_1, \mu), \tilde{z}_1, \mu), z^0, \mu) = z^* - \Phi$. Assumptions following a case number are valid until a new case is encountered.

Proof of Theorem 3.4. From (6.9), if $m = 0$,

$$r(\infty, l_0, 0) = r_1(\infty, l_0, 0)$$

Thus, Theorem 3.4 (1) follows from Lemma 6.1.

From (6.9) again, it is easy to see that (2) follows from (6.5).

We now prove Theorem 3.4 (3). From (6.4) and Lemma 4.1,

$$L_1(t_0, l, m) = \frac{\partial \Phi(t_0, 0, \mu)}{\partial z_0} + \left.\frac{\partial z^s}{\partial z_0}\right|_{z_0 = 0}$$

$$\left|\frac{\partial z^s}{\partial z_0}\right| + \left|\frac{\partial^2 z^s}{\partial z_0\, \partial m}\right| \leqslant C e^{(-\alpha_1 + 2\beta) t_0}, \qquad \alpha_1 > 2\beta > 0$$

$$\frac{\partial \Phi(t_0, 0, \mu)}{\partial z_0} = e^{mt_0}$$

we have

$$\frac{\partial L_1(t_0, l, m)}{\partial m} = t_0 e^{mt_0} + O(e^{(-\alpha_1 + 2\beta) t_0}) \quad .$$

Now that $(\partial/\partial m)(L_1 r_1) = (\partial L_1/\partial m)\, r_1 + L_1(\partial r_1/\partial m)$ and $|\partial r_1/\partial m| \leqslant C$, we have $(\partial/\partial m)(L_1 r_1) = t_0 e^{mt_0} \cdot r_1(t_0, l, m) + O(e^{mt_0})$. Let $\mathcal{O} = \{|l - l_0| < \delta, |m| < \delta\}$ and let $t_0 > 1/\delta$. If $\delta > 0$ is sufficiently small, we have $r_1(t_0, l, m) = C_1 c^*(l_0)$, where $C_1 > \frac{1}{2}$. Thus,

$$\operatorname{sign}\left\{\frac{\partial}{\partial m}(L_1 r_1)\right\} = \operatorname{sign}\{c^*(l_0)\}$$

The assertion in (3) follows by observing that $L \cdot r = L_1 \cdot r1$.

Consider $c^*(l_0) > 0$ and look for SN simple periodic solutions first. We only need to solve (5.11) for $z_1 > 0$ since $G_2(t_0, l, m, z_1)$ is odd in z_1. If z_1 is sufficiently small and t_0 is sufficiently large, we have $r_1(t_0, l, m) > 0$ and $z_0/z_1 > 0$. Thus we need to consider $z_0 > 0$ only. We now look for a solution $z_1 \in (0, \xi)$ with the corresponding $z_0 \in (0, \varepsilon)$, where $(-\varepsilon, \varepsilon)$ is the coordinate chart in the z-axis for $W^c_{\text{loc}}(E)$.

Since $z_0 = 0$ if $z_1 = 0$ and z_0 depends continuously on z_1, we can choose a small constant $\zeta > 0$ so that $z_1 = \zeta$ implies $z_0 < \varepsilon$. If δ, which defines the set $\mathcal{O}$, is small, then either $z_E < \zeta/2, m > 0$, or z_E does not exist $(m \leqslant 0)$. We can choose $\bar{t} > 0$ so large that $t_0 > \bar{t}$ implies that $0 < \Phi < 3\zeta/4$, and $|z^s| < \zeta/4$. The first estimate is based on $\Phi \to z_E < \zeta/2$ or 0 as $t_0 \to \infty$. The second estimate uses Lemma 4.1. Therefore $G_2(t_0, l, m, \zeta) = z^* = \Phi + z^s < \zeta$.

