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CODIAGONALIZATION OF MATRICES AND EXISTENCE OF MULTIPLE HOMOCLINIC SOLUTIONS*

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Abstract The purpose of this paper is twofold. First, we use Lagrange's method and the generalized eigenvalue problem to study systems of two quadratic equations. We find exact conditions so the system can be codiagonalized and can have up to 4 solutions. Second, we use this result to study homoclinic bifurcations for a periodically perturbed system. The homoclinic bifurcation is determined by 3 bifurcation equations. To the lowest order, they are 3 quadratic equations, which can be simplified by the codiagonalization of quadratic forms. We find that up to 4 transverse homoclinic orbits can be created near the degenerate homoclinic orbit.

Keywords Degenerate homoclinic bifurcation, Lyapunov-Schmidt reduction, Lagrange's method, codiagonalization of quadratic forms.

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1. Introduction

We study homoclinic bifurcations of the periodically perturbed system

$$\dot{y}(t) = f(y(t)) + \mu g(y(t), \mu, t), \quad y \in \mathbb{R}^n,\ \mu \in \mathbb{R}. \tag{1.1}$$

The unperturbed autonomous system

$$\dot{y}(t) = f(y(t)), \tag{1.2}$$

satisfies the following assumptions:

(H1) $f \in C^3$, $f(0) = 0$ and $Re(\sigma(Df(0))) \neq 0$.

(H2) Equation (1.2) has a homoclinic solution $\gamma(t)$ asymptotic to the equilibrium $y = 0$.

The variational equation of (1.2) along the homoclinic solution γ is

$$\dot{u}(t) = Df(\gamma(t))u(t). \tag{1.3}$$

System (1.3) has $d \geq 1$ linearly independent bounded solutions, including $\dot{\gamma}$.

We assume g satisfies

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(H3) $g \in C^3$ and $g(y, \mu, t+2) = g(y, \mu, t)$.

By $(H1), y = 0$ is a hyperbolic equilibrium of (1.2). Since the perturbation term $\mu g(x, \mu, t) = O(|\mu|)$, generically, equation (1.1) has a hyperbolic periodic orbit $\theta(t, \mu) := O(|\mu|)$ near 0. Using the change of variable $y = x + \theta(t, \mu)$, we may assume that

(H4) $g(0, \mu, t) = 0$.

By the phase shift of $\gamma(t)$, system (1.2) has a family of homoclinic orbits, among which we assume $x(t)$ is a small perturbation to $\gamma(t-\tau)$, equivalently, $x(t+\tau)$ is a small perturbation of $\gamma(t)$. The parameter τ is determined by a phase condition as follows: If $x(t+\tau) = \gamma(t) + z(t)$ then $z(0) \perp \dot{\gamma}(0)$. Rename $x(t+\tau)$ to $x(t)$, we assume that $x(t)$ is a small perturbation of $\gamma(t)$ and satisfies the following system:

$$\dot{x}(t) = f(x(t)) + \mu g(x(t), \mu, t+\tau). \tag{1.4}$$

From (H4), $x = 0$ is a hyperbolic equilibrium even after small periodic perturbations. Let $W^s(0)$, $W^u(0)$ be the stable and unstable manifolds of $x = 0$ when $\mu = 0$. Clearly, the homoclinic orbit γ lies on $W^s(0) \bigcap W^u(0)$. If (1.3) has d dimension bounded solutions, then $d = \dim(T_{\gamma(0)}W^s(0) \bigcap T_{\gamma(0)}W^u(0))$.

When $\mu \neq 0$, (1.4) may have bifurcations near γ. The case $d = 1$ has been extensively studied. In this case breaking of the homoclinic orbit γ is restored by choosing the parameter τ, as in [5]. Hale [6] proposed to study the degenerate cases where $d \geq 2$. The case $d = 2$ has been considered in [14]. The purpose of the present work is to treat the case $d = 3$. Using the method of Lyapunov-Schmidt reduction, we derive a system of bifurcation functions $H_j, 1 \leq j \leq 3$, the zeros of which correspond to the persistence of homoclinic solutions for (1.4). The last equation $H_3 = 0$ can be dealt with by selecting the parameter τ as usual, while $H_j = 0, j = 1, 2$ can be reduced to a system of quadratic equations. By the Lagrange's method and codiagnalization of quadratic forms, we show that the quadratic system can have up to 4 solutions. Finally, if the solutions to the quadratic system are nondegenerate, then the bifurcation functions have nondegenerate zeros and the perturbed system has transverse homoclinic orbits.

Codiagnalization of matrices has been used by Jibin Li and Lin [12] to study systems of coupled KdV equations. It may also be useful when studinging the 2x2 systems of hyperbolic conservation laws with quadratic nonlinearities [19, 20], base on personal conversation with Shearer. In [14], the method based on circular and hyperbolic rotations, was used to codiagonalize two quadratic forms. The new method in this paper is easier to use if one wants to find conditions for the existence of 4 solutions to quadratic systems.

Given a symmetric real matrix $B \in \mathbb{R}^{2\times 2}$, then

$$F(x_1, x_2) = (x_1, x_2)B(x_1, x_2)^T$$

is a quadratic form associated to B. If B is diagonalized by a nonsingular matrix M: $M^T BM = \mathrm{diag}(d_1, d_2)$, then

$$F(x_1, x_2) = (y_1, y_2)\mathrm{diag}(d_1, d_2)(y_1, y_2)^T = d_1 y_1^2 + d_2 y_2^2,$$

where $(x_1, x_2)^T = M(y_1, y_2)^T$. The symmetric transformation described above is also called the congruence diagonalization. It should not confused with the similarity transformation of B which is defined by $M^{-1}BM$. For example the matrix $\mathrm{diag}(\lambda_1, -\lambda_2), \lambda_j > 0$, can be reduced to $\mathrm{diag}(1, -1)$ by the matrix $M = \mathrm{diag}(1/\sqrt{\lambda_1}, 1/\sqrt{\lambda_2})$, which is a symmetric reduction, not similarity reduction.

In §2, we introduce notations to be used in this paper. We also present the reduced bifurcation functions which, to the lowest degree, represent the breaking of the homoclinc orbits under the periodic perturbations. In §3 we derive the bifurcation equation by using the Lyapunov-Schmidt reduction. To the lowest degree, they reduce to three quadratic equations. In §4, we introduce the Lagrange's method and generalized eigenvalue problems to study solutions of two quadratic forms. The cases when one equation is elliptic are considered in §4.1. The other cases when one equation is hyperbolic and none is elliptic are considered in §4.2. In §4.3, we present the method of codiagonalization of two quadratic equations based on cases studied in §4.1 and §4.2. In §5, we derive the reduced bifurcation function $F(\tau)$. We show a simple zero of F corresponds to the existence of a homoclinic solution near γ. In §6, we present an example showing that our conditions work consistently.

