J. Differential Equations 255 (2013) 58–84

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of standing waves for monostable reaction–convection equations in a large bounded domain with boundary conditions ☆

Xiao-Biao Lin

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, United States

ARTICLE INFO

Article history:
Received 3 November 2012
Revised 24 February 2013
Available online 11 April 2013

Keywords:
Traveling waves
KPP/Fisher equations
Stability on large intervals
Weighted norms
Exponential dichotomies
Lambda Lemma

ABSTRACT

It is well known that the standing wave u_0 for the KPP type convection–diffusion equation is stable if the perturbations of the initial data are in the weighted function spaces proposed by Sattinger. We study boundary conditions so that in a large finite domain, there is a stable standing wave $\tilde{u}$ near u_0. The standing wave $\tilde{u}$ may not be monotone, and the stability is proved by pseudo exponential dichotomies that are weighted both in the spatial variable ξ and in the dual variable s to the time t.

Published by Elsevier Inc.

1. Introduction

This work is motivated by the study of a 1D liquid/vapor phase change model proposed by Haitao Fan. The model consists of a p-system describing the motion of the liquid/vapor mixture coupled with the reaction–diffusion equation describing the change of the percentage λ of the vapor in the mixture. The existence of traveling waves was proved in [6]. In [7], Fan considered a simplified system where the equation for λ is the KPP/Fisher equation:

$$\lambda_t = \lambda_{xx} \pm \lambda(1 - \lambda). \tag{1.1}$$

☆ Research partially supported by NSF grant DMS-1211070.
E-mail address: xblin@math.ncsu.edu.

0022-0396/$ – see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jde.2013.03.010

Here, + (or −) corresponds to evaporation (or condensation) of the fluid mixture. Fan proved that the stability of the whole system is dominated by the stability of the KPP traveling wave: if the traveling wave to the equation for λ is stable, then the traveling wave to the liquid/vapor phase transition system is stable.

The problem discussed here may have practical application to the design of evaporation nozzles. If the liquid/vapor mixture moves to the right with the constant speed c, and relative to the media the traveling wave moves to the left with the same speed, then the evaporation wave appears to be stationary inside the nozzle. We would like to find boundary conditions for the existence and stability of internal layer solutions to the phase transition system. Motivated by the work of Fan, as a first step, we study boundary conditions under which the KPP/Fisher waves with convection are stable in a large finite domain.

Consider the diffusion–convection equation with the KPP/Fisher nonlinearity [8,13]:

$$\begin{gathered} u_t = u_{\xi\xi} - cu_\xi + f(u), \quad c > 0, \\ f(0) = f(1) = 0, \\ f'(0) > 0, \qquad f'(1) < 0, \qquad f''(u) < 0. \end{gathered} \tag{1.2}$$

A typical example is $f(u) = u(1-u)$ where $Df(0) = 1$, $Df(1) = -1$. If $c^2/4 > Df(0)$, then (1.2) has a stationary whole line solution $u_0(\xi)$, $\xi \in \mathbb{R}$ that connects $u = 0$ to $u = 1$. We assume that $\xi = 0$ is the center of the wave, say $u_0(0) = 1/2$. Then u_0 is almost constant for sufficiently large $|\xi|$. The standing wave satisfies

$$u_{\xi\xi} - cu_\xi + f(u) = 0. \tag{1.3}$$

We also consider (1.3) in a finite large domain $J = (a, b)$ where $|a|$ and b are sufficiently large so that $u_0(a) \approx 0$ and $u_0(b) \approx 1$.

The standing wave $\tilde{u}$ for (1.2) is related to a singular perturbation problem. Since $\epsilon := 1/(b-a)$ is a small parameter, using the change of variables $T = \epsilon t$, $X = \epsilon(x - a)$, (1.2) becomes a singularly perturbed equation,

$$\epsilon u_T = \epsilon^2 u_{XX} - \epsilon c u_X + f(u), \quad X \in (0,1).$$

The standing wave becomes an internal layer solution in the bounded domain $(0, 1)$.

Eq. (1.3) also describes traveling wave solutions for the reaction–diffusion equation in the coordinates (x, t) with the wave speed $-c$,

$$u_t = u_{xx} + f(u). \tag{1.4}$$

In the moving coordinate $\xi = x + ct$, the traveling wave $u = u(\xi)$ becomes a standing wave connecting $u = 0$ to $u = 1$. Kolmogorov et al. [13] showed that if $|c| \geqslant 2\sqrt{Df(0)}$ then such traveling wave exists. When we say that the traveling wave (or standing wave with convection) $u_0(\xi)$, $\xi = x + ct$, connects $u = 0$ to $u = 1$, we mean $u_0 \to 0$ as $\xi \to -\infty$ and $u_0 \to 1$ as $\xi \to \infty$. The definition does not depend on $c > 0$ or $c < 0$.

Similar problems have been considered by Beyn and Lorenz [1] for parabolic systems with several unknown variables under the condition that the essential spectrum of the traveling wave lies in the negative complex plane. In this paper, we consider the monostable traveling waves so the condition in [1] is not satisfied. Our system has only one unknown variable which allows us to use slopes of manifolds to describe the boundary conditions. By doing so, we obtained precise boundary conditions that can ensure the existence and stability of the standing waves. Generalization of the results to systems with several unknown variables, in the same spirit of [1] will appear in a separate paper. See also Remark 4.3 in Section 4.

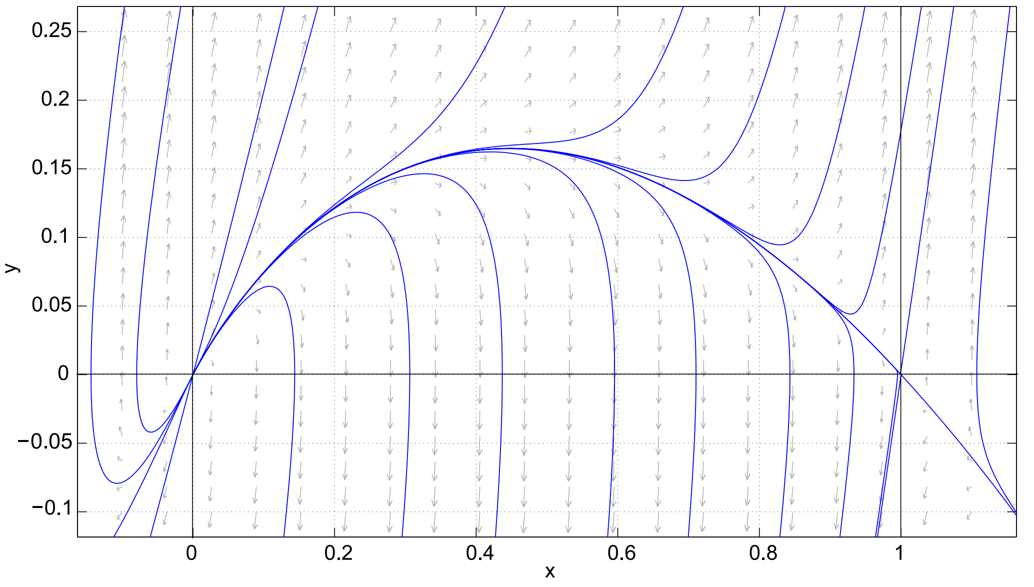

Fig. 1.1. The phase portrait near the heteroclinic solution $q(\xi)$.

Rewrite (1.3) as a first order system

$$u' = v, \qquad v' = cv - f(u), \tag{1.5}$$

of which the eigenvalues at two equilibrium points $E_1 = (0, 0)$ and $E_2 = (1, 0)$ are

$$0 < \lambda_1^- < \lambda_2^-, \quad \text{at } E_1; \qquad \lambda_1^+ < 0 < \lambda_2^+, \quad \text{at } E_2.$$

Associated to the traveling wave u_0 to (1.4), the first order system (1.5) has a node to saddle heteroclinic orbit $q(\xi) = (u_0(\xi), v_0(\xi))$, where $v_0(\xi) = u_0'(\xi)$.

A phase portrait for (1.5) with $c = 3$, $f(u) = 2u(1-u)$ is presented in Fig. 1.1.

In the whole real line, the stability of the KPP traveling wave is usually treated by the weighted norm using the weight function $w(\xi)$ introduced by Sattinger [22]. The weighted norm restricts the allowable initial values on the whole real line so the wave is actually stable under a smaller family of perturbations. In a closed bounded domain, all the continuous functions are bounded even in the weighted norm. So we cannot use weighted norms to select allowable perturbations near the standing wave $u_0(\xi)$. However, we have additional control of the solutions by some boundary conditions so that there exists a unique stable standing wave $\tilde{u}$ for all sufficiently large b and $|a|$. Such boundary conditions will be called "good boundary conditions" for brevity. General statements on good boundary conditions will be given later in this paper. Here are some simple examples where $d > 0$, $d_1 \geqslant 0$ are small constants and $|a|$, b are sufficiently large so that the solution $u(a) \approx 0$ and $u(b) \approx 1$:

(1) $u(a, t) = 0$, $u(b, t) = 1$ are bad boundary conditions for there does not exist any solution near u_0 that satisfies such boundary conditions.
(2) $u_x(a, t) = 0$, $u(b, t) = 1$ are bad boundary conditions for there does not exist any solution near u_0 that satisfies such boundary conditions.
(3) $u(a, t) = d$, $u(b, t) = 1 \pm d_1$ are good boundary conditions.
(4) $u(a, t) = d$, $u_x(b, t) = \pm d_1$ are good boundary conditions.
(5) $u_x(a, t) = d$, $u(b, t) = 1$ are bad boundary conditions. The standing wave $\tilde{u}$ uniquely exists but is unstable.

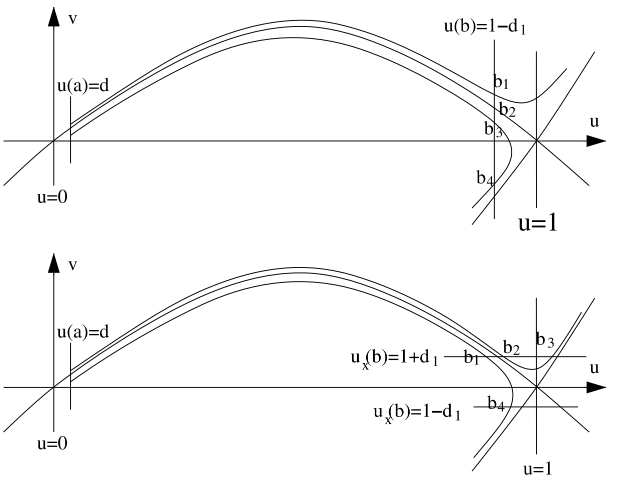

Fig. 1.2. Top: Solutions near the heteroclinic solution $q(\xi)$ that satisfy $u(a) = d$, $u(b) = 1 - d_1$. With the same a, the solutions are determined by the b_j where $u(b_j) = 1 - d_1$. Bottom: Solutions near the heteroclinic solution $q(\xi)$ that satisfy $u(a) = d$, $u_x(b) = 1 \pm d_1$. With the same a, the solutions are determined by the b_j where $u_x(b_j) = 1 \pm d_1$.

For the KPP type scalar equations, the existence of the standing wave $\tilde{u}$ near u_0 can be proved by a phase plan analysis, and in many cases, the stability of $\tilde{u}$ can be proved by monotone/comparison argument. The method in this paper allows us to prove the existence and stability of nearby finite domain standing waves for systems of equations which cannot be obtained by phase plane analysis or comparison argument. Even for scalar equations, examples (3) and (4) show that in some cases, stable standing waves $\tilde{u}$ may not be monotone so comparison argument cannot be used to study the stability of the waves. Several solutions corresponding to boundary conditions (3) and (4) are plotted in Fig. 1.2. The starting time $\xi = a$ is fixed, so the solutions are uniquely determined by the ending time b_j. It is interesting to see that solutions corresponding to different b_j can belong to the same orbit in the phase plane. Also, notice that some solutions may be non-physical for satisfying $u(b) > 1$, but are mathematically valid solutions.

The PDE (1.2) will be considered in the function space $L^2(J \times \mathbb{R}^+)$ where $J \subset \mathbb{R}$ is a bounded or unbounded interval. A solution $u(x,t)$ of (1.2) is in $H^{2,1}(J \times \mathbb{R}^+)$ (i.e., $(u, u_t, u_{xx}) \in L^2(J \times \mathbb{R}^+)$). From the Trace Theorem [15], if $u \in H^{2,1}$, then the mapping

$$x \to \big(u(x,\cdot), u_x(x,\cdot)\big) : J \to H^{3/4}\big(\mathbb{R}^+\big) \times H^{1/4}\big(\mathbb{R}^+\big)$$

is continuous. Let $\hat{u}(x,s)$ be the Fourier–Laplace transform of $u(x,t)$. Then, for each $x \in J$, both $\hat{u}(x,s)(1+|s|^{3/4})$ and $\hat{u}_x(x,s)(1+|s|^{1/4})$ are L^2 functions of ω if $s = i\omega$. The well-posedness of the initial value problem in $H^{2,1}(J \times \mathbb{R}^+)$ can be proved using the weights $(1+|s|^{3/4})$ on $\hat{u}$ and $(1+|s|^{1/4})$ on $\hat{u}_x$, where s is in the interior of a sector in the complex plane that contains the right half plane. See [17]. However, since we only consider eigenvalue problems, it is simpler to weight $\hat{u}(x,s)$ by $(1+|s|^{0.5})$ while $\hat{u}_x(x,s)$ will not be weighted. See Section 4 for details. The factor $(1+|s|^{0.5})$ indicates that $\hat{u}(x,\cdot)$ is more smooth than $\hat{u}_x(x,\cdot)$ and should not be confused with Sattinger's weight $w(\xi)$ that specifies the decay rate of $u(\xi)$, $\xi \to -\infty$. The use of weighted norm can also be achieved by a change of phase variables as in Beyn and Lorenz [1].