Suppose now that $L_1(t_0, l, m)\, r_1(t_0, l, m) > 1$. Then

$$\lim_{z_0 \to 0^+} \frac{z^*}{z^0}\frac{z^0}{z^1} > 1$$

From this, there exists a small $0 < z_1 < \zeta$ such that $G_2(t_0, l, m, z_1) > z_1$. Thus, there exists at least one solution $0 < z_1 < \zeta$ for (5.11) if $Lr = L_1 r_1 > 1$.

In the rest of the proof, we discuss the uniqueness or nonexistence of SN simple periodic solutions. Please refer to Fig. 5 for the flow on $W^c_{\text{loc}}(E)$.

Case (i). $0 < c^*(l_0) < 1$. For any $0 < \eta_1 < 1$, by choosing smaller δ and ε,

$$0 < \eta_1 \leqslant z_0^2/z_1^2 \leqslant 1 - \eta_1 \tag{6.10}$$

Let $m \leqslant 0$ and $z_0 > 0$. Then $0 < \Phi \leqslant z_0$ and $|z^s| < z_0 e^{-\alpha_1 t_0}$. If t_0 is sufficiently large, from Lemma 4.1, we have $(z^*)^2/z_0^2 < 1/(1 - \eta_1)$. Combining this with (6.10), we have $(z^*)^2 < z_1^2$. Therefore, there is no solution for (5.11).

Let $m > 0$ and $z_0 > z_E$. Similar to the previous case, we find no solution for (5.11).

For $m>0$, define

$$z_M=\sup\{z_0 \mid z_0^2 \leqslant (1-\eta)\,\Phi^2,\, 0 \leqslant z_0 \leqslant z_E\}$$

for some $0<\eta<\eta_1$. Clearly $0 \leqslant z_M < z_E$.

Let $m>0$ and $z_0 \in (z_M, z_E)$ we have $z_0^2>(1-\eta)\,\Phi^2$. By choosing a smaller δ, we have $z_0^2>(1-\eta_1)(z^*)^2$. From (6.10), we have $(z^*)^2<z_1^2$. There is no solution for (5.11) in this case.

Let $m>0$, and $z_0 \in (0, z_M]$ if $z_M>0$. At $z_0=z_M$ we have $(\partial/\partial z_0)(\Phi/z_0)<0$, based on Lemma 4.2. We infer that $z_0^2 \leqslant (1-\eta)\,\Phi^2$ for all $z_0 \in (0, z_M]$. Lemma 6.3 implies that (z^*/z_1) is strictly decreasing in that interval. This proves the uniqueness of solutions of (5.11) in this case.

Let $m>0$ and $1 \geqslant L_1(t_0, l, m)\, r_1(t_0, l, m)=\lim_{z_1 \to 0^+}(z^*/z_1)$. In the case $z_0 \in (0, z_M]$, arguing as in the previous case, we have $z^*/z_1<1$. This proves the nonexistence of solutions of (5.11) if $Lr \leqslant 1$.

Case (ii). $C^*(l_0)>1$. For any $0<\eta_1<1$, by choosing smaller ε and δ,

$$z_1^2 \leqslant (1-\eta_1)\, z_0^2 \tag{6.11}$$

Let $m>0$ and $0<z_0 \leqslant z_E$. Since $\Phi \geqslant z_0$ and $|z^s| \leqslant z_0 e^{-\alpha_1 t_0}$, we have $(z^*)^2>(1-\eta_1)\, z_0^2$ if t_0 is large enough. From (6.11), there is no solution to (5.11) in this case.

For $m>0$, define

$$z_m=\inf\{z_0 \mid \Phi^2 \leqslant (1-\eta)\, z_0^2 \text{ and } z_E<z_0 \leqslant \varepsilon\}$$

for some $0<\eta<\eta_1$. From the phase diagram (cf. Fig. 5), $z_E<z_m<\varepsilon$ if η is sufficiently small.

Let $m>0$ and $z_0 \in (z_E, z_m)$. Then $\Phi^2>(1-\eta)\, z_0^2$. We can have $(z^*)^2>(1-\eta_1)\, z_0^2$ if we choose t_0 large enough. Then $z^*>z_1$ based on (6.11). Equation (5.11) has no solution in this case.

