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SADDLE-NODE BIFURCATIONS OF MULTIPLE HOMOCLINIC SOLUTIONS IN ODES

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Abstract. We study codimension 3 degenerate homoclinic bifurcations under periodic perturbations. Assume that among the 3 bifurcation equations, one is due to the homoclinic tangecy along the orbital direction. To the lowest order, the bifurcation equations become 3 quadratic equations. Under generic conditions on perturbations of the normal and tangential directions of the homoclinic orbit, up to 8 homoclinic orbits can be created through saddle-node bifurcations. Our results generate the homoclinic tangency bifurcation in Guckenheimer and Holmes [8].

1. Introduction.

One of the most studied homoclinic bifurcation problems is the periodically perturbed system:

$$\dot{y}(t) = f(y(t)) + \mu g(y(t), t, \mu), \quad y \in \mathbb{R}^n. \tag{1}$$

The unperturbed autonomous equation

$$\dot{y}(t) = f(y(t)), \tag{2}$$

satisfies the following hypotheses:

(H1): $f \in C^3$. $f(0) = 0$ and the eigenvalues of $Df(0)$ lie off the imaginary axis.

(H2): Equation (2) has a homoclinic solution $\gamma(t)$ asymptotic to the hyperbolic equilibrium $y = 0$. That is,

$$\dot{\gamma}(t) = f(\gamma(t)) \text{ and } \lim_{t\to\pm\infty} \gamma(t) = 0.$$

The variational equation of (2) along the homoclinic solution γ is

$$\dot{u}(t) = Df(\gamma(t))u(t). \tag{3}$$

Since $\dot{\gamma}$ is a bounded solution of (3), system (3) has $d \geq 1$ linearly independent bounded solutions.

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We assume g satisfies

(H3): $g \in C^3$, and $g(y, t+1, \mu) = g(y, t, \mu)$.

By $(H1), y = 0$ is a hyperbolic equilibrium of (2). Generically, equation (1) has a hyperbolic periodic orbit $\theta(\mu, t) := O(|\mu|)$ near 0. Using the change of variable $y = x + \theta(\mu, t)$, (1) becomes

$$\dot{x} = f(x) + \mu\bar{g}(x(t), t, \mu)$$

where $\bar{g}$ satisfies $\bar{g}(0, t, \mu) = 0$.

After dropping $-$ on g, we consider an equivalent system to (1):

$$\dot{x}(t) = f(x(t)) + \mu g(x(t), t, \mu), \quad x \in \mathbb{R}^n,\ \mu \in \mathbb{R}. \tag{4}$$

The new system satisfies (H1)-(H3) and

(H4): $g(0, t, \mu) = 0$.

Since $\gamma(t)$ is a homoclinic solution of the autonomous system (2), a time shift of $\gamma(t)$ is also a solution of (2). That is, (2) has a family of homoclinic orbits $\gamma(t-\tau)$ for $\tau \in \mathbb{R}$. We look for a solution whose orbit is close to that of $\gamma(t)$. Our goal is to find a solution $x(t)$ which is a small perturbation of $\gamma(t-\tau)$ for some $\tau \in \mathbb{R}$, where τ is a parameter, equivalently, $x(t+\tau)$ is a small perturbation of $\gamma(t)$. The parameter τ can be determined by a phase condition as follows: Let $x(t+\tau) = \gamma(t) + z(t)$ then $z(0) \perp \dot{\gamma}(0)$. After a time shift, we assume that $x(t)$ is a small perturbation of $\gamma(t)$ and satisfies the following system:

$$\dot{x}(t) = f(x(t)) + \mu g(x(t), t+\tau, \mu). \tag{5}$$

From (H4), $x = 0$ is a hyperbolic equilibrium for small μ. For $\mu = 0$, let $W^s(0)$, $W^u(0)$ be the stable and unstable manifolds of $x = 0$. Clearly, the homoclinic orbit γ lies on $W^s(0) \bigcap W^u(0)$. If $d = \dim(T_{\gamma(0)}W^s(0) \bigcap T_{\gamma(0)}W^u(0))$, then the variational equation of (2) along γ has d dimension bounded solutions.

When $\mu \neq 0$, (5) may have bifurcations near γ. The case $d = 1$ has been extensively studied. In this case breaking of the homoclinic orbit γ is restored by choosing the parameter τ, see [8]. Hale [9] proposed to study the degenerate cases where $d \geq 2$. The case $d = 2$ has been considered in [17]. In this paper we treat the case $d = 3$. Using the method of Lyapunov-Schmidt reduction and exponential dichotomies, we derive a system of bifurcation functions $H_j, 1 \leq j \leq 3$, the zeros of which correspond to the bifurcations of homoclinic solutions for (5) (For the definitions of H_j, see (23)). To the lowest degree of H_j, the bifurcation equations reduce to three quadratic equations $M_j = 0, 1 \leq j \leq 3$ (For the definitions of M_j, see (11)).

The last equation $M_3 = 0$ can be dealt with by selecting the parameter τ as usual, while $M_j = 0, j = 1, 2$ can be simplified by the codiagonalization of quadratic forms. We show that $M_j = 0, j = 1, 2$, can have 4 non-degenerate solutions. Substituting them into the last equation, we show $M_3 = 0$ undergoes the saddle-node bifurcation with respect to the parameter τ. Thus, each of the 4 solutions can generate 2 solutions, and 8 solutions can be obtained for $M_j = 0, 1 \leq j \leq 3$. Finally, as perturbations to $M_j = 0$, the system $H_j = 0, 1 \leq j \leq 3$, can have up to 8 solutions.

Codiagonalization of matrices has been used by Jibin Li and Lin [15] to study systems of coupled KdV equations, and will be one of the main tool used in this paper. Given a symmetric real matrix $B \in \mathbb{R}^{2\times 2}$, then

$$F(x_1, x_2) = (x_1, x_2)B(x_1, x_2)^T$$

is a quadratic form associated to B. If B is diagonalized by a nonsingular matrix M: $M^TBM = \text{diag}(d_1, d_2)$, then

$$F(x_1, x_2) = (y_1, y_2)\text{diag}(d_1, d_2)(y_1, y_2)^T = d_1 y_1^2 + d_2 y_2^2,$$

where $(x_1, x_2)^T = M(y_1, y_2)^T$. The symmetric transformation described above is also called the congruence diagonalization. It should not confused with the similarity transformation of B which is defined by $M^{-1}BM$. For example the matrix $\text{diag}(\lambda_1, -\lambda_2), \lambda_j > 0$, can be reduced to $\text{diag}(1, -1)$ by the matrix $M = \text{diag}(1/\sqrt{\lambda_1}, 1/\sqrt{\lambda_2})$, which is a symmetric reduction, not similarity reduction.

In §2, we introduce notations to be used in this paper. We also present the reduced bifurcation functions (11) which, to the lowest degree, represent the breaking of the homoclinc orbits under the periodic perturbations. In §3, using the Lyapunov-Schmidt reduction, we derive the bifurcation equations (23), which to the lowest degree, become three quadratic equations $M_j = 0, 1 \le j \le 3$. In §4, we introduce conditional max/min problems to codiagonalize two quadratic forms, and obtain general conditions under which the two quadratic equations may have 4 real valued solutions. The case when one equation is elliptic is considered in §4.1 (Theorem 4.2). The other case when both equations are hyperbolic is considered in §4.2 (Theorem 4.2). Conditions for the existence of 4 real valued solutions to quadratic systems after codiagonalization are given in §4.3 (Theorems 4.3 and 4.4). In §5, we prove the existence of homoclinic solutions by solving the bifurcation equations using the contraction mapping principle (Theorems 5.1 and 5.2). In §6 (Theorem 6.1), we prove the transversality of the homoclinic solutions obtained in §5.

2. **Notations and preliminaries. Notations.** Let X, Y be Banach spaces and $L : X \to Y$ be a linear operator. We use $N(L)$ and $R(L)$ to denote the null subspace and range subspace of L, respectively. Since $y = 0$ is a hyperbolic equilibrium, from [6, 20], (3) has exponential dichotomies on $J = \mathbb{R}^\pm$ respectively. Let $U(t)$ be the fundamental matrix of (3). Then there exist projections to the stable and unstable subspaces, $P_s + P_u = I$, and constants $m > 0$, $K_0 \ge 1$ such that

$$\begin{aligned} &(i)\ |U(t)P_sU^{-1}(s)| \le K_0e^{2m(s-t)},\ \text{for } s \le t \text{ on } J,\\ &(ii)\ |U(t)P_uU^{-1}(s)| \le K_0e^{2m(t-s)},\ \text{for } t \le s \text{ on } J. \end{aligned} \tag{6}$$

For the same $m > 0$, define the Banach space

$$\mathcal{Z} = \{z \in C^0(\mathbb{R}, \mathbb{R}^n) : \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|} < \infty\},$$

with the norm $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|}$. The linear variational system

$$Lu := \dot{u} - Df(\gamma)u = h \tag{7}$$

will be considered in $\mathcal{Z}$. The adjoint operator for L is

$$L^*\psi := \dot{\psi} + (Df(\gamma))^*\psi. \tag{8}$$

The domains of (7) and (8) are the dense subset of $\mathcal{Z}$, defined as

$$D(L) := \{u : u, u_t \in \mathcal{Z}\}, \quad D(L^*) := \{\psi : \psi, \psi_t \in \mathcal{Z}\}.$$

From the theory of homoclinic bifurcations [20], $L : \mathcal{Z} \to \mathcal{Z}$ is a Fredholm operator with index 0. The range of L is orthogonal to the null space of L^*. That is

$$h \in R(L) \text{ iff } \int_{-\infty}^{\infty} \langle\psi(t), h(t)\rangle dt = 0,\ \text{for all } \psi \in N(L^*). \tag{9}$$

From $d = 3$, $N(L)$ is three dimensional. Note that $\dot{\gamma} \in N(L)$. Without loss in generality, let (u_1, u_2, u_3) be a basis of $N(L)$, where we choose $u_3 = \dot{\gamma}$. And let (ψ_1, ψ_2, ψ_3) be a basis of $N(L^*)$.

We define some Melnikov types of integrals that will be used in the future. For integers $p, q = 1, 2$ and $i = 1, 2, 3$, define

$$\begin{aligned} b_{pq}^{(i)} &= \int_{-\infty}^{+\infty} \langle \psi_i(t), D^2 f(\gamma(t)) u_p(t) u_q(t) \rangle dt, \ \ p, q = 1, 2, \\ \tilde{a}_i(\tau) &= \int_{-\infty}^{+\infty} \langle \psi_i(t), g(\gamma(t), t + \tau, 0) \rangle dt, \\ \tilde{c}_p^{(i)}(\tau) &= \int_{-\infty}^{+\infty} \langle \psi_i(t), (D^2 f(\gamma(t)) \varphi(t, \tau) + D_1 g(\gamma(t), t + \tau, 0)) u_p(t) \\ &\quad + D_2 g(\gamma(t), t + \tau, 0) \rangle dt, \ \ p = 1, 2, \end{aligned} \tag{10}$$

where $\varphi(t, \tau) = K(I - P) g(\gamma(t), t + \tau, 0)$ (See the definitions of operators K and P in Section 3).