To study the linear variational systems around the traveling/standing waves, we will use the pseudo (or weighted) exponential dichotomies in $\mathbb{R}^{\pm}$. See [10] for the definition of "pseudo exponential trichotomies". The weight of the pseudo exponential dichotomies is closely related to the weight of Sattinger's function spaces, as stated in Proposition 3.3. This will play a key role in this paper.

Using exponential dichotomies, the proof of the stability of standing waves in the finite bounded domain is very similar to the proof of the existence and uniqueness of such waves. Let $\mathbb{C}^+ := \{s \in \mathbb{C}: Re(s) \geqslant 0\}$. For each $s \in \mathbb{C}^+$, we show that $U = 0$ is the only solution that satisfies the boundary conditions. Therefore, s is not an eigenvalue. The method can be used to treat system of KPP type equations.

Some related topics have been considered in earlier papers [1,14]. Sandstede and Scheel [20,21] studied the stability of waves on unbounded and large bounded domains in detail and discovered that remnant and transient instabilities determine the spectral (in)stability of waves under domain truncation. However, the point of view in this project is different. We look for a set of boundary conditions such that there does not exist any eigenvalue in $\mathbb{C}^+$ by checking all the parameters $s \in \mathbb{C}^+$. While in other works, the spectrum sets for unbounded and large bounded domains are compared. Moreover, most of the earlier papers are interested in checking the stability of waves on the whole real line based on information obtained from the stability of waves in large bounded domains, for example, the information from numerical simulation of waves. We are interested in the stability of waves in finite domains so generally speaking, our problem is simpler.

In Section 2, we review the stability for KPP type internal layer solutions on the whole real line. The weighted norms based on Sattinger's weight function are introduced there. In Section 3, we define the pseudo exponential dichotomies and prove some estimates on weighted vectors and weighted functions. In Section 4, we discuss the existence of standing waves in large bounded domains with boundary conditions. The study of the existence of standing waves in this section relies on the existence of pseudo exponential dichotomies for the linear variational system around $q(\xi)$. Although we only discuss the scalar KPP equation in details, our method can be used to treat KPP type systems with several unknown variables. See Remark 4.3 at the end of Section 4.

In Section 5, we prove that the standing wave obtained in Section 4 is stable. A detailed discussion of spectrum equation around $\tilde{u}$ is given showing it has exponential dichotomies closely related to the spectrum projections of the linear system at two equilibria $u = 0$ and $u = 1$. In Section 6, we prove a generalized Lambda Lemma that applies to flows near the unstable node and use it to explore the existence and stability of the standing waves. In particular, the geometric approach allows us to show that some of the finite domain standing waves $\tilde{u}$ are unstable.

Notations. We use the following notations in this paper:

Notations	Meaning
u	solutions to nonlinear scalar equations
$\mathbf{u}$	$(u, v)^\tau$ where $v = u'$
U	solutions to the linearized equation
$\mathbf{U}$	(U, V) where $V = U'$
u_0	the whole line standing wave connecting $u = 0$ to $u = 1$
$\tilde{u}$	standing wave near u_0, satisfying boundary conditions
$q(\xi)$	$(u_0(\xi), v_0(\xi))^\tau$ where $v_0 = u_0'$
$\tilde{q}(\xi)$	$\tilde{q}(\xi) = (\tilde{u}, \tilde{v})$ where $\tilde{v} = \tilde{u}'$
$g(\xi)$	the forcing term to a second order equation
$G(\xi)$	$G = (0, g)^\tau$, the forcing term to a first order system

2. Stability of KPP type internal layer solution on the whole real line

We review some results on the stability of standing waves in the whole real line so they can be used on the stability of standing waves in large bounded domains.

In the whole real line, the spectrum of the standing waves for a diffusion–convection equation is the same as the traveling waves for the corresponding reaction–diffusion equation. The linear variational system around the standing wave $u_0(\xi)$ is

$$U_\tau = LU, \quad \text{where } LU := U_{\xi\xi} - cU_\xi + Df(u_0)U.$$

The spectral equation is

$$(L-s)U = U'' - cU' + \big(Df(u_0) - s\big)U = 0. \tag{2.1}$$

It can be rewritten as a first order system

$$U' = V, \qquad V' = cV - \big(Df(u_0) - s\big)U = 0. \tag{2.2}$$

If the scalar valued function $U(\xi)$ is a solution to (2.1), then the vector valued function $(U(\xi), U'(\xi))$ is a solution to (2.2) and will be denoted by $\mathbf{U}(\xi)$.

With s as a parameter, the eigenvalues for the "spatial differential equations" (2.1) or (2.2) at $u_0(-\infty) = 0$ and $u_0(\infty) = 1$ are

$$\begin{aligned} \lambda_{1,2}^-(s) &= \frac{c}{2} \mp \sqrt{\left(\frac{c}{2}\right)^2 - Df(0) + s}, \\ \lambda_{1,2}^+(s) &= \frac{c}{2} \mp \sqrt{\left(\frac{c}{2}\right)^2 - Df(1) + s}. \end{aligned} \tag{2.3}$$

If $s = 0$, using $Df(0) > 0$, $Df(1) < 0$, it is easy to check that

$$\lambda_1^+(0) < 0 < \lambda_1^-(0) < \lambda_2^-(0) < \lambda_2^+(0).$$

If $s \in \mathbb{C}^+$, then the eigenvalues $\lambda_1^-(s)$ and $\lambda_2^-(s)$ (or $\lambda_1^+(s)$ and $\lambda_2^+(s)$) are in two non-intersecting hyperbolic sectors that satisfy the gap conditions

$$\begin{aligned} Re\,\lambda_1^-(s) \leqslant \lambda_1^-(0) < \lambda_2^-(0) \leqslant Re\,\lambda_2^-(s), \\ Re\,\lambda_1^+(s) \leqslant \lambda_1^+(0) < \lambda_2^+(0) \leqslant Re\,\lambda_2^+(s). \end{aligned} \tag{2.4}$$

The KPP waves in the whole real line are unstable without an appropriate weight function. See for example, Dan Henry [12]. More specifically, without a weight function the spectrum of (2.1) is contained in

$$\sigma_{ess}(L) = \left\{s \in \mathbb{C}\colon Re\sqrt{s - Df(0) + \big(c^2/4\big)} \leqslant c/2\right\}.$$

This set consists of essential spectrum points of the linear operator L and is bounded to the right by a parabola intersecting the real axis at $Df(0)$. It can be verified that both $Re\,\lambda_1^-(s)$ and $Re\,\lambda_2^-(s)$ are positive if $s \in \sigma_{ess}(L)$.

The use of weight functions to treat the stability of KPP waves becomes the standard approach to all the researchers.

Definition 2.1. Define the rate function $r(\xi)$, $\xi \in \mathbb{R}$ as

$$r(\xi) = \begin{cases} e^{\gamma\xi} & \text{if } \xi \leqslant 0, \text{ where } \gamma > 0, \\ 1 & \text{if } \xi \geqslant 0. \end{cases}$$

Let the weight function be $w(\xi) = r(\xi)^{-1}$. Let B_w be the Banach space of scalar or vector valued continuous functions on $\xi \in \mathbb{R}$ of which the following norms are finite:

$$\|u\|_w := \sup\big\{w(\xi)\big|u(\xi)\big|,\ \xi \in \mathbb{R}\big\} < \infty. \tag{2.5}$$

If $u \in B_w$, then $|u(\xi)| \leqslant \|u\|_w \cdot r(\xi)$. The rate function controls the spatial decay rate of $u(\xi)$ as $\xi \to -\infty$.

Remark 2.1. A general discussion of the choice of γ can be found in [23] showing γ should satisfy $\lambda_1^- < \gamma < \lambda_2^-$. In the future we alway assume $\gamma = c/2$.

The weighted norm introduced by Sattinger [22] is

$$\|u\|_w = \sup_{\xi \in \mathbb{R}} \big| u(\xi) w(\xi) \big|, \quad w(\xi) = 1 + e^{-(c/2)\xi}.$$

According to this norm, $|u(\xi)| \leqslant \|u\|_w (1 + e^{-(c/2)\xi})^{-1}$. If $\gamma = c/2$, then the weighted norm defined in this paper is equivalent to the one defined by Sattinger.

In the weighted function space, Sattinger proved that the spectrum of the linearized operator L splits into two subsets in $\mathbb{C}$:

$$\Big\{ s \in \mathbb{C}\colon Re \sqrt{s - Df(1) + \big(c^2/4\big)} \leqslant c/2 \Big\} \cup \Big\{ s \in \mathbb{R}\colon Df(1) \leqslant s \leqslant Df(0) - \big(c^2/4\big) \Big\}.$$

It can be verified that both $Re\,\lambda_{1,2}^+(s) > 0$ if s is in the following set

$$\Big\{ s \in \mathbb{C}\colon Re \sqrt{s - Df(1) + \big(c^2/4\big)} \leqslant c/2 \Big\}.$$

Thus this set consists of essential spectrum points of the linear operator L. The set

$$\Big\{ s \in \mathbb{R}\colon Df(1) \leqslant s \leqslant Df(0) - \big(c^2/4\big) \Big\}$$

consists of (non-isolated) eigenvalues so it is also part of the essential spectrum set. Sattinger proved that if $c^2 > 4Df(0)$, then in the weighted function spaces, the spectrum set is in a negative sector in $\mathbb{C}^-$. He also constructed the resolvent $(L - s)^{-1}$ for the spectral equation and showed that the wave is stable in the weighted space. In particular, any $s \in \mathbb{C}^+$ is not an eigenvalue to the spectral equation.

3. Pseudo exponential dichotomies and estimates on weighted vectors and weighted functions

Since the standing wave $u_0(\xi)$ either stays near $u = 0$ or $u = 1$ for large $|\xi|$, solutions of the ξ-dependent system (2.1) will inherit the exponential dichotomies determined by the eigenvalues $\lambda_{1,2}^{\pm}(s)$ at $u = 0$ and $u = 1$. For the first order system (2.2), let $Y_1^{\pm}(s) = (1, \lambda_1^{\pm}(s))^\tau$ and $Y_2^{\pm}(s) = (1, \lambda_2^{\pm}(s))^\tau$ be eigenvectors corresponding to eigenvalues $\lambda_1^{\pm}(s)$ and $\lambda_2^{\pm}(s)$ at $u_0 = 0$ or $u_0 = 1$. From Hartman [11], or Sattinger [22], if $s \in \mathbb{C}^+$, system (2.2) has two fundamental set of solutions in $\mathbb{R}^{\pm}$ with the following asymptotic properties:

$$\begin{aligned} &Z_1^-(\xi, s) \sim e^{\lambda_1^-(s)\xi} Y_1^-(s), \qquad Z_2^-(\xi, s) \sim e^{\lambda_2^-(s)\xi} Y_2^-(s), \quad \text{for } \xi \to -\infty; \\ &Z_1^+(\xi, s) \sim e^{\lambda_1^+(s)\xi} Y_1^+(s), \qquad Z_2^+(\xi, s) \sim e^{\lambda_2^+(s)\xi} Y_2^+(s), \quad \text{for } \xi \to \infty. \end{aligned} \tag{3.1}$$

Here $Z(\xi, s) \sim e^{\lambda(s)\xi} Y(s)$ means $Z(\xi, s) - e^{\lambda(s)\xi} Y(s) = o(e^{\lambda(s)\xi} Y(s))$. Due to the rate conditions (3.1), the solutions $Z_2^-(\xi, s)$ and $Z_1^+(\xi, s)$ are unique up to constant multiples. The solutions $Z_1^-(\xi, s)$ and $Z_2^+(\xi, s)$ are not unique even after rescaling by constant multiples.

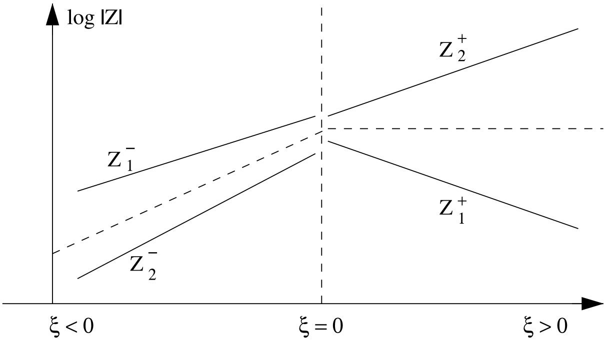

Fig. 3.1. The dichotomy and pseudo dichotomy in $\mathbb{R}^\pm$. The dotted lines indicate possible exponential weights in $\mathbb{R}^\pm$.

The growth and decay rates of $Z^\pm_{1,2}(\xi)$ for $s=0$ are depicted in Fig. 3.1. After normalization we assume $\|Z^\pm_j(0,s)\|=1$. Define the Evans function

$$E(s)=\det\bigl(Z_2^-(0,s),Z_1^+(0,s)\bigr).$$

It is clear that $E(s)=0$ iff s is an eigenvalue for the KPP wave in the weighted space. Sattinger showed that the KPP wave is stable in the weighted space. Therefore $E(s)\neq 0$ for $s\in\mathbb{C}^+$.

In the spaces of bounded continuous functions, $s=0$ is an eigenvalue with $q'(\xi)$ as an eigenfunction for system (2.2). As $\xi\to-\infty$ or $+\infty$, the eigenfunction $q'(\xi)$ has the same asymptotic behavior as $Z_1^-(\xi,0)$ or $Z_1^+(\xi,0)$. We shall assume that $Z_1^-(\xi,0)=q'(\xi)$ for $\xi\leqslant 0$ and $Z_1^+(\xi,0)=q'(\xi)$ for $\xi\geqslant 0$ so $Z_1^-(0-,0)=Z_1^+(0+,0)$. More generally, we can prove the following lemma.