Let $m>0$ and $z_0=z_m$. Then $\Phi^2 \leqslant (1-\eta)\, z_0^2$. This implies that $|\Phi^2-z_E^2| \leqslant (1-\eta)\, |z_0^2-z_E^2|$. We then can show that Φ^2/z_0^2 is decreasing for $z_0 \in [z_m, \varepsilon]$. Thus $\Phi^2 \leqslant (1-\eta)\, z_0^2$ for $z_0 \in [z_m, \varepsilon]$. From Lemma 6.3, z^*/z_1 is strictly decreasing. The solution to (5.11) either is unique or does not exist. We show that $L_1(t_0, l, m)\, r_1(t_0, l, m)>1$ in this case, so that the nonexistence becomes impossible. In fact, if $\varepsilon>0$ is small, then $r_1(t_0, l, m)>1+\eta_2$ for some $\eta_2>0$, due to $c^*(l_0)>1$. Because $m>0$ and $L_1(t_0, l, m)=e^{mt_0}+O(e^{-\alpha_1 t_0})$, let t_0 be sufficiently large, then we have $L_1>(1+\eta_2)^{-1}$. Therefore $L_1 r_1>1$.

For $m \leqslant 0$, define

$$z^m = \inf\{z_0 \mid \Phi^2 \leqslant (1-\eta) z_0^2, 0 \leqslant z_0 \leqslant \varepsilon\}$$

for some $\eta < \eta_1$ and $\eta < 1/4$. Clearly $0 \leqslant z^m < \varepsilon$ if η is sufficiently small.

Let $m \leqslant 0$ and $z_0 \in (0, z^m)$ if $z^m > 0$. Then $\Phi^2 > (1-\eta) z_0^2$. We can make $(z^*)^2 > (1-\eta_1) z_0^2$ by choosing $\varepsilon < 0$ smaller. Therefore $z_1 < z^*$ by (6.11). There is no solution to (5.11).

Let $m \leqslant 0$ and $z_0 = z^m$. Then $\Phi^2 \leqslant (1-\eta) z_0^2$. By Lemma 4.2, Φ^2/z_0^2 is decreasing for $z_0 \in [z^m, \varepsilon)$. Thus $\Phi \leqslant (1-\eta) z_0^2$ for $z_0 \in [z^m, \varepsilon]$. Then z^*/z_1 is strictly decreasing by Lemma 6.3. Either there is no solution or the solution is unique to (5.11) when $z_0 \in [z^m, \varepsilon)$.

Let $m \leqslant 0$ and $0 \leqslant L_1(t_0, l, m)\, r_1(t_0, l, m) \leqslant 1$. By (6.11), $r_1^2 \geqslant (1-\eta_1)^{-1}$. Thus, $L_1^2 < 1-\eta_1$. If we choose $\varepsilon > 0$ small, we have $(z^*)^2/z_0^2 < 1-\eta_2$ for some $0 < \eta_2 < \eta_1$. And also, $\Phi^2/z_0^2 < 1-\eta$ for some $0 < \eta < \eta_2$ in the interval $[z^m, \varepsilon]$. Thus z^*/z_1 is strictly decreasing. Since $\lim_{z_1 \to 0} z^*/z_1 \leqslant 1$, there is no solution to (5.11) in this case.

Case (iii). $C^*(l_0) = 1$. For any $0 < \eta < 1$, we can choose a smaller ε so that

$$1-\eta < (z_0/z_1) < 1+\eta \tag{6.12}$$

If ε is small, then $r_1(t_0, l, m) < 1+\eta$. Let $L_1(t_0, l, m)\, r_1(t_0, l, m) > 1+\delta$ for some $\delta > \eta$. Then

$$L_1 > \frac{1+\delta}{1+\eta} \geqslant 1+\eta_1$$

for some $0 < \eta_1 < \delta$. From $L_1(t_0, l, m) = e^{mt_0} + O(e^{-\alpha_1 t_0})$, if t_0 is large enough, we have $m > 0$, and $mt_0 > \varepsilon_1$ for some $\varepsilon_1 > 0$. From Lemma 4.3, case (a), we have $z_0^2 \leqslant (1-\eta_3)\Phi^2$, or $|\Phi^2 - z_E^2| \leqslant (1-\eta_3)\, |z_0^2 - z_E^2|$ or $e^{mt_0} - 1 \geqslant \eta_3$ for some $\eta_3 > 0$. Therefore z^*/z_1 is strictly decreasing and the solution to (5.11) is unique.