2. Notations and preliminaries

Notations. Since $y = 0$ is a hyperbolic equilibrium, from [17], (1.3) has exponential dichotomies on $J = \mathbb{R}^{\pm}$ respectively. In particular, there exist projections to the stable and unstable subspaces, $P_s + P_u = I$, and constants $m > 0$, $K_0 \geq 1$ such that

$$\begin{aligned} (i)\ \ & |U(t)P_sU^{-1}(s)| \leq K_0 e^{2m(s-t)}, \quad \text{for } s \leqslant t \text{ on } J, \\ (ii)\ \ & |U(t)P_uU^{-1}(s)| \leq K_0 e^{2m(t-s)}, \quad \text{for } t \leqslant s \text{ on } J. \end{aligned} \tag{2.1}$$

For the same $m > 0$, define the Banach space

$$\mathcal{Z} = \{z \in C^0(\mathbb{R}, \mathbb{R}^n) : \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|} < \infty\},$$

with the norm $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|}$. The linear variational system

$$Lu := \dot{u} - Df(\gamma)u = h \tag{2.2}$$

will be considered in $\mathcal{Z}$. The adjoint operator for L is

$$L^*\psi := \dot{\psi} + (Df(\gamma))^*\psi. \tag{2.3}$$

The domains of (2.2) and (2.3) are the dense subset of $\mathcal{Z}$, defined as

$$D(L) := \{u : u, u_t \in \mathcal{Z}\}, \quad D(L^*) := \{\psi : \psi, \psi_t \in \mathcal{Z}\}.$$

From the theory of homoclinic bifurcations [17], $L : \mathcal{Z} \to \mathcal{Z}$ is a Fredholm operator with index 0. The range of L is orthogonal to the null space of L^*. That is

$$h \in R(L) \text{ iff } \int_{-\infty}^{\infty} \langle\psi(t), h(t)\rangle dt = 0, \text{ for all } \psi \in N(L^*). \tag{2.4}$$

From $d = 3$, $N(L)$ is three dimensional. Note that $\dot{\gamma} \in N(L)$. Without loss in generality, let (u_1, u_2, u_3) be a basis of $N(L)$, where we choose $u_3 = \dot{\gamma}$. And let (ψ_1, ψ_2, ψ_3) be a basis of $N(L^*)$.

We define some Melnikov types of integrals [16] that will be used in the future. For integers $p, q = 1, 2$ and $i = 1, 2, 3$, let

$$b_{pq}^{(i)} = \int_{-\infty}^{+\infty} \langle \psi_i(t), \frac{1}{2} D^2 f(\gamma(t)) u_p(t) u_q(t) \rangle dt, \ p, q = 1, 2,$$
$$\tilde{a}_i(\tau) = \int_{-\infty}^{+\infty} \langle \psi_i(t), g(\gamma(t), 0, t+\tau) \rangle dt.$$

We look for conditions so that (1.4) can have homoclinic solutions near γ. Let $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$. We shall use that the reduced bifurcation functions $M_i : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined bellow:

$$M_i(\boldsymbol{\beta}, \tau, \mu) = \sum_{p,q=1}^{2} b_{pq}^{(i)} \beta_p \beta_q + \tilde{a}_i(\tau) \mu, \ i = 1, 2, 3. \tag{2.5}$$

To the lowest degree, (2.5) describes the jump discontinuity $x(0_-) - x(0_+)$ along the direction of $\psi_i(0)$, see [13].

Define the 2×2 matrices $B^{(i)} = (b_{pq}^{(i)})$, $i \in \{1, 2, 3\}$. We need to solve the following system of quadratic equations

$$\boldsymbol{\beta}^T B^{(i)} \boldsymbol{\beta} + \tilde{a}_i(\tau) \mu = 0, \quad i = 1, 2, 3. \tag{2.6}$$

3. Derivation of the bifurcation equations using the Lyapunov-Schimidt reduction

By (H2), system (1.4) with $\mu = 0$ has a homoclinic solution γ. In this section, we will find conditions such that (1.4), with small $\mu \neq 0$, has some homoclinic solution γ_μ satisfying $\|\gamma - \gamma_\mu\| = O(\sqrt{|\mu|})$.

Let $D_i h$ or $D_{ij} h$ denote the derivatives of a multivariate function h with respect to its i-th or the i, j-th variables. With the change of variable $x(t) = \gamma(t) + z(t)$, where $z(0) \perp \dot{\gamma}(0)$, (1.4) is transformed to

$$\dot{z} = Df(\gamma) z + \widetilde{g}(z, \tau, \mu), \tag{3.1}$$

where

$$\widetilde{g}(z, \tau, \mu)(t) = f(\gamma(t) + z(t)) - f(\gamma(t)) - Df(\gamma(t)) z + \mu g(\gamma(t) + z(t), \mu, t + \tau). \tag{3.2}$$

In [14], it is shown that $\widetilde{g}(\cdot, \tau, \mu)$ mapps $\mathcal{Z} \mapsto \mathcal{Z}$ and satisfies the following properties:

$$\begin{aligned} &(1)\ \widetilde{g}(0, \tau, 0) = 0, \ D_1 \widetilde{g}(0, \tau, 0) = 0, \\ &(2)\ D_{11} \widetilde{g}(0, \tau, 0) = D^2 f(\gamma), \\ &(3)\ \frac{\partial \widetilde{g}}{\partial \mu}(0, \tau, 0) = g(\gamma, 0, t + \tau). \end{aligned}$$

Recall that $L(u) = \dot{u} - Df(\gamma) u$ in the Banach space $\mathcal{Z}$. As in [17], we define the subspace of $\mathcal{Z}$, which consists the range of L in $\mathcal{Z}$.

$$\widetilde{\mathcal{Z}} = \{h \in \mathcal{Z} : \int_{-\infty}^{\infty} \langle \psi_i(s), h(s) \rangle ds = 0, i = 1, 2, 3\}.$$

Consider a nonhomogeneous equation

$$\dot{z} - Df(\gamma)z = h, \quad z(0) \perp \dot{\gamma}(0). \tag{3.3}$$

Let $\mathcal{Z}^{\perp}$ be the subspace of $\mathcal{Z}$ consisting of $z(t)$ with $z(0) \perp \dot{\gamma}(0)$. If $h \in \widetilde{\mathcal{Z}}$, using the variation of constants, there exists an operator $K : \widetilde{\mathcal{Z}} \to N(L)^{\perp}$ such that Kh is a solution of (3.3). Clearly, the general bounded solution of (3.3) is $z(t) = \sum_{p=1}^{2} \beta_p u_p(t) + (Kh)(t)$, where $\beta_p \in \mathbb{R}$.