Lemma 3.1. *For $s\in\mathbb{C}^+$, there exist a fundamental set of solutions $Z_{1,2}(\xi,s)$ for (2.2) defined and continuous on $\mathbb{R}$ and satisfying the asymptotic conditions (3.1). In particular*

$$Z_1(0-,s)=Z_1(0+,s),\qquad Z_2(0-,s)=Z_2(0+,s).$$

Proof. Consider the fundamental set of solutions as in (3.1). For each s with $s\in\mathbb{C}^+$, write $Z_1^+(0+,s)$ as

$$Z_1^+(0+,s)=c_1Z_1^-(0-,s)+c_2Z_2^-(0-,s).\tag{3.2}$$

We give an indirect proof that $c_1\neq 0$. If $c_1=0$, then by checking the asymptotic decay rate of $Z_1^+(\xi,s)$ as $\xi\to\pm\infty$, we find that $Z_1^+(\xi,s)$ is an eigenfunction corresponding to the eigenvalue s even in the weighted function space introduced by Sattinger [22]. However, since $s\in\mathbb{C}^+$, this is a contradiction to the stability result proved by Sattinger. Therefore $c_1\neq 0$.

Using c_1 and c_2 obtained before, define a solution to (2.1) as follows:

$$Z_1(\xi,s):=\begin{cases} Z_1^+(\xi,s), & \xi\geqslant 0,\\ c_1Z_1^-(\xi,s)+c_2Z_2^-(\xi,s), & \xi\leqslant 0.\end{cases}$$

Since $c_1\neq 0$, it is easy to show that $Z_1(\xi,s)\sim e^{\lambda_1^-(s)\xi}Y_1^-(s)$ as $\xi\to-\infty$. The property $Z_1(0-,s)=Z_1(0+,s)$ follows from its definition and (3.2).

Similarly, we can construct a solution $Z_2(\xi,s)$, $\xi\in\mathbb{R}$ so that $Z_2(\xi,s)=Z_2^-(\xi,s)$ for $\xi\leqslant 0$, and $Z_2(\xi,s)\sim e^{\lambda_2^+(s)\xi}Y_2^+(s)$ as $\xi\to\infty$. $\square$

It is often more convenient to use pseudo exponential dichotomies to study solutions in $\mathbb{R}^\pm$ than the fundamental set of solutions to (2.2). A general reference for exponential dichotomies is in Coppel [3]. See also the Bohl exponents given in [5]. Let $T(\xi,\eta,s)$ be the principle matrix solution to (2.2) with s as a parameter. Our definition follows from that of Henry [12] which only uses forward flow $T(\xi,\eta,s)$, $\xi \geqslant \eta$ on the stable subspace and backward flow $\xi \leqslant \eta$ on unstable subspace.

Definition 3.1. Let $\mathbb{C}^+ \subset \mathbb{C}$ be the subset of complex numbers whose real parts are nonnegative, and $J \subset \mathbb{R}$ be a bounded or unbounded interval. We say that system (2.1) has a pseudo exponential dichotomy in the interval J for each $s \in \mathbb{C}^+$ if there exist projections $P_s(\xi,s) + P_u(\xi,s) = I$, $\xi \in J$, continuous with respect to ξ, two exponents $\alpha(s) < \beta(s)$ and a constant $K(s) > 0$ such that

$$(1)\quad \begin{cases} T(\xi,\eta,s)P_s(\eta,s) = P_s(\xi,s)T(\xi,\eta,s), & \xi \geqslant \eta \in J, \\ T(\xi,\eta,s)P_u(\eta,s) = P_u(\xi,s)T(\xi,\eta,s), & \xi \leqslant \eta \in J, \end{cases}$$

$$(2)\quad \begin{cases} \left|T(\xi,\eta,s)P_s(\eta,s)\right| \leqslant K(s)e^{\alpha(s)(\xi-\eta)}, & \text{for } \xi \geqslant \eta \in J, \\ \left|T(\eta,\xi,s)P_u(\xi,s)\right| \leqslant K(s)e^{\beta(s)(\eta-\xi)}, & \text{for } \eta \leqslant \xi \in J. \end{cases}$$

The ranges of the projections $P_s(\xi,s)$ and $P_u(\xi,s)$ are called the (relatively) stable and unstable subspaces of the pseudo exponential dichotomy.

We say that the system has a regular exponential dichotomy if the exponents satisfy the condition $\alpha(s) < 0 < \beta(s)$.

Based on the information of the eigenvalues at $u = 0, 1$, the existence of pseudo exponential dichotomies usually can be proved by the property called "the roughness of pseudo exponential dichotomies", see [3,19,10]. The projections to stable and unstable subspaces can also be obtained by using the fundamental set of solutions $Z^\pm_{1,2}(\xi,s)$ and the Evans function [22]. In particular, for the KPP/Fisher equation, if $s \in \mathbb{C}^+$, the linear equation (2.2) has an exponential dichotomy for $\xi \in \mathbb{R}^+$, and has a pseudo exponential dichotomy for $\xi \in \mathbb{R}^-$. In $\mathbb{R}^+$, the exponents are $\lambda^+_1(s) = \alpha^+(s) < \beta^+(s) = \lambda^+_2(s)$. In $\mathbb{R}^-$, the exponents are $\lambda^-_1(s) = \alpha^-(s) < \beta^-(s) = \lambda^-_2(s)$.

Let RP denote the range of the projection P. Using Lemma 3.1, we show that the dichotomies defined in $\mathbb{R}^\pm$ can be combined as in the following lemma.

Lemma 3.2. *For each $s \in \mathbb{C}^+$, we can redefine the dichotomies of system (2.2) on $\mathbb{R}^\pm$ so that the projections $P_s(\xi,s)$ and $P_u(\xi,s)$ are continuous with respect to $\xi \in \mathbb{R}$ and satisfy (1) in Definition 3.1. In particular,*

$$RP_s(0-,s) = RP_s(0+,s), \qquad RP_u(0-,s) = RP_u(0+,s).$$

The exponential rates in each of the intervals $\mathbb{R}^\pm$ are unchanged, that is, property (2) in Definition 3.1 is satisfied with $\alpha(s) = \lambda^-_1(s)$, $\beta(s) = \lambda^-_2(s)$ in $\mathbb{R}^-$, and with $\alpha(s) = \lambda^+_1(s)$, $\beta(s) = \lambda^+_2(s)$ in $\mathbb{R}^+$.

Proof. Using the fundamental set of solutions from Lemma 3.1, let

$$RP_s(\xi,s) := span\{Z_1(\xi,s),\ \xi \in \mathbb{R}\}, \qquad RP_u(\xi,s) := span\{Z_2(\xi,s),\ \xi \in \mathbb{R}\}.$$

Then the projections $P_s(\xi,s) + P_u(\xi,s) = I$ are uniquely defined for $\xi \in \mathbb{R}$, $s \in \mathbb{C}^+$. □

Remark 3.1. We often say that (2.2) has an exponential dichotomy on $\mathbb{R}$ since the projections $P_s(\xi,s)$ and $P_u(\xi,s)$ are defined and continuous for all $\xi \in \mathbb{R}$ and property (1) of Definition 3.1 is satisfied. However, property (2) of Definition 3.1 is only satisfied separately in $\mathbb{R}^-$ with the rate $\alpha^-(s) < \beta^-(s)$, and in $\mathbb{R}^+$ with the rate $\alpha^+(s) < \beta^+(s)$.

In Section 5, we will make a change of variables so that (2.2) becomes (5.4). To that system, we show that if U is weighted by $(1 + |s|^{0.5})$ but $V = U'$ is not weighted, then the spectral system (2.2) can have a regular exponential dichotomy on $\mathbb{R}$ of which the projections are uniformly bounded with respect to $s \in \mathbb{C}^+$. The constant $K(s)$ in the definition can be chosen independent of s.

3.1. Basic estimates on weighted vectors and weighted functions

Definition 3.2. Let $w(\xi)$ be the same weight function as in Definition 2.1 with $\gamma = c/2$. For each $\xi \in R$, defined the Banach space $B_{w(\xi)}$ of vectors $\mathbf{v} \in \mathbb{R}^2$ with the norm $\|\mathbf{v}\|_{w(\xi)} = w(\xi)|\mathbf{v}|$ where $|\mathbf{v}|$ is the Euclidean norm of $\mathbf{v}$.

Thus if $\|\mathbf{v}\|_{w(\xi)} = C$, then $|\mathbf{v}| = Cr(\xi)$ in the Euclidean norm, which is much smaller at $\xi = a$ than at $\xi = b$, if $|a|$ and b are two large constants.

Let $T(\xi, \eta, s)$ be the principle matrix solution of (2.2). From Lemma 3.2, (2.2) has pseudo exponential dichotomies on $\mathbb{R}^\pm$. A simple relationship between the pseudo exponential dichotomies and the weighted vector spaces $B_{w(\xi)}$ is given below.

Proposition 3.3. *The principle matrix solution $T(\xi, \eta, s) : B_{w(\eta)} \to B_{w(\xi)}$ induces a flow on the Banach spaces of vectors $B_{w(\xi)}$, $\xi \in J$. If $T(\xi, \eta, s)$ has a pseudo exponential dichotomy on the interval J, then the induced flow has a regular exponential dichotomy in the vector spaces $B_{w(\xi)}$. In particular, if (2.2) has a pseudo exponential dichotomy in $\xi < 0$ with the exponents $\alpha^- < \beta^-$, then using $\alpha^- < \gamma < \beta^-$, for the induced exponential dichotomy in $B_{w(\xi)}$ the exponents are $\alpha - \gamma < 0 < \beta^- - \gamma$.*

In the following we assume s is fixed and drop the reference to s. So the projections of the dichotomies will be $P_s(\xi)$, $P_u(\xi)$ for $\xi \in J = [a, b]$, and the constant is K. The exponents are $\alpha^- < \beta^-$ for $\xi \leqslant 0$ and $\alpha^+ < \beta^+$ for $\xi \geqslant 0$ respectively.

Lemma 3.4 *(Estimates for the forward and backward flows). Assume*

$$\phi_s \in RP_s(a), \qquad \phi_u \in RP_u(b).$$

Then the function $T(\cdot, a)\phi_s$ satisfies

$$\big\| T(\cdot, a)\phi_s \big\|_w \leqslant K^2 \|\phi_s\|_{w(a)}. \tag{3.3}$$

For the point-wise estimate of the flow from $\xi = a$ to $\xi = b$, we have

$$\big\| T(b, a)\phi_s \big\|_{w(b)} \leqslant K^2 e^{(\gamma - \alpha^-)a + b\alpha^+} \|\phi_s\|_{w(a)}. \tag{3.4}$$

Similarly the function $T(\cdot, b)\phi_u$ satisfies

$$\big\| T(\cdot, b)\phi_u \big\|_w \leqslant K^2 \|\phi_u\|_{w(b)}. \tag{3.5}$$

For the point-wise estimate of the flow from $\xi = b$ to $\xi = a$, we have

$$\big\| T(a, b)\phi_u \big\|_{w(a)} \leqslant K^2 e^{-\beta^+ b + (\beta^- - \gamma)a} \|\phi_u\|_{w(b)}. \tag{3.6}$$

Proof. The norms of $B_{w(a)}$ and $B_{w(b)}$ will be applied to ϕ_s and ϕ_u respectively.

We first give an estimate for the function $T(\xi, a)\phi_s$. For $a \leqslant \xi \leqslant 0$,

$$\big| T(\xi, a)\phi_s \big| \leqslant K e^{(\xi - a)\alpha^-} \|\phi_s\|_{w(a)} e^{\gamma a} \leqslant K e^{\gamma \xi} e^{(\xi - a)(\alpha^- - \gamma)} \|\phi_s\|_{w(a)}.$$

For $0 \leqslant \xi \leqslant b$,

$$\left|T(\xi, a)\phi_s\right| \leqslant K e^{\xi\alpha^+}\left|T(0, a)\phi_s\right| \leqslant K^2 e^{\xi\alpha^+} e^{-(\alpha^- - \gamma)a}\|\phi_s\|_{w(a)}.$$

Combining both estimates, we have (3.3) and (3.4).

Similarly, we derive estimate for $T(\xi, b)\phi_u$. For $0 \leqslant \xi \leqslant b$,

$$\left|T(\xi, b)\phi_u\right| \leqslant K e^{\beta^+(\xi - b)}\|\phi_u\|_{w(b)}.$$

For $a \leqslant \xi \leqslant 0$,

$$\begin{aligned}\left|T(\xi, b)\phi_u\right| &\leqslant K^2 e^{-\beta^+ b} e^{\beta^- \xi}\|\phi_u\|_{w(b)}\\ &\leqslant K^2 e^{\gamma\xi} e^{-\beta^+ b + (\beta^- - \gamma)\xi}\|\phi_u\|_{w(b)}.\end{aligned}$$

Combining both estimates, we have (3.5) and (3.6). □

Lemma 3.5 *(Estimates for the integrals). For $G \in B_W$, which is the weighted Banach space of continuous vector valued functions as in Definition 2.1, let*

$$I_s(\xi) := \int_a^\xi T(\xi, \zeta)P_s(\zeta)G(\zeta)\,d\zeta, \qquad I_u(\xi) := \int_b^\xi T(\xi, \zeta)P_u(\zeta)G(\zeta)\,d\zeta, \tag{3.7}$$

where $\xi \in [a, b]$. Then

$$\|I_s\|_W \leqslant K^2\left(\frac{1}{\gamma - \alpha^-} + \frac{1}{|\alpha^+|}\right)\|G\|_W, \tag{3.8}$$

$$\|I_u\|_W \leqslant K^2\left(\frac{1}{\beta^+} + \frac{1}{\beta^- - \gamma}\right)\|G\|_W. \tag{3.9}$$

Proof. Using the pseudo exponential dichotomies, we have for $a \leqslant \xi \leqslant 0$,

$$\begin{aligned}\left|I_s(\xi)\right| &\leqslant \int_a^\xi K e^{\alpha^-(\xi - \zeta)} e^{\gamma\zeta}\|G\|_W\,d\zeta\\ &= K e^{\gamma\xi}\int_a^\xi e^{-(\gamma - \alpha^-)(\xi - \zeta)}\|G\|_W\,d\zeta\\ &\leqslant K\left(\gamma - \alpha^-\right)^{-1} e^{\gamma\xi}\|G\|_W.\end{aligned}$$

For $0 \leqslant \xi \leqslant b$,

$$\begin{aligned}\left|I_s(\xi)\right| &\leqslant K e^{\alpha^+\xi}\left|I_s(0)\right| + \int_0^\xi K e^{\alpha^+(\xi - \zeta)}\|G\|_W\,d\zeta\\ &\leqslant K^2\left(\left(\gamma - \alpha^-\right)^{-1} + \left|\alpha^+\right|^{-1}\right)\|G\|_W.\end{aligned}$$

All together, we have the estimate (3.8).