Let $L_1(t_0, l, m)\, r_1(t_0, l, m) < 1-\delta$, for some $\delta > \eta$. By a similar argument, $mt_0 < -\varepsilon_1$ for some $\varepsilon_1 > 0$. Also, we must have $m < 0$. From Lemma 4.3, case (b), we then have $\Phi^2 \leqslant (1-\eta_3) z_0^2$ for some $\eta_3 > 0$. Thus z^*/z_1 is strictly decreasing. There is no solution to (5.11) since $\lim_{z_1 \to 0^+} z^*/z_1 = L_1 \cdot r_1 < 1-\delta$.

We have completed the discussion for the case $C^*(l_0) > 0$.

Consider $C^*(l_0) < 0$ and symmetric double periodic SN solutions next. We need to solve (5.11)′. We can divide the case into three subcases—case (iv), $-1 < C^*(l_0) < 0$; case (v), $C^*(l_0) < -1$; and case (vi), $C^*(l_0) = -1$.

They are analogous to cases (i), (ii), and (iii), respectively. The proofs are similar to the previous cases and will not be rendered here.

This completes the proof of Theorem 3.4. □

Proof of Theorem 3.5. From our assumption, $c^*(l_0) = r_1(\infty, l_0, m_0) = 0$ and $(\partial/\partial l)\, r_1(\infty, l_0, m_0) \neq 0$. Using the implicit function theorem we can find a unique C^1 function $l = l^*(m)$ so that $r(\infty, l^*(m), m) = 0$, $|m| < \delta$.

Since $|\tilde{y}(t_0, z_1, \mu)| < Ce^{-\alpha_1 t_0}$ and $|k^*(t_0, l, m, z_1) - k^*(\infty, l, m, z_1)| < Ce^{-\alpha_1 t_0}$, we have $r_1(t_0, l^*(m), m) = O(e^{-\alpha_1 t_0})$. Please refer to (6.1) and (6.2) for the definitions of z_0 and r_1.

Since z_0 is an odd function of z_1, for some $\bar{C} > 0$, we have

$$\left|\frac{z_0}{z_1}\right| \leqslant \bar{C}(e^{-\alpha_1 t_0} + z_1^2 + |l - l^*(m)|)$$

Since $\Phi(t_0, z_0, \mu)$ satisfies the equation $z' = mz - \hat{c}z^3 + \cdots$, we have $|\Phi/z_0| \leqslant e^{mt_0}$ if $|z_0| < \varepsilon$ and $|\Phi| < \varepsilon$. Thus, from Lemma 4.1,

$$\left|\frac{z^*}{z_0}\right| \leqslant e^{mt_0} + Ce^{-\alpha_1 t_0} \leqslant 2e^{mt_0}$$

if $t_0 > \bar{t}$ is sufficiently large. We now choose $\delta(T) = Ce^{-mT}$, where C is a small constant. If $|l - l^*(m)| < \delta(T)$ and $z_1 < (\delta(T))^{1/2}$, then $\bar{C}(e^{-\alpha_1 t_0} + z_1^2 + |l - l^*(m)|) < \frac{1}{2} e^{-mt_0}$. Therefore, $|z^*/z_0|\ |z_0/z_1| < 1$. The bifurcation equation (5.11) or (5.11)′ has no solution in this case. □

7. NUMERICAL TEST ON A PREDATOR–PREY MODEL

The following predator–prey model was proposed by Freedman and Wolkowicz [13] to describe group defense of prey against predatation.