From (2.4), $R(L) \oplus N(L^*) = \mathcal{Z}$. Define a map $P : \mathcal{Z} \to \mathcal{Z}$ such that $N(P) = R(L)$ and $R(P) = N(L^*)$. In particular,

$$h \in N(P) \text{ if and only if } \int_{-\infty}^{\infty} \langle \psi_i(s), h(s) \rangle ds = 0, \ i = 1, 2, 3.$$

As in [17], one can prove that P satisfies the following properties:

Lemma 3.1. (1) *P and $I - P$ are projections.*
(2) *$R(P) \oplus R(L) = \mathcal{Z}$.*
(3) *$R(I - P) = N(P) = R(L) = \widetilde{\mathcal{Z}}$.*

We now use the Lyapunov-Schmidt reduction to (3.1). Applying P and $(I - P)$ on (3.1), we find that (3.1) is equivalent to the following system

$$\dot{z} = Df(\gamma)z - (I - P)\widetilde{g}(z, \tau, \mu), \tag{3.4}$$
$$P\widetilde{g}(z, \tau, \mu) = 0. \tag{3.5}$$

First, we solve (3.4) for $z \in \mathcal{Z}^{\perp}$. Then the bifurcation equations are obtained by substituting the solution z into (3.5).

Lemma 3.2. *There exists a C^2 solution $z = \phi(\boldsymbol{\beta}, \tau, \mu)$ to (3.4) in $\mathcal{Z}^{\perp}$, where ϕ is defined for $|\boldsymbol{\beta}| + |\mu| \le \delta_1, \ \tau \in \mathbb{R}$. Moreover $\phi(\boldsymbol{\beta}, \tau, \mu)$ satisfies $\phi(0, \tau, 0) = 0$ and $(\partial\phi/\partial\beta_p)|_{(0,\tau,0)} = u_p, \ p = 1, 2$.*

Proof. Since $R(I - P) = \widetilde{\mathcal{Z}}$ and $K : \widetilde{\mathcal{Z}} \to N(L)^{\perp}$, we define a C^2 map: $F : \mathcal{Z}^{\perp} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathcal{Z}^{\perp}$ by

$$F(z, \boldsymbol{\beta}, \tau, \mu) = z - (\sum_{p=1}^{2} \beta_p u_p + K(I - P)\widetilde{g}(z, \tau, \mu)), \tag{3.6}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$. From the properties of $\tilde{g}(z, \tau, \mu)$, we have

$$F(0, 0, \tau, 0) = 0, \quad D_1F(0, 0, \tau, 0) = I. \tag{3.7}$$

Since F is periodic in τ, by the Implicit Function Theorem, therre exists a C^2 function $\phi(\boldsymbol{\beta}, \tau, \mu)$ defined for $|\boldsymbol{\beta}| + |\mu| \le \delta_1, \ \tau \in \mathbb{R}$, with the range in $\mathcal{Z}^{\perp}$, and satisfies

$$F(\phi(\boldsymbol{\beta}, \tau, \mu), \boldsymbol{\beta}, \tau, \mu) = 0.$$

Hence,

$$\phi(\boldsymbol{\beta}, \tau, \mu) = \sum_{p=1}^{2} \beta_p u_p + K(I - P)\widetilde{g}(\phi(\boldsymbol{\beta}, \tau, \mu), \tau, \mu). \tag{3.8}$$

Differentiating (3.8) with respect to β_p and evaluating at $(0,\tau,0)$, we get

$$\frac{\partial \phi}{\beta_p}\bigg|_{(0,\tau,0)}(t) = u_p(t), \quad p = 1,2.$$

The proof has been completed. □

Substituting the solution $\phi(\boldsymbol{\beta},\tau,\mu)$ of (3.4) into (3.5), we have the bifurcation equation

$$\begin{aligned} &P\widetilde{g}(\phi(\boldsymbol{\beta},\tau,\mu),\tau,\mu) = 0, \quad \text{equivalently,} \\ &H_i(\boldsymbol{\beta},\tau,\mu) = 0, \quad i = 1,2,3, \end{aligned} \tag{3.9}$$

where

$$H_i(\boldsymbol{\beta},\tau,\mu) := \int_{-\infty}^{+\infty} \langle \psi_i(s), \widetilde{g}(\phi(\boldsymbol{\beta},\tau,\mu),\tau,\mu)(s)\rangle ds. \tag{3.10}$$

We have proved the following important result

Theorem 3.1. *If ϕ satisfies (3.8) and $(\boldsymbol{\beta},\tau,\mu) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ solves (3.9), then $z = \phi$ is a solution of (3.1) and hence the perturbed system (1.4) has a homoclinic orbit $x = \gamma + \phi$.*

Through direct calculations, we can prove the following Lemma.

Lemma 3.3. *For $p, q \in \{1,2\}$, $i \in \{1,2,3\}$, $H_i(\boldsymbol{\beta},\tau,\mu)$ has the following properties:*

(i) *If there are some $(\boldsymbol{\beta},\tau,\mu) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ such that $H_i(\boldsymbol{\beta},\tau,\mu) = 0$, $i = 1,2,3$, then ϕ is a solution of (3.1);*

(ii) $H_i(0,\tau,0) = 0, \quad \frac{\partial H_i}{\partial \beta_p}(0,\tau,0) = 0;$

(iii) $b_{pq}^{(i)} = \frac{1}{2}\frac{\partial^2 H_i}{\partial \beta_p \partial \beta_q}(0,\tau,0) = \int_{-\infty}^{+\infty} \langle \psi_i(t), \frac{1}{2} D^2 f(\gamma(t)) u_p(t) u_q(t)\rangle dt;$

(iv) $\tilde{a}_i(\tau) = \frac{\partial H_i}{\partial \mu}(0,\tau,0) = \int_{-\infty}^{+\infty} \langle \psi_i(t), g(\gamma(t),0,t+\tau)\rangle dt.$

The quadratic functions $M_i : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ given by (2.5) represents the lowest order terms of $H_i(\boldsymbol{\beta},\tau,\mu)$. We are lead to solving the system of quadratic equations (2.6).

4. Codiagonalization and solutions of two quadratic equations

Let $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $F(x,y) = \mathbf{z}^T B \mathbf{z}$. We say that the quadratic equation $F(x,y) = h, h \neq 0$ is of elliptic (or hyperbolic, or line) type if the graph of the equation is an ellipse (or two hyperbolas, or two lines). The graph of two symmetric parallel lines is a special case of two hyperbolas, where the normal direction to two lines replaces the real axis of a hyperbola.

The hyperbolic rotation is well-known for its use in relativity theory [2]. We shall define various transformations that keep a quadratic form $F(x,y) = ax^2+2bxy+cy^2$

invariant. Consider the Hamiltonian system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2b & -2c \\ 2a & 2b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4.1}$$

and its solution mapping $T(t)$.