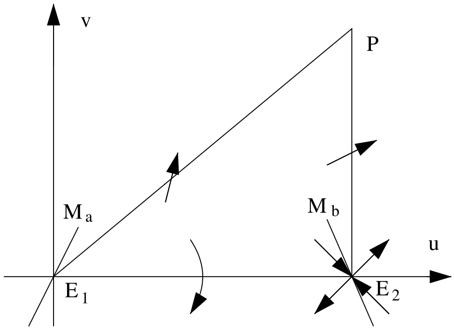

Fig. 4.1. $E_1 = (0, 0)$, $E_2 = (0, 1)$ are equilibrium points. The slope of $\overline{E_1 P}$ is the first positive eigenvalue λ_1^-.

Similarly, for $0 \leqslant \xi \leqslant b$,

$$\left|I_u(\xi)\right| \leqslant \int_{\xi}^{b} K e^{(\xi-\zeta)\beta^+} \|G\|_w \, d\zeta \leqslant K\left(\beta^+\right)^{-1} \|G\|_w.$$

For $a \leqslant \xi \leqslant 0$,

$$\begin{aligned}\left|I_u(\xi)\right| &\leqslant K e^{\xi\beta^-} \left|I_u(0)\right| + \int_{\xi}^{0} K e^{(\xi-\zeta)\beta^-} e^{\gamma\zeta} \|G\|_w \, d\zeta \\ &\leqslant K^2\left(\beta^+\right)^{-1} e^{\gamma\xi} \|G\|_w + K\left(\beta^- - \gamma\right)^{-1} e^{\gamma\xi} \|G\|_w.\end{aligned}$$

Combining both we have the estimate (3.9). □

4. Existence of standing waves in large bounded domains with boundary conditions

In the rest of the paper, we consider the orbit of $q(\xi)$ as the "relatively stable" manifold that passes through $E_1 = (0, 0)$, and denoted by $W^s(E_1)$. In this section, we fixed the value $s = 0$ in (2.1) and (2.2) so the parameter s will be dropped in all the previously defined notations.

First we present a condition under which the standing wave $\tilde{u}$ near u_0 does not exist. Let $P = (1, \lambda_1^-)$ be a point on the phase plane (u, v) where $v = u_\xi$. Observe that the slope of line segment $\overline{E_1 P}$ is λ_1^-, where $E_1 = (0, 0)$. See Fig. 4.1.

In the phase plane $\mathbf{U} = (U, U_\xi)$, the boundary conditions will be defined by $\mathbf{U}(a) \in M_a$, $\mathbf{U}(b) \in M_b$ where the boundary manifolds M_a and M_b are two 1D linear submanifolds of $\mathbb{R}^2$.

Theorem 4.1. *Assume that the boundary manifold M_a is mutually disjoint from the interior of the triangle $\Delta E_1 P E_2$. Then the solution of the boundary value problem that stays near the orbit $q(\xi)$ does not exist.*

Proof. We give an indirect proof. If a solution $\tilde{q}(\xi)$ is near $q(\xi)$ for all $a \leqslant \xi \leqslant b$, then $\tilde{q}(0)$ must be in the triangle $\Delta E_1 P E_2$. From the vector field depicted in Fig. 4.1 along the three sides of $\Delta E_1 P E_2$, the backward orbit $\tilde{q}(\xi)$, $\xi < 0$ stays inside the triangle. Therefore $\tilde{q}(a)$ is in the interior of the triangle $\Delta E_1 P E_2$ and will not be on the boundary manifold M_a, a contradiction to $\tilde{q}(\xi)$ satisfies the boundary condition at a. □

Based on Theorem 4.1, M_a must intersect with some interior point in the triangle $\Delta E_1 P E_2$. Assume that the manifold M_a transversely intersects with the orbit of $q(\xi)$ (the "relatively stable manifold of E_1, $W^s(E_1)$) at a point $P_1 = q(a)$ where $a < 0$ is a large constant so that P_1 is near E_1. Assume also that M_b transversely intersects the unstable manifold $W^u(E_2)$ at P_2 which is close to E_2. Observe that $P_1 \neq E_1$, but $P_2 = E_2$ is allowed. For such $P_1 \in W^s_{loc}(E_1)$ and $P_2 \in W^u_{loc}(E_2)$, define the boundary manifolds as

$$M_a = \{\mathbf{U} \mid (\mathbf{U} - P_1) \cdot \mathbf{n}_a = 0\}, \qquad M_b = \{\mathbf{U} \mid (\mathbf{U} - P_2) \cdot \mathbf{n}_b = 0\}, \tag{4.1}$$

where $\mathbf{n}_a$ and $\mathbf{n}_b$ are normal vectors to M_a and M_b respectively.

The following hypothesis will be assumed throughout this paper:

(H1) Let the angles between M_a and $TW^s(E_1)$ be θ_1 and M_b and $TW^u(E_2)$ be θ_2. We assume that there is a constant $0 < \theta_0 \leqslant \pi/2$ such that $\theta_0 \leqslant \theta_j \leqslant \pi/2$ for $j = 1, 2$.

Remark 4.1. The lower bound θ_0 determines how close P_1, P_2 are to E_1, E_2 and how large $|a|$ and b should be. Notice that the condition P_1 being close to E_1 is the same as $|a|$ being sufficiently large. We state both conditions to make the boundary conditions on both ends look similar.

Also notice that if $a < 0$ is fixed, increasing $b > 0$ can only make $\tilde{u}(b)$ close to P_2. The largeness of $b > 0$ and the closeness of P_2 to E_2 are both necessary to ensure that $\tilde{u}(b)$ is close to E_2.

Let $u(\xi) = q(\xi) + U(\xi)$. Then $U(\xi)$, $a \leqslant \xi \leqslant b$ satisfies the following first order system with boundary conditions:

$$U_\xi = V, \qquad V_\xi = cU_\xi - Df\big(u_0(\xi)\big)U - g(\xi), \tag{4.2}$$

$$q(a) + U(a) \in M_a, \qquad q(b) + U(b) \in M_b. \tag{4.3}$$

Here $g(\xi)$ represents the nonlinear term $N(U(\xi)) = f(q + U) - f(q) - Df(q)U$.

Lemma 4.2. *If $U \in B_w$ with small $\|U\|_w$, then $N(U) \in B_w$ and there exists a constant $C > 0$ such that $\|N(U)\|_w \leqslant C\|U\|_w^2$. Moreover, the linear operator DN maps $B_w \to B_w$ with the operator norm bounded by $\|DN\| \leqslant C\|U\|_w$.*

Proof. From

$$N(U) = f(q + U) - f(q) - Df(q)U = \left(\int_0^1 D^2 f(q + tU)(1 - t)\,dt\right)U^2,$$

and $\|U^2\|_w \leqslant \|U\|_w^2$, we easily find that $\|N(U)\|_w \leqslant C\|U\|_w^2$.

Next for any function $\bar{U} \in B_w$, from $\langle DN(U), \bar{U}\rangle = \langle Df(q + U) - Df(q), \bar{U}\rangle$, we find

$$\big\|\langle DN(U), \bar{U}\rangle\big\|_w \leqslant \big|Df(q + U) - Df(q)\big|\|\bar{U}\|_w \leqslant C\|U\|_w\|\bar{U}\|_w. \qquad \square$$

Theorem 4.3. *Let M_a and M_b be the boundary manifolds defined in (4.1). Assume that P_1 and P_2 are sufficiently close to E_1 and E_2, $M_a \pitchfork W^s_{loc}(E_1)$ and $M_b \pitchfork W^u_{loc}(E_2)$, and the condition (H1) is satisfied. Then for sufficiently large $|a|$ and b, there exists a unique standing wave $\tilde{q}(\xi)$ near $q(\xi)$ for $a \leqslant \xi \leqslant b$ that satisfies the boundary conditions at a and b.*

Proof. First we replace $N(U(\xi))$ by a given function $\tilde{N}(\xi)$ in B_w. We will solve the linear variational equation with the given $\tilde{N}(\xi)$. Then a contraction mapping principle with $\tilde{N} = N(U)$ will determine U for the nonlinear equation.

Let $G(\xi) = (0, \tilde{N}(\xi))^\tau$ be the vector valued function in B_w. Using the pseudo exponential dichotomy, $\mathbf{U}(\xi)$, $a \leqslant \xi \leqslant b$, can be written as

$$\mathbf{U}(\xi) = T(\xi, a)P_s(a)\mathbf{U}(a) + T(\xi, b)P_u(b)\mathbf{U}(b) - I_s(\xi) - I_u(\xi) \tag{4.4}$$

where $I_s(\xi)$, $I_u(\xi)$ are the convolutions of $G(\xi)$ to Green's functions as in (3.7). Let $\delta := \min\{\gamma - \alpha^-, \beta^- - \gamma, |\alpha^+|, \beta^+\} > 0$. From Lemma 3.5 (with $\gamma = c/2$) the integrals $I_s(\xi)$, $I_u(\xi)$ satisfy

$$\|I_s\|_w \leqslant \left(\frac{2K^2}{\delta}\right)\|G\|_w, \qquad \|I_u\|_w \leqslant \left(\frac{2K^2}{\delta}\right)\|G\|_w.$$

From (4.4) the solution $\mathbf{U}$ is determined by the two vectors:

$$\phi_s := P_s(a)\mathbf{U}(a), \qquad \phi_u := P_u(b)\mathbf{U}(b).$$

From Lemma 3.4, in the weighted norms, the functions $T(\xi, a)\phi_s$ and $T(\xi, b)\phi_u$ satisfy

$$\left\|T(\cdot, a)\phi_s\right\|_w \leqslant C\|\phi_s\|_{w(a)}, \qquad \left\|T(\cdot, b)\phi_u\right\|_w \leqslant C\|\phi_u\|_{w(b)}.$$

Notice the weight at $\xi = b$ is $w = 1$. We write $\|\phi_u\|_{w(b)}$ for symmetry only.

Finally $T(b, a)\phi_s$ and $T(a, b)\phi_u$ shall be weighted at b and a by $w(b)$ and $w(a)$ respectively. From Lemma 3.4 again, we have the following decay estimates of vectors in the weighted norms:

$$\left\|T(b, a)\phi_s\right\|_{w(b)} \leqslant Ce^{-\delta(b-a)}\|\phi_s\|_{w(a)}, \qquad \left\|T(a, b)\phi_u\right\|_{w(a)} \leqslant Ce^{-\delta(b-a)}\|\phi_u\|_{w(b)}. \tag{4.5}$$

Denote the function $\mathbf{U}$ in (4.4) by $\mathbf{U} = \mathcal{F}_1(\phi_s, \phi_u, G)$. Then in the weighted norm,

$$\left\|\mathcal{F}_1(\phi_s, \phi_u, G)\right\|_w \leqslant C\big(\|\phi_s\|_{w(a)} + \|\phi_u\|_{w(b)} + \|G\|_w\big). \tag{4.6}$$

If $\|U\|_w$ is sufficiently small, due to Lemma 4.2 $N(U)$ is Lipschitz continuous with a small Lipschitz number. Then the fix point problem

$$\mathbf{U} = \mathcal{F}_1\big(\phi_s, \phi_u, \big(0, N(U)\big)^\tau\big), \tag{4.7}$$

has a unique solution by the contraction mapping principle. Let the solution of the nonlinear equation (4.7) be denoted by

$$\begin{aligned} &\mathbf{U} = \mathcal{F}_2(\phi_s, \phi_u), \quad \text{then} \\ &|\mathbf{U}|_w \leqslant \big|\mathcal{F}_2(\phi_s, \phi_u)\big|_w \leqslant C\big(\|\phi_s\|_{w(a)} + \|\phi_u\|_{w(b)}\big). \end{aligned} \tag{4.8}$$

To satisfy the boundary conditions, (ϕ_s, ϕ_u) must satisfy

$$\begin{aligned} &\phi_s + I_u(a) + q(a) + T(a, b)\phi_u \in M_a, \\ &\phi_u + I_s(b) + q(b) + T(b, a)\phi_s \in M_b. \end{aligned}$$

The above can be expressed as

$$\big(\phi_s + I_u(a) + T(a,b)\phi_u\big)\cdot \mathbf{n}_a = 0,$$
$$\big(\phi_u + I_s(b) + T(b,a)\phi_s + q(b) - P_2\big)\cdot \mathbf{n}_b = 0.$$

Recall that M_a and M_b transversely intersect with $W^s_{loc}(E_1)$ and $W^u_{loc}(E_2)$ respectively and the angles between the manifolds are bounded below by θ_0. If $|a|$ and b are sufficiently large, then M_a and M_b also transversely intersect with $RP_s(a)$ and $RP_u(b)$ respectively. Then $\phi_s \to \phi_s \cdot \mathbf{n}_a$ and $\phi_u \to \phi_u \cdot \mathbf{n}_b$ are two isomorphisms. The inverses to the two mapping are bounded operators and shall be denoted by $\mathcal{K}_a$ and $\mathcal{K}_b$. We then have

$$\phi_s = \mathcal{K}_a(\phi_s\cdot \mathbf{n}_a), \qquad \|\phi_s\|_{w(a)} \leqslant C\big\|(\phi_s\cdot \mathbf{n}_a)\big\|_{w(a)},$$
$$\phi_u = \mathcal{K}_b(\phi_u\cdot \mathbf{n}_b), \qquad \|\phi_u\|_{w(b)} \leqslant C\big\|(\phi_u\cdot \mathbf{n}_b)\big\|_{w(b)}.$$

We are led to the system of fixed point for the mapping $(\phi_s,\phi_u)\to(\phi_s',\phi_u')$ where

$$\begin{aligned}\phi_s' &= -\mathcal{K}_a\big[\big(I_u(a) + T(a,b)\phi_u\big)\cdot \mathbf{n}_a\big],\\ \phi_u' &= -\mathcal{K}_b\big[\big(I_s(b) + q(b) - P_2 + T(b,a)\phi_s\big)\cdot \mathbf{n}_b\big].\end{aligned} \tag{4.9}$$

In the weighted norms the following estimates hold:

$$\big\|\phi_s'\big\|_{w(a)} \leqslant C\big(\|G\|_w + e^{-\delta(b-a)}\|\phi_u\|_{w(b)}\big),$$
$$\big\|\phi_u'\big\|_{w(b)} \leqslant C\big(\|G\|_w + e^{-\delta(b-a)}\|\phi_s\|_{w(a)} + \big\|\big(q(b)-P_2\big)\big\|_{w(b)}\big).$$

Replacing G by $(0, N(U))^\tau$ and using the estimate (4.8), if $b-a$ is sufficiently large then (4.9) defines a contraction mapping from $RP_s(a)\times RP_u(b)$ to itself:

$$(\phi_s,\phi_u)\to\big(\phi_s',\phi_u'\big).$$

The unique fix point $(\phi_s,\phi_u) = (\phi_s',\phi_u')$ determines the solution $\mathbf{U}(\xi)$ that satisfies the boundary conditions at a and b. □

Remark 4.2. Using an iteration method starting with $\phi_s = 0$, $\phi_u = 0$ to approximate the solution of the fixed point problem, at the first approximation, we see $\phi_s^1 = 0$ and $\phi_u^1 = -\mathcal{K}_b((q(b)-P_2)\cdot \mathbf{n}_b)$. The approximation (ϕ_s^1,ϕ_u^1) has an exponentially small error. This suggests that the standing wave solution has a fast boundary layer at near $\xi = b$ if $(q(b)-P_2)\cdot \mathbf{n}_b \neq 0$. This can also be seen from the phase portrait near the equilibrium E_2 in Fig. 1.1.