$$\begin{aligned} \dot{u} &= 2u\left(1 - \frac{u}{k}\right) - 9vp(u) \\ \dot{v} &= v(-\gamma + 11.3p(u)) \qquad (7.1) \\ u &= \text{prey}, \qquad v = \text{predator} \end{aligned}$$

where $p(u) = u/(u^2 + 3.35u + 13.5)$ represents the interaction between prey and predator. For a large range of (γ, k), (7.1) has two interior equilibria $(\bar{u}_0, \bar{v}_0)$ and $(\bar{\bar{u}}_0, \bar{\bar{v}}_0)$, with $\bar{\bar{u}}_0 < \bar{u}_0$. Here $p(\bar{u}_0) = p(\bar{\bar{u}}_0) = \gamma/11.3$, while $\bar{v}_0$, $\bar{\bar{v}}_0$ can be solved from the first equation of (7.1). we are interested in the equilibrium $(\bar{u}_0, \bar{v}_0)$, which is hyperbolic, and shall be denoted $E = E(\gamma, k)$.

Let the Jacobian matrix at E be $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. It is shown in Ref. 25 that $a>0, b>0, c>0$, and $d=0$. Thus, hypothesis H_3 in Section 3 is satisfied.

Freedman and Wolkowicz have discovered a curve $\mathscr{S} \subset \mathbb{R}^2$ such that if $(\gamma, k) \in \mathscr{S}$, then (7.1) possesses a homoclinic solution $q(t)$ asymptotic to the equilibrium $E(\gamma, k)$. Numerical computation shows that the curve $\mathscr{S}$ can be parameterized by $\bar{u}_0$ and is plotted in Fig. 7. For each $(\gamma, k) \in \mathscr{S}$, let γ be fixed and let k vary. Then the homoclinic solution breaks. The derivative of the gap between $W^u(E)$ and $W^s(E)$ with respect to k can be evaluated by the Melnikov integral $M_\gamma(k)$, as in H_4. The Melnikov integral has been computed numerically, and the result is plotted in Fig. 8. Evidently, $M>0$ for all the values considered. Thus, hypotheses H_2 and H_4 in Section 3 are satisfied for those parameter values.

The smooth dependence of $M_\gamma(k)$ on $\bar{u}_0$ indicates that $M>0$ is not a numerical artifice.

In the remainder of this section, we fix $(\bar{u}_0, \gamma, k) = (5.49178,\ 1.0,\ 6.87433)$. After adding diffusions $(d_1 u_{\xi\xi}, d_2 v_{\xi\xi})$, we consider a system of PDEs in the domain $0<\xi<1$ with Neumann boundary conditions; cf. (1.2). Let Γ be the curve in the (d_1, d_2)-plane on which (1.2) has a zero eigenvalue with associated eigenvectors in $\mathscr{X}_1$. Since $bc>0$ and $a>0, \Gamma$ is depicted in Fig. 2, Case 1.

For $(d_1, d_2) \in \Gamma$, we now compute $W^c_{\text{loc}}(E)$ and the flow on it, up to $O(\rho^3)$, where $\rho = |u-\bar{u}_0| + |v-\bar{v}_0|$. Since the boundary conditions are of the Neumann type, we will expand (u, v) into Fourier cosine series. Let $(\bar{u}_0 + \sum_0^\infty u_n \cos n\pi\xi,\ \bar{v}_0 + \sum_0^\infty v_n \cos n\pi\xi) \in W^c_{\text{loc}}(E)$. Let $(u_1 \cos \pi\xi, v_1 \cos \pi\xi)$,

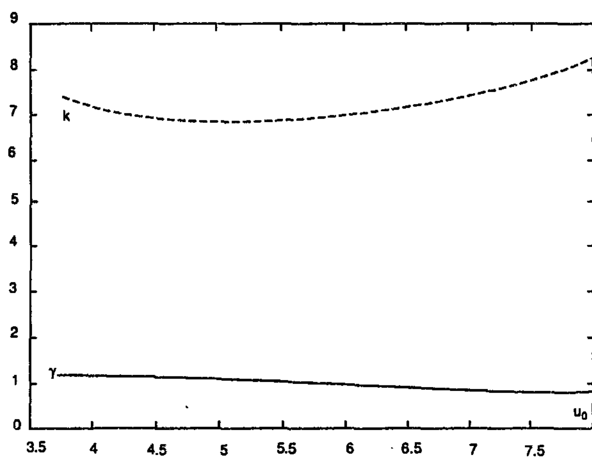


Fig. 7. Values of (γ, k) where a homoclinic orbit to (7.1) exits are plotted, using u_0 as an independent variable.