Definition 4.1. The solution mapping $T(t)$ for (4.1) that maps the ray $\overrightarrow{OP_1}$ to $\overrightarrow{OP_2}$, where $P_2 = T(t)P_1$ will be called *the quadratic rotation by the angle* t. It will also be called the circular, elliptic or hyperbolic rotation if the graph of $F(x, y) = h$ is a circle, ellipse or hyperbola. The angle θ from $\overrightarrow{OP_1}$ to $\overrightarrow{OP_2}$ is defined to be $t \in \mathbb{R}$. On the other hand, if there does not exist any $t \in \mathbb{R}$ with $\overrightarrow{OP_2} = T(t)\overrightarrow{OP_1}$, then the angle between the two rays is undefined.

Just like the polar coordinates, if P_0 is a point on the major axis (or semi-real, or semi-imaginary axis), then we define the angle coordinate of P_0 to be $\theta(P_0) = 0$. For any other $P \in \mathbb{R}^2$, we define its angle coordinate $\theta(P)$ to be the angle from $\overrightarrow{OP_0}$ to $\overrightarrow{OP}$.

Example 4.1. Let $a = c = 1,\ b = 0$ in (4.1). The solution mapping

$$R(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

defines the circular rotation in counter-clockwise direction.

Example 4.2. Let $a = 1,\ c = -1,\ b = 0$ in (4.1). The solution mapping

$$H(t) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

defines the standard hyperbolic rotation in $\mathbb{R}^2$. However, given two rays in $\mathbb{R}^2$, the hyperbolic angle between them can be undefined.

More precisely, the two lines $y = \pm x$ divides $\mathbb{R}^2$ into 4 sectors:

$$S_1 := \{(x, y) : x > 0, |y| < |x|\},\ S_2 := \{(x, y) : y > 0, |x| < |y|\},$$
$$S_3 := \{(x, y) : x < 0, |y| < |x|\},\ S_4 := \{(x, y) : y < 0, |x| < |y|\}.$$

If $(x_0, y_0)^T \in S_j, 1 \le j \le 4$, then $(x(t), y(t))^T \in S_j$ for all $t \in \mathbb{R}$. More precisely, if $(x_0, y_0)^T \in S_1$ or S_3, then there exists an $r_0 > 0$ or $r_0 < 0$ such that $(x_0, y_0) = r_0(\cosh(t_0), \sinh(t_0))$. The hyperbolic rotation simply draws a hyperbola in sector S_1 or S_3,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \cosh(t_0) \\ \sinh(t_0) \end{pmatrix} = r_0 \begin{pmatrix} \cosh(t + t_0) \\ \sinh(t + t_0) \end{pmatrix}.$$

Similarly, if $(x_0, y_0)^T \in S_2$ or S_4, then there exists an $r_0 > 0$ or $r_0 < 0$ such that $(x_0, y_0) = r_0(\sinh(t_0), \cosh(t_0))$. The hyperbolic rotation draws a hyperbola in sector S_2 or S_4,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \sinh(t_0) \\ \cosh(t_0) \end{pmatrix} = r_0 \begin{pmatrix} \sinh(t+t_0) \\ \cosh(t+t_0) \end{pmatrix}.$$

Notice that the circular and standard hyperbolic rotations satisfy:

$$R^*(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H^*(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $T(t)$ is the solution mapping for (4.1), we always have

$$T^*(t) \begin{pmatrix} a & b \\ b & c \end{pmatrix} T(t) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Lemma 4.1. *Let $F(x,y) := (x,y)B(x,y)^T$ be the quadratic form associated to a symmetric matrix $B \in \mathbb{R}^{2\times 2}$.*

(1) If the vector field (4.1) corresponding to $F(x,y)$ satisfies $x' = 0$ on the x-axis, or $y' = 0$ on the y-axis, then the matrix B is diagonal.

(2) If there exist $x_1 \neq 0, y_1 \neq 0$ such that either (i) $F(x_1,y_1) = F(-x_1,y_1)$, or (ii) $F(x_1,y_1) = F(x_1,-y_1)$, then B is diagonal.

Let $B_1, B_2 \in \mathbb{R}^{2\times 2}$ be symmetric, nonzero matrices and let $F_1(x,y) = a_1x^2 + 2b_1xy + c_1y^2 = (x,y)B_1(x,y)^T$, $F_2(x,y) = a_2x^2 + 2b_2xy + c_2y^2 = (x,y)B_2(x,y)^T$. We now study the system of two quadratic equations

$$F_1(x,y) = h_1, \quad F_2(x,y) = h_2. \tag{4.2}$$

(H5) : Assume that the two quadratic forms $F_1(x,y), F_2(x,y)$ are linearly independent, i.e., the two matrices B_1, B_2 are linearly independent.

Consider the conditional maximum/minimum problems:

$$F_1(x,y) = \max \text{ or } \min, \quad \text{subject to } F_2(x,y) = \pm h_2. \tag{4.3}$$

We look for critical points from the Lagranginan:

$$\Lambda(x,y,\lambda) = F_1(x,y) - \lambda F_2(x,y), \quad \nabla_{x,y}\Lambda(\lambda,x,y) = 0. \tag{4.4}$$

To find critical points $P_j = (x_j,y_j)^T, j = 1,2$, of the Lagrangian, we solve the generalized eigenvalue/eigenvector problem

$$(B_1 - \lambda B_2) \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{4.5}$$

4.1. Solutions of (4.2) if one equation is elliptic

In this subsection we assume that $F_2(x,y) = h_2$ is of elliptic type. Hence $b_2^2 - a_2c_2 < 0$. By changing ψ_i to $-\psi_i$, we can change $B^{(i)}$ to $-B^{(i)}$. Hence for elliptic type quadratic forms we assume $a_2 > 0, c_2 > 0$ and $h_2 > 0$.

It is well-known that two symmetric matrices can be simultaneously diagonalized if one of the matrices is positive definite, [8,9]. However, it is not clear if the resulting matrices are real valued.

Theorem 4.1. *Assue that* (4.5) *has two eigenvalues* (λ_1, λ_2) *corresponding to (nonunique) eigenvectors* $(P_1, P_2) = ((x_1, y_1)^T, (x_2, y_2)^T)$. *Rescaling* (P_1, P_2) *so that on both points* $F_2 = h_2$. *We consider three types of systems.*

(i) (EE) type: Assume that F_1 *reaches the minimum* r_1 *at* P_1 *and the maximum* r_2 *at* P_2. *System* (4.2) *has* 4 *solutions if* $r_1 < h_1 < r_2$.