Remark 4.3. With exactly the same method we can treat systems of KPP type equations. Assume that system (1.3), with $u\in\mathbb{R}^n$, has a whole line standing wave solution $u_0(\xi)$ connecting $u = w_1$ to $u = w_2$. The corresponding heteroclinic solution is $q(\xi)$. Assume that the equilibrium $u = w_2$ is stable, i.e., $Re\,\sigma Df(w_2) < 0$. The other equilibrium $u = w_1$ has k unstable modes with real eigenvalues, i.e., if $\sigma Df(w_1) = \{s_j\}_1^n$, then $0 < s_1 \leqslant s_2 \leqslant \cdots \leqslant s_k$ and

$$Re(s_j) < 0, \quad k+1\leqslant j\leqslant n.$$

Assume $c/2 > s_k$. The first order system corresponding to the second order equations has $2n$ eigenvalues. We shall consider the first k positive eigenvalues of the first order system that are bounded by $c/2$ to be "relatively stable". Let the n-D boundary manifold M_a transversely intersect with the local "relatively stable" manifold of $E_1 = (w_1, 0)$ at $q(a) = P_1$; and the n-D boundary manifold M_b

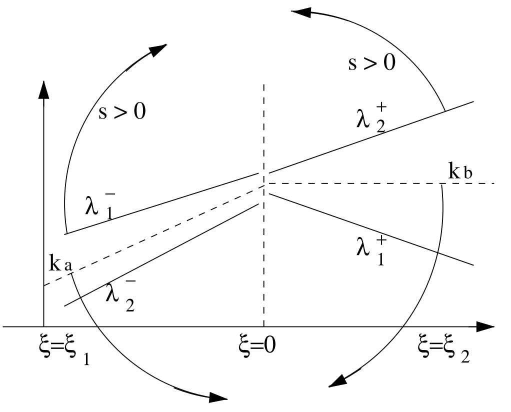

Fig. 5.1. The growth/decay rates of fundamental set of solutions are shown in logarithmic scale. The arrows indicate the ranges of k_a and k_b which are never equal to $\lambda_1^-(s)$ or $\lambda_2^+(s)$ for any $s \geqslant 0$.

transversely intersect with the local unstable manifold of $E_2 = (w_2, 0)$ at P_2. Assume that P_1, P_2 are close to E_1 and E_2 and $|a|$, b are sufficiently large. Then there exists a unique standing wave $\tilde{u}(\xi)$ on $[a, b]$ that satisfies the boundary conditions defined by M_a and M_b. The solution stays close to the whole line standing wave solution $u_0(\xi)$ for all $\xi \in [a, b]$.

5. Stability of the solution in a large bounded domain

Assume all the conditions in Section 4 are satisfied so there exists a unique standing wave solution $\tilde{u}$ that satisfies the boundary conditions defined by M_a and M_b. As before we assume that M_a and M_b pass P_1 and P_2 where $P_1 = (u_1, v_1)$, $P_2 = (u_2, v_2)$. We rewrite M_a, M_b as

$$M_a = \big\{(u, v) \bigm| v - v_1 = k_a(u - u_1)\big\}, \qquad M_b = \big\{(u, v) \bigm| v - v_2 = k_b(u - u_2)\big\}.$$

Recall that $\lambda_{1,2}^{\pm}(0)$ are eigenvalues of (2.2) at $s = 0$, $u = 0, 1$. Beside (H1), the following hypothesis will be assumed in this section:

(H2) $\lambda_1^-(0) < k_a \leqslant \infty$, $-\infty \leqslant k_b < \lambda_2^+(0)$. If $k_a = \infty$ and/or $k_b = -\infty$, we mean the Dirichlet boundary condition $u \equiv u_1$ and/or $u \equiv u_2$ respectively.

The ranges of allowable k_a and k_b and the ranges of $\lambda_1^-(s)$ and $\lambda_2^+(s)$ for $s \geqslant 0$ are depicted in Fig. 5.1.

The linear variational system around the standing wave solution $\tilde{u}$ is

$$\begin{aligned} &\bar{U}_t = \bar{U}_{\xi\xi} - c\bar{U}_\xi + Df(\tilde{u})\bar{U}, \quad a < \xi < b, \\ &\bar{U}_\xi = k_a\bar{U} \quad \text{at } \xi = a, \qquad \bar{U}_\xi = k_b\bar{U} \quad \text{at } \xi = b. \end{aligned} \tag{5.1}$$

Since we deal with linear problem in this section, it is more convenient to use the weight function $e^{-\gamma\xi}$ for both $\xi \leqslant 0$ and $\xi \geqslant 0$. By the change of variable $\bar{U} = e^{\gamma\xi}U$, $\gamma = c/2$, we have the linear boundary value problem for the new variable U,

$$\begin{aligned} &U_t = U_{\xi\xi} + \big(Df(\tilde{u}) - \gamma^2\big)U, \quad a < \xi < b, \\ &U_\xi = (k_a - \gamma)U \quad \text{at } \xi = a, \qquad U_\xi = (k_b - \gamma)U \quad \text{at } \xi = b. \end{aligned} \tag{5.2}$$

Let s be an eigenvalue and U be the corresponding eigenfunction of the linear system. Let $h_a := k_a - \gamma$, $h_b = k_b - \gamma$. Then the spectral equation can be written as a first order system with boundary conditions:

$$\begin{gathered} U_\xi = V, \qquad V_\xi = \big(\gamma^2 + s - Df(\tilde{u})\big)U, \\ V = h_a U \quad \text{at } \xi = a, \qquad V = h_b U \quad \text{at } \xi = b. \end{gathered} \tag{5.3}$$

The boundary condition for the eigenfunction can be expressed as

$$\begin{aligned} &\mathbf{U}(a)\cdot \mathbf{n}_a = 0, \qquad \mathbf{U}(b)\cdot \mathbf{n}_b = 0, \\ &\quad \text{where } \mathbf{n}_a = \left(\frac{h_a}{\sqrt{h_a^2+1}}, \frac{-1}{\sqrt{h_a^2+1}}\right),\ \mathbf{n}_b = \left(\frac{h_b}{\sqrt{h_b^2+1}}, \frac{-1}{\sqrt{h_b^2+1}}\right). \end{aligned}$$

For the Dirichlet boundary conditions at $\xi = a$ and/or $\xi = b$, we let $\mathbf{n}_a = (1,0)$ and/or $\mathbf{n}_b = (1,0)$.

We also consider the linear system with $\tilde{u}(\xi)$ replaced by the whole line standing wave $u_0(\xi)$,

$$U_\xi = V, \qquad V_\xi = \big(\gamma^2 + s - Df(u_0)\big)U. \tag{5.4}$$

The eigenvalues at the limiting states $u = 0$ and $u = 1$ are

$$\begin{aligned} \mu_{1,2}^-(s) &= \mp\sqrt{\gamma^2 - Df(0) + s}, \\ \mu_{1,2}^+(s) &= \mp\sqrt{\gamma^2 - Df(1) + s}. \end{aligned} \tag{5.5}$$

The condition (H2) becomes

(H3) $\mu_1^-(0) < h_a \leqslant \infty$, $-\infty \leqslant h_b < \mu_2^+(0)$. If $h_a = \infty$ and/or $h_b = -\infty$, we mean the Dirichlet boundary conditions at a and/or b.

Before studying the stability problem, we shall state a version of the roughness of the exponential dichotomies which allow us to relate the spectrum at $u = 0$ and $u = 1$ to the exponential dichotomies around u_0 and $\tilde{u}$. Let $T(x, y)$ be the principal matrix solution for $u'(x) = A(x)u(x)$, $x \in I$. Assume the system has an exponential dichotomy on I with projections $P_s(x)$ and $P_u(x)$. Let the constant of the dichotomy be $K_0 \geqslant 1$ and the exponent be $\alpha_0 > 0$. The exponential dichotomy persists under small perturbations.

Theorem 5.1 *(Roughness of exponential dichotomies). Let $T_B(x, y)$ be the principal matrix solution for the following linear system*

$$u'(x) = \big(A(x) + B(x)\big)u(x). \tag{5.6}$$

Assume that the matrix $B(x)$ is piecewise continuous and uniformly bounded with $\delta = \sup\{|B(x)|,\ x \in I\} < \infty$. For any given $0 < \tilde{\alpha} < \alpha_0$, assume that δ is sufficiently small so that

$$C_1\delta < 1, \quad \textit{and} \quad C_2\delta < 1 \quad \textit{where } C_1 = \frac{2K_0}{\alpha_0 - \tilde{\alpha}},\ C_2 = \frac{2K_0^2}{(\alpha_0 - \tilde{\alpha})(1 - C_1\delta)}.$$

Then (5.6) also has an exponential dichotomy on I with projections $\tilde{P}_s(x)$, $\tilde{P}_u(x)$, the constant $\tilde{K}$ and the exponent $\tilde{\alpha}$. Moreover $\tilde{K} = K_0(1 - C_1\delta)^{-1}(1 - C_2\delta)^{-1}$ and

$$\|T_B(x,y)\tilde{P}_s(y)\| \leqslant \tilde{K}e^{-\tilde{\alpha}(x-y)}, \quad y \leqslant x,$$
$$\|T_B(x,y)\tilde{P}_s(y)\| \leqslant \tilde{K}e^{-\tilde{\alpha}(y-x)}, \quad x \leqslant y,$$
$$\|\tilde{P}_s(x) - P_s(x)\| \leqslant \frac{C_2\delta}{1-C_2\delta}.$$

The proof of Theorem 5.1 is in [3,16]. For a shorter proof with almost identical notations, see [18] (simply replace the rate function $a(x)$ by e^x and the decay rate $(a(x)/a(y))^{-\alpha}$ be $e^{-\alpha(x-y)}$).

To treat the linear system (5.3) that depends on the parameter s, following [17], we introduce an s-dependent weighted norm to vectors in $\mathbb{R}^2$ as follows.

Definition 5.1. Consider the Banach space $E^{0.5}(s)$ of vectors $(u,v) \in \mathbb{C}^2$ with the following weighted norms:

$$\|(u,v)^\tau\|_{E^{0.5}(s)} = (1+|s|^{0.5})|u(s)| + |v(s)|,$$

where $|u|$ and $|v|$ are the Euclidean norms.

The principle matrix solution $T(\xi,\eta,s)$ with parameter s of the linear system

$$U_\xi = V, \qquad V_\xi = (sI + A(\xi))U + B(\xi)V \tag{5.7}$$

can be viewed as a linear flow in $E^{0.5}(s)$. Without any change, the results of Theorem 5.1 still hold in $E^{0.5}(s)$.

Definition 5.2. We say that the linear system (5.7), or the flow $T(\xi,\eta,s)$, has an exponential dichotomy in $E^{0.5}(s)$ for $s \in \mathbb{C}^+$ and on the interval $\xi \in J$, if there exist projections $P_s(\xi,s) + P_u(\xi,s) = I$ in $E^{0.5}(s)$, continuous in ξ and uniformly bounded with respect to $s \in \mathbb{C}^+$, such that the property (1) of the following holds. Moreover there exist constant $K > 0$ and exponent $\alpha(1+|s|^{0.5}) > 0$ for $s \in \mathbb{C}^+$, such that the property (2) of the following holds.

$$(1) \quad \begin{cases} T(\xi,\eta,s)P_s(\eta,s) = P_s(\xi,s)T(\xi,\eta,s), & \xi \geqslant \eta \in J, \\ T(\xi,\eta,s)P_u(\eta,s) = P_u(\xi,s)T(\xi,\eta,s), & \xi \leqslant \eta \in J; \end{cases}$$

$$(2) \quad \begin{cases} |T(\xi,\eta,s)P_s(\eta,s)|_{E^{0.5}(s)} \leqslant Ke^{-\alpha(1+|s|^{0.5})(\xi-\eta)}, & \xi \geqslant \eta, \\ |T(\xi,\eta,s)P_u(\eta,s)|_{E^{0.5}(s)} \leqslant Ke^{-\alpha(1+|s|^{0.5})|\xi-\eta|}, & \xi \leqslant \eta. \end{cases}$$

We now consider the existence of exponential dichotomies for the linear systems (5.4) for $\xi \in \mathbb{R}$ and (5.3) for $\xi \in [a,b]$.