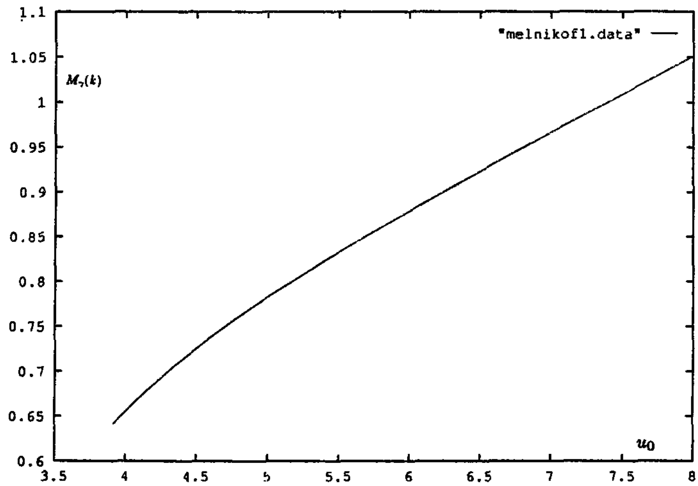


Fig. 8. The Melnikov integral $M_y(k)$ is plotted, using u_0 as an independent variable.

with $(u_1, v_1) = (\eta, 1)$, be the unique eigenvector corresponding to the eigenvalue 0, up to a constant multiple. Then $u_1 = \eta v_1 + \phi(v_1)$, $u_n = u_n^*(v_1)$, and $v_n = v_n^*(v_1)$ for $n \neq 1$, where ϕ, u_n^*, $v_n^* = O(v_1^2)$. This is due to the fact that $W_{\text{loc}}^c(E)$ is tangent to the zero eigenvector corresponding to $(u_1, v_1) = (\eta, 1)$. Because of the R symmetry (Theorem 2.7), we have $\phi = O(\rho^3)$.

The Fourier coefficients (u_n, v_n) are functions of t. They satisfy

$$\begin{aligned} u_n' &= -d_1 n^2\pi^2 u_n - a u_n - b v_n + [f(u, v)]_n + O(\rho^4) \\ v_n' &= -d_2 n^2\pi^2 v_n - c u_n + [g(u, v)]_n + O(\rho^4) \end{aligned} \tag{7.2}$$

where f and g are polynomials of degree 3. For $h \in L^2(0, 1)$, we use $[h]_n$ to denote the nth Fourier cosine coefficient for h. Using some basic trigonometry formulas, we can rewrite $[f(u, v)]_n$ and $[g(u, v)]_n$ in terms of $\{u_n\}_0^\infty$, $\{v_n\}_0^\infty$. Only finitely many terms are needed here since other terms will be included in $O(\rho^4)$.

We can now use the Taylor expansion method in Ref. 2 to obtain a power series expansion of $\phi(v_1)$ and the flow on the center manifold. The function ϕ has the form $\phi(v_1) = c v_1^3 + O(v_1^5)$. And the flow on the center manifold has the form

$$v_1' = \tilde{c} v_1^3 + O(\rho^4)$$

When (d_1, d_2) moves along the curve Γ, values of $\tilde{c}$ have been computed numerically and the results are depicted in Fig. 9, with $\hat{c}$