(ii) (HE) type: Assume that F_1 *reaches a local minimum* $r_1 < 0$ *at* P_1*; and a local maximum* $r_2 > 0$ *at* P_2. *System* (4.2) *has* 4 *solutions if* $r_1 < h_1 < r_2$.

(iii) (LE) type: In this case, the graph of $F_1(x, y) = h_1$ *consists of two parallel lines symmetric about the origin. The eigenvalues are* $\lambda_1 = 0$ *with eigenvector* P_1 *on which* $F_1(x_2, y_2) = 0$*; and* $\lambda_2 \neq 0$ *with the eigenvector* P_2 *that solves the conditional minimum problem with* $F_1 = r_1 < 0$*, or the maximum problem with* $F_1 = r_2 > 0$. *System* (4.2) *has* 4 *solutions if* $r_1 < h_1 < 0$ *or* $0 < h_1 < r_2$.

Proof. Case (EE) type: It is given that $F_1(P_1) = r_1 < h_1 < r_2 = F_1(P_2)$. Between each pair of $(\overrightarrow{OP_1}, \overrightarrow{OP_2})$, there exists a vector $\overrightarrow{OP_0}$ such that$F_2(P_0) = h_2$ and $F_1(P_0) = h_1$. There exist 4 pairs of such $(\overrightarrow{OP_1}, \overrightarrow{OP_2})$ so the total number of solutions is 4. The proof of the other two cases are similar and shall be omitted. □

4.2. Solutions of (4.2) if both equations are hyperbolic

For a give $h_2 \neq 0$, the hyperbola defined by $F_2(x, y) = h_2$ does not circle the origin as the ellips in §4.1. Observe that for the (HH) type systems, the equilibrium $(0, 0)$ of (4.1) is hyperbolic and there exist stable and unstable eigenspaces for the equilibrium $(0, 0)$. Before giving a counter example, we introduce the following definition.

Definition 4.2. Let $L_j^{(i)}, i = 1, 2$, be the stable and unstable eigenspaces of the equilibrium for (4.1), where $(a, b, c,) = (a_j, b_j, c_j)$. They are called the asymptotes for $F_j(x, y) = h_j$. The asymptotes $L_j^{(i)}, i = 1, 2$, divide $\mathbb{R}^2$ into four sectors. We say (x, y) is in the positive (or negative) sector if $F_j(x, y) > 0$ (or $F_j(x, y) < 0$).

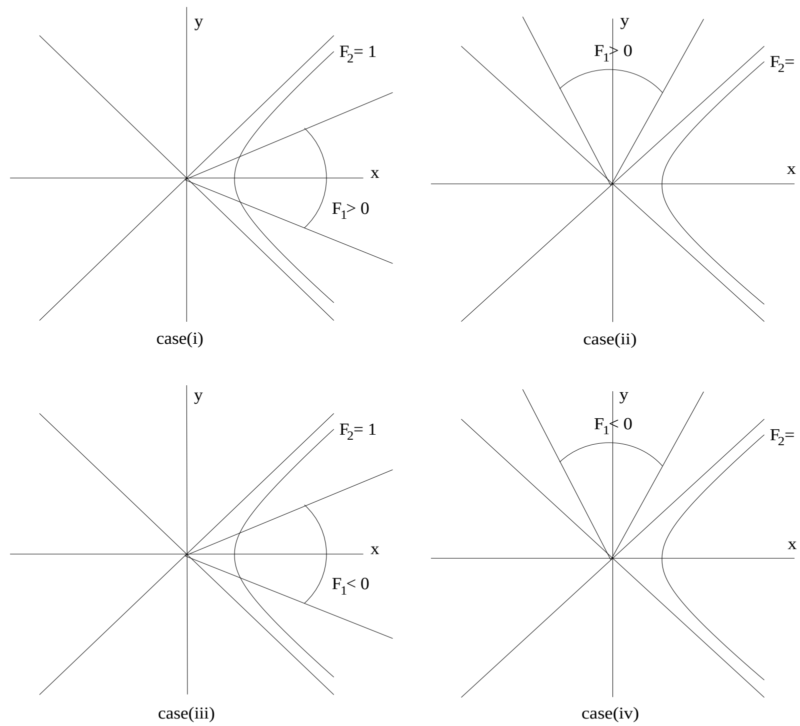
Example 4.3 (A Counter Example)**.** Assume that the asymptotes of two hyperbolas are alternating, for example

$$F_1 = x^2 - y^2,\ F_2 = xy.$$

Following the curve $F_2(x, y) = h_2$, the values of F_1 are not bounded below or above. Therefore, conditional max/min as in §4.1 is not well posed. It is easy to see that in such case, (4.2) has exactly 2 solutions and the two quadratic forms cannot be codiagonalized.

Although the general max/mn problem is not well posed, to each of the cases listed below, it is not hard to find a well posed conditional max/min problem.

Consider 4 sub-cases, as depicted in the four figures: (We skipped part of the graphs can be obtained by symmetry for simplicity.)
(HH i) The two sectors of $F_1 > 0$ are inside the sectors of $F_2 > 0$.
(HH ii) The two sectors of $F_1 > 0$ are inside the sectors of $F_2 < 0$.
(HH iii) The two sectors of $F_1 < 0$ are inside the sectors of $F_2 > 0$.
(HH iv) The two sectors of $F_1 < 0$ are inside the sectors of $F_2 < 0$.



Theorem 4.2. *For cases (HH i) and (HH ii), and $h_2 > 0$ or < 0, consider the conditional maximum problem:*

$$F_1(x,y) = \max, \quad \textit{subject to } F_2(x,y) = h_2. \tag{4.6}$$

Then for (4.6), there exists $r_3 = \max F_1$. System (4.2) has 4 solutions if $h_1 < r_3$.

For cases (HH iii) and (HH iv), and $h_2 > 0$ or < 0, consider the conditional minimum problem:

$$F_1(x,y) = \min, \quad \textit{subject to } F_2(x,y) = h_2. \tag{4.7}$$

Then for (4.7), there exists $r_4 = \min F_1$ System (4.2) has 4 solutions if $r_4 < h_1$.

Finally, after rescalling the generalized eigenvectors (P_1, P_2), we can assume that P_2 solves the max/min problems (4.6) or (4.7). And P_1 solves the complementary max/min problem (4.6) for cases (HH i) and (HH ii), or (4.7*) for cases (HH iii) and (HH iv), defined as:*

$$F_1(x,y) = \max, \quad \textit{subject to } F_2(x,y) = -h_2, \tag{4.6*}$$

$$F_1(x,y) = \min, \quad \textit{subject to } F_2(x,y) = -h_2. \tag{4.7*}$$

Proof. Following the curve $F_2(x,y) = h_2$, or $-h_2$, the range of $F_1(x,y)$ can be bounded above and unbounded below, or bounded below and unbounded above. Therefore, either a conditional max problem or a conditional min problem is well-posed, but not both. □

The (LH) case can be treated just like the (HH) case. Consider 4 sub-cases:
(LH i) $F_1 \leq 0$ and the line $F_1 = 0$ is inside the sectors of $F_2 > 0$.