We look for conditions so that (5) can have homoclinic solutions near γ. Let $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ and $\tilde{\boldsymbol{C}}_i(\tau) = (\tilde{c}_1^{(i)}(\tau), \tilde{c}_2^{(i)}(\tau))$. Consider the reduced bifurcation functions $M_i : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined as

$$M_i(\boldsymbol{\beta}, \tau, \mu) = \frac{1}{2} \sum_{p,q=1}^{2} b_{pq}^{(i)} \beta_p \beta_q + \mu \tilde{a}_i(\tau) + \mu \tilde{\boldsymbol{C}}_i(\tau) \boldsymbol{\beta}, \ \ i = 1, 2, 3, \tag{11}$$

where $b_{pq}^{(i)}, \tilde{a}_i(\tau), \tilde{c}_p^{(i)}(\tau)$ are defined in (10). To the lowest degree, (11) describes the jump discontinuity along the direction of ψ_i, see §3. Define the 2×2 matrices $B^{(i)} = (b_{pq}^{(i)})$, $i \in \{1, 2, 3\}$. We need to solve the following system of quadratic equations

$$\boldsymbol{\beta}^T B^{(i)} \boldsymbol{\beta} = \mu a_i(\tau) + \mu \boldsymbol{C}_i(\tau) \boldsymbol{\beta}, \quad i = 1, 2, 3, \tag{12}$$

where $a_i(\tau) = -2\tilde{a}_i(\tau), \boldsymbol{C}_i(\tau) = -2\tilde{\boldsymbol{C}}_i(\tau)$. The first two equations of (12) form a quadratic system for $\boldsymbol{\beta}$ if τ is given. Geometric method based on circular, elliptic and hyperbolic rotation will be used to codiagonalize the first two equation, which can significantly simplify the system. After codiagonalizing the first two equations, (12) becomes

$$\begin{aligned} d_{11}\alpha_1^2 + d_{12}\alpha_2^2 &= \mu a_1(\tau) + \mu \boldsymbol{C}_1(\tau) \boldsymbol{\alpha}, \\ d_{21}\alpha_1^2 + d_{22}\alpha_2^2 &= \mu a_2(\tau) + \mu \boldsymbol{C}_2(\tau) \boldsymbol{\alpha}, \\ d_{31}\alpha_1^2 + d_{32}\alpha_1\alpha_2 + d_{33}\alpha_2^2 &= \mu a_3(\tau) + \mu \boldsymbol{C}_3(\tau) \boldsymbol{\alpha}. \end{aligned}$$

For each (τ, μ), we first solve the first two equations for $\alpha_1(\tau, \mu), \alpha_2(\tau, \mu)$, where $|\alpha_i| = O(\sqrt{|\mu|})$. By substituting to the third equation, we get a nonlinear equation $G(\tau, \mu) = 0$ (For the definition of G, see (37)). Let $a = d_{11}d_{22} - d_{12}d_{21}$. Expand G in the powers of μ, we have $G(\tau, \mu) = \mu F_1(\tau) + |\mu|^{3/2} F_2(\tau) + O(|\mu|^2)$, where

$$\begin{aligned} F_1(\tau) =& - a a_3(\tau) + d_{31}(a_1(\tau) d_{22} - a_2(\tau) d_{12}) + d_{33}(a_2(\tau) d_{11} - a_1(\tau) d_{21}) \\ &+ d_{32} \sqrt{(a_1(\tau) d_{22} - a_2(\tau) d_{12})(a_2(\tau) d_{11} - a_1(\tau) d_{21})}, \\ F_2(\tau) =& \frac{1}{|a|^{\frac{3}{2}}} \{ a [c_{31}(\tau)(d_{22} a_1(\tau) - d_{12} a_2(\tau))^{\frac{1}{2}} + c_{32}(\tau)(d_{11} a_2(\tau) - d_{21} a_1(\tau))^{\frac{1}{2}}] \end{aligned}$$

$$+ d_{31}f_1(\tau) + d_{33}f_2(\tau) + \frac{d_{32}}{2}[\frac{(d_{22}a_1(\tau) - d_{12}a_2(\tau))^{\frac{1}{2}}f_2(\tau)}{(d_{11}a_2(\tau) - d_{21}a_1(\tau))^{\frac{1}{2}}}$$
$$+ \frac{(d_{11}a_2(\tau) - d_{21}a_1(\tau))^{\frac{1}{2}}f_1(\tau)}{(d_{22}a_1(\tau) - d_{12}a_2(\tau))^{\frac{1}{2}}}]\}.$$

We now assume

- **(H5):** There exist $\tau_0 \in \mathbb{R}$ such that $F_1(\tau_0) = F_1'(\tau_0) = 0, F_1''(\tau_0) \neq 0$ and $F_2(\tau_0) \neq 0$.

Remark 1. In the special case $d = 1$, the term d_{ij} and F_2 do not appear. And $F_1(\tau) = -a(\tau)$, where

$$a(\tau) = \int_{-\infty}^{+\infty} \langle \psi(t), g(\gamma(t), t+\tau, 0)\rangle dt.$$

Then (H5) reduces to the following

- **(H5'):** $a(\tau_0) = a'(\tau_0) = 0,\ a''(\tau_0) \neq 0$.

In this case bifurcations due to homoclinic tangencies may occur, [8].

By changing ψ_i to $-\psi_i$, we can change $B^{(i)}$ to $-B^{(i)}$ without altering the result of the paper. Hence we assume the following conditions are satisfied:

- **(H6):** If the eigenvalus of $B^{(i)}$ satisfy $\lambda_1\lambda_2 > 0$, then $\lambda_1 > 0,\ \lambda_2 > 0$. If the eigenvalues of $B^{(i)}$ satisfy $\lambda_1\lambda_2 = 0$, then $\lambda_1 = 0, \lambda_2 > 0$.

3. Derivation of the bifurcation equations using the Lyapunov-Schimidt reduction.

In this section, we look for conditions such that for small $\mu \neq 0$, (5) may have homoclinic solutions γ_μ with $\|\gamma - \gamma_\mu\| = O(\sqrt{|\mu|})$.

Let $D_i h$ or $D_{ij} h$ denote the derivatives of a multivariate function h with respect to its i-th or the i,j-th variables. With the change of variable $x(t) = \gamma(t) + z(t)$, where $z(0) \perp \dot{\gamma}(0)$, (5) is transformed to

$$\dot{z} = Df(\gamma)z + \widetilde{g}(z, \tau, \mu), \tag{13}$$

where

$$\begin{aligned}\widetilde{g}(z,\tau,\mu)(t) =& f(\gamma(t)+z(t)) - f(\gamma(t)) - Df(\gamma(t))z \\ &+ \mu g(\gamma(t)+z(t), t+\tau, \mu).\end{aligned} \tag{14}$$

Lemma 3.1. *The function $\widetilde{g}(\cdot,\tau,\mu) : \mathcal{Z} \mapsto \mathcal{Z}$ satisfies the following properties:*

$$(1)\ \widetilde{g}(0,\tau,0) = 0,\ D_1\widetilde{g}(0,\tau,0) = 0,$$
$$(2)\ D_{11}\widetilde{g}(0,\tau,0) = D^2 f(\gamma),$$
$$(3)\ \frac{\partial \widetilde{g}}{\partial \mu}(0,\tau,0) = g(\gamma, t+\tau, 0).$$

Proof. It is easy to check from (14) that (1)-(3) hold. We now prove $\widetilde{g}(\cdot,\tau,\mu) : \mathcal{Z} \mapsto \mathcal{Z}$.

Let $\bar{B}_1(0,\delta) \subset \mathbb{R}^n$ and $\bar{B}_2(0,\delta) \subset \mathbb{R}$ be closed balls with radius $\delta > 0$ centered at the origins. For arbitrary $z \in \mathcal{Z}$, we can take a large $\delta > 0$ such that $z(t), \gamma(t), \gamma(t) + z(t) \in \bar{B}_1(0,\delta)$ for $t \in \mathbb{R}$. By (H1) and (H3), there exist a constant A_0 such that

$$|D_1\widetilde{g}(x,\tau,\mu)| < A_0,\ |D_1 g(x, t+\tau, \mu)| < A_0$$

for $(x, \tau, \mu) \in \bar{B}_1(0, \delta) \times \mathbb{R} \times \bar{B}_2(0, \delta)$. Since γ is a homoclinic solution and $z \in \mathcal{Z}$, there is $A_1 > 0$ such that

$$|\gamma(t)| \leq A_1 e^{-m|t|}, \ |z(t)| \leq A_1 e^{-m|t|}.$$

Define a map $\sigma : [0, 1] \to \mathcal{Z}$ by $\sigma(s) = \widetilde{g}(sz, \tau, \mu) - \mu g((1 - s)\gamma, t + \tau, \mu)$. By the smoothness of f, g, we see that $\sigma \in C^1$ and $\sigma(0) = 0$, then

$$\begin{aligned}\widetilde{g}(z, \tau, \mu)(t) &= \sigma(1) - \sigma(0) = \int_0^1 \sigma'(p)dp \\ &= \int_0^1 D_1\widetilde{g}(pz(t), \tau, \mu)z(t) + \mu D_1 g((1 - p)\gamma(t), t + \tau, \mu)\gamma(t)dp.\end{aligned}$$

Therefore

$$\begin{aligned}|\widetilde{g}(z, \tau, \mu)(t)| &\leq |D_1\widetilde{g}(pz(t), \tau, \mu)||z(t)| + |\mu||D_1 g((1 - p)\gamma(t), t + \tau, \mu)||\gamma(t)| \\ &\leq A_0 A_1(1 + |\mu|)e^{-m|t|}, \qquad (15)\end{aligned}$$

which implies that $\widetilde{g}(z, \tau, \mu) \in \mathcal{Z}$. The proof is completed. □

Recall that $L(u) = \dot{u} - Df(\gamma)u$ in the Banach space $\mathcal{Z}$. As in [6, 20], we define the subspace of $\mathcal{Z}$, which consists the range of L in $\mathcal{Z}$.

$$\widetilde{\mathcal{Z}} = \{h \in \mathcal{Z} : \int_{-\infty}^{\infty} \langle \psi_i(s), h(s)\rangle ds = 0, i = 1, 2, 3\}.$$

Consider a nonhomogeneous equation

$$\dot{z} - Df(\gamma)z = h, \quad z(0) \perp \dot{\gamma}(0). \qquad (16)$$

Let $\mathcal{Z}^{\perp}$ be the subspace of $\mathcal{Z}$ consisting of $z(t)$ with $z(0) \perp \dot{\gamma}(0)$. If $h \in \widetilde{\mathcal{Z}}$, using the variation of constants, there exists an operator $K : \widetilde{\mathcal{Z}} \to \mathcal{Z}^{\perp}$ such that Kh is a solution of (16). Clearly, the general bounded solution of (16) is $z(t) = \sum_{p=1}^{2} \beta_p u_p(t) + (Kh)(t)$, where $\beta_p \in \mathbb{R}$.

From (9), $R(L) \oplus N(L^*) = \mathcal{Z}$. Define a map $P : \mathcal{Z} \to \mathcal{Z}$ such that $N(P) = R(L)$ and $R(P) = N(L^*)$. In particular,

$$h \in N(P) \text{ if and only if } \int_{-\infty}^{\infty} \langle \psi_i(s), h(s)\rangle ds = 0, \ i = 1, 2, 3.$$

As in [20], the projection P satisfies the following properties:

Lemma 3.2. (1) *P and $I - P$ are projections.*
(2) *$R(P) \oplus R(L) = \mathcal{Z}$.*
(3) *$R(I - P) = N(P) = R(L) = \widetilde{\mathcal{Z}}$.*

We now use the Lyapunov-Schmidt reduction to (13). Applying P and $(I - P)$ to (13), we have the following equivalent system

$$\dot{z} = Df(\gamma)z - (I - P)\widetilde{g}(z, \tau, \mu), \qquad (17)$$

$$P\widetilde{g}(z, \tau, \mu) = 0. \qquad (18)$$

First, we solve (17) for $z \in \mathcal{Z}^{\perp}$. Then the bifurcation equations are obtained by substituting z into (18).

Lemma 3.3. *There exist open balls $B_1(\delta_0) \subset \mathbb{R}^2$, $B_2(\delta_0) \subset \mathbb{R}$ with radius $\delta_0 > 0$ centered at the origins and a C^2 map $\phi : B_1(\delta_0) \times \mathbb{R} \times B_2(\delta_0) \to \mathcal{Z}$, denoted by $\phi(\boldsymbol{\beta}, \tau, \mu)$, such that $z = \phi(\boldsymbol{\beta}, \tau, \mu)$ is a solution of equation (17). Moreover $\phi(\boldsymbol{\beta}, \tau, \mu)$ satisfies $\phi(0, \tau, 0) = 0$, $(\partial\phi/\partial\beta_p)|_{(0,\tau,0)} = u_p$, $p = 1, 2$ and $(\partial\phi/\partial\mu)|_{(0,\tau,0)} = K(I - P)\varphi$.*

Proof. Since $R(I-P) = \widetilde{\mathcal{Z}}$ and $K : \widetilde{\mathcal{Z}} \to \mathcal{Z}^{\perp}$, (17) can be expressed as a fixed point problem in $\mathcal{Z}^{\perp}$.

$$z = \sum_{p=1}^{2} \beta_p u_p + K(I-P)\widetilde{g}(z, \tau, \mu), \tag{19}$$

Denote the r.h.s. of (19) by $F(z, \boldsymbol{\beta}, \tau, \mu)$. Then $F : \mathcal{Z}^{\perp} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathcal{Z}^{\perp}$ is a C^2 map. From (1) of Lemma 3.1, we have

$$F(0, 0, \tau, 0) = 0, \ D_1F(0, 0, \tau, 0) = 0. \tag{20}$$

By the smoothness of F, given any $\delta > 0$, there exists $c > 0$ such that

$$\|D_2F\| < c, \ \|D_3F\| < c, \ \|D_{11}F\| < c, \ \|D_{12}F\| < c, \ \|D_{13}F\| < c,$$

for $(z, \boldsymbol{\beta}, \tau, \mu) \in \bar{B}(\delta) \times \bar{B}_1(\delta) \times \mathbb{R} \times \bar{B}_2(\delta)$, where $\bar{B}(\delta) \subset \mathcal{Z}$, $\bar{B}_1(\delta) \subset \mathbb{R}^2$, $\bar{B}_2(\delta) \subset \mathbb{R}$ are closed balls of radius δ. Let

$$\delta_1 = min\{\delta, \frac{1}{4c}\}, \ \delta_2 = min\{\delta, \delta_1, \frac{\delta_1}{8c}\}.$$