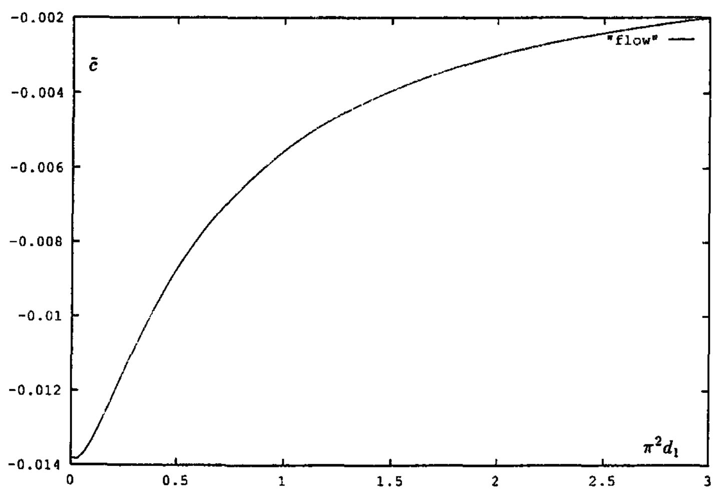


Fig. 9. Values of $\tilde{c}$ are plotted when (d_1, d_2) moves along Γ, which are parameterized by d_1.

against $\pi^2 d_1$. It verifies that $\tilde{c} < 0$ for the portion of Γ under consideration. Since there is a diffeomorphism between v_1 and z, hypothesis H_5 in Section 3 has been verified numerically.

We now compute $c^*(d_1, d_2)$ for $(d_1, d_2) \in \Gamma$, $0 < \pi^2 d_1 < 3$. Again, we fix $u_0 = 5.49178$, $\gamma = 1$, and $k = 6.87433$. Numerical results of c^* are depicted in Fig. 10. We have found a point $d_1^* = 0.183$ such that $c^*(d_1, d_2) < (> \text{ or } =)$ 0 if $d_1 < (> \text{ or } =)\ d_1^*$. The results also show that $\partial c^*(l_0)/\partial l \neq 0$, where $(l_0, 0) \in \Gamma$ corresponds to $d_1 = d_1^*$. Therefore all the twisted, nontwisted, and degenerate cases have be found in Freedman and Wolkowicz's example. However the case $c^*(l_0) \geq 1$ or ≤ -1 has not been found in this example. Numerical and theoretical results also indicate that there is a point $(d_1, d_2) = (0.0093, 0.0093)$ where $c^* = 0$. However, the numerical error near that point is too large to be trustworthy. Thus, we do not include it in Fig. 10.

We end this section by proving Lemma 6.1.

Proof of Lemma 6.1. Recall the definition of $\hat{z}$ in Section 5 and r_1 in (6.1) and (6.2). We need to consider the zth component of $(\partial/\partial z_1)$ $U_*(t_1, \bar{x}, y_1, z_1, \mu)$, with $y_1 = 0$ and $z_1 = 0$. Let $(\partial/\partial z_1)\, U_*(t) = (x(t), y(t), z(t))$, $0 \leq t \leq t_1$. It satisfies the linear variational equation (3.4) and the initial conditions are $x(0) = 0$, $y(0) = 0$, $z(0) = 1$. We now extend the solution $(x(t), y(t), z(t))$ to $t \leq 0$. Notice that we are treating an infinite-dimensional system, so the backward extension of a solution is not

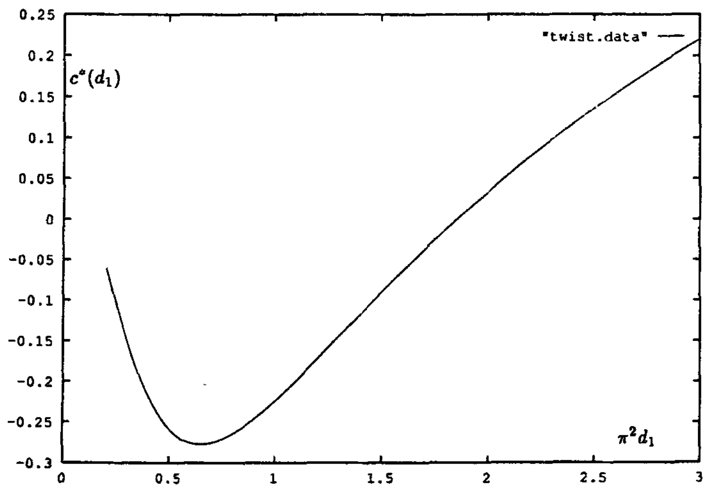


Fig. 10. Values of c^* are plotted when (d_1, d_2) moves along Γ, which are parameterized by d_1.