Lemma 5.2. *Assume that $u_0(\xi)$ is the whole line standing wave connecting $u = 0$ to $u = 1$. Let $\delta_1 = \sup_\xi |Df(u_0(\xi))|$. Assume that $N > 0$ is a large constant and*

$$\delta_2 = \max\{\sup\{|Df(u_0(\xi)) - Df(0)|: \xi \leqslant -N\}, \sup\{|Df(u_0(\xi)) - Df(1)|: \xi \geqslant N\}\}. \tag{5.8}$$

If δ_2 is sufficiently small, then in the space $E^{0.5}(s)$, system (5.4), with $s \in \mathbb{C}^+$, has an exponential dichotomy on $\mathbb{R}$. The projections $P_s(\xi,s)$ and $P_u(\xi,s)$ are uniformly bounded by $K > 0$ that is independent of s. The exponent is $\alpha(1+|s|^{0.5})$ for some $\alpha > 0$.

Moreover, let $P_s(E_1,s)$ and $P_u(E_2,s)$ be the spectral projections at E_1 or E_2. There is a large constant $M > 0$ such that at $\xi = a$ or $\xi = b$, depending on $|s| \geqslant M$ or $|s| < M$, we have

$$\begin{aligned} \left\| P_s(\xi, s) - P_s(E_1, s) \right\| &\leqslant \frac{16K^2\delta_j}{\alpha(1+|s|^{0.5})}, \quad \xi \leqslant -N, \\ \left\| P_u(\xi, s) - P_u(E_2, s) \right\| &\leqslant \frac{16K^2\delta_j}{\alpha(1+|s|^{0.5})}, \quad \xi \geqslant N, \end{aligned} \tag{5.9}$$

where $j = 1$ *for* $|s| \geqslant M$, *and* $j = 2$ *for* $|s| \leqslant M$.

Proof. *Case* 1: *Exponential dichotomies for* $|s| \geqslant M$. Let $M > 0$ be a sufficiently large constant. In the region $\{|s| \geqslant M\} \cap \{s \in \mathbb{C}^+\}$, we treat (5.4) as perturbations to the system

$$U_\xi = V, \qquad V_\xi = (\gamma^2 + s)U.$$

From [17], the system above has an exponential dichotomy in $E^{0.5}(s)$ with the constants K_0 and the exponent $\alpha_0 = \alpha(1 + |s|^{0.5})$.

Although we cannot make $\delta_1 = \sup_\xi |Df(u_0(\xi))|$ small, but the conditions $C_1\delta_1 < 1$ and $C_2\delta_1 < 1$ in Theorem 5.1 can be satisfied if we choose $\tilde{\alpha} = \alpha(1+|s|^{0.5})/2$. Then from $\alpha_0 = \alpha(1+|s|^{0.5})$, $\alpha_0 - \tilde{\alpha} = \alpha(1+|s|^{0.5})/2$ can be large from the condition $|s| \geqslant M$ for a large constant M. If M is sufficiently large then

$$C_1\delta_1 = \frac{4k\delta_1}{\alpha(1+|s|^{0.5})} \leqslant \frac{1}{2}, \qquad C_2\delta_1 \leqslant \frac{8K^2\delta_1}{\alpha(1+|s|^{0.5})} \leqslant \frac{1}{2}.$$

From Theorem 5.1, if $M > 0$ is sufficiently large then system (5.3) has exponential dichotomies in $E^{0.5}(s)$ with the constant $\tilde{K}$ independent of s. The exponent of the dichotomy is $\tilde{\alpha} = \frac{\alpha}{2}(1 + |s|^{0.5})$. The projections satisfy (5.9) with $j = 1$.

Case 2: *Exponential dichotomies for* $|s| \leqslant M$. After $M > 0$ has been determined, we consider the spectral equation in the compact set $\{|s| \leqslant M\} \cap \{s \in \mathbb{C}^+\}$. Assume that in $I_- = (-\infty, -N]$ and $I_+ = [N, \infty)$, $q(\xi)$ is close to E_1 and E_2 respectively. We now replace $Df(u_0(\xi))$ by $Df(0)$ or $Df(1)$ in I_- or I_+. The eigenvalues for the constant system are $\mu^\pm_{1,2}(s)$ as in (5.5).

It is straightforward to show that

$$\begin{aligned} Re\,\mu_1^-(s) \leqslant \mu_1^-(0) < 0 < \mu_2^-(0) \leqslant Re\,\mu_2^-(s), \\ Re\,\mu_1^+(s) \leqslant \mu_1^+(0) < 0 < \mu_2^+(0) \leqslant Re\,\mu_2^+(s). \end{aligned}$$

The two systems with constant coefficients $Df(0)$ and $Df(1)$ have exponential dichotomies on $\mathbb{R}$ with the common exponent $\alpha_0 = Re\,\mu_2^-(s)$, since $Re\,\mu_2^-(s) < Re\,\mu_2^+(s)$. Also the projections depend continuously on s. So in the compact set of s, we can assume that K is independent of s.

Now in I_- or I_+, $u_0(\xi)$ is close to $u = 0$ or $u = 1$, (5.4) is a perturbation to the linear variational system around $u = 0$, $u = 1$ with constant coefficients. If δ_2 as in (5.8) is sufficiently small, then a standard perturbation theory shows that system (5.4) has exponential dichotomies in I_- and I_+. In fact, the unstable (stable) subspace in I_- (or I_+) is uniquely defined while the stable (unstable) subspace in I_- (or I_+) is not unique. For definiteness, assume that $RP_s(-N, s)$ and $RP_u(N, s)$ are the same as the stable eigenspace at $u = 0$ and unstable eigenspace at $u = 1$ of the linear constant system respectively. Then the dichotomies thus obtained are also analytic in s. The dichotomies can be extended to $\xi \in [a, 0]$ and $[0, b]$ by the linear flow. The extended dichotomies have the same exponents but the constant K may be larger.

Since $\{|s| \leqslant M\} \cap \{s \in \mathbb{C}^+\}$ is a compact set, without loss of generality we assume that for system (5.4), the constant of the dichotomy is independent of s and the exponent is $\alpha_1(1 + |s|^{0.5})$ where α_1 is independent of s.

From the proof presented above, if $|s| \geqslant M$, we actually have a unified exponential dichotomy on $\mathbb{R}$, i.e., $P_u(s, 0-) = P_u(s, 0+)$, $P_s(s, 0-) = P_s(s, 0+)$. For $|s| \leqslant M$, the exponential dichotomies related to system (5.4) for $\xi \in (-\infty, 0]$ and $[0, \infty)$ have the following property: $RP_u(s, 0-)$ and $RP_s(s, 0+)$ are linearly independent. This is from the fact that $s \in \mathbb{C}^+$ is not an eigenvalue for the linear variational system around $u_0(\xi)$, due to the assumption that the standing wave solution $u_0(\xi)$ is stable. The linear independence of the subspace allows us to redefine a unified dichotomy for all $\xi \in \mathbb{R}$ if $|s| \leqslant M$, just as in Lemmas 3.1 and 3.2.

If we combine the two cases, unified projections can be defined and are continuous on $\mathbb{R}$. Now select the larger of the two constants K, and reset $\alpha = \min\{\tilde{\alpha}/2, \alpha_1\}$, then system (5.4) has an exponential dichotomy in $E^{0.5}(s)$ for $\xi \in \mathbb{R}$. The constant K is independent of s and the exponent is $\alpha(1 + |s|^{0.5})$. This completes the proof of the lemma. □

Similar results for system (5.3) are stated in the following lemma:

Lemma 5.3. *Assume that $\tilde{u}(\xi)$ is a standing wave solution that is near the whole line standing wave solution $u_0(\xi)$ for all $\xi \in [a, b]$. Let*

$$\delta_3 = \sup\bigl\{\bigl|Df\bigl(\tilde{u}(\xi)\bigr) - Df\bigl(u_0(\xi)\bigr)\bigr|:\ \xi \in [a, b]\bigr\}. \tag{5.10}$$

If δ_3 is sufficiently small, then in the space $E^{0.5}(s)$, system (5.3) has an exponential dichotomy on $[a, b]$. Let the projections of the dichotomy related to $u_0(\xi)$ be denoted by $P_s^0(\xi, s)$ and $P_u^0(\xi, s)$; and the projections of the dichotomy related to $\tilde{u}(\xi)$ be denoted by $P_s(\xi, s)$ and $P_u(\xi, s)$. If the following additional conditions are satisfied:

$$RP_s(\xi, s) = T(\xi, b, s)RP_s^0(b, s),$$
$$RP_u(\xi, s) = T(\xi, b, s)RP_u^0(a, s),$$

then the perturbed system has a unique exponential dichotomy on $[a, b]$. The projections $P_s(\xi, s)$ and $P_u(\xi, s)$ are uniformly bounded by $K > 0$ that is independent of s. The exponent is $\alpha(1 + |s|^{0.5})$ for some $\alpha > 0$.

Moreover, let $P_s(E_1, s)$ and $P_u(E_2, s)$ be the spectral projections at E_1 or E_2. There is a large constant $M > 0$ such that at $\xi = a$ or $\xi = b$, depending on $|s| \geqslant M$ or $|s| < M$, we have

$$\bigl\|P_s(a, s) - P_s(E_1, s)\bigr\| \leqslant \frac{32K^2\delta_j}{\alpha(1 + |s|^{0.5})}, \qquad \bigl\|P_u(b, s) - P_u(E_2, s)\bigr\| \leqslant \frac{32K^2\delta_j}{\alpha(1 + |s|^{0.5})}. \tag{5.11}$$

The constant δ_j in the above is determined as follows: For $|s| \geqslant M$, $\delta_j = \sup_\xi |Df(\tilde{u}(\xi))|$; and for $|s| \leqslant M$, $\delta_j = \delta_4$ where $\delta_4 = \delta_2 + \delta_3$ with δ_2 defined in (5.8).

Proof. The existence of $M > 0$ and the case $|s| \geqslant M$ is proved exactly as in Lemma 5.2.

For the proof of the case $|s| \leqslant M$, if the exponential dichotomy around $u_0(\xi)$ has the exponential $\alpha(1 + |s|^{0.5})$. Then to system (5.3), we let the exponent be $(\alpha/2)(1 + |s|^{0.5})$. If δ_3 is sufficiently small, then Theorem 5.1 can be applied to treat (5.3) as a perturbation of (5.4) for $\xi \in [a, b]$. □

Corollary 5.4. *The projections $P_s(\xi, s)$ and $P_u(\xi, s)$ depend analytically on $s \in \mathbb{C}^+$.*

Proof. It is well known that the uniform limit of a sequence of analytic functions is analytic. The spectral projections for the two systems with constant coefficients $Df(0)$ and $Df(1)$ around $u = 0, 1$ are analytic functions of $s \in \mathbb{C}^+$. The existence of the projections of the exponential dichotomies for systems (5.3) and (5.4) is obtained by a perturbation method. Iteration procedures or the contraction mapping principles are used to find the projection matrices. Therefore, if the equations depend analytically in s, then the projections also depend analytically in s. □

Remark 5.1. The exponents $\alpha(1+|s|^{0.5})$ obtained in Lemma 5.2 and Lemma 5.3 are not the best and can be improved. However, the comparison of projections as in (5.9) and (5.11) is more important in the rest of the paper.

Recall that from (H3), $\mu_1^-(0) < h_a \leqslant \infty$ and $-\infty \leqslant h_b < \mu_2^+(0)$.

Assume that P_1 and P_2 are close to E_1 and E_2 respectively and $|a|$ and b are sufficiently large. For the boundary value problem (5.3), TM_a and TM_b transversely intersect with $RP_s(a,s)$ and $RP_u(b,s)$ respectively. Let $\phi_s \in RP_s(a,s)$, $\phi_u(b,s) \in RP_u(b,s)$. Recall that $\mathbf{n}_a$, $\mathbf{n}_b$ are normal vectors to the boundary manifolds as in (4.1). Then for each $s \in \mathbb{C}^+$, $\phi_s \to \phi_s \cdot \mathbf{n}_a$ and $\phi_u \to \phi_u \cdot \mathbf{n}_b$ are isomorphisms. There exist bounded inverse operators $\mathcal{R}_s(s)$ and $\mathcal{R}_u(s)$ for each $s \in \mathbb{C}^+$ such that:

$$\phi_s = \mathcal{R}_s(s)(\phi_s \cdot \mathbf{n}_a), \qquad \phi_u = \mathcal{R}_u(s)(\phi_u \cdot \mathbf{n}_b). \tag{5.12}$$

Lemma 5.5. *The operators $\mathcal{R}_s(s): \mathbb{R} \to RP_s(a,s)$, $\mathcal{R}_u(s): \mathbb{R} \to RP_u(b,s)$ are bounded operators for each $s \in \mathbb{C}^+$ in the $E^{0.5}(s)$ norm. For the non-Dirichlet boundary conditions, there exists a constant C that only depends on h_a, h_b but is independent of $s \in \mathbb{C}^+$ such that*

$$\|\phi_s\|_{E^{0.5}(s)} \leqslant C\|\phi_s \cdot \mathbf{n}_a\|, \qquad \|\phi_u\|_{E^{0.5}(s)} \leqslant C\|\phi_u \cdot \mathbf{n}_b\|.$$

For the Dirichlet boundary conditions

$$\|\phi_s\|_{E^{0.5}(s)} \leqslant C\big(1+|s|^{0.5}\big)|\phi_s \cdot \mathbf{n}_a|, \qquad \|\phi_u\|_{E^{0.5}(s)} \leqslant C\big(1+|s|^{0.5}\big)|\phi_u \cdot \mathbf{n}_b|.$$

Proof. Assume $k_a \neq \infty$ first. Dirichlet boundary conditions will be treated later.