For any $(z, \boldsymbol{\beta}, \tau, \mu) \in \bar{B}(\delta_1) \times \bar{B}_1(\delta_2) \times \mathbb{R} \times \bar{B}_2(\delta_2)$, define a map $\varphi_1 : [0, 1] \to \mathcal{L}(\mathcal{Z}, \mathcal{Z})$ by $\varphi_1(s) = D_1F(sz, s\boldsymbol{\beta}, \tau, s\mu)$. By the smoothness of F, we see $\varphi_1 \in C^1$. By (20) we know $\varphi_1(0) = 0$, then

$$\begin{aligned}
\|D_1F(z, \boldsymbol{\beta}, \tau, \mu)\| &= \|\varphi_1(1) - \varphi_1(0)\| = \|\int_0^1 \varphi_1'(p)dp\| \\
&\le \|D_{11}F(pz, p\boldsymbol{\beta}, \tau, p\mu)\| \cdot \|z\| \\
&\quad + \|D_{12}F(pz, p\boldsymbol{\beta}, \tau, p\mu)\| \cdot \|\boldsymbol{\beta}\| \\
&\quad + \|D_{13}F(pz, p\boldsymbol{\beta}, \tau, p\mu)\| \cdot \|\mu\| \\
&\le c \cdot \frac{1}{4c} + c \cdot \frac{1}{4c} + c \cdot \frac{1}{4c} = \frac{3}{4}.
\end{aligned} \tag{21}$$

For $(z, \boldsymbol{\beta}, \tau, \mu) \in \bar{B}(\delta_1) \times \bar{B}_1(\delta_2) \times \mathbb{R} \times \bar{B}_2(\delta_2)$, define a map $\varphi_2 : [0, 1] \to \mathcal{Z}$ by $\varphi_2(s) = F(sz, s\boldsymbol{\beta}, \tau, s\mu)$. Clearly $\varphi_2 \in C^1$ and $\varphi_2(0) = 0$, then

$$\begin{aligned}
\|F(z, \boldsymbol{\beta}, \tau, \mu)\| &= \|\varphi_2(1) - \varphi_2(0)\| = \|\int_0^1 \varphi_2'(p)dp\| \\
&\le \|D_1F(pz, p\boldsymbol{\beta}, \tau, p\mu)\| \cdot \|z\| \\
&\quad + \|D_2F(pz, p\boldsymbol{\beta}, \tau, p\mu)\| \cdot \|\boldsymbol{\beta}\| \\
&\quad + \|D_3F(pz, p\boldsymbol{\beta}, \tau, p\mu)\| \cdot \|\mu\| \\
&\le \frac{3}{4}\delta_1 + c \cdot \frac{\delta_1}{8c} + c \cdot \frac{\delta_1}{8c} = \delta_1,
\end{aligned}$$

which implies that $F(\cdot, \boldsymbol{\beta}, \tau, \mu)$ maps $\bar{B}(\delta_1)$ into itself.

For $z_1, z_2 \in \bar{B}(\delta_1)$, $(\boldsymbol{\beta}, \tau, \mu) \in \bar{B}_1(\delta_2) \times \mathbb{R} \times \bar{B}_2(\delta_2)$, define a map $\varphi_3 : [0, 1] \to \mathcal{Z}$ by $\varphi_3(s) = F(sz_1 + (1-s)z_2, \boldsymbol{\beta}, \tau, \mu)$. Then $\varphi_3 \in C^1$ and $\varphi_3(0) = 0$, then

$$\|F(z_1, \boldsymbol{\beta}, \tau, \mu) - F(z_2, \boldsymbol{\beta}, \tau, \mu)\|$$

$$
\begin{aligned}
&= \|\varphi_3(1) - \varphi_3(0)\| = \| \int_0^1 \varphi_3'(p) dp \| \\
&\le \|D_1 F(pz_1 + (1-p) z_2^{(k)}, \boldsymbol{\beta}, \tau, \mu)\| \cdot \|z_1 - z_2\| \\
&\le \frac{3}{4} \|z_1 - z_2\|.
\end{aligned}
$$

Therefore F is a uniform contraction in $\bar{B}(\delta_1)$. By the contraction mapping principle, there are $\delta_{21}, \delta_{22} > 0$ and a C^1 map $\phi : B_1(\delta_{21}) \times \mathbb{R} \times B_2(\delta_{22}) \to B(\delta_1)$ such that $\phi(0, \tau, 0) = 0$ and

$$\phi(\boldsymbol{\beta}, \tau, \mu) = F(\phi(\boldsymbol{\beta}, \tau, \mu), \boldsymbol{\beta}, \tau, \mu).$$

Let $\delta_0 = \min\{\delta_2, \delta_{21}, \delta_{22}\}$. From (19), we have

$$\phi(\boldsymbol{\beta}, \tau, \mu) = \sum_{p=1}^{2} \beta_p u_p + K(I - P)\widetilde{g}(\phi(\boldsymbol{\beta}, \tau, \mu), \tau, \mu). \tag{22}$$

Differentiating (22) with respect to $\boldsymbol{\beta}$, we have

$$
\begin{aligned}
D_1 \phi(\boldsymbol{\beta}, \tau, \mu) = {} & D_1 F(\phi(\boldsymbol{\beta}, \tau, \mu), \boldsymbol{\beta}, \tau, \mu) D_1 \phi(\boldsymbol{\beta}, \tau, \mu) \\
& + D_2 F(\phi(\boldsymbol{\beta}, \tau, \mu), \boldsymbol{\beta}, \tau, \mu).
\end{aligned}
$$

This, together with (21), implies that

$$D_1 \phi = (I - D_1 F(\phi, \boldsymbol{\beta}, \tau, \mu))^{-1} D_2 F(\phi, \boldsymbol{\beta}, \tau, \mu).$$

By the smoothness of F, $D_1\phi$ is a C^1 function. Hence ϕ is C^2 in $\boldsymbol{\beta}$. Similarly, we can prove ϕ is C^2 in μ.

Differentiating (22) with respect to β_p and evaluating at $(0, \tau, 0)$, we get

$$\left.\frac{\partial \phi}{\partial \beta_p}\right|_{(0,\tau,0)} (t) = u_p(t), \ \ p = 1, 2.$$

By direct calculations, we have $(\partial \phi / \partial \mu)|_{(0,\tau,0)} = K(I - P)\varphi$. The proof has been completed. □

By Lemma 3.3, (17) has a solution $\phi(\boldsymbol{\beta}, \tau, \mu)$. Substituting $\phi(\boldsymbol{\beta}, \tau, \mu)$ into (18), we have the system of bifurcation equations

$$P\widetilde{g}(\phi(\boldsymbol{\beta}, \tau, \mu), \tau, \mu) = 0.$$

Equivalently, the above can be recast as

$$
\begin{aligned}
H_i(\boldsymbol{\beta}, \tau, \mu) := \int_{-\infty}^{+\infty} \langle \psi_i(s), \widetilde{g}(\phi(\boldsymbol{\beta}, \tau, \mu), \tau, \mu)(s) \rangle ds \\
= 0, \ \ i = 1, 2, 3. \quad (23)
\end{aligned}
$$

Geometrically, $H_i(\boldsymbol{\beta}, \tau, \mu)$ describes the breaking of the homoclinic orbit under the perturbation of μ along the directions of $\psi_i, i = 1, 2, 3$. That is, even a smooth homoclinic orbit may not exist, a generalized orbits with the jumps $H_i\psi_i, i = 1, 2, 3$ at $t = 0$ always exists. See [16].

We have proved the following important result.

Theorem 3.1. *If ϕ satisfies (22) and $(\boldsymbol{\beta}, \tau, \mu) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ solves (23), then $z = \phi$ is a solution of (13) and hence the perturbed system (5) has a homoclinic orbit $x = \gamma + \phi$.*

Through direct calculations, we can prove the following Lemma.

Lemma 3.4. *For $p, q \in \{1, 2\}$, $i \in \{1, 2, 3\}$, $H_i(\boldsymbol{\beta}, \tau, \mu)$ has the following properties:*

(i) *If there are some $(\boldsymbol{\beta}, \tau, \mu) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ such that $H_i(\boldsymbol{\beta}, \tau, \mu) = 0$, $i = 1, 2, 3$, then ϕ is a solution of (13);*

(ii) $H_i(0, \tau, 0) = 0,\ \dfrac{\partial H_i}{\partial \beta_p}(0, \tau, 0) = 0;$

(iii) $\dfrac{\partial H_i}{\partial \mu}(0, \tau, 0) = \tilde{a}_i(\tau) := \displaystyle\int_{-\infty}^{+\infty} \langle \psi_i(t), g(\gamma(t), t + \tau, 0)\rangle dt,$

(iv) $\dfrac{\partial^2 H_i}{\partial \beta_p \partial \beta_q}(0, \tau, 0) = b_{pq}^{(i)} := \displaystyle\int_{-\infty}^{+\infty} \langle \psi_i(t), D^2 f(\gamma(t)) u_p(t) u_q(t)\rangle dt;$

(v)
$$\frac{\partial^2 H_i}{\partial \beta_p \partial \mu}(0, \tau, 0) = \tilde{c}_p^{(i)}(\tau) := \int_{-\infty}^{+\infty} \langle \psi_i(t), (D^2 f(\gamma(t)) \varphi(t, \tau) + D_1 g(\gamma(t), t + \tau, 0)) u_p(t) + D_2 g(\gamma(t), t + \tau, 0)\rangle dt,\ p = 1, 2,$$

where $\varphi(t, \tau) = K(I - P) g(\gamma(t), t + \tau, 0)$ and where $b_{pq}^{(i)}, \tilde{a}_i(\tau), \tilde{c}_p^{(i)}(\tau)$ are the same as in (10).

Keeping up to quadratic terms of β and μ,

$$H_i(\boldsymbol{\beta}, \tau, \mu) = \frac{1}{2} \boldsymbol{\beta}^T B^{(i)} \boldsymbol{\beta} + \mu \tilde{a}_i(\tau) + \mu \tilde{\boldsymbol{C}}_i(\tau) \boldsymbol{\beta} + \mu^2 \zeta_i(\tau) + O(|\mu|^3 + |\boldsymbol{\beta}|^3).$$

We will see that $\mu^2 \zeta_i(\tau)$ can be dropped in the bifurcation analysis. Let $M_i : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ be given by (11), which retains the quadratic terms of $H_i(\boldsymbol{\beta}, \tau, \mu)$, except for $\mu^2 \zeta_i(\tau)$. Define $a_i(\tau) = -2\tilde{a}_i(\tau)$ and $\boldsymbol{C}_i(\tau) = -2\tilde{\boldsymbol{C}}_i(\tau)$. We need to solve the quadratic system (12).

4. Codiagonalization and solutions of two quadratic equations.

Let $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $F(x, y) = \mathbf{z}^T B \mathbf{z}$. We say that the quadratic equation $F(x, y) = h, h \neq 0$ is of elliptic (or hyperbolic, or line) type if $b^2 - ac < 0$ (or > 0, or $= 0$). In this case the graph of the equation is an ellipse (or two hyperbolas, or two lines).

The graph of the line type equation is a special case of two hyperbolas, where the normal direction to two lines replaces the real axis of a hyperbola. If $b^2 = ac$, and $a, c, h > 0$, then $F(x, y) = (\sqrt{a}x + \sqrt{c}y)^2$. The solution represents two parallel lines $\sqrt{a}x + \sqrt{c}y = \pm\sqrt{h}$, symmetric about the origin.

The hyperbolic rotation is well-known for its use in relativity theory [3]. We shall define various transformations that keep a quadratic form $F(x, y) = ax^2 + 2bxy + cy^2$ invariant. Consider the Hamiltonian system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -b & -c \\ a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{24}$$

and its solution mapping $T(t)$. The values of $F(x, y)$ are invariant under $T(t)$.

Definition 4.1. The solution mapping $T(t)$ for (24) that maps the ray $\overrightarrow{OP_1}$ to $\overrightarrow{OP_2}$, where $P_2 = T(t) P_1$, will be called *the quadratic rotation by the angle* t. It will also be called the circular, elliptic or hyperbolic rotation if the graph of $F(x, y) = h$ is a circle, ellipse or hyperbola. The angle θ from $\overrightarrow{OP_1}$ to $\overrightarrow{OP_2}$ is defined to be $t \in \mathbb{R}$.

On the other hand, if there does not exist any $t \in \mathbb{R}$ with $\overrightarrow{OP_2} = T(t)\overrightarrow{OP_1}$, then the angle between the two rays is undefined.

We can pick any P_0 on a circle and define its angle coordinate to be $\theta(P_0) = 0$. For other quadratic curves, if P_0 is a point on the major axis (or semi-real, or semi-imaginary axis), then we define its angle coordinate to be $\theta(P_0) = 0$. Then for any $P \in \mathbb{R}^2$, we define its angle coordinate $\theta(P)$ to be the angle from $\overrightarrow{OP_0}$ to $\overrightarrow{OP}$.