(LH ii) $F_1 \leq 0$ and the line $F_1 = 0$ is inside the sectors of $F_2 < 0$.
(LH iii) $F_1 \geq 0$ and the line $F_1 = 0$ are inside the sectors of $F_2 > 0$.
(LH iv) $F_1 \geq 0$ and the line $F_1 = 0$ are inside the sectors of $F_2 < 0$.

Theorem 4.3. *For cases (i) and (ii), consider the conditional maximum problem:*

$$F_1(x, y) = \max, \quad \textit{subject to } F_2(x, y) = h_2. \tag{4.8}$$

Then for (4.8)*, there exists* $r_5 = \max F_1$*. System* (4.2) *has* 4 *solutions if* $h_1 < r_5$
For cases (iii) and (iv), consider the conditional minimum problem:

$$F_1(x, y) = \min, \quad \textit{subject to } F_2(x, y) = h_2. \tag{4.9}$$

Then for (4.9)*, there exists* $r_6 = \min F_1$*. System* (4.2) *has* 4 *solutions if* $r_6 < h_1$*.*

Finally, after rescalling the generalized eigenvector (P_1, P_2)*, we can assume that* P_2 *solves the max/min problems* (4.8) *or* (4.9)*. And* P_1 *solves the complementary max/min problem (4.8*) for cases (LH i) and (LH ii), or (4.9*) for cases (LH iii) and (LH iv), defined as:*

$$F_1(x, y) = \max, \quad \textit{subject to } F_2(x, y) = -h_2, \tag{4.8*}$$
$$F_1(x, y) = \min, \quad \textit{subject to } F_2(x, y) = -h_2. \tag{4.9*}$$

For the (LL) case, if two family of lines are not parallel, there are 4 solutions. To simplify the paper, we shall not discuss (LL) case in the sequel.

4.3. Codiagonalization of two quadratic equations

In this subsection, we consider codiagonalization of two quadratic equations, but not the coexistence of real valued solutions. The method is based the generalized eigenvalue/eigenvector problems. For the cases listed in §4.1 and §4.2, we have the following results:

Theorem 4.4. *If one equation of the quadratic system is elliptic, then the two quadratic form can always be codiagonalized by real valued matrices.*

If both equations are hyperbolic, then in all the cases (HH i)-(HH iv), the two quadratic forms can be dociagonalized by real valued matrices.

If $F_1(x, y)$ *is the line type and* $F_2(x, y)$ *is hyperbolic, then in all the cases (LH i)-(LH iv), the two quadratic forms can be cociagonalized by real valued matrices.*

Proof. Let (P_1, P_2) be the generalized eigenvector corresponding to the generalized eigenvalue problem (4.5). After rescaling, assume that P_2 solves the max/min problem. In all the three cases, there exists an angle θ_0 such that $T_2(-\theta_0)P_2$ coincides with the major axis or the minor axis of the graphs of $F_2(x, y) = h_2$.

Based on the results from previous subsections, each generalized eigenvalue problem has two lindearly independent eigenvectors. Thus, the eigenvalus are distinct. This implies

$$\langle P_1, \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} P_2 \rangle = 0.$$

Therefore, in all the cases listed in Theorems 4.1, 4.2 and 4.3, the image of $T_2(-\theta_0)P_1$ should coincide with the minor axis or the major axis of $F_2 = h_2$. Assume that under the rotation $T_2(\theta_0)$, the quadratic form $F_1(x,y) = h_1$ becomes $F_3(x,y) = h_1$ while $F_2(x,y) = h_2$ is unchanged. Now apply a circular rotation $R(-\theta_0')$ to both $F_3(x,y) = h_1$ and $F_2(x,y) = h_2$ so the major axis of $F_2(x,y) = h_2$ is mapped to the x-axis. The matrices that represent the two quadratic forms are

$$R^*(\theta_0')T^*(\theta_0)B_jT(\theta_0)R(\theta_0'),\ j = 1,2.$$

Clearly $F_2(x,y) = h_2$ has been diagonalized. From Lemma 4.1, $F_1(x,y) = h_1$ has also been diagonalized.

□

Lemma 4.2. *The 4 solutions of (4.2) obtained in Theorems 4.1, 4.2 and 4.3 are simple.*

Proof. If not, then the solutions of the system are on the lines spanned by $\overrightarrow{OP_1}$ or $\overrightarrow{OP_2}$ where the graphs are tangent to each other. Contradicting to the fact that the system has 4 solutions. □

5. Coexistence of homoclinic solutions

Assume that the first two equations of (2.6) satisfy the conditions in Theorems 4.1, 4.2 and 4.3. Then they can be codiagonalized by the change of variables $\boldsymbol{\beta} \to \boldsymbol{\alpha}$. And system (2.6) becomes

$$\begin{aligned} &d_{11}\alpha_1^2 + d_{12}\alpha_2^2 - \hat{a}_1(\tau)\mu = 0,\\ &d_{21}\alpha_1^2 + d_{22}\alpha_2^2 - \hat{a}_2(\tau)\mu = 0,\\ &d_{31}\alpha_1^2 + d_{32}\alpha_1\alpha_2 + d_{33}\alpha_2^2 - \hat{a}_3(\tau)\mu = 0.\end{aligned}$$

From (H5), the first two quadratic forms are linearly independent. Using row eliminations to simplify the system, we have

$$\begin{aligned} &\alpha_1^2 - a_1(\tau)\mu = 0,\\ &\alpha_2^2 - a_2(\tau)\mu = 0,\\ &\alpha_1\alpha_2 - a_3(\tau)\mu = 0.\end{aligned} \tag{5.1}$$

For $\mu \in \mathbb{R}$, we look for solutions such that

$$(\alpha_1, \alpha_2) = O(\sqrt{|\mu|}),\ \tau \in \mathbb{R}.$$

Since $\mu < 0$ can be handled similarly, assume $\mu > 0$ and let $x_j = \alpha_j/\sqrt{\mu}$. Then

$$\begin{aligned} &f_1(x_1,x_2,\tau) := x_1^2 - a_1(\tau) = 0,\\ &f_2(x_1,x_2,\tau) := x_2^2 - a_2(\tau) = 0,\\ &f_3(x_1,x_2,\tau) := x_1x_2 - a_3(\tau) = 0.\end{aligned} \tag{5.2}$$

For a fixed $\tau \in \mathbb{R}$, from the first two equations of (5.2), we obtain

$$x_1(\tau) = \pm(a_1(\tau))^{1/2}, \quad x_2(\tau) = \pm(a_2(\tau))^{1/2}. \tag{5.3}$$