Case 1: *Non-Dirichlet boundary conditions.* First consider $\{s \in \mathbb{C}^+\} \cap \{|s| \geqslant M\}$ where $M > 0$ is a large constant. At $\xi = a$, the vector $\phi_s = (u,v)$ is close to the eigenvector $Y_1^-(s) = (1, \lambda_1^-(s))$ and $n_a = (k_a, -1)/\sqrt{k_a^2+1}$. Then

$$\big|Y_1^-(s) \cdot \mathbf{n}_a\big| = \big|k_a - \lambda_1^-(s)\big|/\sqrt{k_a^2+1}.$$

If $M > 0$ is sufficiently large then $|k_a - \lambda_1^-(s)| \geqslant c_1(1+|s|^{0.5})$ for $|s| \geqslant M$. Also $|Y_1^-(s)|_{E^{0.5}(s)} \leqslant c_2(1+|s|^{0.5})$ where the constants $c_1, c_2 > 0$ are independent of s. Therefore

$$\big|Y_1^-(s) \cdot \mathbf{n}_a\big| \geqslant \frac{c_1}{c_2\sqrt{k_a^2+1}}\big|Y_1^-(s)\big|_{E^{0.5}(s)}, \quad \text{for } |s| \geqslant M. \tag{5.13}$$

Next consider s in the compact set $\{|s| \leqslant M\} \cap \{s \in \mathbb{C}^+\}$. It is straightforward to verify that $|k_a - \lambda_1^-(s)| \geqslant c_3$ and $|Y_1^-(s)|_{E^{0.5}(s)} \leqslant c_4$ in such set. Therefore

$$\big|Y_1^-(s) \cdot \mathbf{n}_a\big| \geqslant \frac{c_3}{c_4\sqrt{k_a^2+1}}\big|Y_1^-(s)\big|_{E^{0.5}(s)}, \quad \text{for } |s| \leqslant M. \tag{5.14}$$

From (5.13), (5.14), we have

$$\big|Y_1^-(s) \cdot \mathbf{n}_a\big| \geqslant C\big|Y_1^-(s)\big|_{E^{0.5}(s)}, \quad \text{for } s \in \mathbb{C}^+. \tag{5.15}$$

We now choose $\phi_s(s) = P(a,s)Y_1^-(s) \in RP_s(a,s)$. It is a small perturbation of $Y_1^-(s)$ since

$$\big|\phi_s(s) - Y_1^-(s)\big| = \big\|\big(P(a,s) - P(E_1,s)\big)\big\|\big|Y_1^-(s)\big| \leqslant \frac{16K^2\delta}{\alpha(1+|s|^{0.5})}\big|Y_1^-(s)\big|.$$

The fraction $16K^2\delta/\alpha(1+|s|^{0.5})$ can be made arbitrarily small in two steps as described in Theorem 5.1.

So from (5.15), we have

$$\left|\phi_s(s)\cdot\mathbf{n}_a\right| \geqslant C\left|\phi_s(s)\right|_{E^{0.5}(s)}, \quad \text{for } s\in\mathbb{C}^+.$$

Therefore the inverse operator satisfies the property $\|\mathcal{R}_s(s)\| \leqslant C$ for some constant $C>0$ that is independent of s.

Case 2: *Dirichlet boundary condition.* Next we consider the Dirichlet boundary condition where $k_a=\infty$. In this case $\mathbf{n}_a=(1,0)$ and $|Y_1^-(s)\cdot\mathbf{n}_a|=1$. In the unbounded region $|s|\geqslant M$,

$$\left|Y_1^-(s)\cdot\mathbf{n}_a\right| \geqslant \frac{1}{c_2(1+|s|^{0.5})}\left|Y_1^-(s)\right|_{E^{0.5}(s)}.$$

In the bounded region $|s|\leqslant M$ we have the same estimate with $1/c_2$ replaced by a constant C that is independent of s.

Recall that ϕ_s is a small perturbation of $Y_1^-(s)$. Therefore for the Dirichlet boundary condition at $\xi=a$, we have

$$\left|\phi_s(s)\cdot\mathbf{n}_a\right| \geqslant \frac{C}{1+|s|^{0.5}}\left|\phi_s(s)\right|_{E^{0.5}(s)}, \quad \text{for } s\in\mathbb{C}^+.$$

The inverse operator satisfies the property $\|\mathcal{R}_s(s)\| \leqslant C(1+|s|^{0.5})$ for some constant $C>0$ that is independent of s.

The proof for the boundedness of $\mathcal{R}_u(s)$ in the $E^{0.5}(s)$ norm is similar. □

Theorem 5.6. *Assume that P_1, P_2 are sufficiently close to E_1, E_2 respectively, and $|a|$, b are sufficiently large so that the estimates for the inverse operators $\mathcal{R}_s(s):\mathbb{R}\to RP_s(a,s)$, $\mathcal{R}_u(s):\mathbb{R}\to RP_u(b,s)$ as in Lemma 5.5 hold. Then under these conditions the standing wave $\tilde{u}$ is stable if we assume that $b-a$ is sufficiently large.*

Proof. The proof follows closely to the existence and uniqueness of the standing wave solutions. We show that for any $s\in\mathbb{C}^+$, $(U,V)=0$ is the unique solution for the eigenvalue/eigenfunction equation (5.3). Therefore s is not an eigenvalue.

Using the exponential dichotomy, for $a\leqslant\xi\leqslant b$, we can express the solution of (5.3) as

$$\mathbf{U}(\xi,s)=T(\xi,a)P_s(a,s)\mathbf{U}(a,s)+T(\xi,b,s)P_u(b,s)\mathbf{U}(b,s).$$

The solution is uniquely determined by the unknown vectors

$$\phi_s := P_s(a,s)\mathbf{U}(a,s)\in RP_s(a,s), \qquad \phi_u := P_u(b,s)\mathbf{U}(b,s)\in RP_u(b,s).$$

To satisfy the boundary condition, we require that

$$\left(\phi_s+T(a,b,s)\phi_u\right)\cdot\mathbf{n}_a=0, \qquad \left(\phi_u+T(b,a,s)\phi_s\right)\cdot\mathbf{n}_b=0.$$

We look for the vector (ϕ_s,ϕ_u) which is a fix point to the following system:

$$\begin{aligned}\phi_s' &= -\mathcal{R}_s(s)\left[\left(T(a,b,s)\phi_u\right)\cdot\mathbf{n}_a\right],\\ \phi_u' &= -\mathcal{R}_u(s)\left[\left(T(b,a,s)\phi_s\right)\cdot\mathbf{n}_b\right].\end{aligned} \tag{5.16}$$

From Lemma 5.3 and Lemma 5.5,

$$\|\phi_s'\| + \|\phi_u'\| \leqslant C\big(1+|s|^{0.5}\big)e^{-\alpha(1+|s|^{0.5})(b-a)}\big(\|\phi_s\| + \|\phi_u\|\big).$$

If $b-a$ is sufficiently large then the system of Eqs. (5.16) defines a mapping from $RP_s(a,s) \times RP_u(b,s)$ to itself: $(\phi_s,\phi_u) \to (\phi_s',\phi_u')$, and is a contraction mapping in the space $E^{0.5}(s) \times E^{0.5}(s)$. Therefore (5.16) has a unique solution $(\phi_s,\phi_u) = (0,0)$ for any $s \in \mathbb{C}^+$. Thus corresponding to any $s \in \mathbb{C}^+$, the only solution to the eigenvalue problem is $U(\xi,s) = 0$. Hence any $s \in \mathbb{C}^+$ is not an eigenvalue for the linearized equation with boundary conditions. □

6. Generalized Lambda Lemma and geometric method to the existence and stability problems

A useful tool to show the existence of the standing wave $\tilde{u}$ is the graph transformations commonly known as the Inclination Lemma or the Lambda Lemma [10,2,4]. However, the classical Lambda Lemma does not apply to any neighborhood of E_1 which is not a saddle point. In this section we will present a generalized Lambda Lemma that works in the neighborhood of E_1, and use it to give an alternative proof of the existence of standing waves in large bounded domain with some boundary conditions.

Following [10], we define the so-called u-slice that is transverse to the relatively stable subspace of the dichotomy at $q(\xi)$. Any point in a neighborhood of $q(\xi)$, $\xi \in \mathbb{R}$, can be expressed as $(u,v) = q(\xi) + (\phi_s,\phi_u)$ where $\phi_s \in RP_s(\xi)$, $\phi_u \in RP_u(\xi)$. This servers as a local 2D coordinate system near $q(\xi)$.

Definition 6.1. A C^1 submanifold M_u is said to be a u-slice passing through $q(\xi)$ and of size (ϵ_1, K_1) if

$$M_u := \big\{q(\xi) + (\phi_s,\phi_u)\colon\ \phi_s = h_1(\phi_u),\ |\phi_u|_{w(\xi)} \leqslant \epsilon_1,\ h_1(0) = 0,\ h_1 \in C^1,\ |Dh_1| \leqslant K_1\big\}.$$

Let the flow of (1.3) be $\Phi(\xi,\eta)$. We have the following "Generalized λ Lemma":

Lemma 6.1. *Let $M_u := \{\phi_s = h_1(\phi_u)\}$ be a u-slice passing through the point $q(a)$ and of size (ϵ_1, K_1). Then there exists a constant $\tilde{\epsilon}_1(K_1) > 0$ which depends on K_1 such that the following results hold*:

If $0 < \epsilon_1 \leqslant \tilde{\epsilon}_1$ and if $a < 0$ is sufficiently large, then $\Phi(0,a)M_u$ is a u-slice. The size of the domain of $\Phi(0,a)M_u$ is greater than ϵ_1 in the weighted norm $\|\cdot\|_{w(\xi)}$, and the truncated image

$$\bar{M}_u = \Phi(0,a)M_u \cap \big\{|\phi_u|_{w(0)} \leqslant \epsilon_1\big\},$$

is a u-slice passing through $q(0)$, and of size $(\epsilon_1, CK_1e^{(\beta^- - \alpha^-)a} + C\epsilon_1)$.

Proof. The idea of the proof follows from that of [10] which was adapted from the proof of the Lambda Lemma in [9], although both papers dealt with discrete dynamical systems.

We first show that there is a function $\bar{h}_1 : RP_u(0) \to RP_s(0)$ such that $\Phi(0,a)M_u$ is the graph of the function $\phi_s(0) = \bar{h}_1(\phi_u(0))$.

For any small vector in the 1D subspace: $\phi_u(0) \in RP_u(0)$ with $|\phi_u(0)|_{w(0)} \leqslant \epsilon_1$, we look for the corresponding $\phi_s(0) \in RP_s(0)$ such that $(\phi_s(0),\phi_u(0)) \in \bar{M}_u$. Since $\tilde{M}_0 := \{\mathbf{U} \mid P_u(0)\mathbf{U} = \phi_u(0)\}$ is a manifold transverse to $RP_u(0)$. Just like the proof of Theorem 4.3, with M_u and $\tilde{M}_0$ as two boundary manifolds, we can prove that there exists a unique solution $\mathbf{U}(\xi)$ to the nonlinear BVP which will uniquely determine $\phi_s(0)$. The function $\phi_s(0) = \bar{h}_1(\phi_u(0))$ is defined by changing $\phi_u(0)$ continuously in the ϵ_1-neighborhood of 0 and calculating the corresponding $\phi_s(0)$.

As in (4.8), the solution of the nonlinear BVP $\mathbf{U}(\xi)$ can be expressed by $\phi_u(0)$ and an undetermined $\phi_s(a) := P_s(a)\mathbf{U}(a)$ as

$$\mathbf{U} = \mathcal{F}_2\big(\phi_s(a),\phi_u(0)\big) \quad \text{with } |U|_w \leqslant C\big(\big\|\phi_s(a)\big\|_{w(a)} + \big\|\phi_u(0)\big\|_{w(0)}\big). \tag{6.1}$$

Then $\phi_u(a) := P_u(a)\mathbf{U}(0)$ can be written as

$$\phi_u(a) = T(a,0)\phi_u(0) + \int_0^a T(a,\zeta)P_u(\zeta)G(\zeta)\,d\zeta,$$

where $G(\zeta) = \mathcal{N}(\mathcal{F}_2(\phi_s(a), \phi_u(0)))(\zeta)$ where $\mathcal{N}$ is the higher order term after linearization. Due to the nonlinear term $\mathcal{N}(U)$, the integral term is bounded by $C(\|\phi_s(a)\|^2_{w(a)} + \|\phi_u(0)\|^2_{w(0)})$. At $\xi = a$,

$$\phi_s(a) = h_1\big(\phi_u(a)\big) = h_1\left(T(a,0)\phi_u(0) + \int_0^a T(a,\zeta)P_u(\zeta)G(\zeta)\,d\zeta\right).$$

In the above equation, $\phi_s(a)$ also appears in the right hand side through the small integral term. By the contraction mapping principle one can uniquely find $\phi_s(a)$ as a function of $\phi_u(0)$:

$$\phi_s(a) = \mathcal{H}\big(\phi_u(0)\big), \quad \text{with } \big|\phi_s(a)\big|_{w(a)} \leqslant |Dh_1|\big(\big|T(a,0)P_u(0)\big|\big|\phi_u(0)\big|_{w(0)} + C\big|\phi_u(0)\big|^2_{w(0)}\big).$$

Using this $\phi_s(a)$ to calculate $\phi_s(0)$,

$$\phi_s(0) = P_s(0)\mathbf{U}(0) = T(0,a)\phi_s(a) + \int_a^0 T(0,\zeta)P_s(\zeta)G(\zeta)\,d\zeta.$$

Thus we have an explicit form of $\bar{h}_1 : \phi_u(0) \to \phi_s(0)$:

$$\begin{aligned}
&\phi_s(0) = T(0,a)h_1\big(T(a,0)\phi_u(0) + I_u(a)\big) + I_s(0),\\
&\quad \text{where } I_u(a) = \int_0^a T(a,\zeta)P_u(\zeta)\mathcal{N}\big(\mathcal{F}_2\big(\mathcal{H}\big(\phi_u(0)\big), \phi_u(0)\big)\big)(\zeta)\,d\zeta,\\
&\quad\quad I_s(0) = \int_a^0 T(0,\zeta)P_s(\zeta)\mathcal{N}\big(\mathcal{F}_2\big(\mathcal{H}\big(\phi_u(0)\big), \phi_u(0)\big)\big)(\zeta)\,d\zeta.
\end{aligned} \tag{6.2}$$