Examples. Let $a = c = 1,\ b = 0$ in (24). The solution mapping

$$R(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \quad t \in \mathbb{R}$$

defines the circular rotation in counter-clockwise direction.

Let $a = 1,\ c = -1,\ b = 0$ in (24). The solution mapping

$$H(t) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

defines the standard hyperbolic rotation in 4 invariant sectors of $\mathbb{R}^2$.

More precisely, the lines $y = \pm x$ divide $\mathbb{R}^2$ into 4 invariant sectors:

$$S_1 := \{(x, y) : x > 0, |y| < |x|\},\ S_2 := \{(x, y) : y > 0, |x| < |y|\},$$
$$S_3 := \{(x, y) : x < 0, |y| < |x|\},\ S_4 := \{(x, y) : y < 0, |x| < |y|\}.$$

If $(x_0, y_0)^T \in S_1$ or S_3, then $(x_0, y_0) = r_0(\cosh(t_0), \sinh(t_0)), r_0 \in \mathbb{R}$. And $(x(t), y(t))^T$ remains in S_1 or S_3 with

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \cosh(t_0) \\ \sinh(t_0) \end{pmatrix} = r_0 \begin{pmatrix} \cosh(t + t_0) \\ \sinh(t + t_0) \end{pmatrix}.$$

If $(x_0, y_0)^T \in S_2$ or S_4, then $(x_0, y_0) = r_0(\sinh(t_0), \cosh(t_0)), r_0 \in \mathbb{R}$. And $(x(t), y(t))$ remains in sectors S_2 or S_4 with

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \sinh(t_0) \\ \cosh(t_0) \end{pmatrix} = r_0 \begin{pmatrix} \sinh(t + t_0) \\ \cosh(t + t_0) \end{pmatrix}.$$

Finally, if $T(t)$ is the solution mapping for (24), then

$$T^*(t) \begin{pmatrix} a & b \\ b & c \end{pmatrix} T(t) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Lemma 4.1. *Let $F(x, y) = (x, y)B(x, y)^T$ be the quadratic form associated to a symmetric matrix $B \in \mathbb{R}^{2\times 2}$.*

(1) If the vector field (24) corresponding to $F(x, y)$ satisfies $x' = 0$ on the x-axis, or $y' = 0$ on the y-axis, then matrix B is diagonal.

(2) If there exist $x_1 \neq 0, y_1 \neq 0$ such that either $F(x_1, y_1) = F(-x_1, y_1)$, or $F(x_1, -y_1) = F(x_1, -y_1)$, then B is a diagonal matrix.

Proof. (1) If $x \neq 0, y = 0 \to x' = 0$, from (24), we have $b = 0$. Similarly, if $y \neq 0, x = 0 \to y' = 0$, then $b = 0$.

(2) In both cases, we have

$$ax_1^2 + 2bx_1y_1 + cy_1^2 = ax_1^2 - 2bx_1y_1 + cy_1^2.$$

Thus $b = 0$. □

Let $B_1, B_2 \in \mathbb{R}^{2\times 2}$ be symmetric, nonzero matrices and let $F_1(x,y) = a_1x^2 + 2b_1xy + c_1y^2 = (x,y)B_1(x,y)^T$, $F_2(x,y) = a_2x^2 + 2b_2xy + c_2y^2 = (x,y)B_2(x,y)^T$. We consider the system of two quadratic equations

$$F_1(x,y) = h_1, \quad F_2(x,y) = h_2. \tag{25}$$

(H7): Assume that the two quadratic forms $F_1(x,y), F_2(x,y)$ are linearly independent, i.e., the two matrices B_1, B_2 are linearly independent.

Here is brief outline of the content in §4.1 - §4.3. In §4.1 and §4.2, we use conditional max/min problems to codiagonalize two quadratic equations. The max/min process also provides conditions for the existence of 4 real valued solutions without going through the codiagonalization. In §4.3 we give simple conditions for the existence of 4 real valued solutions on all the systems considered in §4.1and §4.2. However, these conditions are posed on the codiagonalized systems.

4.1. **Codiagonalization and solutions of (25) if one equation is elliptic.** It is well-known that two symmetric matrices can be simultaneously diagonalized if one of the matrices is positive definite, [11, 12]. However, it is not clear if the resulting matrices are real valued.

If $F_2(x,y) = h_2$ is of elliptic type, then $b_2^2 - a_2c_2 < 0$. From (H6), we find that $a_2 > 0, c_2 > 0$ and $h_2 > 0$. We shall use the elliptic rotation $T_2(\theta)$ defined by (24) with $B = B_2$. First we shall find two points $P_j = (x_j, y_j)^T, j = 1, 2$, from the conditional maximum/minimum problems.

$$F_2(x,y) = \text{max or min}, \quad \text{subject to } F_1(x,y) = \pm h_1. \tag{26}$$

We look for the critical points (x,y) of the following Lagranginan:

$$\Lambda(x,y,\lambda) = F_1(x,y) - \lambda F_2(x,y), \quad \nabla_{x,y}\Lambda(x,y,\lambda) = 0. \tag{27}$$

Notice our definition of the Lagrangian is slightly different from those in standard literatures.

To find critical points of the Lagrangian, we look for the generalized eigenvectors of the following system

$$(B_1 - \lambda B_2)\begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{28}$$

We consider three types of systems.

(i) (EE) type: In this case, (28) has two eigenvalues (λ_1, λ_2) with (nonunique) eigenvectors $(P_1, P_2) = ((x_1, y_1)^T, (x_2, y_2)^T)$. Then after rescaling of P_1 and P_2, we assume that on the curve $F_1 = h_1$, F_2 reaches the minimum r_1 at P_1 and the maximum r_2 at P_2. There exists an appropriate angle θ_0 such that $T_2(-\theta_0)P_2$ coincides with the major axis of $F_2(x,y) = h_2$.

(ii) (HE) type: In this case system (28) has two eigenvalues (λ_1, λ_2) with eigenvectors $(P_1, P_2) = ((x_1, y_1)^T, (x_2, y_2)^T)$. Assume that $h_1 > 0$. Then after rescaling, we assume that F_2 reaches a minimum $r_1 > 0$ at P_1, subject to $F_1 = -h_1 < 0$; and a minimum $r_2 > 0$ at P_2 subject to $F_1(x,y) = h_1 > 0$. There exists an appropriate angle θ_0 such that $T_2(-\theta_0)P_2$ coincides with the major axis of $F_2(x,y) = h_2$.

(iii) (LE) type: In this case, the graph of $F_1(x,y) = h_1$ consists of two parallel lines symmetric about the origin. We have $h_1 > 0$ from (H6). The eigenvalues for (28) are (λ_1, λ_2). Then $\lambda_1 = 0$ with eigenvector P_1 on which $F_1(x_2, y_2) = 0$. And $\lambda_2 > 0$ with the eigenvector P_2 that solves the conditional minimization problem with $F_2 = r_2$. There exists an angle θ_0 such that $T_2(-\theta_0)P_2$ coincides with the major axis of $F_2(x,y) = h_2$.

Using the property

$$< P_1, \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} P_2 >= 0,$$

in all the three cases, the image of $T_2(-\theta_0)P_1$ should coincide with the minor axis of $F_2 = h_2$. Also, under the rotation $T_2(\theta_0)$, the quadratic form $F_1(x, y) = h_1$ becomes $F_3(x, y) = h_1$ while $F_2(x, y) = h_2$ is unchanged. Now apply a circular rotation $R(-\theta_0')$ to both $F_3(x, y) = h_1$ and $F_2(x, y) = h_2$ so the major axis of $F_2(x, y) = h_2$ is mapped to the x-axis. The matrices that represent the two quadratic forms are

$$R^*(\theta_0')T_2^*(\theta_0)B_jT_2(\theta_0)R(\theta_0'),\ j = 1, 2.$$

Clearly $F_2(x, y) = h_2$ has been diagonalized. From Lemma 4.1, $F_1(x, y) = h_1$ has also been diagonalized.

We have proved the following results:

Lemma 4.2. *Assume (H1)-(H7) are satisfied. If one equation of the quadratic system is elliptic, then the two quadratic forms can always be codiagonalized by the real valued matrices. More specifically, the codiagonalized graphs of $F_1 = h_1$ are as follows. In the Case (EE) type, the major axis of the ellipse $F_1 = h_1$ is on the x-axis. In the case (HE) type, the real axis of the hyperbola $F_1 = h_1$ is on the x-axis. In the Case (LE) type, the normal direction of the two parallel lines is on the x-axis.*

Theorem 4.1. *The (EE) type of system has 4 solutions if $r_1 < h_2 < r_2$.*

The (HE) type of system has 4 solutions if $h_1 > 0,\ 0 < r_2 < h_2$. The condition becomes $0 < r_1 < h_2$ if $h_1 < 0$.

The (LE) type of system has 4 solutions if $r_2 < h_2$.

Proof. **Case (EE) type.** It is given that $F_2(P_1) = r_1 < h_2 < r_2 = F_2(P_2)$. Let the angle of P_i be θ_i. Then there exists an angle θ_0 between θ_1 and θ_2 such that the corresponding point is P_0 on the graph of $F_1 = h_1$ with $F_2(P_0) = h_2$. There exist 4 pairs of such (P_1, P_2) so the total number of solutions is 4.

Case (HE) type. If $F_2(P_2) = r_2 < h_2$, then as $\theta \to \pm\infty$, the values of F_2 on the graph of $F_1 = h_1$ approach ∞ that is greater than h_2. So there exists two points $P_\pm$ on each of the two branches of $F_1 = h_1$ such that $F_2(P_\pm) = h_2$. The other case $h_1 < 0$ can be proved similarly.

Case (LE) type. At each of the two parallel lines, there exists a point P_2 such that $F_2(P_2) = r_2 < h_2$. Moving away from P_2 on the line $F_1 = h_1$, the value of F_2 gets greater than h_2 in two opposite directions. So on two opposite directions of each line, there exist $P_\pm$ such that $F_2(P_\pm) = h_2$. □

4.2. **Codiagonalization and solutions of (25) if both equations are hyperbolic.** In this subsection we consider the system $F_j(x, y) = h_j, j = 1, 2$, where both equations are of hyperbolic type, denoted by (HH). To simplify the illustration, assume $h_1 > 0, h_2 > 0$. Let $T_2(\theta)$ be the hyperbolic rotation defined by (24) with $B = B_2$.

Unlike the cases studied in §4.1, a general conditional max/min problem is not well posed for (HH) type systems. We can divide the (HH) type system into two sub-cases, and find a suitable conditional max/min problem for each sub-case.

For the (HH) type system, $b_j^2 - a_jc_j > 0, j = 1, 2$, so with $(a, b, c) = (a_j, b_j, c_j)$, the equilibrium $(0, 0)$ of (24) is hyperbolic and there exist stable and unstable eigenspaces for the equilibrium $(0, 0)$.

Definition 4.2. Let $L_j^{(i)}, i = 1,2$, be the stable and unstable eigenspaces of the equilibrium for (24), where $(a,b,c,) = (a_j, b_j, c_j)$. They are called the asymptotes for $F_j(x,y) = h_j$, and $F_j(x,y) = 0$ if $(x,y)^T \in L_j^{(i)}, i = 1,2$. The asymptotes $L_j^{(i)}, i = 1,2$, divide $\mathbb{R}^2$ into four sectors. We say (x,y) is in the positive (or negative) sector if $F_j(x,y) > 0$ (or $F_j(x,y) < 0$).

Consider 4 cases, as depicted in Fig. 1:
(i) The two sectors of $F_1 > 0$ are in the interior of the sectors of $F_2 > 0$.
(ii) The two sectors of $F_1 < 0$ are in the interior of the sectors of $F_2 > 0$.
(iii) The two sectors of $F_1 > 0$ are in the interior of the sectors of $F_2 < 0$.
(iv) The two sectors of $F_1 < 0$ are in the interior of the sectors $F_2 < 0$.