Since the conditions in Theorems 4.1, 4.2 and 4.3 are satisfied, then $a_1(\tau), a_2(\tau) > 0$, and (5.3) represents 4 real valued solutions. From (5.3), the third equation of (5.1) yields the reduced bifurcation equation

$$F(\tau) := \left[\pm(a_1(\tau))^{1/2}\right] * \left[\pm(a_2(\tau))^{1/2}\right] - a_3(\tau) = 0. \tag{5.4}$$

Lemma 5.1. *Assume that τ_0 is a simple zero for $F(\tau)$, and the corresponding (nonzero) x_1 and x_2 are from* (5.3)*. Then at $\tau = \tau_0$,*

$$\left|\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, \tau)}\right| \neq 0.$$

Proof. Observe that

$$\left|\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, \tau)}\right| = \begin{vmatrix} 2x_1 & 0 & -a_1'(\tau_0) \\ 0 & 2x_2 & -a_2'(\tau_0) \\ x_2 & x_1 & -a_3'(\tau_0) \end{vmatrix}.$$

Multiply $-x_2^2 = -a_2$ to row one, $-x_1^2 = -a_1$ to row two and $2x_1x_2 = 2a_3$ to row three, then add. The determinant becomes $2x_1^3x_2^3(a_1'a_2 + a_2'a_1 - 2a_3a_3') = 2x_1^3x_2^3\frac{d}{d\tau}(a_1a_2 - a_3^2)$. But this is nonzero due to $F(\tau_0) = 0$ and $F'(\tau_0) \neq 0$. □

Going back from (5.2) to (5.1), we have proved the following result:

Theorem 5.1. *Assume $\mu \neq 0$. For each particular choice of $\pm$ sign in* (5.4)*, if τ_0 is a simple zero for $F(\tau)$, then* (5.1) *has* 2 *simple solution of the form $(\boldsymbol{\alpha}^{(j)}, \tau^{(j)}, \mu)$. The total number of solutions can be* 2 *or* 4.

Using the the inverse of the codiagonalization of the first two equations, $\boldsymbol{\alpha} \to \boldsymbol{\beta}$, we find that $M_i(\boldsymbol{\beta}, \tau, \mu), i = 1, 2, 3$, defined in (2.5), can have 2 or 4 simple solutions $(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu)$.

Theorem 5.2. *Assume that the conditions (H1)–(H5) are satisfied and the first two equations of* (2.6) *satisfy the conditions in Theorems 4.1, 4.2 or 4.3. Assume that $(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu)$, $1 \leq j \leq 4$ or $1 \leq j \leq 2$ are simple solutions for $M_i(\boldsymbol{\beta}, \tau, \mu), i = 1, 2, 3$. There exists $\bar{\mu} > 0$, independent of j such that if $0 < |\mu| < \bar{\mu}$, then the following is true. For each fixed j, there exists an open region $I_j \subset \mathbb{R}$ containing zero and differentiable functions, $\omega_j : I_j \to \mathbb{R}^2$ and $\eta_j : I_j \to \mathbb{R}$ such that $\omega_j(0) = 0$, $\eta_j(0) = 0$ and $H_i(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau^{(j)} + \eta_j(s)), s^2\mu) = 0$, $i = 1, 2, 3$, for $s \in I_j$ and $s \neq 0$.*

Proof. Let $H = (H_1, H_2, H_3), M = (M_1, M_2, M_3)$. For each fixed j, since $(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu)$ is a simple solution for $M(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu) = 0$, then $D_{(\boldsymbol{\beta},\tau)}M(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu)$ is a 3×3 nonsingular matrix. For each j define a C^2 function $W : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^3$ by

$$W(x, y, s) = \begin{cases} \frac{1}{s^2}H(s(\boldsymbol{\beta}^{(j)} + x), \tau^{(j)} + y, s^2\mu), & \text{for } s \neq 0, \\ M(\boldsymbol{\beta}^{(j)} + x, \tau^{(j)} + y, \mu), & \text{for } s = 0. \end{cases}$$

Clearly, for $s \neq 0$, $H = 0$ if and only if $W = 0$. Through direct calculations, we have $W(0, 0, 0) = 0$ and

$$D_{(x,y)}W(0, 0, 0) = D_{(\boldsymbol{\beta},\tau)}M(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu_0)$$

is a nonsingular matrix. By the implicit function theorem there exist an open region $I_j \subset \mathbb{R}$ containing zero and a differentiable functions, $\omega_j : I_j \to \mathbb{R}^2$ and $\eta_j : I_j \to \mathbb{R}$ such that $\omega_j(0) = 0$, $\eta_j(0) = 0$ and $W(\omega_j(s), \eta_j(s), s) = 0$. Hence

$$H(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau^{(j)} + \eta_j(s)), s^2\mu) = 0 \quad \text{for} \quad s \neq 0.$$

The proof has been completed. □

By Theorem 5.2, the bifurcation function $H = (H_1, H_2, H_3) = 0$ at $(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau^{(j)} + \eta_j(s), s^2\mu)$. Then system (3.1) has the solution $\phi(\boldsymbol{\beta}, \tau, \mu)$. Hence system (1.4) has 2 or 4 homoclinic solutions given by

$$\begin{aligned}\gamma_s^{(j)}(t) =& \gamma(t) + \sum_{p=1}^{2} s(\beta_p^{(j)} + \omega_{jp}(s))u_p(t) \\ &+ K(I-P)\widetilde{g}(\phi, \tau^{(j)} + \eta_j(s), s^2\mu)(t),\end{aligned} \tag{5.5}$$

for $0 \neq s \in I_j$, $1 \leq j \leq 4$ or $1 \leq j \leq 2$. Clearly, $\lim_{s\to 0} \gamma_s^{(j)}(t) = \gamma(t)$.

Remark 5.1. From the construction of $\gamma_s^{(j)}$, we find that the solutions are robust with respect to small perturbation of g. This alone shows that each of the solution obtained is a transversal homoclinic solution. The same argument was used by Mallet-Paret in [15] to show that the homoclinic orbits in some delay equations are transverse.

Alternatively, it is shown in [13] that the functions $H_i, 1 \leq i \leq 3$, as in (3.10), measure the gap between the unstable manifold at $t = 0_-$ and the stable manifold at $t = 0_+$. Since $D_{(\boldsymbol{\beta},\tau)}M(\boldsymbol{\beta}^{(j)}, \tau^{(j)}, \mu_0)$, $1 \leq j \leq 4$, is nonsingular, $D_{(\boldsymbol{\beta},\tau)}H(s(\boldsymbol{\beta}^{(j)} + \omega_j(s)), \tau^{(j)} + \eta_j(s)), s^2\mu)$ is also a nonsingular matrix. Therefore, the intersection of $W^u(0)$ and $W^s(0)$ is transverse.