The integral terms $I_u(0)$ and $I_s(a)$ are small terms as proved in Section 3.1. Using the estimates

$$\big|I_u(a)\big|_{w(a)} + \big|I_s(0)\big|_{w(0)} \leqslant C\big|\phi_u(0)\big|^2_{w(0)},$$

we obtain the following from (6.2):

$$\begin{aligned}
\big|\phi_s(0)\big|_{w(0)} &\leqslant \big|T(0,a)P_s(a)\big||Dh_1|\big(\big|T(a,0)P_u(0)\big|\big|\phi_u(0)\big|_{w(0)} + \big|I_u(a)\big|_{w(a)}\big) + \big|I_s(0)\big|_{w(0)}\\
&\leqslant Ce^{(\beta^- - \alpha^-)a}|Dh_1|\big|\phi_u(0)\big|_{w(0)} + C\big|\phi_u(0)\big|^2_{w(0)}\\
&\leqslant Ce^{(\beta^- - \alpha^-)a}K_1\epsilon_1 + C\epsilon^2.
\end{aligned}$$

From (6.2), we also have

$$\frac{\partial\phi_s(0)}{\partial\phi_u(0)} = T(0,a)P_s(a)Dh_1\left[T(a,0)P_u(0) + \frac{\partial I_u(a)}{\partial\phi_u(0)}\right] + \frac{\partial I_s(0)}{\partial\phi_u(0)},$$

$$\frac{\partial I_u(a)}{\partial\phi_u(0)} = \int_0^a T(a,\zeta)P_u(\zeta)\frac{\partial\mathcal{N}}{\partial U}\left(\frac{\partial\mathcal{F}_2}{\partial\phi_u(0)} + \frac{\partial\mathcal{F}_2}{\partial\phi_s(u)}\frac{\partial\mathcal{H}}{\partial\phi_u(0)}\right)d\zeta.$$

Therefore,

$$\left|\frac{\partial I_u(a)}{\partial\phi_u(0)}\right| \leqslant C\left|\phi_u(0)\right|_{w(0)}.$$

Similarly we can show that

$$\left|\frac{\partial I_s(0)}{\partial\phi_u(0)}\right| \leqslant \left|\phi_u(0)\right|_{w(0)}.$$

We finally have obtain that

$$|D\bar{h}_1| = \left|\frac{\partial\phi_s(0)}{\partial\phi_u(0)}\right| \leqslant Ce^{(\beta^- - \alpha^-)a}|Dh_1| + C\left|\phi_u(0)\right|_{w(0)}$$
$$\leqslant CK_1e^{(\beta^- - \alpha^-)a} + C\epsilon_1. \qquad \square$$

Remark 6.1. In the unweighted norm, the domain of the image of a u-slice is also expanding for $\xi > a$, so a simpler graph transformation lemma on the u-slice using unweighted norm can be proved. However, using the weighted norm allows us to develop a comprehensive theory on the graph transforms near the non-saddle point E_1. In the neighborhood of $q(0)$, we can define the C^1 local submanifold that is transverse to $RP_u(\xi)$ at $\xi = 0$, called the s-slice [10]. Without using the weighted norm, the size of an s-slice shrinks under the backward flow when $\xi \to -\infty$. Using the weighted norms $\|\cdot\|_{w(\xi)}$ in the definition of the s-slice, the backward image of an s-slice appears to be expanding so after truncation it is of the same size in the weighted norm. Although the weighted norm blows up the neighborhood of $q(a)$, it can be shown that as $\xi \to a$, the s-slice C^1 approaches $RP_s(\xi)$ in the weighted norm.

The proof of the part of the Lambda Lemma involving s-slice is more complicated and will not be given in this paper.

We now present a geometric proof of the existence of the standing wave $\tilde{u}$ using the generalized Lambda Lemma (Lemma 6.1).

Recall that $\Phi(\xi,\eta)$ is the flow of (1.3). We will take the forward mapping of M_a from a to $\xi = 0$, the image will be denoted $M'_a = \Phi(0,a)M_a$; and take backwards mapping of M_b from b to $\xi = 0$, the image will be called $M'_b = \Phi(0,b)M_b$. If the two manifolds M'_a and M'_b intersect transversely at $\xi = 0$, then a unique solution $\tilde{q}(\xi)$ near the heteroclinic orbit $q(\xi)$ is determined. See Fig. 6.1.

Because the initial manifold M_a transversely intersects with $RP_s(a)$. The initial manifold can be expressed as $\phi_s(a) = h_1(\phi_u(a))$ where $h_1 : RP_u(a) \to RP_s(a)$ is a mapping from a 1D linear space to another 1D linear space.

From Lemma 6.1, the Generalized λ Lemma, $M'_a = \Phi(0,a)M_a$ can be expressed as $\phi_s(0) = \bar{h}_1(\phi_u(0))$ where $D\bar{h}_1 \to 0$ as $\epsilon_1 \to 0$ and $a \to -\infty$.

The linear variational equation around the heteroclinic orbit has an exponential dichotomy for $\xi \geqslant 0$ and has a pseudo exponential dichotomy for $\xi \leqslant 0$. Using the regular Lambda Lemma [9],

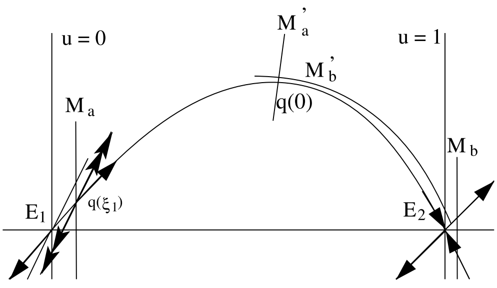

Fig. 6.1. The transverse intersection of M_a' and M_b' uniquely determines the solution of the boundary value problem.

$M_b' = \Phi(0,b)M_b$ is C^1 exponentially close to the stable subspace $RP_s(0)$ of the dichotomy at $0+$. Then one can choose C^1 distance to be small if b is large. From the generalized Lambda Lemma (Lemma 6.1) which shows that $M_a' = \Phi(0,a)M_a$ is C^1 exponentially close to the strong unstable subspace of the dichotomy at $0-$ if $Ce^{(\beta^- - \alpha^-)a} + C\epsilon_1 \ll 1$.

It is known that $RP_u(0)$ and $RP_s(0)$ are linearly independent. Then the strong unstable subspace at $0-$ is transverse to the stable subspace at $0+$. In this example, the stable subspace at $0+$ is the tangent vector of $q'(0)$ while the strong unstable subspace is transverse to $q'(0)$ at $0-$.

On the other hand, if $a < 0$ is sufficiently large, then $M_a' = \Phi(0,a)M_a$ will pass through $q(0)$ and its tangent space TM_a' is close to $RP_u(0)$ which is transverse to $q'(0)$. This shows that M_a' will have a unique nonempty intersection with M_b' at near $q(0)$. The intersection determines $\tilde{u}(0)$, and hence the solution $\tilde{u}(\xi)$ for all $\xi \in J$.

In the rest of this section, we illustrate the usefulness of the graph transformation method. We shall use the linear version of the Lambda Lemma to show that the conditions $k_a > \lambda_1^-$, $k_b < \lambda_2^+$ that define the boundary conditions above are almost necessary for the standing wave solution to be stable.

Theorem 6.2.

(I) *If $-\infty < k_a < \lambda_1^-(0)$ and $-\infty \leqslant k_b < \lambda_2^+(0)$, then the standing wave $\tilde{u}$ is unstable.*
(II) *If $\lambda_1^-(0) < k_a \leqslant \infty$ and $\lambda_2^+(0) < k_b < \infty$, then the standing wave $\tilde{u}$ is unstable.*

Proof. We only need to consider the real eigenvalue $s \geqslant 0$.

(I) As shown in Lemma 5.3, for all real $s \geqslant 0$, the linearized system around $\tilde{u}$ has a pseudo exponential dichotomy on $\mathbb{R}^-$ and $\mathbb{R}^+$ respectively. Although the projections can be defined and continuous on the whole real line $\mathbb{R}$, but the exponential rates on $\mathbb{R}^\pm$ are different. Since $\lambda_2^+(s) \geqslant \lambda_2^+(0)$ for all $s \geqslant 0$, we have $k_b < \lambda_2^+(s)$ for all $s \geqslant 0$. Thus TM_b intersects transversely with the unstable subspace at $\xi = b$. By the regular Lambda Lemma on $\mathbb{R}^+$, $T(0,b,s)$ maps TM_b to TM_b' that is C^1 close to the stable subspace $RP_s(0,s)$ at $\xi = 0$. Since TM_b' intersects with $RP_u(0-)$ transversely, by the linear version of the generalized Lambda Lemma on $\mathbb{R}^-$, which is the same as the pseudo exponential dichotomy on $\mathbb{R}^-$, the backwards flow $T(a,0,s)$ maps TM_b' to its image that is close to the stable subspace of the dichotomy at $\xi = a$. However the stable subspace satisfies $V/U = c/2 - \sqrt{c^2/4 - Df(0) + s}$ which approaches $-\infty$ continuously as $s \to \infty$. Since TM_a satisfies $V/U = k_a$ with $-\infty < k_a < \lambda_1^-(0)$. Then for some real $s_0 > 0$ the backward image of TM_b' will be tangent to TM_a. For such s_0 the tangential intersection of the subspaces determines a (nonzero) eigenfunction. Therefore s_0 is an eigenvalue.

(II) For any $s \geqslant 0$, we have $\lambda_1^-(s) < k_a$. Thus M_a transversely intersects with the stable subspace $RP_s(a,s)$ of the dichotomy at a. The forward flow $T(0,a,s)$ will take TM_a to TM_a', which from the linear version of the generalized Lambda Lemma, is close to the strong unstable subspace $RP_u(0,s)$ at $\xi = 0$. Since the exponential dichotomies have unified projections, $RP_u(0-,s) = RP_u(0+,s)$ at $\xi = 0$. So by the regular Lambda Lemma on $\mathbb{R}^+$ and the fact TM_a' transversely intersects with $RP_s(0+,s)$ at

$\xi = 0$, we conclude that $T(b, 0, s)$ maps TM'_a to a linear space that is close to $RP_u(b, s)$ at b. Now assume that s increases continuously from 0 to ∞, then the slope of $RP_u(b, s)$ starting from near $\lambda_2^+(0)$ monotonically increases to near infinite. For some $s_0 > 0$, the slope of $T(b, 0, s)TM'_a$ will be tangent to M_b. For such s_0 there is a nonzero solution which is the eigenfunction for the eigenvalue s_0. $\square$

Acknowledgment

I am grateful to the referee for careful reading of this paper and many useful comments.

References

[1] W.-J. Beyn, J. Lorenz, Stability of traveling waves: dichotomies and eigenvalue conditions on finite intervals, Numer. Funct. Anal. Optim. 20 (1999) 201–244.
[2] S.-N. Chow, X.-B. Lin, Bifurcation of a homoclinic orbit with a saddle-node equilibrium, Differential Integral Equations 3 (1990) 435–466.
[3] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math., vol. 629, Springer-Verlag, 1978.
[4] B. Deng, Homoclinic bifurcations with nonhyperbolic equilibria, SIAM J. Math. Anal. 21 (1990) 693–720.
[5] Ju.L. Daleckii, M.G. Krein, Stability of Solutions of Differential Equations in Banach Space, Transl. Math. Monogr., vol. 43, American Mathematical Society, 1974.
[6] H. Fan, Traveling waves, Riemann problems and computations of a model of the dynamics of liquid/vapor phase transitions, J. Differential Equations 150 (1998) 385–437.
[7] H. Fan, Convergence to traveling waves in two model systems related to the dynamics of liquid-vapor phase changes, J. Differential Equations 168 (2000) 102–128.
[8] R.A. Fisher, The wave of advance of advantageous genes, Ann. Eugenics 7 (1937) 355–369.
[9] J. Guckenheimer, J. Moser, S.E. Newhouse, Dynamical systems: C.I.M.E. lectures, Bressanone, Italy, June 1978.
[10] J. Hale, X.-B. Lin, Heteroclinic orbits for retarded functional differential equations, J. Differential Equations 65 (1986) 175–202.
[11] P. Hartman, On local homeomorphisms of Euclidean spaces, Bol. Soc. Mat. Mexicana 5 (1960) 220–241.
[12] Dan Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, 1980.
[13] A. Kolmogorov, I. Peterovsky, N. Piskunov, Étude de l'équation de la chaleur avec croissance et son application à un problème biologique, Bull. Univ. État Moscou 6 (1937) 1–25.
[14] Y. Latushkin, A. Sukhtayev, The Evans function and the Weyl–Titchmarsh function, preprint, 2011.
[15] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Springer, 1972.
[16] X.-B. Lin, Exponential dichotomies in intermediate spaces with applications to a diffusively perturbed predator–prey model, J. Differential Equations 108 (1994) 36–63.
[17] X.-B. Lin, Shadowing matching errors for wave-front-like solutions, J. Differential Equations 129 (1996) 403–456.
[18] X.-B. Lin, Algebraic dichotomies with an application to the stability of Riemann solutions of conservation laws, J. Differential Equations 11 (2009) 2924–2965.
[19] K. Palmer, Exponential dichotomies and transversal homoclinic points, J. Differential Equations 55 (1984) 225–256.
[20] Björn Sandstede, Stability of travelling waves, Chapter 18, in: Handbook of Dynamical Systems, vol. 2, 2002, pp. 983–1055.
[21] B. Sandstede, A. Scheel, Absolute and convective instabilities of waves on unbounded and large bounded domains, Phys. D 145 (2000) 233–277.
[22] D.H. Sattinger, On the stability of waves of nonlinear parabolic systems, Adv. Math. 22 (1976) 312–355.
[23] J. Xin, Front propagation in heterogeneous media, SIAM Rev. 42 (2000) 161–230.