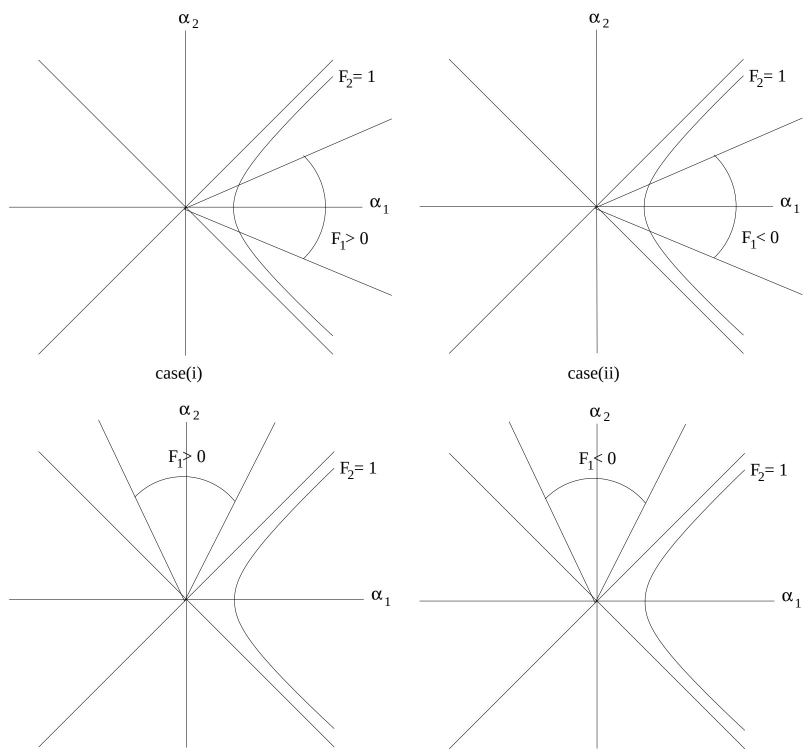

FIGURE 1. Graphs for the (HH) type systems.

Lemma 4.3. *For cases (i) and (ii), consider two conditional max/min problems*

$$\begin{aligned} r_1 &:= F_2(x,y) = \mathit{min}, \quad \textit{subject to } F_1(x,y) = h_1, \\ r_2 &:= F_2(x,y) = \mathit{min}, \quad \textit{subject to } F_1(x,y) = -h_1. \end{aligned} \tag{29}$$

For cases (iii) and (iv), consider two conditional max/min problems

$$\begin{aligned} r_3 &:= F_2(x,y) = \mathit{max}, \quad \textit{subject to } F_1(x,y) = h_1, \\ r_4 &:= F_2(x,y) = \mathit{max}, \quad \textit{subject to } F_1(x,y) = -h_1. \end{aligned} \tag{30}$$

Then the conditional max/min problem in (29) *or* (30) *has exactly* 4 *solutions, each is on a continuous branch of* $F_1(x,y) = h_1$ *or* $-h_1$. *Moreover, for case (i),* $r_2 < 0 < r_1$*; for case (ii),* $r_1 < 0 < r_2$*; for case (iii),* $r_3 < 0 < r_4$*; for case (iv),* $r_4 < 0 < r_3$.

Finally, using the hyperbolic rotation $T_2(\theta)$ *defined by* (24) *with* $B = B_2$, *and the method that proves Lemma 4.2, the system* $F_j = h_j, j = 1, 2$, *can be codiagonalized.*

Proof. **Case (i).** Along the asymptotes of $F_1 = h_1$ or $-h_1$, we find that $F_2(x, y) \to \infty$ as $x^2 + y^2 \to \infty$. The curves for $F_1 = \pm\, h_1$ have 4 continuous branches. Let $(x(t), y(t))$ be the orbit of (24) that is on one of such branches. Then $F_2(x(t), y(t)) \to \infty$ as $t \to \pm\infty$. The search for minimum can be restrict to a compact subinterval of $t \in \mathbb{R}$, on which the continuous function $F_2(x(t), y(t))$ must reach a minimum. Hence there are at least 4 solutions for the max/min problem (29).

It remain to show that (29) cannot have more than 4 solutions. To this end, let $P = (x, y)^T$ be a point where the minimum is reached. Then $(x, y)^T$ is an critical point for the Lagrangian (27) and satisfies the generalized eigenvalue problem (28). There can only be 2 linearly independent vectors $(x, y)^T$. Using the symmetry about the origin, we find that there are exactly 4 such critical points. The assertion $r_2 < 0 < r_1$ is obvious and the proof shall be omitted.

The proofs for cases (ii)-(iv) are similar and will not be given here. □

From a counter example at the end of this subsection, if the asymptotes of $F_1 = h_1$ and $F_2 = h_2$ are alternating, then the system cannot be codiagonalized. So results in Lemma 4.3 are the best we can obtain.

Theorem 4.2. *Let* $r_j, 1 \le j \le 4$ *be defined case by case as in Lemma 4.3. Then the conditions for the following system of quadratic equations*

$$F_1(x, y) = h_1 \ \text{or} \ -h_1, \quad F_2(x, y) = h_2 \tag{31}$$

to have 4 solutions are determined by the asymptotes and positions of positive-negative sectors separated by the asymptotes as follows:

Case (i). The system has 4 solutions provided that $h_1 > 0,\ r_1 < h_2$, *see Fig. 2; or* $-h_1 < 0$ *for any* $r_2 < 0$.

Case (ii). The system has 4 solutions provided that $-h_1 < 0,\ r_2 < h_2$, *see Fig. 2; or* $h_1 > 0$ *for any* $r_1 < 0$.

Case (iii). The system has 4 solutions provided that $-h_1 < 0,\ r_4 > h_2$, *see Fig. 3. It has no solution if* $h_1 > 0$.

Case (iv). The system has 4 solutions provided that $h_1 > 0,\ r_3 > h_2$, *see Fig 3. It has no solution if* $-h_1 < 0$.

Proof. We will prove case (i) only. Let $(x(t), y(t))^T$ be the point on a branch of $F_1(x, y) = h_1$, and the minimum of F_2 is reached at $t = t_0$ with $F_2(x(t_0), y(t_0)) = r_1 < h_2$. It is straightforward to show that $(d/dt)F_2(x(t), y(t)) \neq 0$ if $t \neq t_0$. As $t \to \pm\infty$, $F_2(x(t), y(t)) \to \infty$. Hence there exist exactly two points $t_1 < 0 < t_2$ where $F_2(x(t_j), y(t_j)) = h_2$. Obviously $F_1(x(t_j), y(t_j)) = h_1$.Therefore $(x_j, y_j)^T, j = 1, 2$ are the solutions for $F_1 = h_1, F_2 = h_2$. There are two more solutions on the other branch of $F_1(x, y) = h_1$. So the total number of solutions is 4.

We can similarly consider the $(x(t), y(t))^T$ on a branch of $F_1(x, y) = -h_1$. This time using $r_2 < 0$, we can find two solutions of $F_1 = -h_1, F_2 = h_2$ on each of the two branches. So the total number of solutions is 4. □

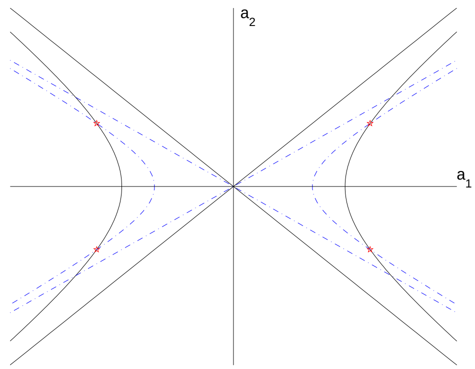

FIGURE 2. In case (4.6-1), if $h_1 > 0$ and $r_1 < r_2$, or in case (4.6-2), if $h_1 < 0$ and $r_1 < r_2$, then the system has 4 solutions.

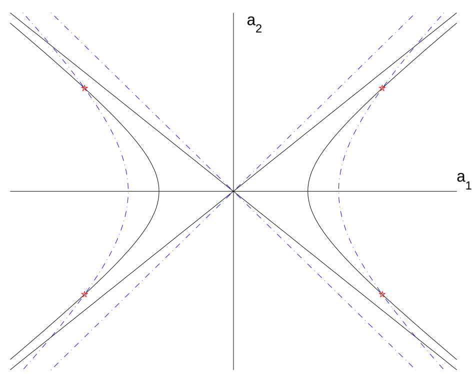

FIGURE 3. In case (4.6-3), if $h_1 < 0$ and $r_1 > r_2$, or in case (4.6-4), if $h_1 > 0$ and $r_1 > r_2$, then the system has 4 solutions.

Before ending this subsection, we present an example from [17] showing that not all the (HH) case can be codiagonalized.

A Counter Example. Assume that the asymptotes of two hyperbolas are alternating. It means that none of the positive or negative sectors of F_1 are inside the positive or negative sectors of F_2 and vise versa. See Figure. 4. In this case the two quadratic forms cannot be codiagonalized.

4.3. **Existence of 4 real valued solutions of (25) for all the possible cases.** After the codiagonalization of two quadratic equations, it is simple to list conditions for the coupled system to have 4 real valued solutions, including all the cases studied in §4.1 and §4.2. After codiagonalization, we have $b_1 = b_2 = 0$, then (25) becomes

$$a_1x^2 + c_1y^2 = h_1, \quad a_2x^2 + c_2y^2 = h_2. \tag{32}$$

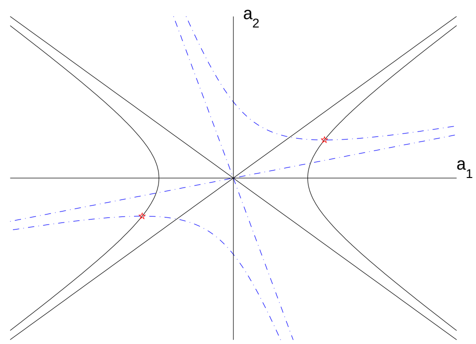

FIGURE 4. If the asymptotes of $F_1 = 0$ and $F_2 = 0$ are alternating, then there always exist exactly two solutions.

In the first quadrant, the graphs of $F_j(x, y) = h_j, j = 1, 2$, may intersect with the boundaries of the first quadrant at certain points, which will simply be called the x-intercept and/or y-intercept, and denoted by ℓ_{jx} and/or ℓ_{jy} respectively.

A hyperbola has two continuous branches. The opening angle of a hyperbola $\mathcal{C}$, denoted by $\Theta(\mathcal{C})$, is defined to be the angle between the two asymptotic lines of any branch of the hyperbola.

More precisely, an ellipse has x and y intercepts with the coordinate lines. A hyperbola has x or y-intercepts but not both. A vertical line has a x-intercept, and a horizontal line has a y-intercept with the coordinate axes. Based on those observations, we list the intersection of curves defined by (32) as follows:
(EE): (i) Both ellipses have x and y-intercepts.
(HE): (i) The hyperbola has x-intercept. (ii) The hyperbola has y intercept.
(LE): (i) The line has x-intercept. (ii) The line has y-intercept.
(HH): (i) One hyperbola has x, the other has y-intercept. (ii) Both hyperbolas have x-intercept. (iii) Both hyperbolas have y-intercept.
(LH): (i) The hyperbola has y and the line has x-intercept. (ii) Both have y-intercepts. (iii) The hyperbola has x and the line has y-intercept. (iv) both have x-intercepts.
(LL): (i) One line has x-intercept and the other line has y-intercept.

All the possible cases, except for case (LL), are plotted in Figures 5-7. Also if two graphs are related by flipping the horizontal and vertical axes, then only one is plotted.

Theorem 4.3. *The conditions to ensure the existence of a positive solution for* (34)*, and hence* 4 *solutions for system* (32) *are as follows.*
(EE): (i) If $\ell_{1x} < (or >)\, \ell_{2x}$ *then* $\ell_{1y} > (or <)\, \ell_{2y}$.
(HE): (i) $\ell_{1x} < \ell_{2x}$. *(ii)* $\ell_{1y} < \ell_{2y}$.
(LE): (i) $\ell_{1x} < \ell_{2x}$. *(ii)* $\ell_{1y} < \ell_{2y}$.
(HH): (i) If $\mathcal{C}_1$ *has* x *(or* y*)-intercept and* $\mathcal{C}_2$ *has* y *(or* x*)-intercept, then* $\Theta(\mathcal{C}_1) + \Theta(\mathcal{C}_2) > \pi$. *(ii) If* $\ell_{1x} < (or >)\, \ell_{2x}$ *then* $\Theta(\mathcal{C}_1) < (or >)\, \Theta(\mathcal{C}_2)$*??. (iii) If* $\ell_{1y} < (or >)\, \ell_{2y}$*, then* $\Theta(\mathcal{C}_1) < (or >)\, \Theta(\mathcal{C}_2)$*??.*
(LH): (i) or (iii) No further condition is needed. (ii) $\ell_{2y} < \ell_{1y}$. *(iv)* $\ell_{2x} < \ell_{1x}$.
(LL): (i) No further condition is needed.