unique. Using the flat coordinates (4.1), in a neighborhood of 0, we write (3.4) as

$$\begin{aligned} x' &= A_1 x + (D_x g_1)\, x + (D_y g_1)\, y + (D_z g_1)\, z \\ y' &= A_2 y + D_y g_2(x_q(t), y_q(t), z_g(t), \mu)\, y \\ z' &= A_3 z + (D_x g_3)\, x + (D_y g_3)\, y + (D_z g_3)\, z \end{aligned} \tag{7.3}$$

where $q(t) = (x_q(t), y_q(t), z_q(t))$ in the flat coordinates. Here we have used the facts that $y_q(t) = 0$, $t \leqslant 0$, and $g_2(x, 0, z, \mu) = 0$ to simplify the second equation of (7.3).

First, let $y(t) \equiv 0$ for $t \leqslant 0$, which solves the second equation. Then $(x(t), z(t))$ can be solved uniquely from (7.3) backward in time. We now show that $x(t) \equiv 0$ for $t \leqslant 0$. If $(x(t), 0, z(t))$ is a solution for (7.3), so is $R(x(t), 0, z(t)) = (x(t), 0, -z(t))$. Thus, $(x(t), 0, 0)$ is a solution of (7.3). Since $x(0) = 0$, solving the one-dimensional ODE for $x(t)$ we have $x(t) \equiv 0$ for $t \leqslant 0$. Observe that $g_1(0, y, z, \mu) = 0$. Thus, the first equation is valid even if $z \neq 0$. The equation for $z(t)$ becomes

$$z' = A_3 z + D_z g_3(x_q(t), y_q(t), z_q(t), \mu)\, z \tag{7.4}$$

with $A_3 = 0$. In our flat coordinates, $z_q(t) = 0$ and $y_q(t) = 0$, we have $D_z g_3(x_q(t), 0, 0, \mu) = D_z g_3(0, 0, 0, \mu) = 0$, since zero is an eigenvalue for

$(d_1, d_2) \in \Gamma$. Here we used the fact that $g_3(x, 0, z, \mu) = g_3(0, 0, z, \mu)$ on W^{cu}_{loc}. Thus, (7.4) becomes $z' = 0$, and $z(t) \equiv 1$ for $t \leqslant 0$. $(x(t), y(t), z(t)) = (0, 0, 1)$ for $t \leqslant 0$.

We now have $(x(0), y(0), z(0)) \in \mathscr{X}_1$ and shall remain in $\mathscr{X}_1$ for $t \geqslant 0$. In particular, $x(t) = 0$ for all $t \in R$. According to Section 3, $(x(t), y(t), z(t)) \to (0, 0, c^*)$ as $t \to +\infty$. However, because the coordinates are flat, $g_3(0, y, z, \mu) = g_3(0, 0, z, \mu)$. Also, $x_q(t) \equiv 0$ for $t \geqslant t_1$. Thus $(\partial/\partial y)$ $g_3(0, y_q(t), 0, \mu) = 0$ for $t \geqslant t_1$. Again, $z(t), t \geqslant t_1$, satisfies (7.4) with $x_q(t) = 0$ and $z_q(t) = 0$. Since $D_z g_3(0, y_q(t), 0, \mu) = D_z g_3(0, 0, 0, \mu) = 0$. We have $z(t) = \text{constant}$ for $t \geqslant t_1$. Thus $z(t_1) = c^*$. This proves Lemma 6.1. □

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