6. An Example

Although the example given in this section is not from applications, it shows that the conditions given in this paper are consistent. Consider the following system

$$\begin{cases} \dot{x}_1 = x_2 + \epsilon x_5 \sin(t - c_1), \\ \dot{x}_2 = x_1 - 2x_1x_5^2 + x_2^2 + \epsilon x_6 \cos(t - c_1), \\ \dot{x}_3 = x_4 + \epsilon k x_6 \cos t, \\ \dot{x}_4 = x_3 - 2x_3x_5^2 + x_2x_4 + \epsilon k x_5 \sin t, \\ \dot{x}_5 = x_6 + \epsilon x_1x_2 \cos(t - c_2), \\ \dot{x}_6 = x_5 - 2x_5^3 + x_3x_4 - \frac{1}{2}\epsilon x_6 \cos(t - c_2). \end{cases} \tag{6.1}$$

The unperturbed system is

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_1 - 2x_1x_5^2 + x_2^2, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = x_3 - 2x_3x_5^2 + x_2x_4, \\ \dot{x}_5 = x_6, \\ \dot{x}_6 = x_5 - 2x_5^3 + x_3x_4. \end{cases} \tag{6.2}$$

It is easy to check that 0 is an equilibrium and the eigenvalues of $Df(0)$ are $\{-1,-1,-1,1,1,1\}$. Hence 0 is a hyperbolic equilibrium. Let $r(t) = \operatorname{sech}(t)$ and $\gamma = (0,0,0,0,r,\dot r)$. By direct calculations, we see that γ is a homoclinic solution to the origin.

Remark 6.1. The example is modefied from [4]. At the first look, it seems to be unnatural to consider a homoclinic orbit with $x_1 = x_2 = x_3 = x_4 = 0$ in $\mathbb{R}^6$. However, if $\gamma(t)$ is a homoclinic orbit that can be embedded in a smooth 2D sub-manifold, by a change of variables, we can assume that $\gamma(t), -\infty < t < \infty$ lies in the (x_5, x_6)-plane.

The variational equation along γ has the following bounded fundamental solutions:

$$u_1 = (r,\dot r,r,\dot r,0,0),\quad u_2 = (0,0,r,\dot r,0,0),\quad u_3 = (0,0,0,0,\dot r,\ddot r).$$

Then we can choose a basis of the bounded fundamental solutions of the adjoint equation as

$$\psi_1 = (-\dot r,r,-\dot r,r,0,0),\quad \psi_2 = (0,0,-\dot r,r,0,0),\quad \psi_3 = (-\dot r,r,0,0,2\ddot r,-2\dot r).$$

By calculations, we have

$$b^{(1)}_{11} = -\frac{\pi}{4},\ b^{(1)}_{22} = -\frac{\pi}{8},\ \ b^{(2)}_{12} = -\frac{\pi}{8},\ b^{(3)}_{11} = \frac{\pi}{8},\ \ b^{(3)}_{22} = \frac{\pi}{4}$$

and $b^{(i)}_{jk} = 0$ for others. Then, up to the quadratic terms, the bifurcation equations are

$$\begin{cases} -\frac{\pi}{4}\beta_1^2 - \frac{\pi}{8}\beta_2^2 = \epsilon\mu_1(\tau - c_1), \\ -\frac{\pi}{8}\beta_1\beta_2 = \epsilon k\mu_2(\tau), \\ \frac{\pi}{8}\beta_1^2 + \frac{\pi}{4}\beta_2^2 = \epsilon\mu_3(\tau - c_2), \end{cases} \tag{6.3}$$

where $\mu_1(\tau), \mu_2(\tau), \mu_3(\tau)$ are given by

$$\begin{aligned} \mu_1(\tau) =& (\sin\tau - \cos\tau)\int_{-\infty}^{\infty} r(t)\dot r(t)\sin t dt \\ &+ \sin\tau \int_{-\infty}^{\infty} r(t)^2\cos t dt - \cos\tau\int_{-\infty}^{\infty}\dot r(t)^2\cos t dt \\ =& -\frac{31\pi}{32}(\sin\tau - \cos\tau) + \pi\sin\tau + \pi\cos\tau\int_{-\infty}^{\infty}\dot r(t)^2\cos t dt, \\ \mu_2(\tau) =& (\sin\tau - \cos\tau)\int_{-\infty}^{\infty} r(t)\dot r(t)\sin t dt = -\frac{31\pi}{32}(\sin\tau - \cos\tau), \\ \mu_3(\tau) =& \sin\tau\int_{-\infty}^{\infty} r(t)^2\cos t dt = \pi\sin\tau. \end{aligned}$$

Consider $\epsilon > 0$ and assume $\beta_1 = \sqrt{\epsilon}x, \beta_2 = \sqrt{\epsilon}y$. Then

$$\begin{aligned}
&-\frac{\pi}{4}x^2 - \frac{\pi}{8}y^2 = \mu_1(\tau - c_1),\\
&-\frac{\pi}{8}xy = k\mu_2(\tau),\\
&\frac{\pi}{8}x^2 + \frac{\pi}{4}y^2 = \mu_3(\tau - c_2),\\
&x^2 = \frac{8}{3\pi}(-2\mu_1(\tau - c_1) - \mu_3(\tau - c_2)),\\
&y^2 = \frac{8}{3\pi}(\mu_1(\tau - c_1) + 2\mu_3(\tau - c_2)),\\
&xy = \pm\frac{8}{\pi}k\mu_2(\tau).
\end{aligned}$$

Finally, the bifurcation equation becomes

$$(-2\mu_1(\tau - c_1) - \mu_3(\tau - c_2))^{1/2} * (\mu_1(\tau - c_1) + 2\mu_3(\tau - c_2))^{1/2} = \pm 3k\mu_2(\tau). \quad (6.4)$$

From its definition, $\mu_2(\tau)$ has simple zeros at $\tau_0 = \pi/4$ and $\tau_0 = 5\pi/4$. If the norms of μ_1 matches that of μ_3, then we can adjust c_1, c_2 so that there exist $a < \tau_0 < b$ such that if $\tau \in [a, b]$ then

$$\begin{aligned}
&2\mu_1(\tau - c_1) + \mu_3(\tau - c_2) < 0,\\
&\mu_1(\tau - c_1) + 2\mu_3(\tau - c_2) > 0.
\end{aligned}$$

If the norms of μ_1, μ_3 differ too much, we can rescale the perturbation terms to make them comparable.

Finally, by choosing k sufficiently large, the functions in the two sides of (6.4) will intersect transversely near τ_0. Finding the parameters (c_1, c_2, k) can be done numerically, and will not be discussed further in this paper.

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