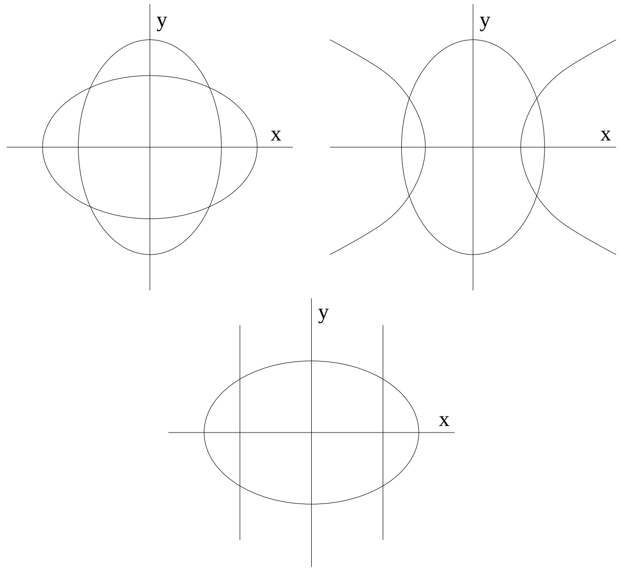

FIGURE 5. Figure (1) is about the (EE) case. Figure (2) is about the (HE) case. The (HE) case where the hyperbola has y intercept is not shown. Figure (3) is about the (LE) case where the lines have x-intercepts. The (LE) case where the lines have y intercepts is not shown.

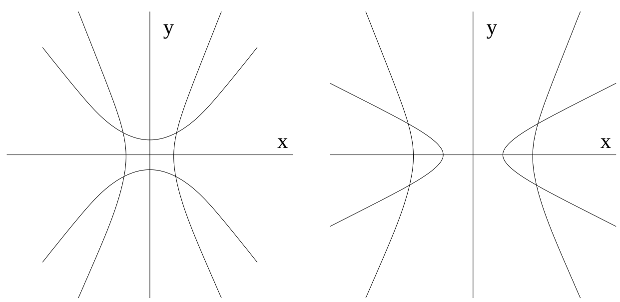

FIGURE 6. Figure (1) is about the (HH) case where one curve has x and the other has y intercepts. Figure (2) is about the (HH) case where both graphs have x-intercepts. The (HH) case where both have y intercepts in not shown.

Proof. The proof can be done by the Intermediate Value Theorem. For example, in the (EE) case, let us follow the $\mathcal{C}_1$ in the first quadrant from the x-intercept to the y-intercept. If $\ell_{1x} < \ell_{2x}, \ell_{1y} > \ell_{2y}$, then $\mathcal{C}_1$ started below, but ended above $\mathcal{C}_2$. Thus the two curves must intersect somewhere in the first quadrant. The (HE) and (LE) cases can be proved similarly.

Alternatively, each case listed in the theorem can be identified with a unique case in Theorems 4.1 and 4.2. And it can be shown that the conditions given there are the same as some conditions in Theorem 4.6. Hence the system has 4 solutions. See Remark 2 for two such examples. $\square$

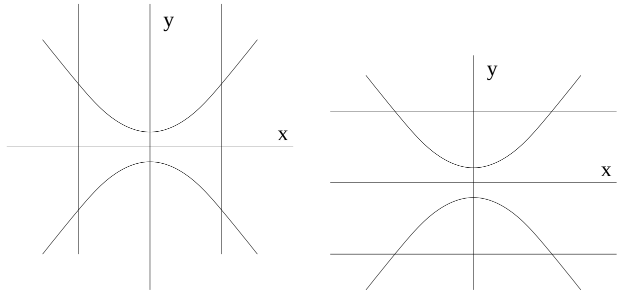

FIGURE 7. Figure (1) is about the (LH) case where the lines have x-intercepts and the hyperbola has y-intercept. Figure (2) is about the (LH) case where both the lines and the hyperbola have y-intercepts. The two (LH) cases where the hyperbola has x-intercepts are not shown.

Remark 2. The conditions listed above look different but are equivalent to the conditions in Theorems 4.1 and 4.2. This can be checked case by case. For example, in the case (HH) (i), assume that $\mathcal{C}_1$ has y-intercept, $\mathcal{C}_2$ has x-intercept, and $\Theta(\mathcal{C}_1)+\Theta(\mathcal{C}_2) > \pi$, as in Theorem 4.3. Then the condition on the opening angles implies that the two asymptotes of $F_1 = h_1$ are in the sector there $F_2 > 0$. And among them, the sectors of $F_1 < 0$ are in the interior of the sectors of $F_2 > 0$. This is exactly the case (ii) considered in Lemma 4.5 and Theorem 4.6. From Theorem 4.6, the system $F_j = h_j, j = 1, 2$, has 4 solutions.

As a second example, in the case (HH) (ii), assume that $\ell_{1x} < \ell_{2x}$ and $\Theta(\mathcal{C}_1) < \Theta(\mathcal{C}_2)$ as in 4.3. Then the condition on the opening angles implies that the sectors of $F_1 > 0$ are in the sector $F_2 > 0$. This case is considered in as case (i) in Lemma 4.5 and Theorem 4.6. Now the condition $\ell_{1x} < \ell_{2x}$ implies that $r_1 < h_2$ as in Theorem 4.6. Therefore the coupled system has 4 solutions.

In following we show the four solutions obtained in Theorem 4.3 are simple.

Theorem 4.4. *Assume that the two quadratic forms are linearly independent as in (H7). Then the four solutions of (25) obtained in Theorem 4.3 are simple.*

Proof. We only give the proof for (EE) type systems for the other cases can be proved similarly. It suffices to consider the two equations after codiagonalization.

$$\begin{aligned} F_1(x,y) - h_1 &:= a_1x^2 + c_1y^2 - h_1, \\ F_2(x,y) - h_2 &:= a_2x^2 + c_2y^2 - h_2. \end{aligned} \tag{33}$$

Under the conditions of (EE) of Theorem 4.1, (33) has four zeros $(x^{(i)}, y^{(i)}), i = 1, 2, 3, 4$. We claim that $x^{(i)} \neq 0, y^{(i)} \neq 0,\ 1 \leq i \leq 4$. For otherwise, the solution is on the major or minor axis of the ellipses. This is a contradiction to $\ell_{1x} \neq \ell_{2x}, \ell_{1y} \neq \ell_{2y}$ as from hypotheses of case (EE) in Theorem 4.3.

The normal directions of F_1 and F_2 at $(x^{(i)}, y^{(i)})$ are $(a_1x^{(i)}, c_1y^{(i)})$ and $(a_2x^{(i)}, c_2y^{(i)})$, respectively. Due to the linear independence of the two quadratic forms, and nonzeroness of $(x^{(i)}, y^{(i)})$,

$$\frac{\partial(F_1, F_2)}{\partial(x, y)}\Big|_{(x^{(i)}, y^{(i)})} = \begin{vmatrix} 2a_1x^{(i)} & 2c_1y^{(i)} \\ 2a_2x^{(i)} & 2c_2y^{(i)} \end{vmatrix} \neq 0,$$

which implies that $(x^{(i)}, y^{(i)}), 1 \leq i \leq 4$, are simple zeros of (33). $\square$

Finally, using the change of variables $u = x^2, v = y^2$, the quadratic system $F_j(x,y) = h_j, j = 1,2$ becomes the following linear system in (u,v):

$$a_1u + c_1v = h_1, \quad a_2u + c_2v = h_2, \quad u > 0, v > 0. \tag{34}$$

Under the conditions $a := a_1c_2 - c_1a_2 \neq 0$, the system has a unique solution

$$u = (c_2h_1 - c_1h_2)/a, \quad v = (a_1h_2 - c_2h_1)/a.$$

Thus $F_j(x,y) = h_j, j = 1,2$ can have 4, possibly complex valued, solutions.

Observe that if ℓ_{jx}, ℓ_{jy} are the x and/or y-intercepts for the jth quadratic equation (32), then $L_{ju} = \ell_{jx}^2, L_jv = \ell_{jy}^2$ are the u and/or v-intercepts for (34). If $k_j, j = 1,2$ are the slopes for the asymptote of the jth hyperbola, then $K_j = k_j^2$ are the slopes for corresponding lines for (34).

If the system on (u,v) has a solution in the first quadrant–$u > 0, v > 0$ then the original system in (x,y) has 4 solutions. Now, under the conditions as in Theorem 4.3 (or equivalently, Theorems 4.1 and 4.2), the 4 solutions are real valued solutions. Under the same conditions (34) has a positive valued solution $u > 0, v > 0$.

5. **Coexistence of homoclinic solutions.** Assume that the first two equations of (12) satisfy the conditions in Theorem 4.3. Then the first two equations can be codiagonalized by the change of variables $\boldsymbol{\beta} \to \boldsymbol{\alpha}$. System (12) becomes

$$\begin{aligned}
f_1(\alpha_1,\alpha_2,\tau,\mu) &:= d_{11}\alpha_1^2 + d_{12}\alpha_2^2 - \mu a_1(\tau) - \mu\boldsymbol{C}_1(\tau)\boldsymbol{\alpha} = 0,\\
f_2(\alpha_1,\alpha_2,\tau,\mu) &:= d_{21}\alpha_1^2 + d_{22}\alpha_2^2 - \mu a_2(\tau) - \mu\boldsymbol{C}_2(\tau)\boldsymbol{\alpha} = 0,\\
f_3(\alpha_1,\alpha_2,\tau,\mu) &:= d_{31}\alpha_1^2 + d_{32}\alpha_1\alpha_2 + d_{33}\alpha_2^2 - \mu a_3(\tau) - \mu\boldsymbol{C}_3(\tau)\boldsymbol{\alpha} = 0.
\end{aligned} \tag{35}$$

Assume $\mu > 0$ for simplicity. Let $\alpha_j = \sqrt{\mu}x_j, j = 1,2$. Then the first two equations become

$$\begin{aligned}
\tilde{f}_1(x_1,x_2,\tau,\mu) &:= d_{11}x_1^2 + d_{12}x_2^2 - a_1(\tau) - \sqrt{\mu}\boldsymbol{C}_1(\tau)(x_1,x_2)^T = 0,\\
\tilde{f}_2(x_1,x_2,\tau,\mu) &:= d_{21}x_1^2 + d_{22}x_2^2 - a_2(\tau) - \sqrt{\mu}\boldsymbol{C}_2(\tau)(x_1,x_2)^T = 0.
\end{aligned}$$

If $\sqrt{\mu} = 0$ and $\tau = \tau_0$, the above system has 4 simple zeros $(x_1^{(i)}, x_2^{(i)}), 1 \le i \le 4$. From the implicit function theorem, for small $\mu = \mu_0$, $\tilde{f}_1, \tilde{f}_2$ has 4 simple zeros $(x_1^{(i)}(\tau,\mu), x_2^{(i)}(\tau,\mu))$. Hence (35) has 4 simple solutions, denoted by $\alpha_j^*(\tau,\mu), j = 1,2$.

Taylor expansions of solutions are as follows

$$\begin{aligned}
\alpha_1^*(\tau,\mu) &= \frac{1}{\sqrt{|a|}}(d_{22}a_1(\tau) - d_{12}a_2(\tau))^{\frac{1}{2}}|\mu|\\
&\quad + \frac{f_1(\tau)}{2|a|(d_{22}a_1(\tau) - d_{12}a_2(\tau))^{\frac{1}{2}}}|\mu|^{\frac{3}{2}} + O(|\mu|^2),\\
\alpha_2^*(\tau,\mu) &= \frac{1}{\sqrt{|a|}}(d_{11}a_2(\tau) - d_{21}a_1(\tau))^{\frac{1}{2}}|\mu|\\
&\quad + \frac{f_2(\tau)}{2|a|(d_{11}a_2(\tau) - d_{21}a_1(\tau)))^{\frac{1}{2}}}\}|\mu|^{\frac{3}{2}} + O(|\mu|^2),
\end{aligned} \tag{36}$$

where $a = d_{11}d_{22} - d_{12}d_{21}$ and

$$\begin{aligned}
f_1(\tau) =& (d_{22}a_1(\tau) - d_{12}a_2(\tau))^{1/2}(d_{22}c_{11}(\tau) - d_{12}c_{21}(\tau))\\
&+ (d_{11}a_2(\tau) - d_{21}a_1(\tau))^{1/2}(d_{22}c_{12}(\tau) - d_{12}c_{22}(\tau)),
\end{aligned}$$

$$f_2(\tau) = (d_{22}a_1(\tau) - d_{12}a_2(\tau))^{1/2}(d_{11}c_{21}(\tau) - d_{21}c_{11}(\tau)) + (d_{11}a_2(\tau) - d_{21}a_1(\tau))^{1/2}(d_{11}c_{22}(\tau) - d_{21}c_{12}(\tau))$$

By substituting $\alpha_1 = \alpha_1^*(\tau, \mu),\ \alpha_2 = \alpha_2^*(\tau, \mu)$ of (36) into the third equation of (35), we have a nonlinear equation $G(\tau, \mu) = 0$. As in §2,

$$G(\tau, \mu) := \mu F_1(\tau) + |\mu|^{3/2} F_2(\tau) + O(|\mu|^2), \tag{37}$$

where $F_1(\tau)$ and $F_2(\tau)$ are also defined in §2. We look for solutions $\tau \approx \tau_0,\ \mu \approx 0$. Since $F_1(\tau_0) = F_1'(\tau_0) = 0$, by the Taylor formula,

$$\begin{aligned} G(\tau, \mu) &= \frac{\mu}{2} F_1''(\tilde{\tau})(\tau - \tau_0)^2 + |\mu|^{\frac{3}{2}} F_2(\tau) + O(|\mu|^2) \\ &= \frac{\mu}{2}[F_1''(\tilde{\tau})(\tau - \tau_0)^2 \pm 2|\mu|^{\frac{1}{2}} F_2(\tau) + O(|\mu|)] \end{aligned} \tag{38}$$

for some $\tilde{\tau}$, where $+$ and $-$ correspond to $\mu > 0$ and $\mu < 0$.

For $\mu < 0$ can be handled similarly, we only give the details for $\mu > 0$. In this case $G(\tau, \mu) = 0$ becomes

$$F_1''(\tilde{\tau})(\tau - \tau_0)^2 + 2\mu^{\frac{1}{2}} F_2(\tau) + O(\mu) = 0. \tag{39}$$

The main part of (39) is $F_1''(\tilde{\tau})(\tau - \tau_0)^2 + 2\mu^{\frac{1}{2}} F_2(\tau)$. From (39), if $F_1''(\tilde{\tau}) \neq 0, F_2(\tau) \neq 0$, then $|\tau - \tau_0|^2 = O(\mu^{\frac{1}{2}})$.

(Case 1.) If $F_1''(\tau_0)F_2(\tau_0) > 0$. By the continuities of F_1'' and F_2, we have $F_1''(\tilde{\tau})F_2(\tau) > 0$. Hence $F_1''(\tilde{\tau})(\tau - \tau_0)^2 + 2\mu^{1/2} F_2(\tau)$ is always positive or negative. Then (39) has no real solutions for small μ.

(Case 2.) If $F_1''(\tau_0)F_2(\tau_0) < 0$. We know that (39) has real solutions iff

$$F_1''(\tilde{\tau})(\frac{\tau - \tau_0}{\mu^{1/4}})^2 + 2F_2(\tau) + O(\mu^{1/2}) = 0, \tag{40}$$

which implies that $(\tau - \tau_0)/\mu^{1/4}$ is bounded when μ is small. That is $(\tau - \tau_0) = O(\mu^{1/4})$. Hence the solutions of $G(\tau, \mu) = 0$ have the form $\tau = \tau_0 + \xi(\mu)\mu^{1/4}$, where ξ is bounded in some neighborhood of $\mu = 0$.

Now we look for a solution of the special form of $\tau = \tau_0 + \xi\mu^{1/4}$. By Taylor formula, we see that $\tilde{\tau} = \tau_0 + \theta\xi\mu^{1/4}$ for some constants $\theta \in [0, 1]$. Substituting into (40), we have

$$H(\xi, \mu) := F_1''(\tau_0 + \theta\xi\mu^{\frac{1}{4}})\xi^2 + 2F_2(\tau_0 + \xi\mu^{\frac{1}{4}}) + O(\mu^{\frac{1}{2}}) = 0.$$

Let

$$\xi_0^{\pm} = \pm\sqrt{-\frac{2F_2(\tau_0)}{F_1''(\tau_0)}}.$$

Clearly, $H(\xi_0^{\pm}, 0) = 0$ and $(\partial H/\partial\xi)(\xi_0^{\pm}, 0) = 2F_1''(\tau_0)\xi_0^{\pm} \neq 0$. The implicit function theory implies that there are $r_1 > 0$ and unique C^1 function $\xi_{\pm} : [0, r_1] \to \mathbb{R}$ such that $\xi_{\pm}(0) = \xi_0^{\pm}$ and $H(\xi_{\pm}(\mu), \mu) = 0$. It is that

$$G(\tau_0 + \xi_{\pm}(\mu)\mu^{\frac{1}{4}}, \mu) = 0 \text{ for } \mu \in (0, r_1].$$

Let $\tau_{\pm}(\mu) = \tau_0 + \xi_{\pm}(\mu)\mu^{1/4}$ for $\mu \in (0, r_1]$. Hence, for $\mu > 0$, if $F_2(\tau_0)/F_1''(\tau_0) > 0$ there is no solution and $F_2(\tau_0)/F_1''(\tau_0) < 0$ there are two solutions. For $\mu < 0$, similar results hold.

Remark 3. From the above discussion, the following facts are clear. When $F_2(\tau_0)/F_1''(\tau_0) > 0$, $G(\tau, \mu) = 0$ has no solutions if $\mu > 0$ and two solutions if $\mu < 0$. Hence $G(\tau, \mu) = 0$ undergoes a saddle-node bifurcation at $\mu = 0$.

When $F_2(\tau_0)/F_1''(\tau_0) < 0$, saddle-node bifurcation similar to the above can happen by reversing μ to $-\mu$.

In the following, we will prove that the solutions obtained above are simple. From (36), we have

$$2\alpha_1^*(\tau,\mu)\frac{\partial}{\partial\tau}\alpha_1^*(\tau,\mu) = \frac{\mu}{a}(d_{22}a_1'(\tau) - d_{12}a_2'(\tau)) + O(\mu^{\frac{3}{2}}),$$
$$2\alpha_2^*(\tau,\mu)\frac{\partial}{\partial\tau}\alpha_2^*(\tau,\mu) = \frac{\mu}{a}(d_{11}a_2'(\tau) - d_{21}a_1'(\tau)) + O(\mu^{\frac{3}{2}}).$$

Therefore

$$c_{13} := 2d_{11}\alpha_1^*\frac{\partial}{\partial\tau}\alpha_1^* + 2d_{12}\alpha_2^*\frac{\partial}{\partial\tau}\alpha_2^* - a_1'(\tau)\mu = 0,$$
$$c_{23} := 2d_{21}\alpha_1^*\frac{\partial}{\partial\tau}\alpha_1^* + 2d_{22}\alpha_2^*\frac{\partial}{\partial\tau}\alpha_2^* - a_2'(\tau)\mu = 0.$$

Clearly, the Jacobian matrix is

$$\begin{aligned}
&\frac{\partial(f_1, f_2, f_3)}{\partial(\alpha_1, \alpha_2, \tau)}|_{(\alpha_1^*(\tau,\mu),\alpha_2^*(\tau,\mu),\tau_\pm(\mu))} \\
&= \begin{pmatrix} 2d_{11}\alpha_1^* & 2d_{12}\alpha_2^* & -a_1'(\tau_\pm)\mu + O(\mu^{\frac{3}{2}}) \\ 2d_{21}\alpha_1^* & 2d_{22}\alpha_2^* & -a_2'(\tau_\pm)\mu + O(\mu^{\frac{3}{2}}) \\ 2d_{31}\alpha_1^* + d_{32}\alpha_2^* & d_{32}\alpha_1^* + 2d_{33}\alpha_2^* & -a_3'(\tau_\pm)\mu + O(\mu^{\frac{3}{2}}) \end{pmatrix} \\
&= \begin{pmatrix} 2d_{11}\alpha_1^* & 2d_{12}\alpha_2^* & c_{13} + O(\mu^{\frac{3}{2}}) \\ 2d_{21}\alpha_1^* & 2d_{22}\alpha_2^* & c_{23} + O(\mu^{\frac{3}{2}}) \\ 2d_{31}\alpha_1^* + d_{32}\alpha_2^* & d_{32}\alpha_1^* + 2d_{33}\alpha_2^* & \frac{\mu}{a}F'(\tau_\pm) + O(\mu^{\frac{3}{2}}) \end{pmatrix} \\
&= \begin{pmatrix} 2d_{11}\alpha_1^* & 2d_{12}\alpha_2^* & O(\mu^{\frac{3}{2}}) \\ 2d_{21}\alpha_1^* & 2d_{22}\alpha_2^* & O(\mu^{\frac{3}{2}}) \\ 2d_{31}\alpha_1^* + d_{32}\alpha_2^* & d_{32}\alpha_1^* + 2d_{33}\alpha_2^* & \frac{\mu}{a}F'(\tau_\pm) + O(\mu^{\frac{3}{2}}) \end{pmatrix}.
\end{aligned}$$

Then we can get that

$$|\frac{\partial(f_1, f_2, f_3)}{\partial(\alpha_1, \alpha_2, \tau)}|_{(\alpha_1^*,\alpha_2^*,\tau_\pm)} = DF''(\tau_0)\frac{\xi_0^\pm}{a}\mu^{\frac{5}{4}} + O(\mu^{\frac{3}{2}}) \neq 0 \text{ for small } \mu.$$

We have proved the following result:

Theorem 5.1. *Assume that the conditions (H1)-(H7) are satisfied and the system*

$$d_{11}\alpha_1^2 + d_{12}\alpha_2^2 = a_1(\tau)\mu, \;\; d_{21}\alpha_1^2 + d_{22}\alpha_2^2 = a_2(\tau)\mu$$

for $\tau = \tau_0$, *and small* $|\mu| > 0$ *have four real simple solutions.*

Assume that $\mu > 0$. *Then if* $F_1''(\tau_0)F_2(\tau_0) > 0$, *there is no homoclinic solution; if* $F_1''(\tau_0)F_2(\tau_0) < 0$, *there exists a small* $\bar{\mu} > 0$ *such that for any* $0 < \mu \le \bar{\mu}$, (35) *has* 8 *simple solutions of the form* $(\boldsymbol{\alpha}_j^*(\tau_\pm(\mu), \mu), \tau_\pm(\mu), \mu)$, *where* $j = 1, 2, 3, 4$ *and* $\tau_\pm(\mu) = \tau_0 + \xi^\pm(\mu)\mu^{1/4}$.

Assume that $\mu < 0$. *Then if* $F_1''(\tau_0)F_2(\tau_0) < 0$, *there is no homoclinic solution; if* $F_1''(\tau_0)F_2(\tau_0) > 0$, *there exists a small* $\tilde{\mu} > 0$ *such that for any* $-\tilde{\mu} < \mu < 0$, (35) *has* 8 *simple solutions of the form* $(\boldsymbol{\alpha}_j^*(\tau_\pm(\mu), \mu), \tau_\pm(\mu), \mu)$, *where* $j = 1, 2, 3, 4$ *and* $\tau_\pm(\mu) = \tau_0 + \xi^\pm(\mu)|\mu|^{1/4}$.

We have constructed 8 simple solutions for (35). Making the change of variable $\boldsymbol{\alpha} \to \boldsymbol{\beta}$, which is the inverse of the codiagonalization of the first two equations, we find that (11) has eight simple solutions $(\boldsymbol{\beta}^{(1)}, \tau_{\pm}^{(1)}, \mu), ..., (\boldsymbol{\beta}^{(4)}, \tau_{\pm}^{(4)}, \mu)$ of $M_i(\boldsymbol{\beta}, \tau, \mu) = 0, i = 1, 2, 3$.

Theorem 5.2. *Assume that* $0 < |\mu_0| < \bar{\mu}$ *and* $(\boldsymbol{\beta}^{(j)}, \tau_{\pm}^{(j)}, \mu_0)$, $1 \leq j \leq 4$ *are 8 simple solutions for* $M_i(\boldsymbol{\beta}, \tau, \mu) = 0, i = 1, 2, 3$. *Then for each fixed* $1 \leq j \leq 4$, *there exists an open region* $I_j \subset \mathbb{R}$ *containing zero and differentiable functions,* $\omega_j : I_j \to \mathbb{R}^2$ *and* $\eta_{\pm}^{(j)} : I_j \to \mathbb{R}$ *such that* $\omega_j(0) = 0$, $\eta_{\pm}^{(j)}(0) = 0$ *and* $H_i(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau_{\pm}^{(j)} + \eta_{\pm}^{(j)}(s)), s^2\mu_0) = 0$, $i = 1, 2, 3$, *for* $s \in I_j$ *and* $s \neq 0$.

Proof. Let $H = (H_1, H_2, H_3), M = (M_1, M_2, M_3)$. For each fixed j, since $(\boldsymbol{\beta}^{(j)}, \tau_{\pm}^{(j)}, \mu_0)$ is a simple solution for $M(\boldsymbol{\beta}^{(j)}, \tau_{\pm}^{(j)}, \mu_0) = 0$, then $D_{(\boldsymbol{\beta},\tau)} M(\boldsymbol{\beta}^{(j)}, \tau_{\pm}^{(j)}, \mu_0)$ is a 3×3 nonsingular matrix. For each j, define a C^2 function $W : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^3$ by

$$W(x, y, s) = \begin{cases} \frac{1}{s^2} H(s(\boldsymbol{\beta}^{(j)} + x), \tau_{\pm}^{(j)} + y, s^2\mu_0), & \text{for } s \neq 0, \\ M(\boldsymbol{\beta}^{(j)} + x, \tau_{\pm}^{(j)} + y, \mu_0), & \text{for } s = 0. \end{cases}$$

Clearly, $H = 0$ if and only if $W = 0$ for $s \neq 0$. Through direct calculations, we have $W(0, 0, 0) = 0$ and

$$D_{(x,y)} W(0, 0, 0) = D_{(\boldsymbol{\beta},\tau)} M(\boldsymbol{\beta}^{(j)}, \tau_{\pm}^{(j)}, \mu_0)$$

is a nonsingular matrix. By the implicit function theorem there exist an open region $I_j \subset \mathbb{R}$ containing zero and a differentiable functions, $\omega_j : I_j \to \mathbb{R}^2$ and $\eta_{\pm}^{(j)} : I_j \to \mathbb{R}$ such that $\omega_j(0) = 0$, $\eta_{\pm}^{(j)}(0) = 0$ and $W(\omega_j(s), \eta_{\pm}^{(j)}(s), s) = 0$. Hence

$$H(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau_{\pm}^{(j)} + \eta_{\pm}^{(j)}(s)), s^2\mu_0) = 0 \text{ for } s \neq 0.$$

The proof has been completed. ∎

By Theorem 5.2, the bifurcation function $H = (H_1, H_2, H_3) = 0$ at $(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau_{\pm}^{(j)} + \eta_{\pm}^{(j)}(s), s^2\mu_0)$. Then system (13) has the solution $\phi(\boldsymbol{\beta}, \tau, \mu)$. Hence system (5) has 8 homoclinic solutions given by

$$\begin{aligned} \gamma_s^{(j)}(t) =& \gamma(t) + \sum_{p=1}^{2} s(\beta_{0p}^{(j)} + \omega_{jp}(s)) u_p(t) \\ & + K(I - P)\widetilde{g}(\phi, \tau_{\pm}^{(j)} + \eta_{\pm}^{(j)}(s), s^2\mu_0)(t), \end{aligned} \tag{41}$$

for $0 \neq s \in I_j$, $1 \leq j \leq 4$. Clearly, $\lim_{s\to 0} \gamma_s^{(j)}(t) = \gamma(t)$.

6. **Transversality of the new homoclinic orbits.** From the construction of $\gamma_s^{(j)}$, we find that the solutions are robust with respect to small perturbation of g. This alone shows that each of the 4 solutions is a transversal homoclinic solution. The same argument was used by Mallet-Paret in [18] to show that the homoclinic orbits in some delay equations are transverse.

The purpose of this section is to give a direct proof of the transversality of the homoclinic solutions bifurcating from γ.

Theorem 6.1. *Assume that* $(H1) - (H7)$ *hold and the conditions of Theorem 4.3 are satisfied. Then the 8 homoclinic solutions* $\gamma_s^{(j)}(t)$, $j = 1, ...4$, *obtained in* (41) *of* §5 *are transverse for each nonzero* $s \in I_j$.

Proof. Observe that

$$\frac{\partial \gamma_s^{(j)}}{\partial s}|_{s=0} = \sum_{p=1}^{2} \beta_{0p}^{(j)} u_p. \tag{42}$$

Since $\gamma_s^{(j)}$ is a solution of (5) with $\mu = s^2\mu_0$, we have

$$\dot{\gamma}_s^{(j)} = f(\gamma_s^{(j)}) + s^2\mu_0 g(\gamma_s^{(j)}, t + \tau_\pm^{(j)} + \eta_\pm^{(j)}(s), s^2\mu_0). \tag{43}$$

Differentiating with respect to t, we have

$$\begin{aligned}\ddot{\gamma}_s^{(j)} =& [Df(\gamma_s^{(j)}) + s^2\mu_0 D_1 g(\gamma_s^{(j)}, t + \tau_\pm^{(j)} + \eta_\pm^{(j)}(s), s^2\mu_0)]\dot{\gamma}_s^{(j)} \\ &+ s^2\mu_0 D_3 g(\gamma_s^{(j)}, t + \tau_\pm^{(j)} + \eta_\pm^{(j)}(s), s^2\mu_0).\end{aligned} \tag{44}$$

The variational equation of (5) along $\gamma_s^{(j)}$ can be written as

$$\dot{u} = [Df(\gamma) + G(s)]u, \tag{45}$$

where

$$G(s) = Df(\gamma_s^{(j)}) - Df(\gamma) + s^2\mu_0 D_1 g(\gamma_s^{(j)}, t + \tau_\pm^{(j)} + \eta_\pm^{(j)}(s), s^2\mu_0).$$

To prove the transvertality of $\gamma_s^{(j)}$, it suffices to show that equation (45) has no nonzero bounded solution. It is easy to check that

$$\begin{aligned} G(0) &= 0 \\ \frac{\partial G}{\partial s}|_{s=0} &= G'(0) = \sum_{p=1}^{2} \beta_{0p}^{(j)} D^2 f(\gamma) u_p. \end{aligned} \tag{46}$$

Applying the projections P and $(I-P)$ on equation (45), we have

$$\dot{u} = Df(\gamma)u + (I-P)G(s)u, \tag{47}$$

$$0 = PG(s)u. \tag{48}$$

The general bounded solution u^* of (47) has the following form

$$u^* = \sum_{q=1}^{3} \eta_q u_q + K(I-P)G(s)u^*,$$

where $\eta_q \in \mathbb{R}$. Since $G(0) = 0$, there exist a small region $\tilde{I}$ around zero such that $(I - K(I-P)G(s))$ is invertible for $s \in \tilde{I}$. We get

$$u^* = [I - K(I-P)G(s)]^{-1} \sum_{q=1}^{3} \eta_q u_q \text{ for } s \in \tilde{I}.$$

Substituting $u = u^*$ into equation (48), we have

$$\begin{aligned} 0 &= PG(s)[I - K(I-P)G(s)]^{-1} \sum_{q=1}^{3} \eta_q u_q \\ &= \sum_{i=1}^{3} \psi_i \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I-P)G(s)]^{-1} \sum_{q=1}^{2} \eta_q u_q \rangle ds \\ &= \sum_{i,q=1}^{3} \psi_i \eta_q \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I-P)G(s)]^{-1} u_q \rangle ds \end{aligned}$$

$$= (\psi_1, \psi_2, \psi_3)V(G(s))(\eta_1, \eta_2, \eta_3),$$

where matrix $V(G(s))$ is given by $V(G(s)) = [v_{iq}(s)]_{3\times 3}$ and

$$v_{iq}(s) = \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I-P)G(s)]^{-1}u_q\rangle dt. \tag{49}$$

Note that ψ_1, ψ_2, ψ_3 are linearly independent. If we can prove that $V(G(s))$ is a nonsingular matrix, then $\eta_1 = \eta_2 = \eta_3 = 0$. Thus the only bounded solution for the linear variational equation along $\gamma_s^{(i)}$ is $u^* = 0$. Therefore $\gamma_s^{(i)}$ is a transverse homoclinic solution of (5).

It remains to show $V(G(s))$ is nonsingular. By (46), (49) and (54), we have $v_{iq}(0) = 0$ and

$$\begin{aligned}\frac{\partial v_{iq}}{\partial s}|_{s=0} &= \sum_{p=1}^{2} \beta_{0p}^{(j)} \int_{-\infty}^{+\infty} \langle \psi_i, D^2 f(\gamma)u_q u_p\rangle dt.\\ &= \sum_{p=1}^{2} b_{pq}^{(i)}\beta_{0p}^{(j)},\end{aligned}$$

where $q \neq 3$. We have the following approximation of $v_{iq}(s)$:

$$v_{iq}(s) = s\sum_{p=1}^{2} b_{pq}^{(i)}\beta_{0p}^{(j)} + O(s^2), \tag{50}$$

where $i = 1, 2, 3, q = 1, 2$. When $q = 3$, by (49)

$$v_{i3}(s) = \int_{-\infty}^{+\infty} \langle \psi_i, G(s)[I - K(I-P)G(s)]^{-1}\dot{\gamma}\rangle dt. \tag{51}$$

Then from (44), we see that $\dot{\gamma}_s^{(j)}$ is a bounded solution of

$$\dot{u} = Df(\gamma)u + G(s)\dot{\gamma}_s^{(j)} + s^2\mu_0 D_3 g(\gamma_s^{(j)}, t + \tau_\pm^{(j)} + \eta_\pm^{(j)}(s), s^2\mu_0). \tag{52}$$

Hence $\dot{\gamma}_s^{(j)}$ can be expressed as

$$\dot{\gamma}_s^{(j)} = \sum_{q=1}^{3} u_q c_q(s) + K(I-P)(G(s)\dot{\gamma}_s^{(j)} + s^2\mu_0 D_3 g(\gamma_s^{(j)}, t + \tau_\pm^{(j)} + \eta_\pm^{(j)}(s), s^2\mu_0)),$$

for some smooth functions $c_i : \mathbb{R} \to \mathbb{R}$. From (43), we have

$$\dot{\gamma}_s^{(j)}|_{s=0} = \sum_{q=1}^{3} u_q c_q(0) = \dot{\gamma}$$

and hence

$$\dot{\gamma}_s^{(j)} = \dot{\gamma} + s\sum_{q=1}^{3} u_q c_q'(0) + K(I-P)G(s)\dot{\gamma}_s^{(j)} + O(s^2).$$

This implies that

$$[I - K(I-P)G(s)]\dot{\gamma}_s^{(j)} = \dot{\gamma} + s\sum_{q=1}^{3} u_q c_q'(0) + O(s^2).$$

Note that map $I-K(I-P)G(s)$ is invertible for $s\in(-s_1,s_1)$. So

$$[I-K(I-P)G(s)]^{-1}\dot{\gamma}=\dot{\gamma}_s^{(j)}-s\sum_{q=1}^{3}u_q c_q'(0)+O(s^2).$$

We get

$$\begin{aligned}G(s)[I-K(I-P)G(s)]^{-1}\dot{\gamma}&=G(s)\dot{\gamma}_s^{(j)}-s^2G'(0)\sum_{q=1}^{3}u_q c_q'(0)+O(s^3)\\&=\ddot{\gamma}_s^{(j)}-Df(\gamma)\dot{\gamma}_s^{(j)}-s^2\mu_0 D_3 g(\gamma,t+\tau_0,0)\\&\quad-s^2\sum_{p=1}^{2}\beta_{0p}^{(j)}D^2f(\gamma)u_p\dot{\gamma}-s^2\sum_{p,q=1}^{2}\beta_{0p}^{(j)}D^2f(\gamma)u_pu_qc_q'(0)+O(s^3),\end{aligned}\tag{53}$$

where (46) and (52)are used.

By substituting u_p, $1\le p\le 3$, into equation (3) and differentiating respect to t, we get

$$\ddot{u}_p=D^2f(\gamma)\dot{\gamma}u_p+Df(\gamma)\dot{u}_p.$$

We have $D^2f(\gamma)\dot{\gamma}u_p=L(\dot{u}_p)\in R(L)$ and hence

$$\int_{-\infty}^{+\infty}\langle\psi_i,D^2f(\gamma)\dot{\gamma}u_p\rangle dt=0.\tag{54}$$

From (51), (53) and (54), we get

$$\begin{aligned}&\int_{-\infty}^{+\infty}\langle\psi_i,G(s)[I-K(I-P)G(s)]^{-1}\dot{\gamma}\rangle dt\\&=-s^2\mu_0\int_{-\infty}^{+\infty}\langle\psi_i,D_3g(\gamma,t+\tau_0,0)\rangle dt\\&-s^2\sum_{p,q=1}^{2}\beta_{0p}^{(j)}c_q'(0)\int_{-\infty}^{+\infty}\langle\psi_i,D^2f(\gamma)u_pu_q\rangle dt+O(s^3).\end{aligned}$$

Hence

$$v_{i\,3}(s)=-s^2\mu_0 a_i'(\tau_0)-s^2\sum_{p,q=1}^{2}\beta_{0p}^{(j)}c_q'(0)b_{pq}^{(i)}+O(s^3).\tag{55}$$

From (50) and (55), we have

$$\begin{aligned}\det(V(G(s)))&=-s^4\det\left(\frac{\partial(M_1,M_2,M_3)}{\partial(\beta_1,\beta_2,\tau)}(\tau_0,\boldsymbol{\beta}_0^{(j)},\mu_0)\right)+O(s^5),\\&=s^4\det(D_{(\boldsymbol{\beta},\tau)}M(\boldsymbol{\beta}_0^{(j)},\tau_0,\mu_0))+O(s^5).\end{aligned}$$

Note that $D_{(\boldsymbol{\beta},\tau)}M(\boldsymbol{\beta}_0^{(j)},\tau_0,\mu_0)$ is nonsingular. Then there exists a region $\hat{I}$, $\hat{I}\subset\tilde{I}$ such that $V(G(s))$ is nonsingular when $0\ne s\in\hat{I}$. Then the variational equation along $\gamma_s^{(j)}$ has no nonzero bounded solutions. So $\gamma_s^{(j)}$ is a transverse homoclinic solution of (5). The proof has been completed. □

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